THE ITERATION PROCESS FOR THE NONLINEAR TWO-DIMENSIONAL OSCILLATION PROBLEM

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Abstract. The initial boundary value problem for an integro-differential Kirchhoff equation is considered in the case of a square domain. To find an approximate solution, step-by-step discretization is performed with respect to spatial variables and a time argument. The obtained cubic system is solved by the iteration method. The method error is estimated.

Key words: Kirchhoff equation, Jacobi nonlinear iteration process, error estimate

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Let us consider the following initial boundary value problem

$$w_{tt} - \Big(\lambda + \frac{4}{\pi^2}\int_\Omega (w_x^2 + w_y^2)dx\,dy\Big)(w_{xx} + w_{yy}) = 0, \tag{1}$$

$$(x,y) \in \Omega, \quad 0 < t \le T,$$

$$w(x,y,0) = w^{(0)}(x,y), \quad w_t(x,y,0) = w^{(1)}(x,y), \quad (x,y) \in \Omega, \tag{2}$$

$$w(x,y,t) = 0, \quad (x,y) \in \partial\Omega, \quad 0 \le t \le T, \tag{3}$$

where $\Omega = \{(x,y)\,|\ 0 < x < \pi,\ 0 < y < \pi\}$, $\partial\Omega$ is the boundary of the domain Ω, $w^{(0)}(x,y)$ and $w^{(1)}(x,y)$ are given functions, $\lambda > 0$ and T are given constants.

Equation (1) is a two-dimensional analogue of the well-known Kirchhoff equation [1]

$$w_{tt} - \Big(\lambda + \frac{2}{\pi}\int_0^\pi w_x^2 dx\Big)w_{xx} = 0 \tag{4}$$

for string vibration. The studies of many researchers are devoted to Kirchhoff type equations (for the bibliography see, e.g., [2], [3]).

We present here one numerical method of the solution of problem (1)–(3). An approximate solution will be sought for as a finite sum

$$w_n(x,y,t) = \sum_{i,j=1}^{n} w_{nij}(t)\sin ix\,\sin jy,$$

where the coefficients $w_{nij}(t)$ are calculated from the Galerkin system

$$w''_{nij}(t) + \Big[\lambda + \sum_{p,l=1}^{n}(p^2 + l^2)w_{npl}^2(t)\Big](i^2 + j^2)w_{nij}(t) = 0, \tag{5}$$

$$\begin{gathered} i,j=1,2,\dots,n, \\ w_{nij}(0)=\frac{4}{\pi^2}\int_\Omega w^{(0)}(x,y)\sin ix\,\sin jy\,dx\,dy, \\ w'_{nij}(0)=\frac{4}{\pi^2}\int_\Omega w^{(1)}(x,y)\sin ix\,\sin jy\,dx\,dy,\quad i,j=1,2,\dots,n. \end{gathered} \tag{6}$$

We replace problem (5), (6) by the problem of finding functions $\underline{w}_{nij}(t)$, $i,j=1,2,\dots,n$, where the relation between the functions $w_{nij}(t)$ and $\underline{w}_{nij}(t)$ is such that

$$w_{nij}(t)=\frac{1}{\sqrt{i^2+j^2}}\,\underline{w}_{nij}(t). \tag{7}$$

We have to solve the following problem

$$\underline{w}''_{nij}(t)+\Big[\lambda+\sum_{p,l=1}^{n}\underline{w}^2_{npl}(t)\Big](i^2+j^2)\underline{w}_{nij}(t)=0, \tag{8}$$

$$\begin{gathered} i,j=1,2,\dots,n, \\ \underline{w}_{nij}(0)=\frac{4}{\pi^2}\sqrt{i^2+j^2}\int_\Omega w^{(0)}(x,y)\sin ix\,\sin jy\,dx\,dy, \\ \underline{w}'_{nij}(0)=\frac{4}{\pi^2}\sqrt{i^2+j^2}\int_\Omega w^{(1)}(x,y)\sin ix\,\sin jy\,dx\,dy, \\ i,j=1,2,\dots,n. \end{gathered} \tag{9}$$

To solve the obtained Cauchy problem (8), (9) we will use a difference scheme of symmetrical type. To this end, we introduce the grid $\{t_m\,|\,0=t_0<t_1<\ldots<t_M=T\}$ with a constant step $\tau=t_m-t_{m-1}$. The approximate values $\underline{w}_{nij}(t_m)$ are denoted by $\underline{w}^m_{nij}$, $i,j=1,2,\dots,n$, $m=0,1,\dots,M$.

