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Hierarchical Models for Prismatic Shells with Mixed Conditions on Face Surfaces

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I. Vekua constructed hierarchical models for elastic prismatic shells, in particular, plates of variable thickness, when on the face surfaces either stresses or displacements are known. In the present paper other hierarchical models for cusped, in general, elastic prismatic shells are constructed and analyzed, namely, when on the face surfaces (i) a normal to the projection of the prismatic shell component of a stress vector and parallel to the projection of the prismatic shell components of a displacement vector, (ii) a normal to the projection of the prismatic shell component of the displacement vector and parallel to the projection of the prismatic shell components of the stress vector are prescribed. Besides we construct hierarchical models, when on the one face surface conditions (i) and on the other one conditions (ii) are known and also models, when on the upper face surface stress vector and on the lower face surface displacements and vice versa are known. In the zero approximations of the models under consideration peculiarities (depending on sharpening geometry of the cusped edge) of correct setting boundary conditions at edges are investigated. In concrete cases some boundary value problems are solved in an explicit form.

Keywords: Hierarchical models; Elastic cusped prismatic shells; Cusped prismatic shells; Mixed conditions on face surfaces.

AMS Subject Classification: 74K20, 74K25, 74B05, 35J70, 35L80.

1. Introduction

Let $Ox_1x_2x_3$ be an anticlockwise-oriented rectangular Cartesian frame of origin O. We conditionally assume the x_3-axis vertical. The elastic body is called a prismatic shell [1]-[3] if it is bounded above and below by, respectively, the surfaces (so called face surfaces)

$$x_3 = \overset{(+)}{h}(x_1, x_2) \ \text{ and } \ x_3 = \overset{(-)}{h}(x_1, x_2), \quad (x_1,\, x_2) \in \omega,$$

laterally by a cylindrical surface Γ of generatrix parallel to the x_3-axis and its vertical dimension is sufficiently small compared with other dimensions of the body. $\overline{\omega} := \omega \cup \partial\omega$ is the so-called projection of the prismatic shell on $x_3 = 0$.

Let the thickness of the prismatic shell be

$$2h(x_1, x_2) := \overset{(+)}{h}(x_1, x_2) - \overset{(-)}{h}(x_1, x_2) \ \begin{cases} > 0 & \text{for} \quad (x_1, x_2) \in \omega, \\ \geq 0 & \text{for} \quad (x_1, x_2) \in \partial\omega \end{cases}$$

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and

$$2\widetilde{h}(x_1, x_2) := \overset{(+)}{h}(x_1, x_2) + \overset{(-)}{h}(x_1, x_2).$$

If the thickness of the prismatic shell vanishes on some subset of $\partial\omega$, it is called a cusped one.

Let us note that the lateral boundary of the standard shell is orthogonal to the "middle surface" of the shell, while the lateral boundary of the prismatic shell is orthogonal to the prismatic shell's projection on $x_3 = 0$.

I. Vekua [1], [2] constructed hierarchical models for elastic prismatic shells, in particular, plates of variable thickness, when on the face surfaces either stresses (Model I) or displacements (Model II) are known. He and his followers have investigated various aspects of the first model (for a survey see the introduction in [4]). The up-dated survey of results concerning cusped elastic prismatic shells in the cases of the first and second models is given in [3]. In the present paper six hierarchical models for cusped, in general, elastic prismatic shells are constructed and analyzed, when on the face surfaces (i) a normal to the projection of the prismatic shell component $Q_{\overset{(\pm)}{\nu}3}$ of a stress vector and parallel to the projection of the prismatic shell components $u_\alpha(x_1, x_2, \overset{(\pm)}{h}, t)$, $\alpha = 1, 2$, of a displacement vector (Model III), (ii) a normal to the projection of the prismatic shell component $u_3(x_1, x_2, \overset{(\pm)}{h}, t)$ of the displacement vector and parallel to the projection of the prismatic shell components $Q_{\overset{(\pm)}{\nu}\alpha}$, $\alpha = 1, 2$, of the stress vector (Model IV) are known. $\overset{(+)}{\nu}$ and $\overset{(-)}{\nu}$ denote outward normals to the face surfaces $\overset{(+)}{h}$ and $\overset{(-)}{h}$, respectively. Hierarchical Models will be called Model V and Model VI, when on the one face surface conditions (i) and on the other one conditions (ii) are prescribed. Besides we construct hierarchical models when on the upper face surface stress vector and on the lower face surface displacements (Model VII) and vice versa (Model VIII) are prescribed. In the zero approximations of the models under consideration peculiarities (depending on sharpening geometry of the cusped edge) of correct setting boundary conditions at edges are investigated. In concrete cases some boundary value problems are solved in an explicit form. In what follows X_{ij} and e_{ij} are the stress and strain tensors, respectively, u_i are the displacements, Φ_i are the volume force components, ρ is the density, λ and μ are the Lamé constants, δ_{ij} is the Kronecker delta, subscripts preceded by a comma mean partial derivatives with respect to the corresponding variables. Moreover, repeated indices imply summation (Greek letters run from 1 to 2 and Latin letters run from 1 to 3).

According to I.Vekua's [1, 2] dimension reduction method, in order to construct hierarchical models for elastic prismatic shells we multiply the basic equations of linear three-dimensional elasticity,

Motion Equations

$$X_{ij,i} + \Phi_j = \rho \frac{\partial^2 u_j}{\partial t^2}(x_1, x_2, x_3, t), \quad (x_1, x_2, x_3) \in \Omega \subset \mathbb{R}^3, \quad t > t_0, \quad j = 1, 2, 3; \quad (1)$$

Generalized Hooke's law (isotropic case)

$$X_{ij} = \lambda\theta\delta_{ij} + 2\mu e_{ij}, \quad i, j = 1, 2, 3, \quad \theta := e_{ii}; \quad (2)$$

Kinematic Relations

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad i,j = 1,2,3, \tag{3}$$

by Legendre polynomials $P_r(ax_3 - b)$, where

$$a(x_1,x_2) := \frac{1}{h(x_1,x_2)}, \quad b(x_1,x_2) := \frac{\widetilde{h}(x_1,x_2)}{h(x_1,x_2)},$$

and then we integrate with respect to the thickness variable x_3 within the limits $\overset{(-)}{h}(x_1,x_2)$ and $\overset{(-)}{h}(x_1,x_2)$. By these calculations prescribed on upper and lower face surfaces components are assumed as known, while the values of other components on the face surfaces are calculated from their Fourier-Legendre series expansions on the segment $x_3 \in \left[\overset{(-)}{h}(x_1,x_2),\ \overset{(+)}{h}(x_1,x_2)\right]$. So, we get the equivalent infinite system of relations with respect to the so called l-th order moments

$$\Big(X_{ijl},\, e_{ijl},\, u_{il}\Big)(x_1,x_2,t) := \int_{\overset{(-)}{h}}^{\overset{(+)}{h}} \Big(X_{ij},\, e_{ij},\, u_i\Big)(x_1,x_2,x_3,t)P_l(ax_3 - b)\; dx_3,$$
$$i,j = 1,2,3, \;\; l = 0,1, \cdots .$$

Then, having followed the usual procedure used in the theory of elasticity, we get an equivalent infinite system, consisting of three groups corresponding to each $j = 1,2,3$ (to this end, see (1)), with respect to the l-th order moments u_{il}. After this if we assume that the moments whose subscripts, indicating moments' order, are greater than N equal zero and consider for each $j = 1,2,3$ only the first $N+1$ equations in the obtained infinite system of equations with respect to the l-th order moments u_{il} we obtain the Nth order approximation (hierarchical model) governing system with respect to $\overset{N}{u}_{il}$ or

$$\overset{N}{v}_{il} := \frac{\overset{N}{u}_{il}}{h^{l+1}}, \quad i = 1,2,3, \quad l = \overline{0,N}$$

(roughly speaking $\overset{N}{u}_{il}$ is an "approximate value" of u_{il}).

2. Construction of Models

For the sake of transparency we confine ourselves to immediate deriving the $N=0$ approximation. To this end, we will need only integration of the basic $3D$ relations of the theory of elasticity keeping in mind quantities prescribed on the face surfaces.

Integrating basic equations (1)-(3) with respect to x_3 from $\overset{(-)}{h}(x_1,x_2)$ to

$\overset{(+)}{h}(x_1,x_2)$, we get

$$\begin{aligned}
&X_{j\beta 0,\beta}(x_1,x_2,t)-X_{j\beta}(x_1,x_2,\overset{(+)}{h}(x_1,x_2),t)\overset{(+)}{h}_{,\beta}\\
&+X_{j3}(x_1,x_2,\overset{(+)}{h}(x_1,x_2),t)+X_{j\beta}(x_1,x_2,\overset{(-)}{h}(x_1,x_2),t)\overset{(-)}{h}_{,\beta}\\
&-X_{j3}(x_1,x_2,\overset{(-)}{h}(x_1,x_2),t)+\Phi_{j0}(x_1,x_2,t)=\rho\frac{\partial^2 u_{j0}(x_1,x_2,t)}{\partial t^2},\ j=1,2,3; \quad (4)
\end{aligned}$$

$$X_{ij0}(x_1,x_2,t)=\lambda e_{kk0}(x_1,x_2,t)\delta_{ij}+2\mu e_{ij0}(x_1,x_2,t),\quad i,j=1,2,3; \qquad (5)$$

$$e_{i\beta 0}(x_1,x_2,t)=\frac{1}{2}\Big[u_{i0,\beta}(x_1,x_2,t)+u_i(x_1,x_2,\overset{(-)}{h},t)\overset{(-)}{h}_{,\beta}-u_i(x_1,x_2,\overset{(+)}{h},t)\overset{(+)}{h}_{,\beta}$$

$$+\left\{\begin{array}{l} u_{\beta 0,\alpha}(x_1,x_2,t)-u_\beta(x_1,x_2,\overset{(+)}{h},t)\overset{(+)}{h}_{,\alpha}+u_\beta(x_1,x_2,\overset{(-)}{h},t)\overset{(-)}{h}_{,\alpha},\ \ i=\alpha,\\ u_\beta(x_1,x_2,\overset{(+)}{h},t)-u_\beta(x_1,x_2,\overset{(-)}{h},t),\ \ i=3\end{array}\right.\Bigg], \quad (6)$$

$$i=1,2,3,\quad \beta=1,2,$$

$$e_{330}(x_1,x_2,t)=u_3(x_1,x_2,\overset{(+)}{h},t)-u_3(x_1,x_2,\overset{(-)}{h},t). \qquad (7)$$

Model III. *On the face surfaces*

$$\begin{aligned}
&Q_{\overset{(\pm)}{\nu}3}(x_1,x_2,\overset{(\pm)}{h}(x_1,x_2),t)\\
&\qquad=X_{3\beta}(x_1,x_2,\overset{(\pm)}{h}(x_1,x_2),t)\overset{(\pm)}{\nu}_\beta+X_{33}(x_1,x_2,\overset{(\pm)}{h}(x_1,x_2),t)\overset{(\pm)}{\nu}_3, \quad (8)\\
&u_\alpha(x_1,x_2,\overset{(\pm)}{h}(x_1,x_2),t),\quad \alpha=1,2,
\end{aligned}$$

are known.

