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Quarter-Symmetric Non-Metric Connection on Pseudosymmetric Kenmotsu Manifolds

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In this paper we shall introduce a quarter-symmetric non-metric connection in a pseudosymmetric Kenmotsu manifold and find out some of its properties. We shall show the existence of quarter-symmetric non-metric connection on Kenmotsu manifold. Also we state the definitions of Weyl-pseudosymmetric Kenmotsu manifold and Ricci pseudosymmetric Kenmotsu manifold with respect to quarter-symmetric non-metric connection. Next we show some results on Weyl-pseudosymmetric Kenmotsu manifold and partially Ricci pseudosymmetric Kenmotsu manifold with respect to quarter-symmetric non-metric connection and η-Einstein manifold. At the end we show an example of pseudosymmetric Kenmotsu manifold with respect to quarter-symmetric non-metric connection.

Keywords: Kenmotsu manifold, Quarter-Symmetric Non-Metric connection, Pseudosymmetric Kenmotsu Manifolds, Weyl-pseudosymmetric, Ricci pseudosymmetric.

AMS Subject Classification: 53C05, 53C15, 53D15.

1. Introduction

In 1987, M.C. Chaki and B. Chaki [11] studied pseudosymmetric manifolds with semisymmetric connection and many authors studied properties on this manifold. Also R. Deszcz et. al. studied Ricci-pseudosymmetric manifolds and pseudosymmetric manifolds [2], [3], [6], [7]. The conceptions of pseudosymmetric manifold are different with the above authors. In 2008, C. S. Bagewadi and et. al. studied pseudosymmetric Lorentzian $\alpha-$Sasakian manifolds in the Deszcz sense [10]. We shall study the properties of pseudosymmetric Kenmotsu manifolds and Ricci-pseudosymmetric Kenmotsu manifolds with respect to quarter-symmetric non-metric connection in the Deszcz sense.

A Riemannian manifold (M, g) of dimension n is called pseudosymmetric if the Riemannian curvature tensor R satisfies the conditions [1], [4], [7]

$$1. \quad (R(X,Y).R)(U,V,W) = L_R[((X \wedge Y) \circ R)(U,V,W)] \tag{1}$$

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for all vector fields X, Y, U, V, W on M , where$L_R \in C^\infty(M)$, $R(X,Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z$ and $X \wedge Y$ is an endomorphism defined by

$$(X \wedge Y)Z = g(Y,Z)X - g(X,Z)Y \tag{2}$$

2. $$(R(X,Y).R)(U,V,W) = R(X,Y)(R(U,V)W)$$

$$-R(R(X,Y)U,V)W - R(U,R(X,Y)V)W - R(U,V)(R(X,Y)W), \tag{3}$$

3. $$((X \wedge Y).R)(U,V,W) = (X \wedge Y)(R(U,V)W)$$

$$-R((X \wedge Y)U,V)W - R(U,(X \wedge Y)V)W - R(U,V)((X \wedge Y)W). \tag{4}$$

M is said to be pseudosymmetric of constant type if L is constant. A Riemannian manifold (M, g) is called quarter-symmetric if $R.R = 0$, where $R.R$ is the derivative of R by R.

Remark 1 : From [4], [5] we know that the $(0, k+2)$ tensor fields $R.T$ and $Q(g,T)$ are defined by
$(R.T)(X_1, ..., X_k; X, Y) = (R(X,Y).T)(X_1, ..., X_k)$
$= -T(R(X,Y)X_1, ..., X_k) - ... - T(X_1, ..., R(X,Y)X_k)$
$Q(g,T)(X_1, ..., X_k; X, Y) = -((X \wedge Y).T)(X_1, ..., X_k)$
$= T((X \wedge Y)X_1, ..., X_k) + ... + T(X_1, ..., (X \wedge Y)X_k)$, where T is a $(0, k)$ tensor field.