The scheme has the form

$$\begin{gathered} \frac{\underline{w}^m_{nij}-2\underline{w}^{m-1}_{nij}+\underline{w}^{m-2}_{nij}}{\tau^2}+\frac{1}{2}\sum_{u=-1,0}\Big\{\Big[\lambda+\sum_{p,l=1}^{n}\frac{(\underline{w}^{m+u}_{npl})^2+(\underline{w}^{m+u-1}_{npl})^2}{2}\Big] \\ \times(i^2+j^2)\frac{\underline{w}^{m+u}_{nij}+\underline{w}^{m+u-1}_{nij}}{2}\Big\}=0, \\ i,j=1,2,\dots,n,\quad m=2,3,\dots,M, \end{gathered} \tag{10}$$

$$\begin{aligned} &\underline{w}^0_{nij}=\underline{w}_{nij}(0), \\ &\underline{w}^1_{nij}=\underline{w}_{nij}(0)+\frac{\tau}{2}\,\underline{w}'_{nij}(0) \\ &\quad+\frac{2\tau^2}{\pi^2}\Big[\lambda+\frac{4}{\pi^2}\int_\Omega((w^{(0)}_x)^2+(w^{(0)}_y)^2)dx\,dy\Big]\times \\ &\quad\times\int_\Omega(w^{(0)}_{xx}+w^{(0)}_{yy})\sin ix\,\sin jy\,dx\,dy,\quad i,j=1,2,\dots,n. \end{aligned} \tag{11}$$

From system (10) we write a subsystem for fixed m, $2 \le m \le M$. Then

$$\frac{8}{\tau^2(i^2+j^2)}\,\underline{w}^m_{nij} + \Big\{2\lambda + \sum_{p,l=1}^{n}[(\underline{w}^m_{npl})^2 + (\underline{w}^{m-1}_{npl})^2]\Big\}$$
$$\times(\underline{w}^m_{nij} + \underline{w}^{m-1}_{nij}) = \frac{8}{\tau^2(i^2+j^2)}\,\underline{f}^m_{nij}, \quad i,j = 1,2,\ldots,n, \tag{12}$$

where

$$\underline{f}^m_{nij} = 2\underline{w}^{m-1}_{nij} - \underline{w}^{m-2}_{nij} - \tfrac{\tau^2}{2}\Big(\lambda + \textstyle\sum_{p,l=1}^{n}\frac{(\underline{w}^{m-1}_{npl})^2+(\underline{w}^{m-2}_{npl})^2}{2}\Big)\times$$
$$\times(i^2+j^2)\frac{\underline{w}^{m-1}_{nij}+\underline{w}^{m-2}_{nij}}{2}, \quad i,j=1,2,\ldots,n.$$

Let us agree that for the solution of problem (10), (11) we will use iteration layerwise, more exactly, knowing the approximate solutions $\underline{w}^{m-2}_{nij}$ and $\underline{w}^{m-1}_{nij}$, $i,j=1,2,\ldots,n$, from (12) we find $\underline{w}^m_{nij}$, $i,j=1,2,\ldots,n$, by iteration. As to the initial values, i.e. values at the zero and the first layer, formulas (11) make it possible to find $\underline{w}^0_{nij}$ and $\underline{w}^1_{nij}$, $i,j=1,2,\ldots,n$.

Denote the k-th iteration approximation $\underline{w}^m_{nij}$ by $\underline{w}^m_{nij,k}$, $i,j=1,2,\ldots,n$, $k=0,1,\ldots$.