Model IV.*On the face surfaces*

$$\begin{aligned}
&Q_{\overset{(\pm)}{\nu}\alpha}(x_1,x_2,\overset{(\pm)}{h}(x_1,x_2),t)\\
&\qquad=X_{\alpha\beta}(x_1,x_2,\overset{(\pm)}{h}(x_1,x_2),t)\overset{(\pm)}{\nu}_\beta+X_{\alpha 3}(x_1,x_2,\overset{(\pm)}{h}(x_1,x_2),t)\overset{(\pm)}{\nu}_3, \quad (9)\\
&u_3(x_1,x_2,\overset{(\pm)}{h}(x_1,x_2),t),\quad \alpha=1,2,
\end{aligned}$$

are known.

Model V.

$$\textit{On the upper surface}\ \overset{(+)}{h}\ \textit{the quantities (8) are known,} \qquad (10)$$

$$\textit{on the lower surface } \overset{(-)}{h} \textit{ the quantities (9) are known.} \tag{11}$$

Model VI. *On* $\overset{(+)}{h}$ *the quantities (9) and on* $\overset{(-)}{h}$ *the quantities (8) are known.*

Model VII. *On the upper face surface stress vector components* $Q_{\overset{(+)}{\nu}i}$, $i = 1, 2, 3,$ *and on the lower face surface displacements are known.*

Model VIII. *On the upper face surface displacements and on the lower face surface stress vector components* $Q_{\overset{(-)}{\nu}i}$, $i = 1, 2, 3,$ *are known.*

In the $N = 0$ approximation it is assumed that

$$u_i(x_1, x_2, x_3, t) \cong \frac{u_{i0}(x_1, x_2, t)}{2h} =: \frac{1}{2} v_{i0}(x_1, x_2, t), \quad i = 1, 2, 3, \tag{12}$$

$$X_{ij}(x_1, x_2, x_3, t) \cong \frac{X_{ij0}(x_1, x_2, t)}{2h}, \quad i = 1, 2, 3, \tag{13}$$

In the case of Model III, taking into account (12), (13), from (4), (7), (6) we obtain correspondingly

$$\begin{aligned} X_{3\beta 0,\beta} &+ Q_{\overset{(+)}{\nu}3} \sqrt{(\overset{(+)}{h}_{,1})^2 + (\overset{(+)}{h}_{,2})^2 + 1} \\ &+ Q_{\overset{(-)}{\nu}3} \sqrt{(\overset{(-)}{h}_{,1})^2 + (\overset{(-)}{h}_{,2})^2 + 1} + \Phi_{30} = \rho \frac{\partial^2 u_{30}}{\partial t^2}, \end{aligned} \tag{14}$$

$$X_{\alpha\beta 0,\beta} - \frac{1}{2h}\Big[X_{\alpha\beta 0}\Big(\overset{(+)}{h}_{,\beta} - \overset{(-)}{h}_{,\beta}\Big) - X_{\alpha 30} + X_{\alpha 30}\Big] + \Phi_{\alpha 0} = \rho \frac{\partial^2 u_{\alpha 0}}{\partial t^2}, \quad \alpha = 1, 2,$$

i.e.,

$$X_{\alpha\beta 0,\beta} - (\ln h)_{,\beta}\, X_{\alpha\beta 0} + \Phi_{\alpha 0} = \rho \frac{\partial^2 u_{\alpha 0}}{\partial t^2}, \quad \alpha = 1, 2; \tag{15}$$

$$\begin{aligned} e_{330} &= u_3(x_1, x_2, \overset{(+)}{h}, t) - u_3(x_1, x_2, \overset{(-)}{h}, t) \\ &=: \Psi_{33} = \frac{1}{2} v_{30}(x_1, x_2, t) - \frac{1}{2} v_{30}(x_1, x_2, t) = 0, \end{aligned} \tag{16}$$

$$e_{3\beta 0} = \frac{1}{2}\Big\{u_{30,\beta} + \Big[u_\beta(x_1, x_2, \overset{(+)}{h}, t) - u_\beta(x_1, x_2, \overset{(-)}{h}, t)\Big] + \frac{1}{2} v_{30}\Big(\overset{(-)}{h}_{,\beta} - \overset{(+)}{h}_{,\beta}\Big)\Big\} \tag{17}$$

$$= \frac{1}{2}\Big[(h v_{30})_{,\beta} - h_{,\beta}\, v_{30}\Big] + \frac{1}{2}\Big[u_\beta(x_1, x_2, \overset{(+)}{h}, t) - u_\beta(x_1, x_2, \overset{(-)}{h}, t)\Big]$$

$$= \frac{1}{2} h v_{30,\beta} + \frac{1}{2}\Big[u_\beta(x_1, x_2, \overset{(+)}{h}, t) - u_\beta(x_1, x_2, \overset{(-)}{h}, t)\Big], \quad \beta = 1, 2;$$

$$e_{\alpha\beta 0} = \frac{1}{2}\Big[u_{\alpha 0,\beta} + u_{\beta 0,\alpha} - u_\beta(x_1, x_2, \overset{(+)}{h}, t) \overset{(+)}{h}_{,\alpha} + u_\beta(x_1, x_2, \overset{(-)}{h}, t) \overset{(-)}{h}_{,\alpha} \tag{18}$$

$$+ u_\alpha(x_1, x_2, \overset{(-)}{h}, t) \overset{(-)}{h}_{,\beta} - u_\alpha(x_1, x_2, \overset{(+)}{h}, t) \overset{(+)}{h}_{,\beta} \Big] = \frac{1}{2}\big(u_{\alpha 0,\beta} + u_{\beta 0,\alpha}\big) + \Psi_{\alpha\beta}$$

$$= \frac{1}{2}\Big[(h v_{\alpha 0})_{,\beta} + (h v_{\beta 0})_{,\alpha} \Big] + \Psi_{\alpha\beta}, \quad \alpha, \beta = 1, 2,$$

where

$$\Psi_{\alpha\beta} := \frac{1}{2}\Big[u_\beta(x_1, x_2, \overset{(-)}{h}, t) \overset{(-)}{h}_{,\alpha} - u_\beta(x_1, x_2, \overset{(+)}{h}, t) \overset{(+)}{h}_{,\alpha}$$

$$+ u_\alpha(x_1, x_2, \overset{(-)}{h}, t) \overset{(-)}{h}_{,\beta} - u_\alpha(x_1, x_2, \overset{(+)}{h}, t) \overset{(+)}{h}_{,\beta} \Big].$$

Substituting (16)-(18) into (5), we get

$$X_{330}(x_1, x_2, t) = \lambda\Big(u_{\gamma 0,\gamma} + \Psi_{\gamma\gamma}\Big) = \lambda (h v_{\gamma 0})_{,\gamma} + \lambda \Psi_{\gamma\gamma}, \tag{19}$$

$$X_{3\beta 0}(x_1, x_2, t) = \mu h v_{30,\beta} + \mu\Big[u_\beta(x_1, x_2, \overset{(+)}{h}, t) - u_\beta(x_1, x_2, \overset{(-)}{h}, t)\Big], \quad \beta = 1, 2, \tag{20}$$

$$\begin{aligned} X_{\alpha\beta 0}(x_1, x_2, t) = \lambda (h v_{\gamma 0})_{,\gamma}\, \delta_{\alpha\beta} + \lambda \Psi_{\gamma\gamma} \delta_{\alpha\beta} + \mu\Big[(h v_{\alpha 0})_{,\beta} + (h v_{\beta 0})_{,\alpha} \Big] \\ + 2\mu \Psi_{\alpha\beta}, \quad \alpha, \beta = 1, 2. \end{aligned} \tag{21}$$

So, when normal to the projection of the prismatic shell components $Q_{\overset{(+)}{\nu}3}$ and $Q_{\overset{(-)}{\nu}3}$ of a stress vector and parallel to the projection of the prismatic shell components $u_\alpha(x_1, x_2, \overset{(+)}{h}, t)$ and $u_\alpha(x_1, x_2, \overset{(-)}{h}, t)$, $\alpha = 1, 2$, of a displacement vector are known on $\overset{(+)}{h}$ and $\overset{(-)}{h}$, respectively, substituting (19)-(21) into (15), (14) the governing system for the weighted zero moments v_{j0}, $j = 1, 2, 3$, (superscript $N = 0$ is omitted below) has the following form

$$\begin{aligned} \mu (h v_{\alpha 0})_{,\beta\beta} + (\lambda + \mu)(h v_{\gamma 0})_{,\gamma\alpha} - (\ln h)_{,\beta} \left\{ \lambda \delta_{\alpha\beta} (h v_{\gamma 0})_{,\gamma} + \mu \left[(h v_{\alpha 0})_{,\beta} + (h v_{\beta 0})_{,\alpha} \right] \right\} \\ + 2\mu\, \Psi_{\alpha\beta,\beta} + \lambda\, \Psi_{\gamma\gamma,\alpha} - (\ln h)_{,\beta} \left[\lambda \delta_{\alpha\beta} \Psi_{\gamma\gamma} + 2\mu\, \Psi_{\alpha\beta} \right] + \Phi_{\alpha 0} = \rho h \frac{\partial^2 v_{\alpha 0}}{\partial t^2}, \\ \alpha = 1, 2, \end{aligned} \tag{22}$$

$$
\begin{aligned}
&\mu(hv_{30,\beta})_{,\beta} + \mu\left[u_\beta(x_1,x_2,\overset{(+)}{h},t) - u_\beta(x_1,x_2,\overset{(-)}{h},t)\right]_{,\beta}\\
&+Q_{\overset{(+)}{\nu}3}\sqrt{(\overset{(+)}{h}_{,1})^2 + (\overset{(+)}{h}_{,2})^2 + 1} + Q_{\overset{(-)}{\nu}3}\sqrt{(\overset{(-)}{h}_{,1})^2 + (\overset{(-)}{h}_{,2})^2 + 1}\\
&+\Phi_{30} = \rho h \frac{\partial^2 v_{30}}{\partial t^2}.
\end{aligned}
\tag{23}
$$

In this model $\Psi_{33} \equiv 0$ in contrast to the model, when on the face surfaces only displacements are prescribed (Model II).

