λ-CENTRAL BMO ESTIMATES FOR MULTILINEAR COMMUTATORS OF LITTLEWOOD-PALEY OPERATOR ON CENTRAL MORREY SPACES

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Abstract: In this paper, we establish λ-central BMO estimates for the multilinear commutator related to the Littlewood-Paley operator in central Morrey spaces, and so does the multilinear commutator of the fractional maximal operator.

Key words: λ-central space; Multilinear commutator; Littlewood-Paley operator; Central Morrey spaces.

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1. Introduction

Let $b \in BMO(R^n)$ and T be the Calderón-Zygmund operator, the commutator $[b, T]$ generated by b and T is defined by

$$[b, T](f) = bT(f) - T(bf).$$

A classical result of Coifman, Rochberb and Weiss (see [2]) proved that the commutator $[b, T]$ is bounded on $L^p(R^n)$, $(1 < p < \infty)$. Since $BMO \subset \bigcap_{q>1} CBMO^q$ (see [3]), if we only assume $b \in CBMO^q$, or more generally $b \in CBMO^{q,\lambda}$ with $q > 1$, then $[b, T]$ may not be a bounded operator on $L^p(R^n)$. However, it has some boundedness properties on other spaces. As a matter of fact, Grafakos, Li and Yang (see [4]) considered the commutator with $b \in CBMO^q$ on Herz spaces for the first time. Alvarez, Guzmán-Partida and Lakey (see [1]) and Komori (see [6]) have obtained the λ-central BMO estimates for the commutators of a class of singular integral operators on central Morrey spaces. Motivated by these results, in this paper, we will establish λ-central BMO estimates for the multilinear commutator related to the Littlewood-Paley operator in central Morrey spaces. And the multilinear commutator of the fractional maximal operator is also discussed.

2. Preliminaries and Theorem

First, let us introduce some notations. Let M_δ be the fractional maximal operator, which is

$$M_\delta(f)(x) = \sup_{B \ni x} |B|^{\delta/n-1} \int_B |f(y)|dy, \quad 0 < \delta/n < 1, \tag{1}$$

and $M_\delta^{\vec{b}}$ be the multilinear commutator of the fractional maximal operator which is defined as follows:

$$M_\delta^{\vec{b}}(f)(x) = \sup_{x\in B} |B|^{\delta/n-1} \int_B |\prod_{j=1}^m (b_j(x) - b_j(y))f(y)|dy. \qquad (2)$$

For $b_j \in CBMO^{p_{j+1},\lambda_{j+1}}(R^n)(j = 1, \cdots, m)$, set

$$||\vec{b}||_{CBMO^{\vec{p},\vec{\lambda}}} = \prod_{j=1}^m ||b_j||_{CBMO^{p_{j+1},\lambda_{j+1}}}.$$

Given a positive integer m and $1 \le j \le m$, we denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1), \cdots, \sigma(j)\}$ of $\{1, \cdots, m\}$ of j different elements. For $\sigma \in C_j^m$, set $\sigma^c = \{1, \cdots, m\} \setminus \sigma$. For $\vec{b} = (b_1, \cdots, b_m)$ and $\sigma = \{\sigma(1), \cdots, \sigma(j)\} \in C_j^m$, set $\vec{b}_\sigma = (b_{\sigma(1)}, \cdots, b_{\sigma(j)})$, $b_\sigma = b_{\sigma(1)} \cdots b_{\sigma(j)}$ and $||\vec{b}_\sigma||_{CBMO^{\vec{p},\vec{\lambda}}} = ||b_{\sigma(1)}||_{CBMO^{p_2,\lambda_2}} \cdots ||b_{\sigma(j)}||_{CBMO^{p_{j+1},\lambda_{j+1}}}$.

Definition 1. Let $0 < \lambda < \delta/n$, $0 < \delta < n$ and $1 < q < \infty$. A function $f \in L^q_{loc}(R^n)$ is said to belong to the λ-central bounded mean oscillation space $CBMO^{q,\lambda}(R^n)$ if

$$||f||_{CBMO^{q,\lambda}} = \sup_{r>0} \left(\frac{1}{|B(0,r)|^{1+\lambda q}} \int_{B(0,r)} |f(x) - f_{B(0,r)}|^q dx \right)^{1/q} < \infty. \qquad (3)$$

where, $B = B(0, r) = \{x \in R^n : |x| < r\}$ and $f_{B(0,r)}$ is the mean value of f on $B(0, r)$.

Remark 1. If two functions which differ by a constant are regarded as a function in the space $CBMO^{q,\lambda}$ becomes a Banach space. The space $CBMO^{q,\lambda}(R^n)$ when $\lambda = 0$ is just the space $CBMO(R^n)$ defined as follows:

$$||f||_{CBMO_q} = \sup_{r>0} \left(\frac{1}{|B(0,r)|} \int_{B(0,r)} |f(x) - f_{B(0,r)}|^q dx \right)^{1/q} < \infty.$$

Apparently, (3) is equivalent to the following condition (see [6]):

$$||f||_{CBMO^{q,\lambda}} = \sup_{r>0} \inf_{c\in \mathbf{C}} \left(\frac{1}{|B(0,r)|^{1+\lambda q}} \int_{B(0,r)} |f(x) - c|^q dx \right)^{1/q} < \infty.$$