From our algorithmic approach it follows that (12) is a system of equations with respect to $\underline{w}^m_{nij}$, $i,j=1,2,\ldots,n$. To solve this system we use the nonlinear Jacobi iteration process. From (12) it follows that the process has the form

$$\underline{w}^3_{nij,k+1} + a_{ij}\,\underline{w}^2_{nij,k+1} + b_{ij}\,\underline{w}_{nij,k+1} + c_{ij} = 0, \tag{13}$$

where

$$a_{ij} = \underline{w}^{m-1}_{nij}, \quad b_{ij} = d_{ij} + (\underline{w}^{m-1}_{nij})^2 + \frac{8}{\tau^2(i^2+j^2)},$$
$$c_{ij} = (d_{ij} + (\underline{w}^{m-1}_{nij})^2)\underline{w}^{m-1}_{nij} - \frac{8}{\tau^2(i^2+j^2)}\,\underline{f}^m_{nij}, \tag{14}$$
$$d_{ij} = 2\lambda + \sum_{\substack{p,l=1\\ p\ne i\\ l\ne j}}^{n}[(\underline{w}^m_{npl,k})^2 + (\underline{w}^{m-1}_{npl})^2], \quad k=0,1,\ldots$$

In addition to (14) we need the values

$$r_{ij} = d_{ij} + \frac{2}{3}\,(\underline{w}^{m-1}_{nij})^2 + \frac{8}{\tau^2(i^2+j^2)},$$
$$s_{ij} = \frac{2}{3}\Big(d_{ij} + \frac{10}{9}\,(\underline{w}^{m-1}_{nij})^2\Big)\underline{w}^{m-1}_{nij} - \tag{15}$$
$$-\frac{8}{\tau^2(i^2+j^2)}\Big(\frac{1}{3}\,\underline{w}^{m-1}_{nij} + \underline{f}^m_{nij}\Big), \quad i,j=1,2,\ldots,n.$$

Using the Cardano formulas [4], by (13) we write $\underline{w}^m_{nij,k+1}$ in the explicit form

$$\underline{w}^m_{nij,k+1} = \psi_{ij,k}\,, \tag{16}$$

where

$$\psi_{ij,k} = -\frac{a_{ij}}{3} + \sigma_{ij,1} - \sigma_{ij,2}, \quad i,j = 1,2,\dots,n, \tag{17}$$

$$\sigma_{ij,v} = \Big[(-1)^v \frac{s_{ij}}{2} + \Big(\frac{s^2_{ij}}{4} + \frac{r^3_{ij}}{27}\Big)^{\frac{1}{2}}\Big]^{\frac{1}{3}}, \quad v = 1,2, \quad i,j = 1,2,\dots,n. \tag{18}$$

To establish the convergence conditions for the iteration process used, we consider the Jacobi matrix

$$J = \Big\{\frac{\partial \psi_{ij,k}}{\partial \underline{w}_{ni_1j_1,k}}\Big\}, \quad i,j = 1,2,\dots,n, \quad i_1,j_1 = 1,2,\dots,n. \tag{19}$$

Here ij and i_1j_1 are respectively the numbers of a row and a column of matrix (19).

By (14)–(18) on the principal diagonal of the matrix J we find zeros. As to nondiagonal elements, $i \neq i_1$, $j \neq j_1$, we find

$$\begin{aligned}\frac{\partial \psi_{ij,k}}{\partial \underline{w}^m_{ni_1j_1,k}} = &-\frac{1}{9}\,\underline{w}^m_{ni_1j_1,k} \sum_{v=1}^{2} \frac{1}{\sigma^2_{ij,v}}\Big[2\underline{w}^{m-1}_{nij}\\ &+(-1)^v\Big(\underline{w}^{m-1}_{nij} s_{ij} + \frac{1}{3}\,r^2_{ij}\Big)\Big(\frac{s^2_{ij}}{4} + \frac{r^3_{ij}}{27}\Big)^{-\frac{1}{2}}\Big].\end{aligned} \tag{20}$$

