CLASSIFICATION OF A WIDE SET OF GEOMETRIC FIGURES

I. TAVKHELIDZE

I. Vekua Institute of Applied Mathematics,
Iv. Javakhishvili Tbilisi State University
University Street 2 Tbilisi 0143, Georgia
e-mail: i.tavkhelidze@math.sci.tsu.ge

ABSTRACT. Aim of this article is the analytical representation of a class of geometric figures, surfaces and lines. This class of surfaces includes the surfaces appearing in some problems of Shell Theory or problems of spreading of smoke-rings; furthermore, the lines of this class can be used for describing the complicated orbit of some celestial objects. In previous articles [1-5] sets of Generalized Möbius Listing's bodies, which are a particular case of this class in static case, have been already defined. In particular cases, this analytic representation gives back many classical objects (torus, helicoid, helix, Möbius strip ... etc.). In present paper was studied some relations between set of GML_2^n (Generalized Möbius -Listing's surfases) and sets of Knots and Links . Also, here was defined classes of DML_2^n (Degenerated Möbius -Listing's surfases)and SML_2^n (Singular Möbius -Listing's surfases).

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Notations
In this article we use following notations:

- X, Y, Z, or x, y, z - is the ordinary notation for coordinates;
- t - time value, $t \in [0, +\infty)$, but in the present article we give parametric representations of a wide set of geometric figures, where every function does not depend on the time argument t. General case can be founded in [4] or [6].
- τ, ψ, θ - are space values (local coordinates or parameters in parallelogram):

$$\begin{array}{ll} 1. & \tau \in [\tau_*, \tau^*], \text{ where } \tau_* \text{ and } \tau^* \text{ are some non-negative constants;} \\ 2. & \psi \in [0, 2\pi]; \\ 3. & \theta \in [0, 2\pi h], \text{ where } h \in \mathbf{Z} \text{ (integer);} \end{array} \tag{1}$$

- $PR_m \equiv A_1A_2 \ldots A_mA'_1A'_2 \ldots A'_m$ denotes an orthogonal prism, whose ends $A_1A_2 \ldots A_m$ and $A'_1A'_2 \ldots A'_m$ are "regular polygons" P_m and m is the number of its angles or vertices. In general case - edges of "regular polygons" are not always straight lines (A_iA_{i+1} may be, for example: edge of epicycloid or edge of hypocycloid);
  - PR_0 - is a segment and P_0 is a point;
  - PR_1 - is an orthogonal cylinder, which cross section is a a P_1 - plane

figure without symmetry ;
- $PR_2 \equiv A_1A_2A_1'A_2'$ is a rectangle, if $P_2 \equiv A_1A_2$ is a segment of straight line;
- PR_∞ - is an orthogonal cylinder, which cross section is a P_∞-circle.

$$x = p(\tau, \psi), \quad z = q(\tau, \psi) \tag{2}$$

or

$$\begin{array}{c} x = p(\tau, \theta)\cos\psi \\ z = p(\tau, \theta)\sin\psi \end{array} \tag{2*}$$

are analytic representations of the "regular polygon" P_m, usually $p(0,0) = q(0,0) = 0$ and the point $(0,0)$ is the center of symmetry of this polygon. In general the "shape" of prism PR_m, may be changeable and his cross section, which is defined by (2*) depends on place of the cross section (θ).

- $D(p,q)$ or $D(p)$ - diameter of P_m, "regular polygon" or cross section.
- OO' - axis of symmetry of the prism PR_m;
- L_ρ - Family of lines situated on the plane, which parametric representation are

$$L_\rho = \begin{cases} X = f_1(\rho, \theta) \\ Y = f_2(\rho, \theta) \end{cases} \qquad \rho \in [0, \rho^*), \quad \theta \in [0, 2\pi h], \quad h \in \mathbf{Z} \tag{3}$$

or

$$L_\rho = \begin{cases} X(\rho, \theta) = \rho_1(\theta)\cos\theta \\ Y(\rho, \theta) = \rho_2(\theta)\sin\theta \end{cases} \tag{3*}$$

We assume the following hypotheses:
i) For any parameters $\rho_1, \rho_2 \in [0, \rho^*]$, $\rho_1 \neq \rho_2$, the lines L_{ρ_1} and L_{ρ_2} have not common points.
ii) If L_ρ is a closed curve, then for every fixed $\rho \in [0, \rho^*]$ f_i - are $2p$-periodic functions $f_i(\rho, \theta + 2\pi) = f_i(\rho, \theta), (i = 1, 2)$.