In Model II $\Psi_{33} \not\equiv 0$, in general (see system (55) below).

Now, we consider Model IV. In the case of Model IV taking into account (12), (13) from (4), (7), (6) we have correspondingly

$$
\begin{aligned}
X_{\alpha\beta0,\beta} + Q_{\overset{(+)}{\nu}\alpha}\sqrt{(\overset{(+)}{h}_{,1})^2 + (\overset{(+)}{h}_{,2})^2 + 1} + Q_{\overset{(-)}{\nu}\alpha}\sqrt{(\overset{(-)}{h}_{,1})^2 + (\overset{(-)}{h}_{,2})^2 + 1} + \Phi_{\alpha0}\\
= \rho h \frac{\partial^2 v_{\alpha0}}{\partial t^2}, \quad \alpha = 1,2,
\end{aligned}
\tag{24}
$$

$$
X_{3\beta0,\beta} - \frac{1}{2h}\left[X_{3\beta0}\left(\overset{(+)}{h}_{,\beta} - \overset{(-)}{h}_{,\beta}\right) - X_{330} + X_{330}\right] + \Phi_{30} = \rho h \frac{\partial^2 v_{30}}{\partial t^2},
$$

i.e.,

$$
X_{3\beta0,\beta} - (\ln h)_{,\beta}\, X_{3\beta0} + \Phi_{30} = \rho h \frac{\partial^2 v_{30}}{\partial t^2};
\tag{25}
$$

$$
e_{330} = u_3(x_1,x_2,\overset{(+)}{h},t) - u_3(x_1,x_2,\overset{(-)}{h},t) =: \Psi_{33},
\tag{26}
$$

$$
\begin{aligned}
e_{3\beta0} &= \frac{1}{2}\Big[\ u_{30,\beta} + u_\beta(x_1,x_2,\overset{(+)}{h},t) - u_\beta(x_1,x_2,\overset{(-)}{h},t)\\
&\qquad + u_3(x_1,x_2,\overset{(-)}{h},t)\overset{(-)}{h}_{,\beta} - u_3(x_1,x_2,\overset{(+)}{h},t)\overset{(+)}{h}_{,\beta}\Big]\\
&= \frac{1}{2}\Big[\ (hv_{30})_{,\beta} + \frac{u_{\beta0}(x_1,x_2,t)}{2h} - \frac{u_{\beta0}(x_1,x_2,t)}{2h}\\
&\qquad + u_3(x_1,x_2,\overset{(-)}{h},t)\overset{(-)}{h}_{,\beta} - u_3(x_1,x_2,\overset{(+)}{h},t)\overset{(+)}{h}_{,\beta}\Big]
\end{aligned}
$$

$$
= \frac{1}{2}\left[(hv_{30})_{,\beta} + u_3(x_1,x_2,\overset{(-)}{h},t)\overset{(-)}{h}_{,\beta} - u_3(x_1,x_2,\overset{(+)}{h},t)\overset{(+)}{h}_{,\beta}\right], \quad \beta = 1,2,
\tag{27}
$$

$$e_{\alpha\beta 0}=\frac{1}{2}\Big[u_{\alpha 0,\beta}+u_{\beta 0,\alpha}-u_\beta(x_1,x_2,\overset{(+)}{h},t)\overset{(+)}{h}_{,\alpha}+u_\beta(x_1,x_2,\overset{(-)}{h},t)\overset{(-)}{h}_{,\alpha}$$

$$+u_\alpha(x_1,x_2,\overset{(-)}{h},t)\overset{(-)}{h}_{,\beta}-u_\alpha(x_1,x_2,\overset{(+)}{h},t)\overset{(+)}{h}_{,\beta}\Big]$$

$$=\frac{1}{2}\Big[(hv_{\alpha 0})_{,\beta}+(hv_{\beta 0})_{,\alpha}-v_{\beta 0}h_{,\alpha}-v_{\alpha 0}h_{,\beta}\Big]=\frac{1}{2}h\Big(v_{\alpha 0,\beta}+v_{\beta 0})_{,\alpha}\Big),\quad \alpha=1,2. \quad (28)$$

Substituting (26)-(28) into (5), we get

$$X_{330}=\lambda\Big[hv_{\gamma 0,\gamma}+\Psi_{33}\Big]+2\mu\Psi_{33}, \quad (29)$$

$$X_{3\beta 0}=\mu\Big[(hv_{30})_{,\beta}+u_3(x_1,x_2,\overset{(-)}{h},t)\overset{(-)}{h}_{,\beta}-u_3(x_1,x_2,\overset{(+)}{h},t)\overset{(+)}{h}_{,\beta}\Big],\quad \beta=1,2, \quad (30)$$

$$X_{\alpha\beta 0}=\lambda\Big[hv_{\gamma 0,\gamma}+\Psi_{33}\Big]\delta_{\alpha\beta}+\mu h(v_{\beta 0,\alpha}+v_{\alpha 0,\beta}),\quad \alpha,\beta=1,2. \quad (31)$$

So, when normal to the projection of the prismatic shell components $u_3(x_1,x_2,\overset{(+)}{h},t)$ and $u_3(x_1,x_2,\overset{(-)}{h},t)$ of the displacement vector and parallel to the projection of the prismatic shell components $Q_{\overset{(+)}{\nu}\alpha}$ and $Q_{\overset{(-)}{\nu}\alpha}$, $\alpha=1,2$, of the stress vector are prescribed on $\overset{(+)}{h}$ and $\overset{(-)}{h}$, respectively, substituting (29)-(31) into (24), (25) the governing system for the weighted zero moments v_{j0}, $j=1,2,3$, has the following form

$$\mu(hv_{\alpha 0,\beta})_{,\beta}+\mu(hv_{\beta 0,\alpha})_{,\beta}+\lambda(hv_{\gamma 0,\gamma})_{,\alpha}+\lambda\Psi_{33,\alpha}+Q_{\overset{(+)}{\nu}\alpha}\sqrt{(\overset{(+)}{h}_{,1})^2+(\overset{(+)}{h}_{,2})^2+1}$$

$$+Q_{\overset{(-)}{\nu}\alpha}\sqrt{(\overset{(-)}{h}_{,1})^2+(\overset{(-)}{h}_{,2})^2+1}+\Phi_{\alpha 0}=\rho h\frac{\partial^2 v_{30}}{\partial t^2},\quad \alpha=1,2, \quad (32)$$

$$\mu(hv_{30})_{,\beta\beta}-\mu(\ln h)_{,\beta}(hv_{30})_{,\beta}+\mu\Big[u_3(x_1,x_2,\overset{(-)}{h},t)\overset{(-)}{h}_{,\beta}-u_3(x_1,x_2,\overset{(+)}{h},t)\overset{(+)}{h}_{,\beta}\Big]_{,\beta}$$

$$-\mu(\ln h)_{,\beta}\Big[u_3(x_1,x_2,\overset{(-)}{h},t)\overset{(-)}{h}_{,\beta}-u_3(x_1,x_2,\overset{(+)}{h},t)\overset{(+)}{h}_{,\beta}\Big]+\Phi_{30}=\rho h\frac{\partial^2 v_{30}}{\partial t^2}. \quad (33)$$

In this model $\Psi_{33}\not\equiv 0$, in general (it is identically zero if $u_3(x_1,x_2,\overset{(+)}{h},t)=u_3(x_1,x_2,\overset{(-)}{h},t)$), in contrast to the Model I (see system (53) below), when on the face surfaces only stress vectors are known (in the last case $\Psi_{33}\equiv 0$).

In a similar way the governing system for Model V can be constructed .

Indeed, for the Model V, taking into account (12), (13), from (4), (7), (6) we obtain correspondingly

$$X_{3\beta 0,\beta}+\frac{1}{2h}\Big(X_{3\beta 0}\overset{(-)}{h}_{,\beta}-X_{330}\Big)$$
$$+Q_{\overset{(+)}{\nu}3}\sqrt{(\overset{(+)}{h}_{,1})^2+(\overset{(+)}{h}_{,2})^2+1}+\Phi_{30}=\rho\frac{\partial^2 u_{30}}{\partial t^2}, \tag{34}$$

$$X_{\alpha\beta 0,\beta}-\frac{1}{2h}\Big(X_{\alpha\beta 0}\overset{(+)}{h}_{,\beta}-X_{\alpha 30}\Big)$$
$$+Q_{\overset{(-)}{\nu}\alpha}\sqrt{(\overset{(-)}{h}_{,1})^2+(\overset{(-)}{h}_{,2})^2+1}+\Phi_{\alpha 0}=\rho\frac{\partial^2 u_{\alpha 0}}{\partial t^2},\quad \alpha=1,2; \tag{35}$$

$$e_{330}=u_3(x_1,x_2,\overset{(+)}{h},t)-u_3(x_1,x_2,\overset{(-)}{h},t)$$
$$=\frac{u_{30}}{2h}-u_3(x_1,x_2,\overset{(-)}{h},t)=\frac{v_{30}}{2}-u_3(x_1,x_2,\overset{(-)}{h},t) \tag{36}$$

$$e_{3\beta 0}=\frac{1}{2}\Big[u_{30,\beta}+u_\beta(x_1,x_2,\overset{(+)}{h},t)-u_\beta(x_1,x_2,\overset{(-)}{h},t)$$
$$+u_3(x_1,x_2,\overset{(-)}{h},t)\overset{(-)}{h}_{,\beta}-u_3(x_1,x_2,\overset{(+)}{h},t)\overset{(+)}{h}_{,\beta}\Big]$$
$$=\frac{1}{2}\Big[u_{30,\beta}+u_\beta(x_1,x_2,\overset{(+)}{h},t)-\frac{1}{2h}u_{\beta 0}+u_3(x_1,x_2,\overset{(-)}{h},t)\overset{(-)}{h}_{,\beta}-\frac{\overset{(+)}{h}_{,\beta}}{2h}u_{30}\Big]$$
$$=\frac{1}{2}\Big[\big(hv_{30}\big)_{,\beta}-\frac{1}{2}\big(v_{\beta 0}+\overset{(+)}{h}_{,\beta}v_{30}\big)\Big]+\chi_{3\beta},\qquad \beta=1,2. \tag{37}$$