Let S and r denote the Ricci tensor and the scalar curvature tensor of M respectively. The operator Q and the $(0,2)-$tensor S^2 are defined by

$$S(X,Y) = g(QX,Y) \tag{5}$$

and

$$S^2(X,Y) = S(QX,Y) \tag{6}$$

The Weyl conformal curvature operator C is defined by

$$C(X,Y) = R(X,Y) - \frac{1}{n-2}[X \wedge QY + QX \wedge Y - \frac{r}{n-1}X \wedge Y]. \tag{7}$$

If $C = 0$, $n \geq 4$ then M is called conformally flat. If the tensor $R.C$ and $Q(g, C)$ are linearly dependent then M is called Weyl-pseudosymmetric. This is equivalent to

$$R.C(U,V,W;X,Y) = L_C[((X \wedge Y).C)(U,V)W], \tag{8}$$

holds on the set $U_C = \{x \in M : C \neq 0 \text{ at } x\}$, where L_C is defined on U_C. If $R.C = 0$, then M is called Weyl-semi-symmetric. If $\nabla C = 0$, then M is called conformally symmetric [10].

2. Preliminaries:

Let M be an almost contact metric manifold of dimension $2n+1$ with an almost contact metric structure (ϕ, ξ, η, g) where ϕ is $(1,1)$ tensor field, ξ is a contravariant vector field, η is a 1-form and g is an associated Riemannian metric such that,

$$\phi^2 = -I + \eta \otimes \xi, \tag{9}$$

$$\eta(\xi) = 1, \qquad \phi\xi = 0, \qquad \eta \circ \phi = 0, \tag{10}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{11}$$

and

$$g(X, \xi) = \eta(X), \tag{12}$$

$\forall \quad X, \quad Y \in \chi(M)$, then M is called a *Kenmotsu manifold* provided,

$$(\nabla_X \phi)(Y) = -g(X, \phi Y)\xi - \eta(Y)\phi X \tag{13}$$

and

$$\nabla_X \xi = X - \eta(X)\xi) \tag{14}$$

holds, where ∇ is affine connection on M [8], [9].

On a Kenmotsu manifold, it can be shown that

$$(\nabla_X \eta)Y = g(\phi X, \phi Y), \tag{15}$$

$$F(X, Y) = -F(Y, X), \tag{16}$$

where $F(X, Y) = g(\phi X, Y)$, is a fundamental 2-form.

Further on a Kenmotsu manifold the following relations hold, [8]

$$\eta(R(X, Y)Z) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X), \tag{17}$$

$$R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi, \tag{18}$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{19}$$

$$S(\xi, X) = S(X, \xi) = -2n\eta(X), \tag{20}$$

$$Q\xi = -2n\xi. \tag{21}$$

3. Quarter-symmetric non-metric connection on Kenmotsu manifold:

Let M be a Kenmotsu manifold with Levi-Civita connection ∇ and X, $Y \in \chi(M)$. We define a linear connection D on M by

$$D_X Y = \nabla_X Y + \eta(Y)\phi(X), \tag{22}$$

where η is a $1-$form and ϕ is a tensor field of type $(1,1)$. D is said to be quarter symmetric connection if $\bar{T}$, the torsion tensor with respect to the connection D, satisfies

$$\bar{T}(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y. \tag{23}$$

D is said to be non-metric connection if $(Dg) \neq 0$. Using (16) we have

$$(D_X g)(Y,Z) = -\{\eta(Y)g(\phi X, Z) + \eta(Z)g(\phi X, Y)\}. \tag{24}$$

A linear connection D is said to be a quarter-symmetric non-metric connection if it satisfies (22), (23) and (24).

Now we shall show the existence of the quarter-symmetric non-metric connection D on a Kenmotsu manifold M.

Theorem 3.1: *Let X, Y, Z be any vectors fields on a Kenmotsu manifold M with an almost structure (ϕ, ξ, η, g). Let us define a connection D by*

$$\begin{aligned}2g(D_X Y, Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y)\\ &+g([X,Y],Z) - g([Y,Z],X) + g([Z,X],Y)\\ &+g(\eta(Y)\phi X - \eta(X)\phi Y, Z) + g(\eta(X)\phi Z\\ &-\eta(Z)\phi X, Y) + g(\eta(Y)\phi Z + \eta(Z)\phi Y, X).\end{aligned} \tag{25}$$

Then D is a quarter-symmetric non-metric connection on M.