Definition 2. Let $\lambda \in \mathbf{R}$ and $1 < q < \infty$. The central Morrey space $\dot{B}^{q,\lambda}(R^n)$ is defined by

$$||f||_{\dot{B}^{q,\lambda}} = \sup_{r>0} \left(\frac{1}{|B(0,r)|^{1+\lambda q}} \int_{B(0,r)} |f(x)|^q dx \right)^{1/q} < \infty. \qquad (4)$$

Remark 2. It follows from (3) and (4) that $\dot{B}^{q,\lambda}(R^n)$ is a Banach space continuously included in $CBMO^{q,\lambda}(R^n)$. We denote by $CMO^{q,\lambda}(R^n)$ and

$B^{q,\lambda}(R^n)$ the inhomogeneous versions of the λ-central bounded mean oscillation space and the central Morrey space by taking the supremum over $r \geq 1$ in Definition 1 and Definition 2 instead of $r > 0$ there. Obviously, $CBMO^{q,\lambda}(R^n) \subset CMO^{q,\lambda}(R^n)$ for $\lambda < \delta/n$ and $1 < q < \infty$, and $\dot{B}^{q,\lambda}(R^n) \subset B^{q,\lambda}(R^n)$ for $\lambda \in \mathbf{R}$ and $1 < q < \infty$.

Remark 3. When $\lambda_1 < \lambda_2$, it follows from the property of monotone functions that $B^{q,\lambda_1}(R^n) \subset B^{q,\lambda_2}(R^n)$ and $CMO^{q,\lambda_1}(R^n) \subset CMO^{q,\lambda_2}(R^n)$ for $1 < q < \infty$. If $1 < q_1 < q_2 < \infty$, then by Hölder's inequality, we know that $\dot{B}^{q_2,\lambda}(R^n) \subset \dot{B}^{q_1,\lambda}(R^n)$ for $\lambda \in \mathbf{R}$ and $CBMO^{q_2,\lambda} \subset CBMO^{q_1,\lambda}$, $CMO^{q_2,\lambda}(R^n) \subset CMO^{q_1,\lambda}(R^n)$ for $0 < \lambda < \delta/n$.

Definition 3. Let $1 \leq q < \infty$, $\alpha \in R$. The central Campanato space is defined by(see [17])

$$CL_{\alpha,\ q}(R^n) = \{f \in L^q_{loc}(R^n) : ||f||_{CL_{\alpha,\ q}} < \infty\},$$

where

$$||f||_{CL_{\alpha,\ q}} = \sup_{r>0} |B(0,r)|^{-\alpha} \left(\frac{1}{|B(0,r)|} \int_{B(0,r)} |f(x) - f_{B(0,r)}|^q dx \right)^{1/q}.$$

Definition 4. Fix $\delta > 0$. Let ψ be a fixed function which satisfies the following properties:

(1) $\int_{R^n} \psi(x)dx = 0$;

(2)$|\psi(x)| \leq C(1 + |x|)^{-(n+1-\delta)}$;

(3)$|\psi(x + y) - \psi(x)| \leq C|y|^{\epsilon}(1 + |x|)^{-(n+\epsilon-\delta)}$ when $2|y| < |x|$.

We denote that $\Gamma(x) = \{(y,t) \in R^{n+1}_+ : |x - y| < t\}$ and the characteristic function of $\Gamma(x)$ by $\chi_{\Gamma(x)}$. The Littlewood-Paley multilinear commutator is defined by

$$S^{\vec{b}}_{\delta}(f)(x) = \left[\int\int_{\Gamma(x)} |F^{\vec{b}}_t(f)(x,y)|^2 \frac{dydt}{t^{n+1}} \right]^{1/2}, \tag{5}$$

where

$$F^{\vec{b}}_t(f)(x,y) = \int_{R^n} \prod_{j=1}^{m} (b_j(x) - b_j(z))\psi_t(y - z)f(z)dz, \tag{6}$$

and $\psi_t(x) = t^{-n+\delta}\psi(x/t)$ for $t > 0$. We also define that

$$S_{\delta}(f)(x) = \left(\int\int_{\Gamma(x)} |f * \psi_t(y)|^2 \frac{dydt}{t^{n+1}} \right)^{1/2}, \tag{7}$$

which is the Littlewood-Paley operator(see [16]).

Let H be the space $H = h : ||h|| = (\int\int_{R^{n+1}_+} |h(y,t)|^2 \frac{dydt}{t^(n+1)})^{1/2} < \infty$, then, for each fixed $x \in R_n$, $F^{\vec{b}}_t(f)(x,y)$ may be viewed as a mapping from $[0, +\infty)$ to H, and it is clear that

$$S_{\delta}(f)(x) = ||\chi_{\Gamma(x)} F_t(f)(x)||$$

and

$$S_\delta^{\vec{b}}(f)(x) = ||\chi_{\Gamma(x)} F_t^{\vec{b}}(f)(x,y)||.$$

Note that when $b_1 = \cdots = b_m$, $S_\delta^{\vec{b}}$ is just the commutator of order m. It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors(see [5][7-10][12-15]). Our main purpose is to study the boundedness properties for the multilinear commutator on central Morrey spaces.

Now we state our theorems as following.

