ON FINITE AND INFINITE *OT*-SETS

A. Kharazishvili

A. Razmadze Mathematical Institute
University Street, 2, Tbilisi 0186, Georgia
I. Vekua Institute of Applied Mathematics
University Street, 2, Tbilisi 0186, Georgia
e-mail: kharaz2@yahoo.com

Abstract: A set X in the Euclidean space $\mathbf{R}^m$ is said to be an *ot*-set if every three-point subset of X forms an obtuse-angled triangle. Some properties of finite and infinite *ot*-sets are considered. In particular, under the Continuum Hypothesis, it is demonstrated that in the plane $\mathbf{R}^2$ there exists an uncountable set of points in general position, no uncountable subset of which is an *ot*-set.

Key words: *ot*-set, Ramsey's theorem, Sierpiński set, density point

MSC 2010: 03E75, 28A05, 52A10, 52C10

Numerous interesting problems and open questions in the geometry of Euclidean spaces can be indicated, which are of set-theoretical or combinatorial character. As a rule, they are closely connected with point set theory, finite or discrete systems, and the structure of convexity (see [1], [3], [4], [5], [6], [7], [9]). A number of problems of such a kind was raised by P. Erdös in a series of his publications (cf., for example, [3], [4]).

Here we would like to discuss some questions of this type. First of all, let us introduce the notation and several definitions.

Below, the symbol $\mathbf{N}$ denotes the set of all natural numbers. The cardinality of $\mathbf{N}$ is denoted by ω (which is usually identified with $\mathbf{N}$).

$\mathbf{R}$ is the real line and, for any natural number m, the symbol $\mathbf{R}^m$ denotes the m-dimensional Euclidean space (consequently, $\mathbf{R} = \mathbf{R}^1$).

$\mathbf{c}$ is the cardinality of the continuum, i.e., $\mathbf{c} = \mathrm{card}(\mathbf{R}) = 2^\omega$.

ω_1 denotes, as usual, the least uncountable ordinal (cardinal) number.

Let X be a subset of $\mathbf{R}^m$. We shall say that X is an *ot*-set if every three-element subset of X forms an obtuse-angled triangle.

The following two properties are directly implied by the above definition:

(i) any subset of an *ot*-set is also an *ot*-set;

(ii) if $\{X_j : j \in J\}$ is a directed (with respect to the standard inclusion relation) family of *ot*-sets, then $\cup\{X_j : j \in J\}$ is also an *ot*-set.

Example 1. In the space $\mathbf{R}^m$, where $m \geq 2$, consider the curve given by the formula

$$t \to (t, t^2, ..., t^m) \qquad (t \in [0, 1]).$$

It is easy to verify that the range of this curve is an *ot*-set, all whose points are in general position (i.e., no $m + 1$ of them lie in an affine hyperplane of

$\mathbf{R}^m$). The above-mentioned curve plays an important role in the theory of convex polyhedra, because for $m \geq 4$ it provides various examples of so-called Carathéodory-Gale polyhedra (see, e.g., [5]).

We shall say that an *ot*-set $X \subset \mathbf{R}^m$ is maximal if there is no *ot*-set in $\mathbf{R}^m$ properly containing X.

The following statement is a geometric corollary of general set-theoretical concepts.

Theorem 1. *Any ot-set in the Euclidean space* $\mathbf{R}^m$ *is contained in some maximal ot-subset of* $\mathbf{R}^m$.

Indeed, it can readily be seen that the following two relations for a set $Z \subset \mathbf{R}^m$ are equivalent:

(1) Z is an *ot*-set;

(2) every finite subset of Z is an *ot*-set.

In other words, the property of being an *ot*-set is of finite character (see, e.g., [11]). Thus, the assertion of Theorem 1 trivially follows from the Kuratowski-Zorn lemma or, more precisely, from its consequence concerning any property of finite character.

As far as we know, the following problem remains unsolved.

Problem 1. Give a characterization of all maximal *ot*-subsets of $\mathbf{R}^m$.

Here are two simple (and constructive) examples of maximal *ot*-sets in the Euclidean plane $\mathbf{R}^2$.

Example 2. Let X be a half-open unit semi-circumference in the plane $\mathbf{R}^2$, i.e.,

$$X = \{(\cos(\phi), \sin(\phi)) : 0 \leq \phi < \pi\}.$$

Then X is a maximal *ot*-subset of $\mathbf{R}^2$.

