THE SOLUTION OF THE FIRST BOUNDARY VALUE PROBLEM OF ELASTOSTATICS FOR THE DOUBLE POROUS PLANE WITH A CIRCULAR HOLE

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Abstract: In the present work we solve explicitly the first boundary value problem of elastostatics for the double porous plane with a circular hole by means of absolutely and uniformly convergent series. For the particular boundary value problem the numerical results are obtained.

Key words: porous media, double porosity, the first boundary value problem, numerical solution

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1. Introduction

Porous materials play an important role in many branches of engineering, e.g., the petroleum industry, chemical engineering, geomechanics. The study of an elastic joint body consisting of porous and permeable beams separated from each other by a system of cracks (such, for example, as bone, granite) is of practical interest. At every point of the medium are introduced two pressures: liquid pressure in pores and that in cracks. For such a body, called a medium with double porosity [1,2], in the Aifantis theory of consolidation the problems of the theory of elasticity are formulated with the following boundary conditions: there are given the values of the displacement (or stress) vector and those of mean (or normal derivative) pressures of liquid in pores. From the point of view of applications, very actual is the construction of solutions explicitly which allows one to perform numerical analysis of the problem under investigation.

In this paper the Aifantis theory of elasticity for isotropic solids with double porosity is considered [1]. The explicit solution of the first boundary value problem of elastostatics for the double porous plane with a circular hole is constructed by means of absolutely and uniformly convergent series and the numerical results are obtained.

2. Basic equations and boundary value problem

We consider the plane D with a circular hole. Let R be the radius of the boundary S of D. The system of equations of the theory of elastostatics for isotropic solids with double porosity can be written as [1]:

$$\begin{gathered} \mu\Delta\, u(x) + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\, u(x) = \mathrm{grad}\,[\beta_1 p_1(x) + \beta_2 p_2(x)], \\ (m_1\Delta - k)\, p_1(x) + k\, p_2(x) = 0,\ \ k\, p_1(x) + (m_2\Delta - k)\, p_2(x) = 0,\ \ x\in D, \end{gathered} \tag{1}$$

where $\lambda, \mu, m_1, m_2, k, \beta_1$ and β_2 are the elastic and physical constants [1, 2]; $u(x) = ((u_1(x), u_2(x))$ is the displacement of the point x; p_1 and p_2 are the fluid pressures in the primary and secondary pores, respectively; Δ is the Laplacian operator.

The first boundary value problem. Find a regular solution $U = (u_1, u_2, p_1, p_2)$ of system (1) satisfying the boundary conditions

$$u(z) = f(z), \quad p_1(z) = f_3(z), \quad p_2(z) = f_4(z), \quad z \in S, \tag{2}$$

where $f = (f_1, f_2)$; f_1, f_2, f_3, f_4 are the given functions on the circumference S, and vector $U(x)$ satisfies the following conditions at infinity:

$$U(x) = o(1), \quad \partial_{x_j} U(x) = O(1), \quad j = 1, 2.$$

3. Explicit solution of the boundary value problem

On the basic of the system (1), we can write

$$p_1(x) = a_1\varphi_1(x) + a_2\varphi_2(x), \qquad p_2(x) = a_3\varphi_1(x) + a_4\varphi_2(x), \tag{3}$$

where $\Delta\varphi_1(x) = 0$, $(\Delta + \lambda_0^2)\varphi_2(x) = 0$, $\lambda_0^2 = -\dfrac{k(m_1 + m_2)}{m_1 m_2}$, $a_1 = a_3 = \dfrac{2}{m_1 + m_2}$, $a_2 = -\dfrac{m_1 - m_2}{m_1(m_1 + m_2)}$, $a_4 = \dfrac{m_1 - m_2}{m_2(m_1 + m_2)}$, $k, m_1, m_2 > 0$, $x \in D$.

The functions φ_1 and φ_2 in formulas (3) are unknown. From the boundary conditions (2) we can write

$$\varphi_1(z) = \Omega_1(z), \quad \varphi_2(z) = \Omega_2(z), \quad z \in S,$$

where $\Omega_1(z) = \dfrac{d_1}{d}$, $\Omega_2(z) = \dfrac{d_2}{d}$, $d = a_1 a_4 - a_2^2$, $d_1 = a_4 f_3 - a_2 f_4$, $d_2 = a_1 f_4 - a_2 f_3$.

For the harmonic function $\varphi_1(x)$ we have:

$$\varphi_1(x) = A_0 + \sum_{m=1}^{\infty} \left(\frac{R}{r}\right)^m [A_m \cos(m\psi) + B_m \sin(m\psi)], \tag{4}$$

where $x = (r, \psi)$, $r^2 = x_1^2 + x_2^2$,

$$\begin{aligned} A_0 &= \frac{1}{2\pi}\int_0^{2\pi} \Omega_1(\theta)d\theta, \quad A_m = \frac{1}{\pi}\int_0^{2\pi} \Omega_1(\theta)\cos(m\theta)d\theta, \\ B_m &= \frac{1}{\pi}\int_0^{2\pi} \Omega_1(\theta)\sin(m\theta)d\theta, \quad m = 1, 2, \cdots. \end{aligned} \tag{5}$$

The metaharmonic function $\varphi_2(x)$ is defined by the series [3]:

$$\varphi_2(x) = K_0(\lambda_0 r)C_0 + \sum_{m=1}^{\infty} K_m(\lambda_0 r)\left[C_m \cos(m\psi) + D_m \sin(m\psi)\right], \qquad (6)$$

where $K_0(\lambda_0 r)$ and $K_m(\lambda_0 r)$ are the Bessel's functions with an imaginary argument; $C_0 = \frac{1}{2\pi}\int_0^{2\pi} \Omega_2(\theta)d\theta$, $C_m = \frac{1}{\pi}\int_0^{2\pi} \Omega_2(\theta)\cos m\theta d\theta$, $D_m = \frac{1}{\pi}\int_0^{2\pi} \Omega_2(\theta) sinm\theta d\theta$, $m = 1, 2, \cdots$.