Theorem 1. Let $0 < \delta < n$, $1 < p < n/\delta$, $1/q = 1/p - \delta/n$. If $\lambda_1 < -\delta/n$ and $\lambda_2 = \lambda_1 + \delta/n$, then S_δ is bounded from $\dot{B}^{p,\lambda_1}(R^n)$ to $\dot{B}^{q,\lambda_2}(R^n)$.

Theorem 2. Let $0 < \delta < n$, $1 < p < n/\delta$, $1/q = 1/p - \delta/n$. If $\lambda_1 < -\delta/n$ and $\lambda_2 = \lambda_1 + \delta/n$, then M_δ is bounded from $\dot{B}^{p,\lambda_1}(R^n)$ to $\dot{B}^{q,\lambda_2}(R^n)$.

Theorem 3. Let $0 < \delta < n$, $1 < p_u < n/\delta (1 \le u \le m+1)$, $1/p_1 + 1/p_2 + \cdots 1/p_{m+1} < 1$ and $1/q = 1/p_1 + 1/p_2 + \cdots + 1/p_{m+1} - \delta/n$. Suppose $0 < \lambda_i < \delta/n (i = 2, 3, \cdots, m+1)$, $\lambda_1 < -\lambda_2 - \lambda_3 - \cdots - \lambda_{m+1} - \delta/n$ and $\lambda = \lambda_1 + \lambda_2 + \cdots \lambda_{m+1} + \delta/n$. If $b_j \in CBMO^{p_{j+1}, \lambda_{j+1}}(R^n)$, for $j = 1, \cdots, m$, then $S_\delta^{\vec{b}}$ is bounded from $\dot{B}^{p_1,\lambda_1}(R^n)$ to $\dot{B}^{q,\lambda}(R^n)$, and the following inequality holds:

$$||S_\delta^{\vec{b}}(f)||_{\dot{B}^{q,\lambda}} \le C||\vec{b}||_{CBMO^{\vec{p},\vec{\lambda}}} ||f||_{\dot{B}^{p_1,\lambda_1}}.$$

Theorem 4. Let $0 < \delta < n$, $1 < p_u < n/\delta (1 \le u \le m+1)$, $1/p_1 + 1/p_2 + \cdots 1/p_{m+1} < 1$ and $1/q = 1/p_1 + 1/p_2 + \cdots + 1/p_{m+1} - \delta/n$. Suppose $0 < \lambda_i < \delta/n (i = 2, 3, \cdots, m+1)$, $\lambda_1 < -\lambda_2 - \lambda_3 - \cdots - \lambda_{m+1} - \delta/n$ and $\lambda = \lambda_1 + \lambda_2 + \cdots \lambda_{m+1} + \delta/n$. If $b_j \in CBMO^{p_{j+1}, \lambda_{j+1}}(R^n)$, for $j = 1, \cdots, m$, then $M_\delta^{\vec{b}}$ is bounded from $\dot{B}^{p_1,\lambda_1}(R^n)$ to $\dot{B}^{q,\lambda}(R^n)$, and the following inequality holds:

$$||M_\delta^{\vec{b}}(f)||_{\dot{B}^{q,\lambda}} \le C||\vec{b}||_{CBMO^{\vec{p},\vec{\lambda}}} ||f||_{\dot{B}^{p_1,\lambda_1}}.$$

3. Proof of Theorems.

To prove the theorems, we need the following lemmas.

Lemma 1.(see [16]) Let $0 < \delta < n$, $1 < p < n/\delta$ and $1/q = 1/p - \delta/n$. Then S_δ is bounded from $L^p(R^n)$ to $L^q(R^n)$.

Lemma 2.(see [11]) Let $0 < \delta < n$, $1 < p < n/\delta$ and $1/q = 1/p - \delta/n$. Then M_δ is bounded from $L^p(R^n)$ to $L^q(R^n)$.

Lemma 3. Let $0 < \delta < n$, $1 < p < n/\delta$, $\lambda > 0$. Suppose $b \in CBMO^{p,\lambda}(R^n)$, then

$$|b_{2^{k+1}B} - b_B| \le C||b||_{CBMO^{p,\lambda}} k|2^{k+1}B|^\lambda \text{ for } k \ge 1 .$$

Proof.

$$|b_{2^{k+1}B} - b_B| \le \sum_{j=0}^{k} |b_{2^{j+1}B} - b_{2^jB}| \le \sum_{j=0}^{k} \frac{1}{|2^jB|} \int_{2^jB} |b(y) - b_{2^{j+1}B}| dy$$

$$\leq C\sum_{j=0}^{k}\left(\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|b(y)-b_{2^{j+1}B}|^p dy\right)^{1/p} \leq C||b||_{CBMO^{p,\lambda}}\sum_{j=0}^{k}|2^{j+1}B|^{\lambda}$$
$$\leq C||b||_{CBMO^{p,\lambda}}(k+1)|2^{k+1}B|^{\lambda} \leq C||b||_{CBMO^{p,\lambda}}k|2^{k+1}B|^{\lambda}.$$