$$\chi_{3\beta}:=\frac{1}{2}\big[u_\beta(x_1,x_2,\overset{(+)}{h},t)+u_3(x_1,x_2,\overset{(-)}{h},t)\overset{(-)}{h}_{,\beta}\big],$$

$$e_{\alpha\beta 0}=\frac{1}{2}\Big[u_{\alpha 0,\beta}+u_{\beta 0,\alpha}-u_\beta(x_1,x_2,\overset{(+)}{h},t)\overset{(+)}{h}_{,\alpha}$$
$$+u_\beta(x_1,x_2,\overset{(-)}{h},t)\overset{(-)}{h}_{,\alpha}+u_\alpha(x_1,x_2,\overset{(-)}{h},t)\overset{(-)}{h}_{,\beta}-u_\alpha(x_1,x_2,\overset{(+)}{h},t)\overset{(+)}{h}_{,\beta}\Big]$$
$$=\frac{1}{2}\Big[u_{\alpha 0,\beta}+u_{\beta 0,\alpha}-u_\beta(x_1,x_2,\overset{(+)}{h},t)\overset{(+)}{h}_{,\alpha}+\frac{\overset{(-)}{h}_{,\alpha}}{2h}u_{\beta 0}(x_1,x_2,t)$$
$$+\frac{\overset{(-)}{h}_{,\beta}}{2h}u_{\alpha 0}(x_1,x_2,t)-u_\alpha(x_1,x_2,\overset{(+)}{h},t)\overset{(+)}{h}_{,\beta}\Big]$$
$$=\frac{1}{2}\Big[(hv_{\alpha 0})_{,\beta}+(hv_{\beta 0})_{,\alpha}+\frac{\overset{(-)}{h}_{,\alpha}}{2}v_{\beta 0}+\frac{\overset{(-)}{h}_{,\beta}}{2}v_{\alpha 0}\Big]-\chi_{\alpha\beta}, \tag{38}$$

$$\chi_{\alpha\beta} := \frac{1}{2}\big[u_\beta(x_1,x_2,\overset{(+)}{h},t)\overset{(+)}{h}_{,\alpha} + u_\alpha(x_1,x_2,\overset{(+)}{h},t)\overset{(+)}{h}_{,\beta}\big], \qquad \alpha,\beta=1,2;$$

Substituting (36)-(38) into (5), we get

$$\begin{aligned}
X_{330} &= \lambda\Big[(hv_{\gamma0})_{,\gamma} + \frac{1}{2}\overset{(-)}{h}_{,\gamma}v_{\gamma0} - \chi_{\gamma\gamma}\Big] + (\lambda+2\mu)\Big[\frac{1}{2}v_{30} - u_3(x_1,x_2,\overset{(-)}{h},t)\Big] \\
&= \lambda\Big[(hv_{\gamma0})_{,\gamma} + \frac{1}{2}\overset{(-)}{h}_{,\gamma}v_{\gamma0}\Big] + \frac{\lambda+2\mu}{2}v_{30} \\
&-\lambda\chi_{\gamma\gamma} - (\lambda+2\mu)u_3(x_1,x_2,\overset{(-)}{h},t),
\end{aligned} \tag{39}$$

$$X_{3\beta0} = \mu\Big[(hv_{30})_{,\beta} - \frac{1}{2}(v_{\beta0} + \overset{(+)}{h}_{,\beta}v_{30})\Big] + 2\mu\chi_{3\beta}, \quad \beta=1,2; \tag{40}$$

$$\begin{aligned}
X_{\alpha\beta0} &= \lambda\Big[(hv_{\gamma0})_{,\gamma} + \frac{1}{2}\overset{(-)}{h}_{,\gamma}v_{\gamma0} - \chi_{\gamma\gamma} + \frac{\lambda}{2}v_{30} - \lambda u_3(x_1,x_2,\overset{(-)}{h},t)\Big]\delta_{\alpha\beta} \\
&+\mu\Big[(hv_{\alpha0})_{,\beta} + (hv_{\beta0})_{,\alpha} + \frac{\overset{(-)}{h}_{,\alpha}}{2}v_{\beta0} + \frac{\overset{(-)}{h}_{,\beta}}{2}v_{\alpha0}\Big] - 2\mu\chi_{\alpha\beta} \\
&= \lambda\Big[(hv_{\gamma0})_{,\gamma} + \frac{\overset{(-)}{h}_{,\gamma}}{2}v_{\gamma0} + \frac{v_{30}}{2}\Big]\delta_{\alpha\beta} + \mu\Big[(hv_{\alpha0})_{,\beta} + (hv_{\beta0})_{,\alpha} + \frac{\overset{(-)}{h}_{,\alpha}}{2}v_{\beta0} \\
&+\frac{\overset{(-)}{h}_{,\beta}}{2}v_{\alpha0}\Big] - \lambda\Big[\chi_{\gamma\gamma} + u_3(x_1,x_2,\overset{(-)}{h},t)\Big]\delta_{\alpha\beta} - 2\mu\chi_{\alpha\beta}, \quad \alpha,\beta=1,2.
\end{aligned} \tag{41}$$

Substituting (39)-(41) into (35), (34) we have the following governing system for Model V

$$\begin{aligned}
&\mu(hv_{\alpha0})_{,\beta\beta} + \mu(hv_{\beta0})_{,\alpha\beta} + \lambda(hv_{\gamma0})_{,\gamma\alpha} + \frac{\lambda}{2}\Big(\overset{(-)}{h}_{,\gamma}v_{\gamma0}\Big)_{,\alpha} + \frac{\mu}{2}\Big(\overset{(-)}{h}_{,\alpha}v_{\beta0}\Big)_{,\beta} \\
&+\frac{\mu}{2}\Big(\overset{(-)}{h}_{,\beta}v_{\alpha0}\Big)_{,\beta} + \frac{\lambda}{2}v_{30,\alpha} - \frac{1}{2h}\Big\{\Big[\lambda\Big((hv_{\gamma0})_{,\gamma} + \frac{\overset{(-)}{h}_{,\gamma}}{2}v_{\gamma0} + \frac{v_{30}}{2}\Big)\delta_{\alpha\beta} \\
&+\mu\Big((hv_{\alpha0})_{,\beta} + (hv_{\beta0})_{,\alpha} + \frac{\overset{(-)}{h}_{,\alpha}}{2}v_{\beta0} + \frac{\overset{(-)}{h}_{,\beta}}{2}v_{\alpha0}\Big)\Big]\overset{(+)}{h}_{,\beta} \\
&-\mu\Big[(hv_{30})_{,\alpha} - \frac{1}{2}(v_{\alpha0} + \overset{(+)}{h}_{,\alpha}v_{30})\Big]\Big\} + Q_{\overset{(-)}{\nu}\alpha}\sqrt{(\overset{(-)}{h}_{,1})^2 + (\overset{(-)}{h}_{,2})^2 + 1} \\
&-\lambda\Big[\chi_{\gamma\gamma} + u_3(x_1,x_2,\overset{(-)}{h},t)\Big]_{,\alpha} - 2\mu\chi_{\alpha\beta,\beta} + \frac{\lambda}{2h}\Big[\Big(\chi_{\gamma\gamma} + u_3(x_1,x_2,\overset{(-)}{h},t)\Big)\delta_{\alpha\beta} \\
&-2\mu\chi_{\alpha\beta}\Big]\overset{(+)}{h}_{,\beta} + \frac{\mu}{h}\chi_{3\alpha} + \Phi_{\alpha0} = \rho h\frac{\partial^2 v_{\alpha0}}{\partial t^2}, \qquad \alpha=1,2,
\end{aligned} \tag{42}$$

$$\mu(hv_{30})_{,\beta\beta} - \frac{\mu}{2}v_{\beta0,\beta} - \frac{\mu}{2}\Big(\overset{(+)}{h}_{,\beta}v_{30}\Big)_{,\beta} + \frac{1}{2h}\Big\{\mu\Big[(hv_{30})_{,\beta} - \frac{1}{2}(v_{\beta0} + \overset{(+)}{h}_{,\beta}v_{30})\Big]\overset{(-)}{h}_{,\beta}$$
$$-\lambda\Big[(hv_{\gamma0})_{,\gamma} + \frac{1}{2}\overset{(-)}{h}_{,\gamma}v_{\gamma0}\Big] - \frac{\lambda+2\mu}{2}v_{30}\Big\} + Q_{\overset{(+)}{\nu}3}\sqrt{(\overset{(+)}{h}_{,1})^2 + (\overset{(+)}{h}_{,2})^2 + 1} \tag{43}$$
$$+2\mu\chi_{3\beta,\beta} + \frac{\mu}{h}\chi_{3\beta}\overset{(-)}{h}_{,\beta} + \frac{1}{2h}\Big[\lambda\chi_{\gamma\gamma} + (\lambda+2\mu)u_3(x_1,x_2,\overset{(-)}{h},t)\Big] + \Phi_{30} = \rho h\frac{\partial^2 v_{30}}{\partial t^2}.$$

Taking into account conditions of Model VII and (12), (13), from (4), (7), (6) we obtain correspondingly

$$X_{i\beta0,\beta} + Q_{\overset{(+)}{\nu}i}\sqrt{(\overset{(+)}{h}_{,1})^2 + (\overset{(+)}{h}_{,2})^2 + 1}$$
$$+\frac{1}{2h}\Big(X_{i\beta0}\overset{(-)}{h}_{,\beta} - X_{i30}\Big) + \Phi_{i0} = \rho\frac{\partial^2 u_{i0}}{\partial t^2}, \qquad i = 1,2,3; \tag{44}$$

$$e_{330} = \frac{u_{30}}{2h} - u_3(x_1,x_2,\overset{(-)}{h},t) = \frac{v_{30}}{2} - u_3(x_1,x_2,\overset{(-)}{h},t), \tag{45}$$

$$e_{3\beta0} = \frac{1}{2}\Big[u_{30,\beta} + \frac{u_{\beta0}}{2h} - u_\beta(x_1,x_2,\overset{(-)}{h},t) + u_3(x_1,x_2,\overset{(-)}{h},t)\overset{(-)}{h}_{,\beta}$$
$$-\frac{u_{30}}{2h}\overset{(+)}{h}_{,\beta}\Big] = \frac{1}{2}\Big[(hv_{30})_{,\beta} + \frac{v_{\beta0}}{2} - \frac{v_{30}}{2}\overset{(+)}{h}_{,\beta}\Big] - \frac{1}{2}\Big[u_\beta(x_1,x_2,\overset{(-)}{h},t)$$
$$-u_3(x_1,x_2,\overset{(-)}{h},t)\overset{(-)}{h}_{,\beta}\Big], \qquad \beta = 1,2, \tag{46}$$

$$e_{\alpha\beta0} = \frac{1}{2}\Big[u_{\alpha0,\beta} + u_{\beta0,\alpha} - \overset{(+)}{h}_{,\alpha}\frac{u_{\beta0}}{2h} + u_\beta(x_1,x_2,\overset{(-)}{h},t)\overset{(-)}{h}_{,\alpha}$$
$$+u_\alpha(x_1,x_2,\overset{(-)}{h},t)\overset{(-)}{h}_{,\beta} - \overset{(+)}{h}_{,\beta}\frac{u_{\alpha0}}{2h}\Big]$$
$$= \frac{1}{2}\Big[(hv_{\alpha0})_{,\beta} + (hv_{\beta0})_{,\alpha} - \frac{\overset{(+)}{h}_{,\alpha}}{2}v_{\beta0} - \frac{\overset{(+)}{h}_{,\beta}}{2}v_{\alpha0}\Big]$$
$$+\frac{1}{2}\Big[u_\beta(x_1,x_2,\overset{(-)}{h},t)\overset{(-)}{h}_{,\alpha} + u_\alpha(x_1,x_2,\overset{(-)}{h},t)\overset{(-)}{h}_{,\beta}\Big], \quad \alpha,\beta = 1,2. \tag{47}$$