- $$g(\theta) : [0, 2\pi] \longrightarrow [0, 2\pi] \tag{4}$$

be an arbitrary function and for every $\Theta \in [0, 2\pi]$ there exists $\theta \in [0, 2\pi]$, such that $\Theta = g(\theta)$;

- $mod_m(n)$ - natural number $< m$; for every two numbers $m \in \mathbf{N}$ (natural) and $n \in \mathbf{Z}$ (integer) there exists a unique representation $n = km + j \equiv km + mod_m(n)$, where $k \in \mathbf{Z}$ and $j \equiv mod_m(n) \in \mathbf{N} \cup \{0\}$;

- $$\mu \equiv \begin{cases} \dfrac{n}{m}, & \text{when} \quad m \in \mathbf{N} \quad \text{and} \quad n \in \mathbf{Z} \\ n & \text{when} \quad m = \infty \quad \text{and} \quad n \in \mathbf{Z} \quad (\text{or } n \in \mathbf{R}(\text{Real})) \end{cases} \tag{5}$$

- GML_m^n - **Generalized Möbius Listing's body** - is obtained by identifying of the opposite ends of the prism PR_m in such a way that:

A) For any integer $n \in \mathbf{Z}$ and $i = 1, \ldots, m$ each vertex A_i coincides with $A'_{i+n} \equiv A'_{mod_m(i+n)}$, and each edge A_iA_{i+1} coincides with the edge

$$A'_{i+n}A'_{i+n+1} \equiv A'_{mod_m(i+n)}A'_{mod_m(i+n+1)}$$

correspondingly; **B)** The integer $n \in \mathbf{Z}$ is the number of rotations of the end of the prism with respect to the axis OO' before the identification. If $n > 0$ the rotations are counter-clockwise, and if $n < 0$ then rotations are clockwise. Some particular examples of GML_m^n and its graphical realizations can be founded in [6].

Example 1. If $m = 2$, then:
GML_1^2 becomes a classical (or regular) Möbius band (see e.g. [1-6] and fig. 3 a.));
GML_0^2 are surfaces:

- in regular cases: ring, cylinder, or frustrum of a cone (see also fig.1 b.), fig.2 a.), and c.));
- in degenerated cases: disk, surface of cone, two cone (see also fig.1 a.), fig.2 b.)).

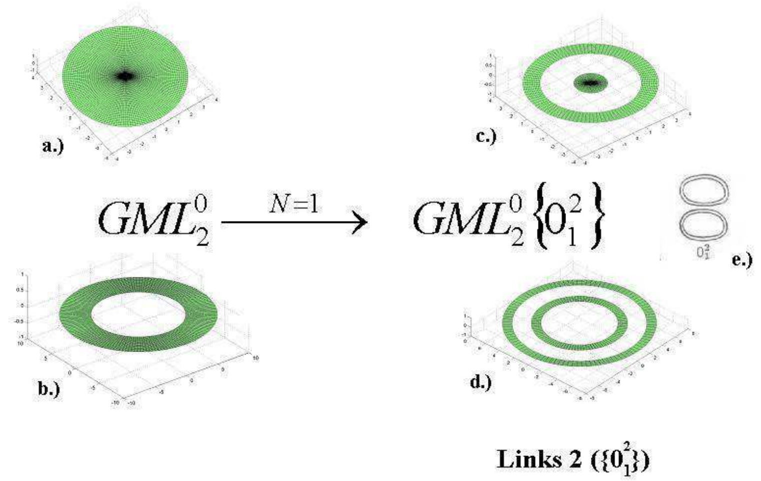


Fig.1.