Proof of Theorem 1. Let f be a function in $\dot{B}^{p,\lambda_1}(R^n)$. For fixed $r>0$, set $B=B(0,r)$. We consider

$$\left(\frac{1}{|B|^{1+\lambda_2 q}}\int_B |S_\delta(f)(x)|^q dx\right)^{1/q}$$
$$\leq \left(\frac{1}{|B|^{1+\lambda_2 q}}\int_B |S_\delta(f\chi_B)(x)|^q dx\right)^{1/q} + \left(\frac{1}{|B|^{1+\lambda_2 q}}\int_B |S_\delta(f\chi_{B^c})(x)|^q dx\right)^{1/q}$$
$$= I + II.$$

For I, by the boundedness of S_δ from $L^p(R^n)$ to $L^q(R^n)$, we have

$$\begin{aligned} I &\leq C|B|^{-1/q-\lambda_2}\left(\int_B |f(x)|^p dx\right)^{1/p} \\ &\leq C|B|^{-1/q-\lambda_2}|B|^{1/p+\lambda_1}||f||_{\dot{B}^{p,\lambda_1}} \\ &\leq C||f||_{\dot{B}^{p,\lambda_1}}. \end{aligned}$$

For II, using Minkowski's inequality, we have

$$\begin{aligned} S_\delta(f\chi_{B^c})(x) &\leq \int_{B^c} f(z)\left(\int\int_{\Gamma(x)}\frac{t^{1-n}}{(t+|x-z|)^{2(n+1-\delta)}}dydt\right)^{1/2}dz \\ &\leq \int_{B^c} f(z)\left(\int_0^\infty \frac{tdt}{(t+|x-z|)^{2(n+1-\delta)}}\right)^{1/2}dz \\ &\leq \int_{B^c} f(z)|x-z|^{-n+\delta}dz, \end{aligned}$$

thus, using Hölder's inequality and note that $x\in B$, we get

$$\begin{aligned} |S_\delta(f\chi_{B^c})(x)| &\leq \sum_{k=1}^{\infty}\int_{2^{k+1}B\setminus 2^k B}|x-z|^{-n+\delta}|f(z)|dz \\ &\leq C\sum_{k=0}^{\infty}|2^k B|^{\delta/n-1}\left(\int_{2^{k+1}B}|f(z)|^p dz\right)^{1/p}|2^{k+1}B|^{1-1/p} \\ &\leq C\sum_{k=0}^{\infty}|2^k B|^{\delta/n-1}|2^{k+1}B|^{1/p+\lambda_1}||f||_{\dot{B}^{p,\lambda_1}}|2^{k+1}B|^{1-1/p} \\ &\leq C||f||_{\dot{B}^{p,\lambda_1}}|B|^{\delta/n+\lambda_1}, \end{aligned}$$

therefore, we deduce

$$II \leq C||f||_{\dot{B}^{p,\lambda_1}}|B|^{\delta/n+\lambda_1}|B|^{-1/q-\lambda_2}|B|^{1/q} \leq C||f||_{\dot{B}^{p,\lambda_1}}.$$

This completes the proof of Theorem 1.

Proof of Theorem 2. Set

$$\widetilde{S}_\delta(f)(x)=\int_{R^n}|x-z|^{-n+\delta}|f(z)|dz,$$

it is easy to know the Theorem 1 is also true for the operator $\widetilde{S}_\delta(f)(x)$. Since

$$M_\delta(f)(x)\le \widetilde{S}_\delta(f)(x),$$

thus, Theorem 2 can be easily deduced. We omit the details here.

Proof of Theorem 3. Let f be a function in $\dot{B}^{p_1,\lambda_1}(R^n)$. When $m=1$, set $(b_1)_B=|B|^{-1}\int_B b_1(x)dx$ and note that

$$S_\delta^{b_1}(f)(x)=(b_1(x)-(b_1)_B)S_\delta(f)(x)-S_\delta((b_1-(b_1)_B)f)(x).$$

We have

$$\begin{aligned}
&\left(\frac{1}{|B|^{1+\lambda q}}\int_B|S_\delta^{b_1}(f)(x)|^q dx\right)^{1/q}\\
\le\ &\left(\frac{1}{|B|^{1+\lambda q}}\int_B|(b_1(x)-(b_1)_B)(S_\delta(f\chi_B))(x)|^q dx\right)^{1/q}\\
&+\left(\frac{1}{|B|^{1+\lambda q}}\int_B|(b_1(x)-(b_1)_B)(S_\delta(f\chi_{(B)^c}))(x)|^q dx\right)^{1/q}\\
&+\left(\frac{1}{|B|^{1+\lambda q}}\int_B|S_\delta((b_1-(b_1)_B)f\chi_B)(x)|^q dx\right)^{1/q}\\
&+\left(\frac{1}{|B|^{1+\lambda q}}\int_B|S_\delta((b_1-(b_1)_B)f\chi_{(B)^c})(x)|^q dx\right)^{1/q}\\
=\ &J_1+J_2+J_3+J_4.
\end{aligned}$$