Example 3. Let $f : \mathbf{R} \to \mathbf{R}$ be a function, $G(f)$ denote the graph of f and suppose that the following conditions are satisfied:

(a) f is increasing and continuous;

(b) all points of $G(f)$ are in general position in the plane $\mathbf{R}^2$ (i.e., no three points of $G(f)$ are collinear).

Then $G(f)$ is a maximal *ot*-set in $\mathbf{R}^2$.

Less trivial examples of maximal *ot*-subsets of the space $\mathbf{R}^m$, are presented in [8].

It is easy to show that no finite *ot*-subset of $\mathbf{R}^m$ ($m \geq 2$) can be maximal. More precisely, if $X \subset \mathbf{R}^m$ is a finite *ot*-set in $\mathbf{R}^m$ ($m \geq 2$), then there exists a point $x \in \mathbf{R}^m \setminus X$ such that the set $X \cup \{x\}$ is an *ot*-set, too.

On the other hand, the following statement was proved in the same work [8].

Theorem 2. *There exists a countable locally finite maximal ot-subset of the plane* $\mathbf{R}^2$.

Notice that the statement analogous to Theorem 2 can be established for the m-dimensional Euclidean space $\mathbf{R}^m$, where $m > 2$. In other words, we have the following

Theorem 3. *If $m > 2$, then there exists a countable locally finite maximal ot-set in the space $\mathbf{R}^m$.*

The proof of this statement can be carried out in the same manner as for the Euclidean plane $\mathbf{R}^2$ (cf. the argument given in [8]). Some additional purely technical details occur, but they are not connected with substantial difficulties.

Examples 2, 3 and Theorems 2, 3 show that there exist maximal *ot*-sets whose cardinalities are equal to $\mathbf{c}$ and ω respectively. Keeping in mind these examples and theorems, it is natural to formulate the second unsolved problem concerning *ot*-sets in Euclidean space.

Problem 2. Let $m \geq 2$ be a natural number and let κ be a cardinal number from the open interval $]\omega, \mathbf{c}[$. Does there exist a maximal *ot*-set in $\mathbf{R}^m$ whose cardinality is equal to κ?

Obviously, this problem becomes trivial under the Continuum Hypothesis.

The next example is taken from [8] and vividly shows that the maximality of an *ot*-set essentially depends on the dimension of Euclidean space which contains the *ot*-set.

Example 4. Consider the three-dimensional Euclidean space $\mathbf{R}^3$ and its two-dimensional vector subspace $\mathbf{R}^2 \times \{0\}$. Let S be a closed semi-circumference in $\mathbf{R}^2 \times \{0\}$ whose end-points are y and z. Let $l(y, z)$ denote the straight line passing through y and z. Take any point x on $l(y, z)$ not belonging to the line segment $[y, z]$ and put

$$X = (S \setminus \{y\}) \cup \{(x, 1)\}.$$

It is not hard to check that X is an *ot*-set in $\mathbf{R}^3$. Also, as we already know, $S \setminus \{y\}$ is a maximal *ot*-subset of the plane $\mathbf{R}^2 \times \{0\}$ (see Example 2). Since X properly contains $S \setminus \{y\}$, we conclude that $S \setminus \{y\}$ is not a maximal *ot*-set in the space $\mathbf{R}^3$. Moreover, we cannot even assert that X is a maximal *ot*-subset of $\mathbf{R}^3$. For instance, if the semi-circumference S is such that

$$(\forall t \in S)(||t - x|| < 1),$$

then the set

$$X' = (S \setminus \{y\}) \cup \{(x, 1)\} \cup \{(x, -1)\} = X \cup \{(x, -1)\}$$

turns out to be an *ot*-subset of $\mathbf{R}^3$ which properly contains X.

The following question naturally arises: is it true, for a finite set $Z \subset \mathbf{R}^m$ containing sufficiently many points no three of which are collinear, that there exists an *ot*-set $Y \subset Z$ containing the prescribed number of points?

The answer to this question is positive. The proof of this fact is based on widely known Ramsey's theorem (see [13], [11]) and on one interesting result of combinatorial geometry.

Theorem 4. *Let X be a subset of $\mathbf{R}^m$ such that any three points from X form either acute-angled or right-angled triangle. Then* $\mathrm{card}(X) \leq 2^m$.

For the proof of Theorem 4 and further comments on this statement, see, e.g., [1].

In connection with Theorem 4, it is reasonable to consider the next example which shows that, for an infinite-dimensional separable Hilbert space over the field $\mathbf{R}$, the situation is radically different.