Substituting (45)-(47) into (5), we get

$$X_{330} = \lambda\Big[(hv_{\gamma0})_{,\gamma} - \frac{1}{2}\overset{(+)}{h}_{,\gamma}v_{\gamma0} + \frac{v_{30}}{2}\Big] + \lambda\Big[u_\gamma(x_1,x_2,\overset{(-)}{h},t)\overset{(-)}{h}_{,\gamma}$$
$$-u_3(x_1,x_2,\overset{(-)}{h},t)\Big] + \mu v_{30} - 2\mu u_3(x_1,x_2,\overset{(-)}{h},t), \tag{48}$$

$$
\begin{aligned}
X_{3\beta 0} = {} & \mu\Big[(hv_{30})_{,\beta} + \frac{1}{2}v_{\beta 0} - \frac{\overset{(+)}{h}_{,\beta}}{2}v_{30}\Big] \\
& -\mu\Big[u_\beta(x_1, x_2, \overset{(-)}{h}, t) - u_3(x_1, x_2, \overset{(-)}{h}, t)\overset{(-)}{h}_{,\beta}\Big] \qquad \beta = 1, 2,
\end{aligned}
\tag{49}
$$

$$
\begin{aligned}
X_{\alpha\beta 0} = {} & \lambda\Big[(hv_{\gamma 0})_{,\gamma} - \frac{1}{2}\overset{(+)}{h}_{,\gamma}v_{\gamma 0} + \frac{v_{30}}{2}\Big]\delta_{\alpha\beta} + \lambda\Big[u_\gamma(x_1, x_2, \overset{(-)}{h}, t)\overset{(-)}{h}_{,\gamma} \\
& -u_3(x_1, x_2, \overset{(-)}{h}, t)\Big]\delta_{\alpha\beta} + \mu\Big[(hv_{\alpha 0})_{,\beta} + (hv_{\beta 0})_{,\alpha} - \frac{\overset{(+)}{h}_{,\alpha}}{2}v_{\beta 0} - \frac{\overset{(+)}{h}_{,\beta}}{2}v_{\alpha 0}\Big] \\
& +\mu\Big[u_\beta(x_1, x_2, \overset{(-)}{h}, t)\overset{(-)}{h}_{,\alpha} + u_\alpha(x_1, x_2, \overset{(-)}{h}, t)\overset{(-)}{h}_{,\beta}\Big], \qquad \alpha, \beta = 1, 2.
\end{aligned}
\tag{50}
$$

Substituting (48)-(50) into (44), we have the following system for Model VII

$$
\begin{aligned}
& \mu(hv_{\alpha 0})_{,\beta\beta} + \mu(hv_{\beta 0})_{,\alpha\beta} + \lambda(hv_{\gamma 0})_{,\gamma\alpha} - \frac{\lambda}{2}(\overset{(+)}{h}_{,\gamma}v_{\gamma 0})_{,\alpha} + \frac{\lambda}{2}v_{30,\alpha} - \frac{\mu}{2}(\overset{(+)}{h}_{,\alpha}v_{\beta 0})_{,\beta} \\
& -\frac{\mu}{2}(\overset{(+)}{h}_{,\beta}v_{\alpha 0})_{,\beta} + \frac{1}{2h}\Big\{\Big[\lambda\Big((hv_{\gamma 0})_{,\gamma} - \frac{1}{2}\overset{(+)}{h}_{,\gamma}v_{\gamma 0} + \frac{v_{30}}{2}\Big)\delta_{\alpha\beta} \\
& +\mu\Big((hv_{\alpha 0})_{,\beta} + (hv_{\beta 0})_{,\alpha} - \frac{\overset{(+)}{h}_{,\alpha}}{2}v_{\beta 0} - \frac{\overset{(+)}{h}_{,\beta}}{2}v_{\alpha 0}\Big)\Big]\overset{(-)}{h}_{,\beta} \\
& -\mu\Big((hv_{30})_{,\alpha} + \frac{1}{2}v_{\alpha 0} - \frac{\overset{(+)}{h}_{,\alpha}}{2}v_{30}\Big)\Big\} + Q_{\overset{(+)}{\nu}\alpha}\sqrt{(\overset{(+)}{h}_{,1})^2 + (\overset{(+)}{h}_{,2})^2 + 1} \\
& +\lambda\Big[u_\gamma(x_1, x_2, \overset{(-)}{h}, t)\overset{(-)}{h}_{,\gamma} - u_3(x_1, x_2, \overset{(-)}{h}, t)\Big]_{,\alpha} \\
& +\mu\Big[u_\beta(x_1, x_2, \overset{(-)}{h}, t)\overset{(-)}{h}_{,\alpha} + u_\alpha(x_1, x_2, \overset{(-)}{h}, t)\overset{(-)}{h}_{,\beta}\Big]_{,\beta} \\
& +\frac{1}{2h}\Big[\lambda\Big(u_\gamma(x_1, x_2, \overset{(-)}{h}, t)\overset{(-)}{h}_{,\gamma} - u_3(x_1, x_2, \overset{(-)}{h}, t)\Big)\delta_{\alpha\beta}
\end{aligned}
$$

$$
\begin{aligned}
& +\mu\Big(u_\beta(x_1, x_2, \overset{(-)}{h}, t)\overset{(-)}{h}_{,\alpha} + u_\alpha(x_1, x_2, \overset{(-)}{h}, t)\overset{(-)}{h}_{,\beta}\Big)\Big]\overset{(-)}{h}_{,\beta} \\
& +\frac{\mu}{2h}\Big(u_\alpha(x_1, x_2, \overset{(-)}{h}, t) - u_3(x_1, x_2, \overset{(-)}{h}, t)\overset{(-)}{h}_{,\alpha}\Big) + \Phi_{\alpha 0} = \rho h\frac{\partial^2 v_{\alpha 0}}{\partial t^2}, \quad \alpha = 1, 2,
\end{aligned}
\tag{51}
$$

$$\mu(hv_{30})_{,\beta\beta}+\frac{\mu}{2}v_{\beta 0,\beta}-\frac{\mu}{2}\Big(\overset{(+)}{h}_{,\beta}\,v_{30}\Big)_{,\beta}+\frac{1}{2h}\Big\{\mu\Big[(hv_{30})_{,\beta}+\frac{1}{2}v_{\beta 0}-\frac{\overset{(+)}{h}_{,\beta}}{2}v_{30}\Big]\overset{(-)}{h}_{,\beta}$$
$$-\lambda\Big[(hv_{\gamma 0})_{,\gamma}-\frac{1}{2}\overset{(+)}{h}_{,\gamma}v_{\gamma 0}\Big]-\frac{\lambda+2\mu}{2}v_{30}\Big\}+Q_{\overset{(+)}{\nu}3}\sqrt{(\overset{(+)}{h}_{,1})^2+(\overset{(+)}{h}_{,2})^2+1}$$
$$-\mu\Big[u_\beta(x_1,x_2,\overset{(-)}{h},t)-u_3(x_1,x_2,\overset{(-)}{h},t)\overset{(-)}{h}_{,\beta}\Big]_{,\beta}$$
$$-\frac{1}{2h}\Big\{\mu\Big[u_\beta(x_1,x_2,\overset{(-)}{h},t)-u_3(x_1,x_2,\overset{(-)}{h},t)\overset{(-)}{h}_{,\beta}\Big]\overset{(-)}{h}_{,\beta}$$
$$-\lambda\Big[u_\gamma(x_1,x_2,\overset{(-)}{h},t)\overset{(-)}{h}_{,\gamma}-u_3(x_1,x_2,\overset{(-)}{h},t)\Big]+2\mu u_3(x_1,x_2,\overset{(-)}{h},t)\Big\}$$
$$+\Phi_{30}=\rho h\frac{\partial^2 v_{30}}{\partial t^2}. \qquad (52)$$

Model VI and Model VIII follow from the Model V and Model VII after changing direction of the x_3–axis.