For J_1, taking $1<p_1<n/\delta$ and t such that $1/t=1/p_1-\delta/n$, choosing $1/q=1/p_2+1/t$, by Hölder's inequality and the boundedness of S_δ from $L^{p_1}(R^n)$ to $L^t(R^n)$, we know

$$\begin{aligned}
J_1\ &\le\ |B|^{-1/q-\lambda}\left(\int_B|b_1(x)-(b_1)_B|^{p_2}dx\right)^{1/p_2}\left(\int_B|S_\delta(f\chi_B)(x)|^t\right)^{1/t}\\
&\le\ C|B|^{-1/q-\lambda}|B|^{1/p_2+\lambda_2}||b_1||_{CBMO^{p_2,\lambda_2}}\left(\int_B|f(x)|^{p_1}dx\right)^{1/p_1}\\
&\le\ C|B|^{-1/q-\lambda}|B|^{1/p_2+\lambda_2}||b_1||_{CBMO^{p_2,\lambda_2}}|B|^{1/p_1+\lambda_1}||f||_{\dot{B}^{p_1,\lambda_1}}\\
&\le\ C||b_1||_{CBMO^{p_2,\lambda_2}}||f||_{\dot{B}^{p_1,\lambda_1}}.
\end{aligned}$$

For J_2, using the fact $|S_\delta(f\chi_{B^c})(x)|\le C||f||_{\dot{B}^{p,\lambda_1}}|B|^{\delta+\lambda_1}$ from the proof of Theorem 1 and by Hölder' inequality, we get

$$J_2\ \le\ C|B|^{-1/q-\lambda}|B|^{\delta/n+\lambda_1}||f||_{\dot{B}^{p,\lambda_1}}\left(\int_B|b_1(x)-(b_1)_B|^{p_2}dx\right)^{1/p_2}|B|^{1/q-1/p_2}$$

$$
\begin{aligned}
&\le C|B|^{-1/q-\lambda}|B|^{\delta/n+\lambda_1}||f||_{\dot{B}^{p,\lambda_1}}|B|^{1/p_2+\lambda_2}||b_1||_{CBMO^{p_2,\lambda_2}}|B|^{1/q-1/p_2}\\
&\le C||b_1||_{CBMO^{p_2,\lambda_2}}||f||_{\dot{B}^{p_1,\lambda_1}}.
\end{aligned}
$$

For J_3, taking $1 < l < n/\delta$ and q such that $1/q = 1/l - \delta/n$, choosing $1/l = 1/p_1 + 1/p_2$, by the boundedness of S_δ from L^l to L^q and Hölder's inequality, we have

$$
\begin{aligned}
J_3 &\le C|B|^{-1/q-\lambda}\left(\int_B |(b_1(x)-(b_1)_B f(x)|^l dx\right)^{1/l}\\
&\le C|B|^{-1/q-\lambda}\left(\int_B |b_1(x)-(b_1)_B|^{p_2} dx\right)^{1/p_2}\left(\int_B |f(x)|^{p_1} dx\right)^{1/p_1}\\
&\le C|B|^{-1/q-\lambda}|B|^{1/p_2+\lambda_2}||b_1||_{CBMO^{p_2,\lambda_2}}||f||_{\dot{B}^{p_1,\lambda_1}}|B|^{1/p_1+\lambda_1}\\
&\le C||b_1||_{CBMO^{p_2,\lambda_2}}||f||_{\dot{B}^{p_1,\lambda_1}}.
\end{aligned}
$$

For J_4, note that $x \in B$, using Hölder's inequality, Lemma 3, noticing that $\lambda_2 > 0$ and $\lambda_1 < -\lambda_2 - \delta/n$, we have

$$
\begin{aligned}
&|S_\delta((b_1-(b_1)_B)f\chi_{(B)^c})(x)|\\
\le\ & C\sum_{k=0}^{\infty}|2^kB|^{\delta/n-1}\left(\int_{2^{k+1}B}|b_1(z)-(b_1)_B|^{p_2}dz\right)^{1/p_2}\\
\times\ & \left(\int_{2^{k+1}B}|f(z)|^{p_1}dz\right)^{1/p_1}|2^{k+1}B|^{1-1/p_1-1/p_2}\\
\le\ & C||f||_{\dot{B}^{p_1,\lambda_1}}\sum_{k=0}^{\infty}|2^kB|^{\delta/n-1}|2^{k+1}B|^{1/p_1+\lambda_1}|2^{k+1}B|^{1-1/p_1-1/p_2}\\
&\times\Big[\Big(\int_{2^{k+1}B}|b_1(z)-(b_1)_{2^{k+1}B}|^{p_2}dz\Big)^{1/p_2}+|(b_1)_{2^{k+1}B}-(b_1)_B||2^{k+1}B|^{1/p_2}\Big]\\
\le\ & C||b_1||_{CBMO^{p_2,\lambda_2}}||f||_{\dot{B}^{p_1,\lambda_1}}\sum_{k=0}^{\infty}k2^{kn(\lambda_1+\lambda_2+\delta/n)}|B|^{\lambda_1+\lambda_2+\delta/n}\\
\le\ & C||b_1||_{CBMO^{p_2,\lambda_2}}||f||_{\dot{B}^{p_1,\lambda_1}}|B|^{\lambda_1+\lambda_2+\delta/n},
\end{aligned}
$$

thus, we get

$$
\begin{aligned}
J_4 &\le C||b_1||_{CBMO^{p_2,\lambda_2}}||f||_{\dot{B}^{p_1,\lambda_1}}|B|^{\lambda_1+\lambda_2+\delta/n}|B|^{-1/q-\lambda}|B|^{1/q}\\
&\le C||b_1||_{CBMO^{p_2,\lambda_2}}||f||_{\dot{B}^{p_1,\lambda_1}}.
\end{aligned}
$$