From (18) we obtain the formulas

$$\sigma_{ij,1}\sigma_{ij,2} = \frac{r_{ij}}{3}\,, \quad \sigma^3_{ij,2} - \sigma^3_{ij,1} = s_{ij}, \quad \Big(\frac{s^2_{ij}}{4} + \frac{r^3_{ij}}{27}\Big)^{\frac{1}{2}} = \frac{\sigma^3_{ij,1} + \sigma^3_{ij,2}}{2}\,,$$

which together with (20) give

$$\frac{\partial \psi_{ij,k}}{\partial \underline{w}^m_{ni_1j_1,k}} = \psi^{(1)}_{iji_1j_1} + \psi^{(2)}_{iji_1j_1}\,, \tag{21}$$

where

$$\begin{aligned}\psi^{(v)}_{iji_1j_1} = \quad &\Big(\frac{2}{3}\,\underline{w}^m_{ni_1j_1,k}\Big)(s_{ij})^{v-1}\Big(-\frac{2}{3}\,\underline{w}^{m-1}_{nij}\Big)^{2-v}\\ &\times\Big(\sigma^{2v}_{ij,1} + \Big(-\frac{r_{ij}}{3}\Big)^v + \sigma^{2v}_{ij,2}\Big)^{-1}, \quad v = 1,2.\end{aligned} \tag{22}$$

Let us estimate $|\psi^{(v)}_{iji_1j_1}|$, $v = 1,2$. To this end, we consider the functions

$$\psi^{(u)}(z) = (-r)^u + \sum_{v=1}^{2}\,[z + (-1)^v(z^2 + r^3)^{\frac{1}{2}}]^{\frac{2u}{3}},$$
$$-\infty < z < \infty, \quad r = \text{const} > 0,$$

$u=1,2$. Each function $\psi^{(u)}(z)$, $u=1,2$, is even and increasing for any $z\geq 0$ and therefore

$$\min_{-\infty<z<\infty}|\psi^{(u)}(z)|=\psi^{(u)}(0)=(2u-1)r^u,\quad u=1,2.$$

The latter relation, (14), (15) and (22) imply

$$|\psi^{(u)}_{iji_1j_1}|\leq\frac{[\tau^2(i^2+j^2)]^u}{26u-20}\,|s_{ij}|^{u-1}|\underline{w}^{m-1}_{nij}|^{2-u}|\underline{w}^{m}_{ni_1j_1,k}|,\ \ u=1,2. \tag{23}$$

From formulas (21), (23) and (14), (15) it follows that

$$\Big|\frac{\partial\psi_{ij,k}}{\partial\underline{w}_{ni_1j_1,k}}\Big|\leq\frac{1}{4}\,\tau^2(i^2+j^2)|\underline{w}^m_{ni_1j_1,k}|\Big\{\frac{1}{6}\,\tau^2(i^2+j^2)|\underline{w}^{m-1}_{nij}|\Big[\lambda+ \\ +\sum_{p,l=1}^{n}\Big(\frac{1}{2}(\underline{w}^m_{npl,k})^2+\frac{5}{9}(\underline{w}^{m-1}_{npl})^2\Big)\Big]+|\underline{w}^{m-1}_{nij}|+|\underline{f}^m_{nij}|\Big\}. \tag{24}$$

Let us introduce the vectors

$$\underline{w}^{m-1}_n=(\underline{w}^{m-1}_{nij})^n_{i,j=1},\ \ \underline{w}^m_{n,k}=(\underline{w}^m_{nij,k})^n_{i,j=1},\ \ \underline{f}^m_n=(\underline{f}_{nij})^n_{i,j=1}.$$

Besides, we also need vector and matrix norms. For the vector $\mu=(\mu_s)^N_{s=1}$ and the matrix $G=(g_{rs})^N_{r,s=1}$ we define $\|\mu\|_1=\sum\limits_{s=1}^N|\mu_s|$ and $\|G\|_1=\max\limits_{1\leq s\leq N}\sum\limits_{r=1}^N|g_{rs}|$.