The well-known governing systems of Model I and Model II have the forms [1-3]

$$\mu\big[(hv_{\alpha 0,\beta})_{,\beta}+(hv_{\beta 0,\alpha})_{,\beta}\big]+\lambda(hv_{\gamma 0,\gamma})_{,\alpha}+Q_{\overset{(+)}{\nu}\alpha}\sqrt{(\overset{(+)}{h}_{,1})^2+(\overset{(+)}{h}_{,2})^2+1}$$
$$+Q_{\overset{(-)}{\nu}\alpha}\sqrt{(\overset{(-)}{h}_{,1})^2+(\overset{(-)}{h}_{,2})^2+1}+\Phi_{\alpha 0}=\rho h\frac{\partial^2 v_{\alpha 0}}{\partial t^2},\quad \alpha=1,2, \qquad (53)$$

$$\mu(hv_{30,\beta})_{,\beta}+Q_{\overset{(+)}{\nu}3}\sqrt{(\overset{(+)}{h}_{,1})^2+(\overset{(+)}{h}_{,2})^2+1}$$
$$+Q_{\overset{(-)}{\nu}3}\sqrt{(\overset{(-)}{h}_{,1})^2+(\overset{(-)}{h}_{,2})^2+1}+\Phi_{30}=\rho h\frac{\partial^2 v_{30}}{\partial t^2}, \qquad (54)$$

and [3]

$$\mu(hv_{\alpha 0})_{,\beta\beta}+(\lambda+\mu)(hv_{\gamma 0})_{,\gamma\alpha}-(\ln h)_{,\beta}\{\lambda\delta_{\alpha\beta}(hv_{\gamma 0})_{,\gamma}$$
$$+\mu[(hv_{\alpha 0})_{,\beta}+(hv_{\beta 0})_{,\alpha}]\}+2\mu\,\Psi_{\alpha\beta,\beta}(x_1,x_2,t)+\lambda\,\Psi_{kk,\alpha}(x_1,x_2,t)-(\ln h)_{,\beta} \qquad (55)$$
$$\times[\lambda\delta_{\alpha\beta}\Psi_{kk}(x_1,x_2,t)+2\mu\,\Psi_{\alpha\beta}(x_1,x_2,t)]+\Phi_{\alpha 0}(x_1,x_2,t)=\rho h\frac{\partial^2 v_{\alpha 0}}{\partial t^2},\quad \alpha=1,2;$$
$$\mu(hv_{30})_{,\beta\beta}-(\ln h)_{,\beta}\,\mu(hv_{30})_{,\beta}+2\mu\,\Psi_{3\beta,\beta}(x_1,x_2,t)$$
$$-2\mu(\ln h)_{,\beta}\,\Psi_{3\beta}(x_1,x_2,t)+\Phi_{30}(x_1,x_2,t)=\rho h\frac{\partial^2 v_{30}}{\partial t^2}, \qquad (56)$$

where

$$\Psi_{3\beta}\,(x_1,x_2,t):=$$

$$\frac{1}{2}\Big[u_3\left(x_1,x_2,\overset{(-)}{h},t\right)\overset{(-)}{h}_{,\beta}-u_3\left(x_1,x_2,\overset{(+)}{h},t\right)\overset{(+)}{h}_{,\beta}$$

$$+u_\beta\left(x_1,x_2,\overset{(+)}{h},t\right)-u_\beta\left(x_1,x_2,\overset{(-)}{h},t\right)\Big],$$

respectively.

Similarly, the following models can be constructed:

Model IX.

$$On\ the\ upper\ surface\ \overset{(+)}{h}\ the\ stress\ vector\ is\ known,$$

$$on\ the\ lower\ surface\ \overset{(-)}{h}\ the\ quantities\ (8)\ are\ known.$$

Model X.

$$On\ the\ upper\ surface\ \overset{(+)}{h}\ the\ stress\ vector\ is\ known,$$

$$on\ the\ lower\ surface\ \overset{(-)}{h}\ the\ quantities\ (9)\ are\ known.$$

Model XI.

$$On\ the\ upper\ surface\ \overset{(+)}{h}\ the\ displacements\ are\ known,$$

$$on\ the\ lower\ surface\ \overset{(-)}{h}\ the\ quantities\ (8)\ are\ known.$$

Model XII.

$$On\ the\ upper\ surface\ \overset{(+)}{h}\ the\ displacements\ are\ known,$$

$$on\ the\ lower\ surface\ \overset{(-)}{h}\ the\ quantities\ (9)\ are\ known.$$

Model XIII.

$$On\ the\ upper\ surface\ \overset{(+)}{h}\ the\ quantities\ (8)\ are\ known,$$

$$on\ the\ lower\ surface\ \overset{(-)}{h}\ the\ stress\ vector\ is\ known.$$

Model XIV.

$$On\ the\ upper\ surface\ \overset{(+)}{h}\ the\ quantities\ (9)\ are\ known,$$

$$on\ the\ lower\ surface\ \overset{(-)}{h}\ the\ stress\ vector\ is\ known.$$

Model XV.

$$On\ the\ upper\ surface\ \overset{(+)}{h}\ the\ quantities\ (8)\ are\ known,$$

$$on\ the\ lower\ surface\ \overset{(-)}{h}\ the\ displacements\ are\ known.$$

Model XVI.

$$On\ the\ upper\ surface\ \overset{(+)}{h}\ the\ quantities\ (9)\ are\ known,$$

$$on\ the\ lower\ surface\ \overset{(-)}{h}\ the\ displacements\ are\ known.$$

Similar hierarchical models can be constructed for composite prismatic shells as well.

3. Analysis of the Constructed Models

Models III-VIII are suggested in the present paper for the first time and, therefore, are not studied at all. Model II, actually, is not investigated. Model I is studied sufficiently well even in the case of cusped prismatic shells, i.e., in the case of prismatic shells with a cusped edge $\omega_0 \subseteq \partial\omega$, where the thickness $2h(x_1, x_2)$ vanishes:

$$\omega_0 := \{(x_1, x_2) \in \partial\omega : 2h(x_1, x_2) = 0\}.$$

Evidently, ω_0 is a closed set.

Dirichlet Problem. Find a solution $v_{i0} \in C^2(\omega) \cap C(\overline{\omega})$, $i = 1, 2, 3$, of the governing system in ω, satisfying the boundary conditions

$$v_{i0}(x_1, x_2) = \varphi_i(x_1, x_2), \quad (x_1, x_2) \in \partial\omega, \quad i = 1, 2, 3,$$

where φ_i, $i = 1, 2, 3$, are given continuous on $\partial\omega$ functions.

Keldysh Problem. Find a bounded solution $v_{i0} \in C^2(\omega) \cap C(\omega \backslash \omega_0)$, $i = 1, 2, 3$, of the governing system in ω, satisfying the boundary conditions

$$v_{i0}(x_1, x_2) = \varphi_i(x_1, x_2), \quad (x_1, x_2) \in \partial\omega \backslash \omega_0, \quad i = 1, 2, 3,$$

where φ_i, $i = 1, 2, 3$, are given continuous on $(\partial\omega) \setminus \omega_0$ functions.

Here C^2 is a class of twice continuously differentiable functions in the domain under consideration; C is a class of continuous functions on the sets under consideration.

Considering cusped prismatic shells within the framework of models I-XVI, governing systems are systems of partial differential equations with order degeneration. Since the governing systems are not degenerate with respect to t by $t = 0$, setting initial conditions do not posses peculiarities in contrast to setting boundary conditions with respect to space variables at cusped edges.

The question of consideration of prismatic shells with cusped edges within the framework of Model I was raised by I. Vekua [1, 2], concerning studies of the governing system of the general N-th order approximation see [3] and references therein. System (53),(54) was explored in [5,6] (see also [3] and references therein), where the main peculiarities of the well-posedeness of boundary conditions in displacements (i.e. for $v_{i0}, \quad i = 1, 2, 3$) are established:

on the non-cusped edge $\partial\omega\backslash\omega_0$ boundary conditions can always be prescribed;

on the cusped edge ω_0

(i) boundary conditions should be prescribed (Dirichlet type problem) if

$$\frac{\partial h}{\partial \nu} = +\infty; \tag{57}$$

(ii) boundary conditions should not be prescribed (Keldysh type problem) if

$$\frac{\partial h}{\partial \nu} \geq 0, \tag{58}$$

where ν is the inward normal to $\omega_0 \subseteq \partial\omega$.

Criterion in the integral form is given in [7]:

on the cusped edge ω_0

(i) boundary conditions should be prescribed if

$$\int_P^Q h^{-1} d\nu < +\infty; \tag{59}$$

(ii) boundary conditions should not be prescribed if

$$\int_P^Q h^{-1} d\nu = +\infty, \tag{60}$$

where $P \in \omega_0, \ Q \in \omega$.

The last criterion in the N-th approximation looks like:

on the cusped edge boundary conditions for

$$v_{ir}, \quad i = 1, 2, 3, \quad r = 0, 1, ..., N,$$

should be prescribed if

$$\int_P^Q h^{-2r-1} d\nu < +\infty$$

and should not be prescribed if

$$\int_P^Q h^{-2r-1} d\nu = +\infty.$$

Let us note that as it is easily seen from systems (53), (54); (55), (56); (22), (23), and (32), (33) systems (53), (54),(22) and (32) with respect to $v_{\alpha 0}$, $\alpha = 1, 2$, and equations (54), (56),(23) and (33) with respect to v_{30} we can consider separately.

If the thickness has the form

$$2h(x_1, x_2) = h_0 x_2^{\kappa}, \quad h_0 = const > 0, \quad \kappa = const \geq 0, \tag{61}$$

and $\partial\omega$ contains a segment of the x_1-axis, then (57), (59) mean that $0 \leq \kappa < 1$, while (58), (60) mean that $\kappa \geq 1$. Evidently, if $0 \leq \kappa < 1$, a profile (a normal cross-section of the prismatic shell at the cusped edge) has a smooth boundary, while if $\kappa \geq 1$, the profile is not smooth, namely, ends with an angle $\varphi \in [0, \pi[$ at cusped edge.

Cusped prismatic shells of the form (61) are investigated at most (see [3,8-10] and references therein). When ω is a half-plane $x_2 \geq 0$, the Flamant, Cerutti, and Carothers type problems are solved in explicit forms, which in the particular case $\kappa = 0$ coincide with the classical Flamant, Cerutti, and Carothers formulas for the plate of constant thickness [11-14].

Elastic equilibrium of a cusped prismatic shell under action of an arbitrary stress vector concentrated along the cusped edge is solved in the explicit form (quadratures) [15].

System (32) (Model IV) provided $\Psi_{33} \equiv 0$ and system (53) (Model I) coincide.

System (55) (Model II) provided $\Psi_{33} \equiv 0$ and system (22) (Model III) coincide.

So, systems (32) and (53) coincide up to the known summand Ψ_{33}. Therefore, all the results obtained for system (53) can appropriately be reformulated for system (32).

Note that systems (32) and (53) for the symmetric ($\overset{(-)}{h}(x_1, x_2) = -\overset{(+)}{h}(x_1, x_2)$) prismatic shells of constant thickness $2h = const$ coincide with the system of the plane strain.

Systems (22) and (55) coincide up to the known summand $\Psi_{33,\alpha}$, $\alpha = 1, 2$. Therefore, systems (22) and (55) can be studied by the same methods and the results will be qualitatively the same (e.g. in the sense of well-posedeness of boundary value problems).

Equation (23) (Model III) and equation (54) (Model I) coincide up to the known summand

$$\mu[u_\beta(x_1, x_2, \overset{(+)}{h}, t) - u_\beta(x_1, x_2, \overset{(-)}{h}, t)]_{,\beta}.$$

Hence, all the results obtained for equation (54) can appropriately be reformulated for equation (23).