This completes the proof of the case $m = 1$.
When $m > 1$, set $\vec{b}_B = ((b_1)_B, \cdots, (b_m)_B)$, where $(b_j)_B = |B|^{-1}\int_B |b_j(x)|dx$, $1 \le j \le m$, we have

$$
F_t^{\vec{b}}(f)(x,y) = \sum_{j=0}^{m}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x)-(b)_B)_\sigma
$$

$$
\begin{aligned}
&\times \int_{R^n}(b(z)-(b)_B)_{\sigma^c}\psi_t(y-z)f(z)dz \\
&= \prod_{j=1}^m (b_j(x)-(b_j)_B)F_t(f)(y)+(-1)^m F_t(\prod_{j=1}^m (b_j-(b_j)_B))f)(y) \\
&+ \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x)-(b)_B)_\sigma \int_{R^n}(b(z)-b(x))_{\sigma^c}\psi_t(y-z)f(z)dz \\
&= \prod_{j=1}^m (b_j(x)-(b_j)_B)F_t(f)(y)+(-1)^m F_t(\prod_{j=1}^m (b_j-(b_j)_B))f)(y) \\
&+ \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(b(x)-(b)_B)_\sigma F_t^{\vec{b}_{\sigma^c}}(f)(x,y),
\end{aligned}
$$

thus,

$$
\begin{aligned}
&S_\delta^{\vec{b}}(f)(x) = ||\chi_{\Gamma(x)}F_t^{\vec{b}}(f)(x)|| \\
&\leq ||\chi_{\Gamma(x)}\prod_{j=1}^m (b_j(x)-(b_j)_B)F_t(f)(x)|| \\
&+ \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||\chi_{\Gamma(x)}(b(x)-(b)_B)_\sigma F_t^{\vec{b}_{\sigma^c}}(f)(x)|| \\
&+||\chi_{\Gamma(x)}F_t(\prod_{j=1}^m (b_j-(b_j)_B)f)(x)|| \\
&\leq \prod_{j=1}^m (b_j(x)-(b_j)_B)S_\delta(f)(x)+(-1)^m S_\delta(\prod_{j=1}^m (b_j-(b_j)_B))_B)f)(x) \\
&+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x)-b_B)_\sigma S_\delta((b-b_B)_{\sigma^c}f)(x).
\end{aligned}
$$

We consider

$$
\begin{aligned}
&\left(\frac{1}{|B|^{1+\lambda q}}\int_B |S_\delta^{\vec{b}}(f)(x)|^q dx\right)^{1/q} \\
&\leq \left(\frac{1}{|B|^{1+\lambda q}}\int_B |\prod_{j=1}^m (b_j(x)-(b_j)_B)(S_\delta(f\chi_B))(x)|^q dx\right)^{1/q} \\
&+\left(\frac{1}{|B|^{1+\lambda q}}\int_B |\prod_{j=1}^m (b_j(x)-(b_j)_B)(S_\delta(f\chi_{B^c}))(x)|^q dx\right)^{1/q} \\
&+\left(\frac{1}{|B|^{1+\lambda q}}\int_B |S_\delta(\prod_{j=1}^m (b_j-(b_j)_B)f\chi_B)(x)|^q dx\right)^{1/q}
\end{aligned}
$$

$$+\left(\frac{1}{|B|^{1+\lambda q}}\int_B |S_\delta(\prod_{j=1}^m (b_j-(b_j)_B)f\chi_{B^c})(x)|^q dx\right)^{1/q}$$

$$+\left(\frac{1}{|B|^{1+\lambda q}}\int_B \Big|\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(b(x)-b_B)_\sigma S_\delta((b-b_B)_{\sigma^c}f\chi_B)(x)\Big|^q dx\right)^{1/q}$$

$$+\left(\frac{1}{|B|^{1+\lambda q}}\int_B \Big|\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(b(x)-b_B)_\sigma S_\delta((b-b_B)_{\sigma^c}f\chi_{B^c})(x)\Big|^q dx\right)^{1/q}$$

$$= \nu_1+\nu_2+\nu_3+\nu_4+\nu_5+\nu_6.$$

For ν_1, taking $1<p_1<n/\delta$ and t such that $1/t=1/p_1-\delta/n$, choosing $1/q=1/p_2+\cdots+1/p_{m+1}+1/t$, by Hölder's inequality and the boundedness of S_δ from $L^{p_1}(R^n)$ to $L^t(R^n)$, we have

$$\begin{aligned}
\nu_1 &\le |B|^{-1/q-\lambda}\prod_{j=1}^m\left(\int_B |b_j(x)-(b_j)_B|^{p_{j+1}}dx\right)^{1/p_{j+1}}\left(\int_B |(S_\delta(f\chi_B))(x)|^t dx\right)^{1/t}\\
&\le C|B|^{-1/q-\lambda}\prod_{j=1}^m |B|^{1/p_{j+1}+\lambda_{j+1}}||b_j||_{CBMO^{p_{j+1},\lambda_{j+1}}}\left(\int_B |f(x)|^{p_1}dx\right)^{1/p_1}\\
&\le C|B|^{-1/q-\lambda}\prod_{j=1}^m |B|^{1/p_{j+1}+\lambda_{j+1}}||b_j||_{CBMO^{p_{j+1},\lambda_{j+1}}}|B|^{1/p_1+\lambda_1}||f||_{\dot{B}^{p_1,\lambda_1}}\\
&\le C||\vec{b}||_{CBMO^{\vec{p},\vec{\lambda}}}||f||_{\dot{B}^{p_1,\lambda_1}}.
\end{aligned}$$