Let us consider the sums $\sum\limits_{i,j=1}^n(i^2+j^2)^u$, $u=1,2$. Taking into account that [5]

$$\sum_{l=1}^n l^{2u}=\frac{n(n+1)(2n+1)}{6}\Big(\frac{3n^2+3n-1}{5}\Big)^{u-1},\ \ u=1,2,$$

we obtain

$$\sum_{i,j=1}^n(i^2+j^2)^u\leq\frac{n^2(n+1)(2n+1)}{3}\Big(\frac{6n^2+6n-2}{5}\Big)^{u-1},\ u=1,2. \tag{25}$$

Note that for $u=1$ in (25) we have the equality.

Let us estimate the norm of the Jacobi matrix (19). By virtue of (24) and (25)

$$\begin{aligned}\|J\|_1\quad&\leq\frac{\tau^2n^2(n+1)(2n+1)}{12}\\ &\times\max_{i_1,j_1}|\underline{w}^m_{ni_1j_1,k}|\Big\{\frac{\tau^2(3n(n+1)-1)}{15}\|\underline{w}^{m-1}_n\|_1\times\\ &\times\Big[\lambda+\sum_{i,j=1}^n\Big(\frac{1}{2}(\underline{w}^m_{nij,k})^2+\frac{5}{9}(\underline{w}^{m-1}_{nij})^2\Big)\Big]+\|\underline{w}^{m-1}_n\|_1+\|\underline{f}^m_n\|_1\Big\},\\ &\qquad i_1,j_1=1,2,\ldots,n.\end{aligned} \tag{26}$$

Applying the principle of compressed mapping [6], we assume that

$$\begin{aligned} &\|J\|_1 \le q, \quad 0<q<1, \\ &\|\underline{w}^m_{n,k} - \underline{w}^m_{n,0}\|_1 \le \frac{1}{1-q}\,\|\underline{w}^m_{n,1} - \underline{w}^m_{n,0}\|_1, \ \ k=1,2,\dots, \end{aligned} \tag{27}$$

is fulfilled. From (26) it follows that for (27) to be hold, it is sufficient that the following biquadratic inequality

$$\alpha\,\tau^4 + \beta\,\tau^2 - \gamma \le 0 \tag{28}$$

be fulfilled with respect to the step τ. Here

$$\begin{aligned} \alpha &= \frac{3n(n+1)-1}{15}\|\underline{w}^{m-1}_n\|_1\Big[\lambda + \frac{1}{2}\Big(\|\underline{w}^m_{n,0}\|_1 + \\ &+ \frac{1}{1-q}\|\underline{w}^m_{n,1} - \underline{w}^m_{n,0}\|_1\Big)^2 + \frac{5}{9}\sum_{i,j=1}^{n}|\underline{w}^{m-1}_{nij}|^2\Big], \\ \beta &= \|\underline{w}^{m-1}_n\|_1 + \|\underline{f}^m_n\|_1, \\ \gamma &= \frac{12q}{n^2(n+1)(2n+1)}\Big(\|\underline{w}^m_{n,0}\|_1 + \frac{1}{1-q}\|\underline{w}^m_{n,1} - \underline{w}^m_{n,0}\|_1\Big)^{-1}. \end{aligned}$$

Thus we come to a conclusion that if the parameter q and the step τ satisfy inequality (28), then system (12) has a unique solution with respect to the unknowns $\underline{w}^m_{nij}$, $i,j=1,2,\dots,n$. The vector $\underline{w}^m_n = (\underline{w}^m_{nij})^n_{i,j=1}$ consisting of the components of solutions $\underline{w}^m_{nij}$ is the limit of a sequence of vectors $\underline{w}^m_{n,k}$, as $k\to\infty$. Moreover, the estimate

$$\|\underline{w}^m_{n,k} - \underline{w}^m_n\|_1 \le \frac{q^k}{1-q}\|\underline{w}^m_{n,1} - \underline{w}^m_{n,0}\|_1, \ \ k=0,1,\dots,$$

is true.

Formula (7) enables us to construct with the aid of $\underline{w}^m_{nij,k}$ approximate solutions of the function $w_{nij}(t)$ at the grid points.

The problem considered here for the one-dimensional equation (4) is studied in [7].

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