Equations (33) (Model IV) and (56) (Model II) coincide up to their known parts. Therefore, they can be studied by the same methods and the results will be qualitatively the same (e.g. in the sense of well-posedeness of boundary value problems).

Similar analysis can be carried out with respect to Models V-XVI.

Considering models I-XVI in the case (61) with $\partial\omega$ containing a segment of the x_1-axis, we can take arbitrarily functions $\overset{(+)}{h}(x_1,x_2)$ and $\overset{(-)}{h}(x_1,x_2)$, provided their difference $\overset{(+)}{h}(x_1,x_2) - \overset{(-)}{h}(x_1,x_2)$, i.e., thickness varies according to (61). In particular, (61) will be realized if we assume

$$\overset{(+)}{h}(x_1,x_2) = \overset{(+)}{h}_0 x_2^{\kappa}, \quad \overset{(-)}{h}(x_1,x_2) = \overset{(-)}{h}_0 x_2^{\kappa}, \quad \kappa = const \geq 0,$$

$$\overset{(+)}{h}_0, \overset{(-)}{h}_0 = const, \quad \overset{(+)}{h}_0 - \overset{(-)}{h}_0 > 0.$$

If $\kappa > 0$, we have to do with a cusped prismatic shell with a cusped edge by $x_2 = 0$. If $\kappa = 0$, the prismatic shell will be of constant thickness h_0, which in the symmetric case $\overset{(-)}{h} = -\overset{(+)}{h}$ will be a plate.

The following theorem is true [16] (compare with [17], where $m_1 = 0$)

Theorem 3.1 : *If the coefficients a_α, $\alpha = 1,2$, and c of the equation*

$$x_2^{m_\alpha} u,_{\alpha\alpha} + a_\alpha(x_1,x_2)u,_\alpha + c(x_1,x_2)u = 0, \quad c \leq 0, \quad m_\alpha = const \geq 0, \quad \alpha = 1,2,$$

are analytic in $\overline{\omega}$ bounded by a sufficiently smooth arc $(\partial\omega \setminus \omega_0)$ lying in the half-plane $x_2 \geq 0$ and by a segment ω_0 of the x_1-axis, then

(i) if either $m_2 < 1$, or $m_2 \geq 1$, $a_2(x_1,x_2) < x_2^{m_2-1}$ in $\bar{I}_\delta$ for some $\delta = const > 0$, where

$$I_\delta := \{(x_1,x_2) \in \omega \, : \, 0 < x_2 < \delta\},$$

the Dirichlet problem is correct;

(ii) if $m_2 \geq 1$, $a_2(x_1,x_2) \geq x_2^{m_2-1}$ in I_δ and $a_1(x_1,x_2) = O(x_2^{m_1})$, $x_2 \to 0_+$ (O is the Landau symbol), the Keldysh problem is correct.

Remark 1 : If $1 < m_2 < 2$, $b(x,0) \leq 0$, the Dirichlet problem is correct.

Using the method applied in [18] (see pages 58, 68-74), it is not difficult to verify that the theorem is also true for Hölder continuous c and a_α, $\alpha = 1,2$, on $\overline{\omega}$, provided:

(i) $\lim\limits_{x_2 \to 0_+} x_2^{1-m_2} a_2(x_1,x_2) = a_0 = const < 1$ for $(x_1,0) \in \omega_0$ when $0 \leq m_2 < 1$;

(ii) if $a_2(x_1^0,0) = 0$ for a fixed $(x_1^0,0) \in \omega_0$ when $1 < n < 2$, then there exists such a $\delta = const > 0$ that $a_2(x_1^0,x_2) = \kappa(x_1^0,x_2) \cdot x_2$ with bounded $\kappa(x_1^0,x_2)$ for $0 \leq x_2 < \delta$.

Since in each of Models I and II qualitative properties by setting boundary conditions for v_{10}, v_{20}, and v_{30} are the same, in order to compare Model I and Model

II for the sake of clearness and simplicity we assume $v_{10} \equiv 0,\ (u_1 \equiv 0)$, $v_{20} \equiv 0,\ (u_2 \equiv 0),\ v_{30} = v_{30}(x_2)$, then we get

$$\left(x_2^\kappa v_{30,2}\right)_{,2} + T_1(x_2) = 0, \quad x_2 \in]0, L[\ , \quad L = const > 0, \tag{62}$$

for Model I (by the way also for Model III),

$$\left(x_2^\kappa v_{30}\right)_{,22} - \kappa x_2^{-1}\left(x_2^\kappa v_{30}\right)_{,2} + T_2(x_2) = 0, \quad x_2 \in]0, L[\ , \quad L = const > 0, \tag{63}$$

for Model II (by the way also for Model IV), where

$$\mu\frac{h_0}{2}T_1(x_2) := Q_{\overset{(+)}{\nu}3}\sqrt{(\overset{(+)}{h}_{,2})^2 + 1} + Q_{\overset{(-)}{\nu}3}\sqrt{(\overset{(-)}{h}_{,2})^2 + 1} + \Phi_{30},$$

$$\mu\frac{h_0}{2}T_2(x_2) := \mu\Big[u_3(x_2, \overset{(-)}{h}, t)\overset{(-)}{h}_{,2} - u_3(x_2, \overset{(+)}{h}, t)\overset{(+)}{h}_{,2}\Big]_{,2}$$
$$-\mu\kappa x_2^{-1}\Big[u_3(x_2, \overset{(-)}{h}, t)\overset{(-)}{h}_{,2} - u_3(x_2, \overset{(+)}{h}, t)\overset{(+)}{h}_{,2}\Big] + \Phi_{30}$$

are known.

Note that for Model V and Model VII we have

$$\left(x_2^\kappa v_{30}\right)_{,22} - \frac{1}{h_0}(\overset{(+)}{h}_{,2}v_{30})_{,2}$$
$$+\frac{x_2^{-\kappa}}{\mu h_0^2}\Big\{\mu\big[h_0\left(x_2^\kappa v_{30}\right)_{,2} - \overset{(+)}{h}_{,2}v_{30}\big]\overset{(-)}{h}_{,2} - \frac{\lambda + 2\mu}{2}v_{30}\Big\} + T_3(x_2) = 0,$$

where

$$\mu\frac{h_0}{2}T_3(x_2) := \mu\Big[u_3(x_2, \overset{(-)}{h}, t)\overset{(-)}{h}_{,2}\Big]_{,2} + Q_{\overset{(+)}{\nu}3}\sqrt{(\overset{(+)}{h}_{,2})^2 + 1}$$
$$+\frac{x_2^{-\kappa}}{h_0}\Big[\mu u_3(x_2, \overset{(-)}{h}, t)\big(\overset{(-)}{h}_{,2}\big)^2 + (\lambda + 2\mu)u_3(x_2, \overset{(-)}{h}, t)\Big] + \Phi_{30}$$

is known. In particular,

$$x_2^2 v_{30,22} + \kappa x_2 v_{30,2} + \Big\{\kappa\Big[\frac{\overset{(+)}{h_0}}{h_0}\Big(1 - \kappa\frac{\overset{(-)}{h_0}}{h_0}\Big) - 1\Big]$$
$$-\frac{\lambda + 2\mu}{\mu h_0^2}x_2^{2(1-\kappa)}\Big\}v_{30} - x_2^{2-\kappa}T_3(x_2) = 0,$$

$$\mu\frac{h_0}{2}T_3(x_2) := \mu\Big[u_3(x_2, \overset{(-)}{h}, t)\kappa\overset{(-)}{h}_0\chi_2^{\kappa-1}\Big]_{,2} + Q_{\overset{(+)}{\nu}3}\sqrt{(\kappa\overset{(+)}{h}_0)^2\chi_2^{2(\kappa-1)}+1}$$

$$+\frac{\mu(\kappa\overset{(-)}{h}_0)^2 x_2^{\kappa-2}}{h_0}u_3(x_2, \overset{(-)}{h}, t) + (\lambda+2\mu)\frac{x_2^{-\kappa}}{h_0}u_3(x_2, \overset{(-)}{h}, t) + \Phi_{30}.$$

General solutions of equations (62) and (63) have the forms

$$v_{30}^{I}(x_2) = \frac{1}{\kappa-1}\int\limits_{x_2^0}^{x_2}(x_2^{-\kappa}-\xi^{-\kappa})T_1(\xi)d\xi + c_1x_2^{1-\kappa} + c_2 \qquad \text{for} \qquad \kappa \neq 1 \tag{64}$$

$$v_{30}^{I}(x_2) = \int\limits_{x_2^0}^{x_2}T_1(\xi)ln\frac{\xi}{x_2}d\xi + c_1lnx_2 + c_2 \qquad \text{for} \qquad \kappa = 1 \tag{65}$$

and

$$v_{30}^{II}(x_2) = \frac{x_2^{-\kappa}}{\kappa+1}\int\limits_{x_2^0}^{x_2}(x_2^{\kappa+1}-\xi^{\kappa+1})\xi^{-\kappa}T_2(\xi)d\xi + c_1x_2 + c_2x_2^{-\kappa}, \tag{66}$$

respectively, where $x_2 \in]0, L[$ and c_α, $\alpha = 1, 2$, are arbitrary constants.

Under some restriction on $T_\alpha(x_2)$, $\alpha = 1, 2$, the integral summands in (64) and (65) become continuous on $[0, L]$. (64)-(66) can always satisfy the boundary condition

$$v_{30}(L) = v_{30}^{L} = const \tag{67}$$

at the non-cusped edge $x_2 = L$. Taking into account (67), from (64)-(66) it follows that either

$$c_1 = L^{\kappa-1}\Big[v_{30}^{L} - c_2 - \frac{1}{\kappa-1}\int\limits_{x_2^0}^{L}(L^{-\kappa}-\xi^{-\kappa})T_1(\xi)d\xi\Big] \quad for\ \ 0 \le \kappa < 1 \ \ and \ \ for\ \kappa > 1, \tag{68}$$

$$c_1 = \ln^{-1}L\Big[v_{30}^{L} - c_2 - \int\limits_{x_2^0}^{L}T_1(\xi)\ln\frac{\xi}{L}d\xi\Big] \ \ for\ \ \kappa = 1, \tag{69}$$

and

$$c_1 = L^{-1}\Big[v_{30}^L - c_2L^{-\kappa} - \frac{L^{-\kappa}}{\kappa+1}\int\limits_{x_2^0}^{L}(L^{\kappa+1}-\xi^{\kappa+1})T_2(\xi)d\xi\Big] \ \ for \ \ \kappa \geq 0, \tag{70}$$

respectively, or

$$c_2 = v_{30}^L - c_1L^{1-\kappa} - \frac{1}{\kappa-1}\int\limits_{x_2^0}^{L}(L^{-\kappa}-\xi^{-\kappa})T_1(\xi)d\xi \ \ for \ \ 0 \leq \kappa < 1 \ \ and \ \ for \ \ \kappa > 1, \tag{71}$$

$$c_2 = v_{30}^L - c_1L^{1-\kappa} - \int\limits_{x_2^0}^{L}T_1(\xi)\ln\frac{\xi}{L}d\xi \ \ for \ \ \kappa = 1, \tag{72}$$

and

$$c_2 = L^{\kappa}\Big[v_{30}^L - c_1L - \frac{L^{-\kappa}}{\kappa+1}\int\limits_{x_2^0}^{L}(L^{\kappa+1}-\xi^{\kappa+1})T_2(\xi)d\xi\Big] \ \ for \ \ \kappa \geq 0, \tag{73}$$

respectively.