For ν_2, using the fact $|S_\delta(f\chi_{(B)^c})(x)|\le C||f||_{\dot{B}^{p,\lambda_1}}|B|^{\delta/n+\lambda_1}$ from the proof of Theorem 1 and the Hölder's inequality, we get

$$\begin{aligned}
\nu_2 &\le C|B|^{-1/q-\lambda}||f||_{\dot{B}^{p_1,\lambda_1}}|B|^{\delta/n+\lambda_1}\prod_{i=1}^m\left(\int_B |b_i(x)-(b_i)_B|^{p_{i+1}}dx\right)^{1/p_{i+1}}\\
&\times |B|^{1/q-1/p_2-\cdots-1/p_{m+1}}\\
&\le C|B|^{-1/q-\lambda}||f||_{\dot{B}^{p_1,\lambda_1}}|B|^{\delta/n+\lambda_1}\prod_{i=1}^m |B|^{1/p_{i+1}+\lambda_{i+1}}||b_i||_{CBMO^{p_{i+1},\lambda_{i+1}}}\\
&\times |B|^{1/q-1/p_2-\cdots-1/p_{m+1}}\\
&\le C||\vec{b}||_{CBMO^{\vec{p},\vec{\lambda}}}||f||_{\dot{B}^{p_1,\lambda_1}}.
\end{aligned}$$

For ν_3, taking $1<p_1<n/\delta$ and q such that $1/q=1/l-\delta/n$, choosing $1/l=1/p_1+\cdots+1/p_{m+1}$, by the boundedness of S_δ from L^l to L^q and Hölder's inequality, we have

$$\nu_3 \le C|B|^{-1/q-\lambda}\left(\int_B |\prod_{j=1}^m (b_j(x)-(b_j)_B)f(x)|^l dx\right)^{1/l}$$

$$
\begin{aligned}
&\leq C|B|^{-1/q-\lambda}\prod_{j=1}^{m}\left(\int_B |b_j(x)-(b_j)_B|^{p_{j+1}}dx\right)^{1/p_{j+1}}\left(\int_B |f(x)|^{p_1}dx\right)^{1/p_1}\\
&\leq C|B|^{-1/q-\lambda}\prod_{j=1}^{m}|B|^{1/p_{j+1}+\lambda_{j+1}}||b_i||_{CBMO^{p_{j+1},\lambda_{j+1}}}|B|^{1/p_1+\lambda_1}||f||_{\dot{B}^{p_1,\lambda_1}}\\
&\leq C||\vec{b}||_{CBMO^{\vec{p},\vec{\lambda}}}||f||_{\dot{B}^{p_1,\lambda_1}}.
\end{aligned}
$$

For ν_4, note that $x \in B$, by Hölder's inequality, Lemma 3 and noticing that $\lambda_j > 0 (2 \leq j \leq m+1)$, $\lambda_1 < -\lambda_2 - \cdots - \lambda_{m+1} < \delta/n$, we have

$$
\begin{aligned}
&|S_\delta(\prod_{j=1}^{m}(b_j-(b_j)_B)f\chi_{(B)^c})(x)|\\
\leq\ & \sum_{k=0}^{\infty}\int_{2^{k+1}B\setminus 2^kB}|\prod_{j=1}^{m}(b_j(z)-(b_j)_B)||x-z|^{-n+\delta}|f(z)|dz\\
\leq\ & C\sum_{k=0}^{\infty}|2^kB|^{\delta/n-1}\left(\int_{2^{k+1}B}|b_1(z)-(b_1)_B|^{p_2}dz\right)^{1/p_2}\\
\cdots\ & \left(\int_{2^{k+1}B}|b_m(z)-(b_m)_B|^{p_{m+1}}dz\right)^{1/p_{m+1}}\\
\times\ & \left(\int_{2^{k+1}B}|f(z)|^{p_1}dz\right)^{1/p_1}|2^{k+1}B|^{1-1/p_1-1/p_2\cdots-1/p_{m+1}}\\
\leq\ & C||f||_{\dot{B}^{p_1,\lambda_1}}\sum_{k=0}^{\infty}|2^kB|^{\delta/n-1}|2^{k+1}B|^{1/p_1+\lambda_1}|2^{k+1}B|^{1-1/p_1-1/p_2-\cdots1/p_{m+1}}\\
\times\ & k|2^{k+1}B|^{1/p_2+\lambda_2}||b_1||_{CBMO^{p_2,\lambda_2}}\cdots k|2^{k+1}B|^{1/p_{m+1}+\lambda_{m+1}}||b_m||_{CBMO^{p_{m+1},\lambda_{m+1}}}\\
\leq\ & C||\vec{b}||_{CBMO^{\vec{p},\vec{\lambda}}}||f||_{\dot{B}^{p_1,\lambda_1}}|B|^{\lambda_1+\lambda_2+\cdots+\lambda_{m+1}+\delta/n},
\end{aligned}
$$

thus, we obtain

$$
\begin{aligned}
\nu_4 &\leq C||\vec{b}||_{CBMO^{\vec{p},\vec{\lambda}}}||f||_{\dot{B}^{p_1,\lambda_1}}|B|^{\lambda_1+\lambda_2+\cdots+\lambda_{m+1}+\delta/n}|B|^{-1/q-\lambda}|B|^{1/q}\\
&\leq C||\vec{b}||_{CBMO^{\vec{p},\vec{\lambda}}}||f||_{\dot{B}^{p_1,\lambda_1}}.
\end{aligned}
$$