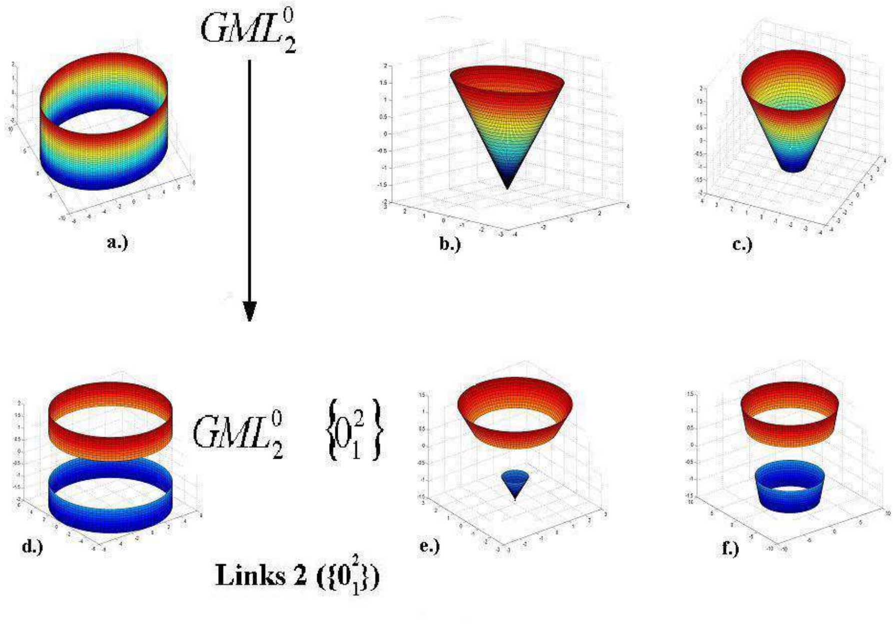


Fig.2.

I. - Generalized Möbius -Listing's surfases

In this part of the article we give parametric representations of a wide set of geometric figures, surfaces and lines under the following restrictions:
1) The OO'-axis of symmetry (middle line) of the prism is transformed into some "basic line" L_ρ;
2) Rotation of the end of the prism is semi-regular along the middle line OO'.
3) Every function does not depend on the time argument t.

Under the above restrictions the analytic representation of this class is given by the following formulas

$$\begin{aligned} X(\tau,\psi,\theta) &= f_1\left(\left[R + p(\tau,\psi)\cos(\mu g(\theta)) - q(\tau,\psi)\sin(\mu g(\theta))\right],\theta\right) \\ Y(\tau,\psi,\theta) &= f_2\left(\left[R + p(\tau,\psi)\cos(\mu g(\theta)) - q(\tau,\psi)\sin(\mu g(\theta))\right],\theta\right) \\ Z(\tau,\psi,\theta) &= K(\theta) + p(\tau,\psi)\sin(\mu g(\theta)) + q(\tau,\psi)\cos(\mu g(\theta))\,, \end{aligned} \tag{6}$$

or

$$\begin{aligned} X(\tau,\psi,\theta) &= [\rho_1(\theta) + p(\tau,\theta)\cos(\psi + \mu g(\theta))]\cos(\theta) \\ Y(\tau,\psi,\theta) &= [\rho_2(\theta) + p(\tau,\theta)\cos(\psi + \mu g(\theta))]\sin(\theta) \\ Z(\tau,\psi,\theta) &= K(\theta) + p(\tau,\theta)\sin(\psi + \mu g(\theta))\,, \end{aligned} \tag{6*}$$

where, respectively
- the arguments (τ,ψ,θ) are defined in (1);
- the functions f_1 and f_2 or $\rho_1(\theta)$ and $\rho_2(\theta)$ are the "shape of plane basic line", defined by (3) or (3*);
- the functions $p(\tau,\psi)$ and $q(\tau,\psi)$ or $p(\tau,\theta)$ denote the "shape of the radial cross section", defined by (2) or (2*);
- the function $g(\theta)$ is the "rule of twisting around basic line", defined by (4);
- $K(\theta)$ is the "Law of vertical stretching of figure" - an arbitrary smooth function;
- μ - is the "number of rotation", defined by (5);
- R is the "radius" - some fixed real number.