Example 5. Let H denote an infinite-dimensional separable Hilbert space (over $\mathbf{R}$). It can be shown that there exists a set $X \subset H$ such that:

(a) the cardinality of X is equal to $\mathbf{c}$;

(b) any three distinct points of X form an acute-angled triangle.

The existence of X follows directly from the well-known result of infinite combinatorics stating that there is an almost disjoint family of infinite subsets of $\mathbf{N}$, whose cardinality is equal to $\mathbf{c}$. Indeed, without loss of generality, we may identify H with the standard Hilbert space

$$\mathbf{l}_2 = \{t \in \mathbf{R}^{\mathbf{N}} : \sum\{(t(n))^2 : n \in \mathbf{N}\} < +\infty\}.$$

Let $\{N_j : j \in J\}$ be a family of infinite subsets of $\mathbf{N}$ such that:

(1) $\mathrm{card}(J) = \mathbf{c}$;

(2) $\mathrm{card}(N_j \cap N_{j'})$ is finite for any two distinct indices $j \in J$ and $j' \in J$.

Now, for each $j \in J$, define the element $x_j \in \mathbf{l}_2$ by the formula

$$x_j(n) = (1/2^n)\chi_j(n) \qquad (n \in \mathbf{N}),$$

where χ_j denotes the characteristic function of the set $N_j \subset \mathbf{N}$.

Putting $X = \{x_j : j \in J\}$, it is easy to check that any three distinct points of X form an acute-angled triangle (cf. [9] where a more complicated argument for establishing the existence of X with the above-mentioned properties (a) and (b) is presented).

Combining Theorem 4 with Ramsey's theorem, it is not difficult to prove the following statement.

Theorem 5. *Let k be an arbitrary natural number. There exists a natural number $p = p(k, m)$ having the following property:*

for any set $X \subset \mathbf{R}^m$ with $\mathrm{card}(X) \geq p$, no three points of which are collinear, there is an ot-set $Y \subset X$ with $\mathrm{card}(Y) = k$.

The infinite (countable) version of Ramsey's theorem yields an analogue of Theorem 5 for infinite (countable) sets.

Theorem 6. *Let X be an arbitrary infinite subset of $\mathbf{R}^m$ no three points of which are collinear. Then there exists an infinite ot-set $Y \subset X$.*

The last statement does not admit a generalization to the case of uncountable sets in $\mathbf{R}^m$. More precisely, if $X \subset \mathbf{R}^m$ is an uncountable set no three points of which are collinear, then we cannot assert (in general) that X contains an uncountable *ot*-subset. In order to show this, we need a certain geometric property of two-dimensional Sierpiński sets.

Recall that in 1924 Sierpiński constructed (by starting with the Continuum Hypothesis and utilizing the method of transfinite recursion) a certain subset of the real line $\mathbf{R}$, which later became a standard tool for producing various counterexamples in general topology, classical measure theory, and real analysis. This set is now called a Sierpiński set (in $\mathbf{R}$). Its definition looks as follows (see, e.g., [12]).

A set $S \subset \mathbf{R}$ is a Sierpiński set if S is uncountable and intersects each Lebesgue measure zero subset of $\mathbf{R}$ in countably many points.

It readily follows from this definition that S is of first category (i.e. small from the Baire category viewpoint) but is extremely bad from the Lebesgue measure standpoint. The latter means that every uncountable subset of S is nonmeasurable in the Lebesgue sense and, consequently, has strictly positive outer Lebesgue measure.

As was mentioned above, Sierpiński sets play an important role in many topics of general topology and real analysis (see, for instance, [10], [12]). Here we wish to give an application of two-dimensional Sierpiński sets to combinatorial geometry of the Euclidean plane.

Let λ (respectively, λ_2) denote the standard one-dimensional (respectively, two-dimensional) Lebesgue measure on $\mathbf{R}$ (respectively, on $\mathbf{R}^2$).

If X is any λ-measurable subset of $\mathbf{R}$, then according to the classical Lebesgue theorem, λ-almost all points of X are its density points (see, e.g., [12]). The analogous statement is true for sets measurable with respect to λ_2.

Now, let L be an arbitrary straight line in $\mathbf{R}^2$ and let $Z \in \mathrm{dom}(\lambda_2)$.

For each point $z \in \mathbf{R}^2$, let us denote by $L(z)$ the straight line in $\mathbf{R}^2$ passing through z and parallel to L.