Now we try to satisfy the boundary condition

$$v_{30}(0) = v_{30}^0 = const \tag{74}$$

at the cusped edge $x_2 = 0$. As is obvious, (64)-(66) are unbounded as $x_2 \to 0+$ by $\kappa > 1$, $\kappa = 1$ and $\kappa > 0$, respectively, and in these cases (74) can not be fulfilled. For boundedness of solutions we have to take $c_1 = 0$ and $c_2 = 0$ for (64), (65) and (66), respectively. So, Keldysh type problem can be well-posed.

Unique bounded solutions $v_{30} \in C^2(]0, L[)$ of the Keldysh type problems (62),(67) with $\kappa \geq 1$ and (63), (67) with $\kappa > 0$, by virtue of (70)-(72), have the forms

$$v_{30}^I(x_2) = \frac{1}{\kappa-1}\int\limits_0^{x_2}(x_2^{-\kappa}-\xi^{-\kappa})T_1(\xi)d\xi + v_{30}^L - \frac{1}{\kappa-1}\int\limits_0^{L}(L^{-\kappa}-\xi^{-\kappa})T_1(\xi)d\xi, \quad \kappa > 1,$$

(provided $T_1(x_2)$ is integrable on $]0, L[$ and $T_1(x_2) = O(x_2^{\delta})$, $\delta > \kappa-1$, $x_2 \to 0+$)

$$v_{30}^I(x_2) = \int\limits_0^{x_2}T_1(\xi)\ln\frac{\xi}{x_2}d\xi + v_{30}^L - \int\limits_0^{L}T_1(\xi)\ln\frac{\xi}{L}d\xi, \quad \kappa = 1,$$

(provided $T_1(x_2)$ is integrable and bounded on $]0, L[$) and

$$v_{30}^{II}(x_2) = \frac{x_2^{-\kappa}}{\kappa+1}\int\limits_{x_2^0}^{x_2}(x_2^{\kappa+1} - \xi^{\kappa+1})\xi^{-\kappa}T_2(\xi)d\xi$$

$$+L^{-1}\Big[v_{30}^L - \frac{L^{-\kappa}}{\kappa+1}\int\limits_{x_2^0}^{L}(L^{\kappa+1} - \xi^{\kappa+1})T_2(\xi)d\xi\Big]x_2, \quad \kappa > 0,$$

(provided $T_2(\xi)$ is integrable on $]0, L[$ and for $\kappa \geq 2$

$$T_2(x_2) = O(x_2^{\delta}), \ \ \delta > \kappa - 2, \ \ x_2 \to 0+).$$

(64) and (66) can fulfil (74) if correspondingly $0 \leq \kappa < 1$ and $\kappa = 0$. In these cases from (74) with $x_2^0 = 0$, taking into account (68), (69), there follows that

$$c_2 = v_{30}^0,$$

provided $T_\alpha(x_2), \quad \alpha = 1, 2$, are integrable on $[0, L]$.

Thus, unique solutions $v_{30} \in C^2(]0, L[) \cap C([0, L])$ of the Dirichlet type problems (62), (67), (70) with $0 \leq \kappa < 1$ and (63), (67), (70) with $\kappa = 0$, have the forms

$$v_{30}^I(x_2) = \frac{1}{\kappa-1}\int\limits_0^{x_2}(x_2^{-\kappa} - \xi^{-\kappa})T_1(\xi)d\xi$$

$$+L^{\kappa-1}\Big[v_{30}^L - v_{30}^0 - \frac{1}{\kappa-1}\int\limits_0^{L}(L^{-\kappa} - \xi^{-\kappa})T_1(\xi)d\xi\Big]x_2^{1-\kappa} + v_{30}^0, \ \ 0 \leq \kappa < 1,$$

and

$$v_{30}^{II}(x_2) = \int\limits_0^{x_2}(x_2 - \xi)T_2(\xi)d\xi + L^{-1}\Big[v_{30}^L - v_{30}^0 - \int\limits_0^{L}(L - \xi)T_2(\xi)d\xi\Big]x_2 + v_{30}^0,$$

respectively.

So, we arrive at the following conclusion:

(i) If $0 \leq \kappa < 1$, *solution (64) is continuous on* $[0, L]$, *therefore boundary conditions can be satisfied on both the non-cusped* $(x_2 = L)$ *and cusped* $(x_2 = 0)$ *edges and by means of (64) a unique solution of the Dirichlet type boundary value problem can be written in the explicit form;*

(ii) If $\kappa \geq 1$, *taking* $c_1 = 0$ *we avoid unboundedness of the solution (64) as* $x_2 \to 0+$ *and, in view of (64), a unique solution can be constructed by means of a boundary condition at the non-cusped edge (Keldysh type boundary value problem);*

(iii) For any $\kappa > 0$ *solution (65) is unbounded as* $x_2 \to 0+$ *unless* $c_2 = 0$ *and if* $c_2 = 0$, *by virtue of (65), we can construct in the explicit form a unique solution of the Keldysh problem with a prescribed value at the non-cusped edge* $(x_2 = L)$*;*

(iv) Unbounded solutions of Keldysh type problems (62), (67) and (63), (67) are defined up to the summand $c_1 x_2^{1-\kappa}$ for $\kappa > 1$ ($c_1 \ln x_2$ for $\kappa = 1$) and $c_2 x_2^{-\kappa}$, respectively, where c_α, $\alpha = 1, 2$, are arbitrary constants.

From conclusions (i)-(iii) it is clear that the presence of a cusped edge depending, on sharpening geometry at a cusped edge, causes change of the Dirichlet type problem by the Keldysh type problem for ensuring well-posedeness of boundary value problems in displacements. Moveover, in contrast to Model I, where depending on sharpening geometry of the cusped edge (in other words of the kind of degeneration of equations under consideration) arise both the Dirichlet and Keldysh type problems, in Model II presence of cusped edges always demands consideration of the Keldysh type problem.

From (64), (65) it is easily seen that under obvious restrictions on $T_1(x_2)$ the boundary value problem, when on the non-cusped edge boundary conditon (67) and on the cusped edge boundary conditon

$$X_{320}(0) = \lim_{x_2 \to 0+} \mu h v_{30,\beta} = X_0 = const$$

are prescribed, is correct. The solution v_{30}^{I} can be represented in the explicit form. If on the non-cusped edge we set boundary conditon

$$X_{320}(L) = X_L = const,$$

then a solution v_{30}^{I} will be determined up to the additive constant. If $0 \le \kappa < 1$, the last constant can be uniquely determined by boundary condition (74). Hence, the last boundary value problem will be correct.

From (66) it is easily seen that under obvious restrictions on $T_2(x_2)$ the boundary value problem, when on the non-cusped edge boundary condition

$$X_{320}(L) = \mu \Big[\frac{h_0}{2} \big(x_2^{\kappa} v_{30} \big)_{,2} \Big|_{x_2 = L} + u_3 \Big(x_1, L, \overset{(-)}{h}_0 L^{\kappa} \Big) \kappa \overset{(-)}{h}_0 L^{\kappa-1}$$
$$- u_3 \Big(x_1, L, \overset{(+)}{h}_0 L^{\kappa} \Big) \kappa \overset{(+)}{h}_0 L^{\kappa-1} \Big] = X_L$$

is prescribed, a solution v_{30}^{II} can be determined up to the addend $c_2 x_2^{-\kappa}$ and is unbounded unless $c_2 = 0$.

$$X_{320}(0) = \mu \lim_{x_2 \to 0+} \Big\{ \frac{h_0}{2} x_2^{\kappa} \int_{x_2^0}^{x_2} \xi^{-\kappa} T_2(\xi) d\xi$$
$$+ \Big[u_3 \big(x_1, x_2, \overset{(-)}{h}_0 x_2^{\kappa} \big) \overset{(-)}{h}_0 - u_3 \big(x_1, x_2, \overset{(+)}{h}_0 x_2^{\kappa} \big) \overset{(+)}{h}_0 \Big] \kappa x_2^{\kappa-1} \Big\}$$

and can never be prescribed arbitrarily, i.e. in Model II cusped edges should be released from boundary conditions.

4. Conclusions

1) Hierarchical models for cusped, in general, elastic prismatic shells under mixed conditions on the face surfaces are constructed. Such models are important in applications, e.g., in the study of seismic processes [19].

2) In models I-IV equations charachterizing tension-compression (see, correspondingly, systems (53), (55), (22), and (32)) and bending (more precisely deflection in $N = 0$ approximation (see, correspondingly, equations (54), (56), (23), and (33)) can be considered separately. This is not the fact in the case of models V-VIII.

3) Setting boundary conditions at edges in displacements depends on what is considered to be known on the face surfaces, stresses or displacements; moreover, assuming as known on the face surfaces at least one of the stress vector components, it will depend on the sharpening geometry of the cusped edge. The criteria, when the BCs are classical (Dirichlet type) or nonclassical (Keldysh type), are established (compare with results in [3]). If on both the face surfaces only displacements are assumed as known, at cusped edges neither displacement nor integrated stress vectors can be prescribed. In other words, cusped edge should be released from boundary conditions.

4) In the static case, in the $N = 0$ approximation, when the cusped prismatic shell-like body projection is a half-plane $x_2 \geq 0$, the thickness $2h = h_0 x_2^{\kappa}$, $h_0 =$ const > 0, $\kappa =$ const ≥ 0, and at the edge $x_2 = 0$ integrated stress vector (either distributed but concentrated along the cusped edge or concentrated at point of the cusped edge load) is applied, the problem is solved in the explicit form (in quadratures). For $\kappa > 0$ prismatic shell-like body is cusped one; for $\kappa = 0$, in particular, we have a plate of constant thickness.

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