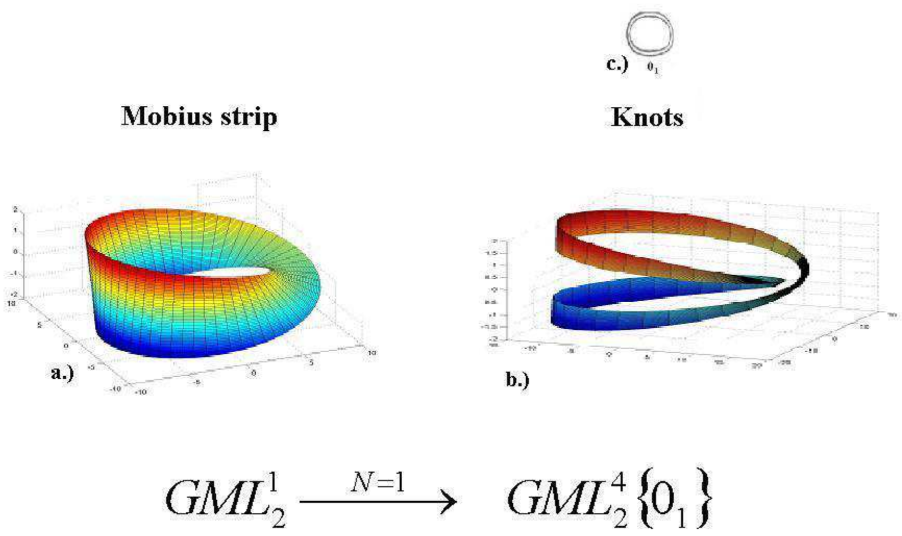
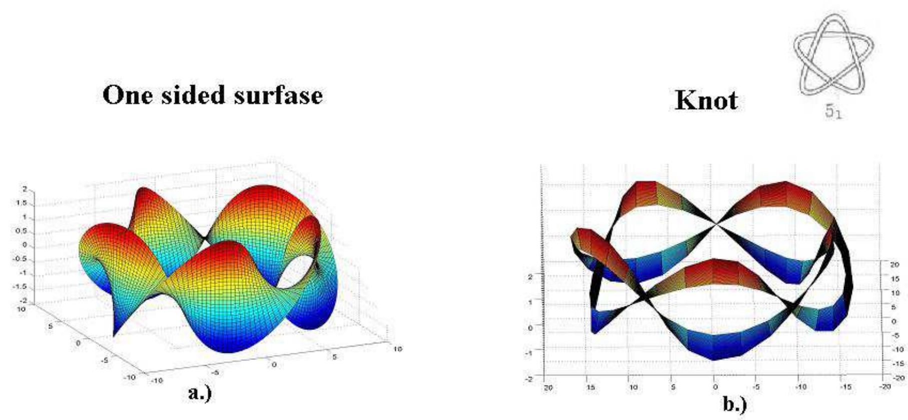


Fig.3.

II. - General remarks Some general remarks about figures, which appear with these analytic representation can be founded in [2-6], but in present article are considered some particular cases:

- "Basic line" L_ρ is a closed line, the "Law of vertical stretching of figure" $K(\theta) \equiv 0$ and $m = 2$ is simultaneously: the number of angles or vertices of P_m "regular polygon" ("shape of the radial cross section") is a straight line and the denominator in (5) defining μ, then equations (6) and (6*) define a set of surfaces, in particular the Generalized Möbius - Listing bodies GML_2^n are included;
- Here in (4) $g(\theta) \equiv \theta$, and restriction **I. - 2)** has the following formulation - Rotation (twisting) of the end of the prism, (before the identification, since the "basic line" L_ρ is a closed line), is called *regular* along the middle line OO'. In this case (6) and (6*) define set of figures with "regular twisting" along the "basic line" L_ρ;
- In particular case one of simplest form of analytic representation (6) or (6*) is a form (9) (see for example [1-3]).
- If in (1) $\tau_* = 0$, then (6) or (6*) or corresponding form (9) give the analytic representation of Generalized Möbius - Listing bodies GML_2^n with "basic line" L_ρ , examples see in fig1, a.), b.), fig.2 a.),b.),c.), fig.3 a.), fig.4 a.) and fig.5 a.);
- If in (1) $\tau_* > 0$, then (6) or (6*) or corresponding form (9) give the analytic representation of a figure, which, in some cases, may be interpreted as a Generalized Möbius - Listing bodies GML_2^n cutting belong the "basic line " L_ρ, with thickness $\tau^* - \tau_*$ examples see in fig1, c.), d.), fig.2 d.),e.),f.), fig.3 b.), fig.4 b.) and fig.5 b.);

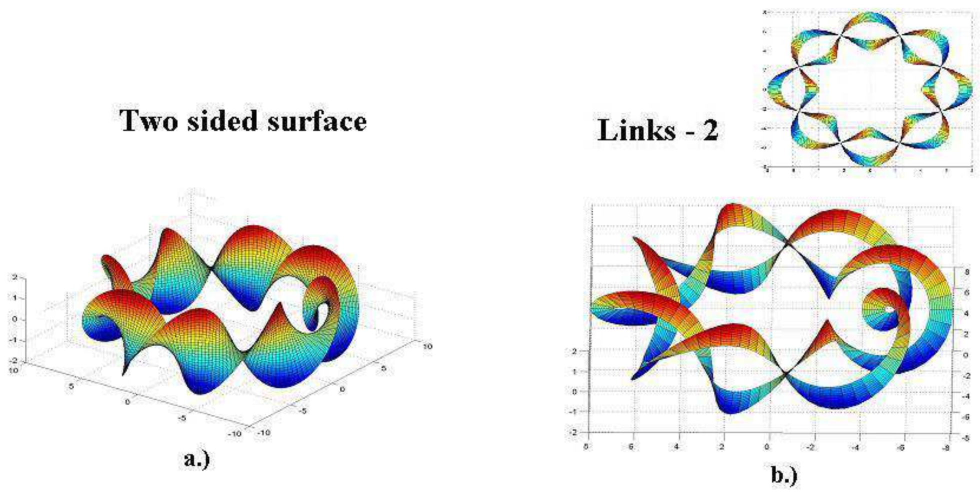


$$GML_2^5 \xrightarrow{N=1} GML_2^{12}\{5_1\}$$

Fig.4.

III. - Relations between Generalized Möbius-Listing's Surfaces and Knots and Links-2 Classifications

It is well known Knots and Links-2 Classifications (see for example [7,8,9]), but these geometric objects are usually lines. In this paper are considered Knots and Links-2 classification to the Generalized Möbius - Listing's surfaces GML_2^n, i.e. the objects of classification are closed surfaces GML_2^n with corresponding number n. Analytic representation (6), (6*) or (9) give us possibility to discover following



$$GML_2^8 \xrightarrow{N=1} GML_2^8\{8_1^2\}$$

Fig.5.

Theorem 1. *Let GML_2^n Generalized Möbius - Listing's bodies ($R > \tau^*$) and if we cut these surfaces belong the "basic line " L_ρ, then*

- *if number n is a even (i.e. $n = 2j$), then after cutting belong the "basic line " L_ρ, appear objects "links 2" and both components are GML_2^n and topological group is a $\{(n)_1^2\}$ (see classification in [7], or [8]), i.e.*

$$GML_2^{2j} \longrightarrow \text{Links} - 2 \text{ of } GML_2^{2j} \quad \text{group } \{(2j)_1^2\}; \tag{7}$$

- *if number n is a odd (i.e. $n = 2j+1$), then after cutting belong the "basic line " L_ρ, appear objects - "Knots" to the GML_2^{2n+2} and topological group is a $\{(n)_1\}$ (see classification in [7], or [9]), i.e*

$$GML_2^{2j+1} \longrightarrow \text{Knots of } GML_2^{4j+4} \quad \text{group } \{(2j+1)_1\}. \tag{8}$$

In the fig. 1-5 the illustration of this theorem are presented in the particular cases.