For ν_5, taking $1 < s < n/\delta$ and t such that $1/t = 1/s - \delta/n$, choosing $1/q = 1/p_3 + 1/t$, $1/s = 1/p_1 + 1/p_2$, by the boundedness of S_δ from L^t to L^s and Hölder's inequality, we have

$$
\begin{aligned}
\nu_5 \ \leq\ & C|B|^{-1/q-\lambda}\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\left(\int_B |(b(x)-b_B)_\sigma|^{p_3}dx\right)^{1/p_3}\\
&\times\left(\int_B |S_\delta((b-b_B)_{\sigma^c}f\chi_B)(x)|^t dx\right)^{1/t}
\end{aligned}
$$

$$
\begin{aligned}
&\le C|B|^{-1/q-\lambda}\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\left(\int_B|(b(x)-b_B)_\sigma|^{p_3}dx\right)^{1/p_3}\\
&\quad\times\left(\int_B|(b-b_B)_{\sigma^c}|^{p_2}dx\right)^{1/p_2}\left(\int_B|f(x)|^{p_1}dx\right)^{1/p_1}\\
&\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}|B|^{-1/q-\lambda}|B|^{1/p_3+\lambda_3}||\vec{b}_\sigma||_{CBMO^{p_3,\lambda_3}}|B|^{1/p_2+\lambda_2}\\
&\quad\times||\vec{b}_{\sigma^c}||_{CBMO^{p_2,\lambda_2}}|B|^{1/p_1+\lambda_1}||f||_{\dot{B}^{p_1,\lambda_1}}\le C||\vec{b}||_{CBMO^{\vec{p},\vec{\lambda}}}||f||_{\dot{B}^{p_1,\lambda_1}}.
\end{aligned}
$$

For ν_6, note that $x\in B$, by Hölder's inequality , Lemma 3 and noticing that $\lambda_j>0(2\le j\le m+1)$, $\lambda_1<-\lambda_2-\delta/n$, we have

$$
\begin{aligned}
&|S_\delta((b-b_B)_{\sigma^c}f\chi_{B^c})(x)|\\
&\le C\sum_{k=0}^{\infty}|2^kB|^{\delta/n-1}\left(\int_{2^{k+1}B}|(b(z)-b_B)_{\sigma^c}|^{p_2}dz\right)^{1/p_2}\\
&\quad\times\left(\int_{2^{k+1}B}|f(z)|^{p_1}dz\right)^{1/p_1}|2^{k+1}B|^{1-1/p_1-1/p_2}\\
&\le C||\vec{b}_{\sigma^c}||_{CBMO^{p_2,\lambda_2}}||f||_{\dot{B}^{p_1,\lambda_1}}\sum_{k=1}^{\infty}|2^kB|^{\delta/n-1}|2^{k+1}B|^{1/p_1+\lambda_1}k^m|2^{k+1}B|^{1/p_2+\lambda_2}\\
&\quad\times|2^{k+1}B|^{1-1/p_1-1/p_2}\le C||\vec{b}_{\sigma^c}||_{CBMO^{p_2,\lambda_2}}||f||_{\dot{B}^{p_1,\lambda_1}}|B|^{\lambda_1+\lambda_2+\delta/n},
\end{aligned}
$$

thus, we get

$$
\begin{aligned}
\nu_6 &\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}|B|^{-1/q-\lambda}||\vec{b}_{\sigma^c}||_{CBMO^{p_2,\lambda_2}}||f||_{\dot{B}^{p_1,\lambda_1}}|B|^{\lambda_1+\lambda_2+\delta/n}\\
&\quad\times\left(\int_B|(b(x)-b_B)_\sigma|^{p_3}dx\right)^{1/p_3}|B|^{1/q-1/p_3}\\
&\le C||\vec{b}||_{CBMO^{\vec{p},\vec{\lambda}}}||f||_{\dot{B}^{p_1,\lambda_1}}.
\end{aligned}
$$

This completes the total proof of the Theorem 3.

Proof of Theorem 4. Set

$$
\widetilde{S_\delta^{\vec{b}}}(f)(x)=\int_{R^n}|x-z|^{-n+\delta}|\prod_{j=1}^{m}(b_j(x)-b_j(z))f(z)|dz,
$$

it is easy to know the Theorem 3 is also true for the commutator $\widetilde{S_\delta^{\vec{b}}}(f)(x)$. Since

$$
M_\delta^{\vec{b}}(f)(x)\le\widetilde{S_\delta^{\vec{b}}}(f)(x),
$$

thus, Theorem 4 can be easily deduced. We omit the details here.

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