Thus by means of (4), (6) and (3), the functions φ_1, φ_2 and p_1, p_2 are defined explicitly, respectively.

The solution of the first equation of the system (1) with the boundary conditions (2) is given by the sum

$$u(x) = v_0(x) + v(x), \qquad (7)$$

where v_0 is the particular solution of equation$(1)_1$ and has the following form

$$v_0(x) = \frac{1}{\lambda + 2\mu}\mathrm{grad}\left(-\frac{a}{\lambda_0^2}\varphi_2 + b\,\varphi_0\right); \qquad (8)$$

φ_0 is the biharmonic function, $\Delta\varphi_0 = \varphi_1$,

$$\varphi_0(x) = \frac{R^2}{4}\sum_{m=2}^{\infty}\left\{\frac{1}{1-m}\left(\frac{R}{r}\right)^{m-2}\left[A_m\cos(m\psi) + B_m\sin(m\psi)\right]\right\} + \frac{A_0}{4}r^2, \quad (9)$$

$a = (\beta_1 + \beta_2)a_1$, $b = \beta_1 a_2 + \beta_2 a_4$; A_m and B_m are given by (5). The vector v is the solution of the homogeneous equation which can be found by means of the formula

$$\begin{aligned} v_1(x) &= \partial_{x_1}\left[\Phi_1(x) + \Phi_2(x)\right] - \partial_{x_2}\Phi_3(x),\\ v_2(x) &= \partial_{x_2}\left[\Phi_1(x) + \Phi_2(x)\right] + \partial_{x_1}\Phi_3(x), \end{aligned} \qquad (10)$$

where $\triangle\Phi_1(x) = 0$, $\triangle\triangle\Phi_2(x) = 0$, $\triangle\triangle\Phi_3(x) = 0$,

$$\Phi_1(x) = \sum_{m=0}^{\infty}\left(\frac{R}{r}\right)^m (X_{m1}\cdot\nu_m(\psi)), \quad \Phi_2(x) = \sum_{m=0}^{\infty}\left(\frac{R}{r}\right)^{m-2} R^2(X_{m2}\cdot\nu_m(\psi)),$$

$$\Phi_3(x) = \frac{R^2(\lambda+2\mu)}{\mu}\sum_{m=0}^{\infty}\left(\frac{R}{r}\right)^{m-2}(X_{m2}\cdot s_m(\psi)), \quad \nu_m(\psi) = (\cos m\psi, \sin m\psi),$$

$$s_m(\psi) = (-\sin m\psi, \cos m\psi), \quad X_{01} = \frac{\alpha_0 R}{4}, \quad X_{02} = \frac{\beta_0 R}{4},$$

$$X_{m1} = \frac{R(\alpha_m + \beta_m)}{2m(\lambda + 3\mu)}[2\mu + (\lambda+\mu)m] - \frac{R\alpha_m}{m},$$

$$X_{m2} = \frac{\mu(\alpha_m + \beta_m)}{2(\lambda+3\mu)R}, \quad m = 1, 2, \cdots,$$

α_m and β_m are the Fourier coefficients of the normal and tangential components of the vector-function $\Psi(z) = f(z) - v_0(z)$, $z \in S$, respectively.

For the numerical solution there is the program. The functions $p_1(x)$ and $p_2(x)$ are calculated from (3), (4) and (6); $u_1(x)$ and $u_2(x)$ are calculated from (7), where $v_0(x)$ calculated from (8), (6) and (9), and the vector-function $v(x)$ is calculated from (10).

Let us consider a particular case with the following conditions:

$$R = 2; \quad r = 2.5; \quad \psi = 45^\circ; \quad \lambda = 7.28 \cdot 10^6; \quad \mu = 3.5 \cdot 10^6;$$

$$m_1 = 0.88; \quad m_2 = 0.22; \quad k = 1; \quad \beta_1 = 0.3; \quad \beta_2 = 0.4;$$

$$f_1(\theta) = \frac{R}{2}(\cos\theta - \frac{1}{3}) \cdot 10^{-4}; \quad f_2(\theta) = R(\sin\theta + 3) \cdot 10^{-4};$$

$$f_3(\theta) = \frac{R}{3}(\cos\theta + 2) \cdot 10^{-5}; \quad f_4(\theta) = R(\sin\theta - 1) \cdot 10^{-5}, \quad 0 \le \theta \le 2\pi.$$

We obtain that:

$$u_1 = -1.719 \cdot 10^{-4}; \quad u_2 = -6.862 \cdot 10^{-3}; \quad p_1 = -4.59 \cdot 10^{-3}; \quad p_2 = 0.018.$$

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