Remark 1. *Note that the corresponding relation (6), (6*) or (9) is a one to one transformation of stripe $\mathbf{T}^{2\pi}_{-\tau^*,\tau^*} \equiv \{[-\tau^*,\tau^*] \times [0, 2\pi]\}$ to the GML_2^n and also it is a "width-preserving" transformation (details can be founded in [2].)*

Proof: Since the previous remark, cutting the GML_2^n can be interpreted how the corresponding cutting of stripe

$$\mathbf{T}^{2\pi}_{-\tau^*,\tau^*} \equiv \mathbf{T}^{2\pi}_{-\tau_*,\tau_*} \cup \mathbf{T}^{2\pi}_{\tau_*,\tau^*} \cup \mathbf{T}^{2\pi}_{-\tau^*,-\tau_*}.$$

- if $n = 2j$ - is even number, then after transformation (6) of the $\mathbf{T}^{2\pi}_{\tau_*,\tau^*}$ (or correspondingly $\mathbf{T}^{2\pi}_{-\tau^*,-\tau_*}$) is a similar GML^n_2 and we have two object, which are linked as a link -2 of a group $\{(n)^2_1\}$;
- if $n = 2j + 1$ - is odd number, then after transformation (6) of the $\mathbf{T}^{2\pi}_{\tau_*,\tau^*}$ we have non-closed surface. Only if we consider transformation (6) together to $\mathbf{T}^{2\pi}_{\tau_*,\tau^*} \cup \mathbf{T}^{2\pi}_{-\tau^*,-\tau_*}$ appear one new "4π-periodic" geometric object GML^{2n+2}_2 and also this object is a Knot of a group $\{(n)_1\}$;

IV. - Degenerated Möbius-Listing's Surfaces

Generalized Möbius-Listing's Surfaces $GML - 2^n$ are presented with formula (6) or (6*) and if "radial cross section" is a straight line , then in this particular case analytic representation (6) or (6*) have following simple form (here usually $\mu = n/2$)

$$\begin{aligned} X(\tau,\psi,\theta) &= R(\theta)\cos(\theta) + \tau\cos(\psi + \mu g(\theta))\cos(\theta) \\ Y(\tau,\psi,\theta) &= R(\theta)\sin(\theta) + \tau\cos(\psi + \mu g(\theta))\sin(\theta) \\ Z(\tau,\psi,\theta) &= \tau\sin(\psi + \mu g(\theta)), \end{aligned} \tag{9}$$

and transformation (9) is a one to one transformation and one of its physical meaning if following: - ***This is a Trajectory of displacement of segment*** $[-\tau^*, \tau^*]$***, when its origin*** $\tau = 0$ ***always move belong the "basic line"*** L_R ***and this segment make corresponding rotation around "basic line"*** (for example: Trajectory of airline's nose bland when airline fly belong the plane closed line). But if we make in (9) very small changes

$$\begin{aligned} X(\tau,\psi,\theta) &= R(\theta)\cos(\theta) - \tau\cos(\psi + \mu g(\theta))\sin(\theta) \\ Y(\tau,\psi,\theta) &= R(\theta)\sin(\theta) + \tau\cos(\psi + \mu g(\theta))\cos(\theta) \\ Z(\tau,\psi,\theta) &= \tau\sin(\psi + \mu g(\theta)) \end{aligned} \tag{10}$$

then appear new class of geometric objects which we call **" Degenerated Möbius-Listing's Surfaces"**, this transformation not always one to one transformation (i.e. impossible to prepare this objects to the strip). But one of physical meaning of this transformation following: - ***This is a Trajectory of displacement of segment*** $[-\tau^*, \tau^*]$***, when its origin*** $\tau = 0$ ***always move belong the "basic line"*** L_R ***and this segment make corresponding rotation around "basic line" but this segment always is a tangent of orthogonal cylinder with cross section "basic line "*** L_R***.*** (for example: Trajectory of helicopters tail bland when helicopter fly belong the plane closed line).