We shall say that a point $z \in \mathbf{R}^2$ is a linear density point of Z in direction L if this z is a density point of $L(z) \cap Z \subset L(z)$.

The set of all linear density points of Z in direction L will be denoted below by the symbol $D(Z, L)$.

Lemma 1. *For any $Z \in \mathrm{dom}(\lambda_2)$, the set $D(Z, L)$ is λ_2-measurable.*

We omit the standard proof of Lemma 1.

Lemma 2. *For any set $Z \in \mathrm{dom}(\lambda_2)$, we have*

$$\lambda_2(Z \setminus D(Z, L)) = 0,$$

i.e., λ_2-almost all points of Z are its linear density points in direction D.

The proof of Lemma 2 easily follows from the above-mentioned Lebesgue theorem, Lemma 1 and Fubini's theorem.

Lemma 3. *Let L_1 and L_2 be any two straight lines in $\mathbf{R}^2$ and let Z be an arbitrary λ_2-measurable set with $\lambda_2(Z) > 0$. Then there exists a point $z \in Z$ which simultaneously is a linear density point of Z in direction L_1 and a linear density point of Z in direction L_2.*

Clearly, Lemma 3 is a direct consequence of Lemma 2. In its turn, Lemma 3 easily implies the next auxiliary statement of geometric character.

Lemma 4. *Let $\triangle$ be a triangle in $\mathbf{R}^2$, let a and b be the lengths of two sides of $\triangle$, and let γ denote the internal angle of $\triangle$ formed by these sides. Fix a real $\varepsilon > 0$ and take any set $Z \in \mathrm{dom}(\lambda_2)$ with $\lambda_2(Z) > 0$. Then there exist three points x, y and z in Z such that:*

(1) the internal angle in the triangle $[x, y, z]$ formed by the line segments $[x, y]$ and $[x, z]$ is equal to γ;

(2) $|a/b - ||x - y||/||x - z||| < \varepsilon$.

It follows from Lemma 4 that, for any triangle $\triangle$, the set Z contains the vertices of a triangle which is almost similar to $\triangle$. Therefore, taking an arbitrary acute-angled triangle $\triangle$, we obtain that Z contains the vertices of some triangle $\triangle^*$ which is almost similar to $\triangle$, so one may assume that this $\triangle^*$ is acute-angled as well.

The just mentioned geometric fact will be crucial for our further consideration.

A set $S \subset \mathbf{R}^2$ is a Sierpiński set if S is uncountable and intersects each λ_2-measure zero subset of $\mathbf{R}^2$ in countably many points.

Lemma 5. *Assuming the Continuum Hypothesis, there exists a Sierpiński set S in $\mathbf{R}^2$, all points of which are in general position.*

Proof. The argument is fairly standard (cf. [12], [9]). Let $\{B_\xi : \xi < \omega_1\}$ denote the ω_1-sequence of all those Borel subsets of $\mathbf{R}^2$ which have λ_2-measure zero. By using the method of transfinite recursion, we construct a family $\{s_\xi : \xi < \omega_1\}$ of points of $\mathbf{R}^2$.

Suppose that, for an ordinal $\xi < \omega_1$, the partial family $\{s_\zeta : \zeta < \xi\}$ has already been defined and put

$$S(\xi) = \{s_\zeta : \zeta < \xi\}.$$

For any two distinct points x and y from $S(\xi)$, denote by $L(x, y)$ the straight line passing through x and y. Now, consider the set

$$P(\xi) = (\cup\{B_\zeta : \zeta < \xi\}) \cup (\cup\{L(x, y) : x \in S(\xi), y \in S(\xi), x \neq y\}).$$

Evidently, $\lambda_2(P(\xi)) = 0$. Therefore, we may take a point $s \in \mathbf{R}^2 \setminus P(\xi)$ and put $s_\xi = s$.

Proceeding in this manner, we are able to construct the required family $\{s_\xi : \xi < \omega_1\} \subset \mathbf{R}^2$. Finally, denoting

$$S = \{s_\xi : \xi < \omega_1\},$$

we conclude that S is a Sierpiński set in $\mathbf{R}^2$ and all points of S are in general position. This completes the proof of Lemma 5.

Now, we are ready to prove the following statement.