Remark 2. *Only one case* $n = 0$ *and* $\psi = 0$*, when both surfaces* GML^0_2 *and* DML^0_2 *coincide and they are cylinder.*

Remark 3. *Usually relation (10) is not one to one transformation and if* $n \neq 0$ *it have a singular points (see for example fig. 6 a.) and b.))*

Similarly we may defined **"Singular Möbius-Listing's Surfaces"-** SML^n_2, which one of physical meaning are following: - ***This is a Trajectory of displacement of segment*** $[-\tau^*, \tau^*]$***, when its origin*** $\tau = 0$ ***always move belong the "basic line"*** L_R ***and this segment make corresponding rotation around "basic line" but this segment and "basic line" always are in the one plane*** (for example: Trajectory of helicopters basic bland when helicopter fly belong the plane closed line). Analytic representation of this set have a following form

$$\begin{aligned} X(\tau,\psi,\theta) &= R(\theta)\cos(\theta) + \tau\cos(\psi + \mu g(\theta)) \\ Y(\tau,\psi,\theta) &= R(\theta)\sin(\theta) + \tau\cos(\psi + \mu g(\theta)) \\ Z(\tau,\psi,\theta) &= \text{const.}. \end{aligned} \tag{11}$$

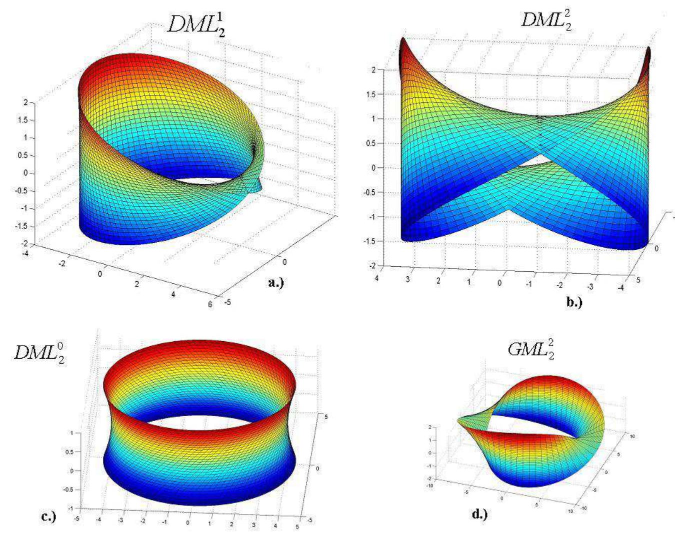


Fig.6.

Remark 4. *Usually relation (11) is not one to one transformation and it have a singular points (some particular cases of these trajectory we can see in fig.7*

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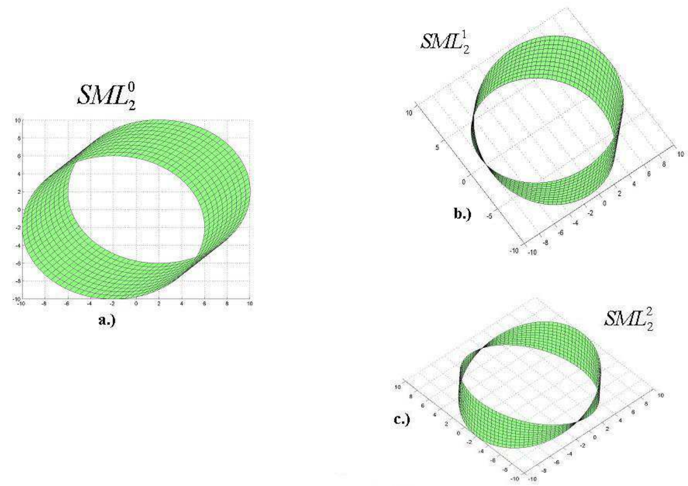

Fig.7.

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