Theorem 7. *Under the Continuum Hypothesis, there exists an uncountable set $S \subset \mathbf{R}^2$ of points in general position, such that no uncountable subset of S is an ot-set.*

Proof. Let S be as in Lemma 5 and let us verify that for this S the assertion of Theorem 7 is valid. Suppose otherwise, i.e., suppose that there exists an uncountable $S' \subset S$ which is an *ot*-set. According to the definition of Sierpiński sets, S' has strictly positive outer λ_2-measure. Consider the closure $\mathrm{cl}(S')$ of S'. Clearly, the set $\mathrm{cl}(S')$ is λ_2-measurable and $\lambda_2(\mathrm{cl}(S')) > 0$.

Now, one of the consequences of Lemma 4 says that $\mathrm{cl}(S')$ contains the vertices of an acute-angled triangle. On the other hand, since all neighbourhoods of any point of $\mathrm{cl}(S')$ contain some points of S', and S' is an *ot*-set by our assumption, it readily follows that any three distinct points of $\mathrm{cl}(S')$ must form either obtuse-angled or right-angled triangle. We thus obtain a contradiction which ends the proof.

Remark 1. Similarly to the proof of Theorem 7, it can be established that, under Martin's Axiom, there exists a set $S \subset \mathbf{R}^2$ with $\mathrm{card}(S) = \mathbf{c}$ such that:

(a) all points of S are in general position;

(b) there is no *ot*-subset of S whose cardinality is equal to $\mathbf{c}$.

Indeed, the role of S can be played by a generalized Sierpiński set in $\mathbf{R}^2$ no three points of which are collinear.

Remark 2. As is known, the dual objects to Sierpiński sets are Luzin sets (see, e.g., [12] where the Sierpiński-Erdös duality principle and Luzin sets are discussed with some applications). For proving Theorem 7, we could start with a Luzin set $Z \subset \mathbf{R}^2$, all points of which are in general position (the existence of such an Z follows from the corresponding analogue of Lemma 5). Arguing in this manner, we get a simpler proof of Theorem 7 which does not need Lemmas 1 - 4. However, Lemmas 1 - 4 are of their own interest and can be applied to other combinatorial and set-theoretical questions of Euclidean geometry.

Remark 3. The well-known Dushnik-Miller theorem [2] from infinite combinatorics states that if G is an uncountable complete graph whose edges are coloured with two colours, say, 0 and 1, then either there exists an uncountable subgraph of G all whose edges are coloured with 0 or there exists an infinite subgraph of G all whose edges are coloured with 1 (see also [9] where much stronger versions of the result are presented). This is one of possible generalizations of Ramsey's theorem to the case of uncountable sets. Theorem 7 shows in an implicit manner that if we replace edges (i.e., two-element subsets) by three-element subsets, then the corresponding combinatorial statement fails to be true in general.

References

[1] V.G. Boltyanskii, I.Ts. Gokhberg, Theorems and Problems from Combinatorial Geometry, Izd. Nauka, Moscow, 1965 (Russian).

[2] B. Dushnik, E.W. Miller, Partially ordered sets, American Journal of Mathematics, 63 (1941), 600-610.

[3] P. Erdös, Problems and results in combinatorial geometry. Discrete geometry and convexity, Ann. New York Acad. Sci., v. 440, 1982 (1985), 1-11.

[4] P. Erdös, G. Szekeres, A combinatorial problem in geometry, Compos. Math., 2 (1935), 463-470.

[5] B. Grünbaum, Convex Polytopes, Wiley Interscience, New York, 1967.

[6] H. Hadwiger, H. Debrunner, V. Klee, Combinatorial geometry in the plane, New York, 1964.

[7] A.B. Kharazishvili, Selected Topics in Geometry of Euclidean spaces, Izd. Tbil. Gos. Univ., Tbilisi, 1978 (Russian).

[8] A.B. Kharazishvili, On maximal *ot*-subsets of the Euclidean plane, Georgian Mathematical Journal, 10, no. 1 (2003), 127 - 131.

[9] P. Komjáth, V. Totik, Problems and Theorems in Classical Set Theory, Springer-Verlag, New York, 2006.

[10] K. Kunen, J.E. Vaughan (editors), Handbook of Set-Theoretic Topology, North-Holland Publ. Co., Amsterdam, 1984.

[11] O. Ore, Theory of Graphs, Amer. Math. Soc., Rhode Island, 1962.

[12] J.C. Oxtoby, Measure and Category, Springer-Verlag, New York, 1971.

[13] F. P. Ramsey, On a problem of formal logic, Proc. London Math. Soc., 30 (1930), 264–286.

Received August,26,2011; Revised December 18, 2011; Accepted December, 20, 2011.