Proof: It can be verified that $D : (X,Y) \to D_X Y$ satisfies the following equations:

$$D_X(Y+Z) = D_X Y + D_X Z \tag{26}$$

$$D_{X+Y} Z = D_X Z + D_Y Z \tag{27}$$

$$D_{fX} Y = f D_X Y \tag{28}$$

$$D_X(fY) = f(D_X Y) + (Xf)Y \tag{29}$$

for all $X, Y, Z \in \chi(M)$ and for all f, all differentiable functions on M.
From (26), (27), (28) and (29) we can conclude that D is a linear connection on M. From (25) we have,
$D_X Y - D_Y X - [X, Y] = \eta(Y)\phi X - \eta(X)\phi Y$
or,

$$\bar{T}(X, Y) = \eta(Y)\phi X - \eta(X)\phi Y. \tag{30}$$

Again from (25) we get,
$2g(D_X Y, Z) + 2g(D_X Z, Y)$

$$= 2Xg(Y, Z) + 2\eta(Y)g(\phi X, Z) + 2\eta(Z)g(\phi X, Y)$$

$$(D_X g)(Y, Z) = -\{\eta(Y)g(\phi X, Z) + \eta(Z)g(\phi X, Y)\}. \tag{31}$$

This shows that D is a quarter-symmetric non-metric connection on M. ◇

Theorem 3.2: *Let D be a linear connection on a Kenmotsu manifold M, given by*

$$D_X Y = \nabla_X Y + H(X, Y), \tag{32}$$

where $H(X, Y)$ is a $(1, 2)$ tensor field and ∇ is Levi-Civita connection, satisfying (24). Then $H(X, Y) = \eta(Y)\phi(X)$.

Proof: Using (32) in the definition of torsion tensor, we get

$$\bar{T}(X, Y) = H(X, Y) - H(Y, X). \tag{33}$$

From (32), we have

$$g(H(X, Y), Z) + g(H(X, Z), Y) = -(D_X g)(Y, Z). \tag{34}$$

From (24), (32), (33) and (34) we have
$g(\bar{T}(X, Y), Z) + g(\bar{T}(Z, Y), X) + g(\bar{T}(Z, X), Y)$

$$= 2g(H(X, Y), Z) - (D_Z g)(X, Y) + (D_Y g)(X, Z) + (D_X g)(Y, Z).$$

We get from the above equation,
$g(H(X, Y), Z) = \frac{1}{2}[g(\bar{T}(X, Y), Z) + g(\bar{T}(Z, Y), X)$

$$+g(\bar{T}(Z, X), Y)] + [\eta(Y)g(\phi X, Z) + \eta(X)g(\phi Y, Z)].$$

Thus, we get
$H(X, Y) = \frac{1}{2}[\bar{T}(X, Y) + \tilde{T}(X, Y) + \tilde{T}(Y, X)] + [\eta(Y)\phi X + \eta(X)\phi Y]$,
where $\tilde{T}$ is a tensor field of type (1,2) defined by

$$g(\tilde{T}(X, Y), Z) = g(\bar{T}(Z, X), Y).$$

Thus $H(X, Y) = \eta(Y)\phi X$.
Hence $D_X Y = \nabla_X Y + \eta(Y)\phi X$. ◇

4. Curvature tensor and Ricci tensor with respect to quarter-symmetric non-metric connection D in a Kenmotsu manifold

Let $\bar{R}(X,Y)Z$ and $R(X,Y)Z$ be the curvature tensors on a Kenmotsu manifold M with respect to the quarter-symmetric non-metric connection D and with respect to the Riemannian connection ∇ respectively. A relation between the curvature tensors of M with respect to the quarter-symmetric non-metric connection D and the Riemannian connection ∇ is given by
$\bar{R}(X,Y)Z = R(X,Y)Z + 2\eta(Z)g(\phi X,Y)\xi$

$$+ g(X,Z)\phi Y - g(Y,Z)\phi X. \tag{35}$$

Also from (35) we obtain

$$\bar{S}(X,Y) = S(X,Y) + g(\phi X,Y), \tag{36}$$

where $\bar{S}$ and S are the Ricci tensors of the connections D and ∇ respectively. Contracting (36), we get

$$\bar{r} = r, \tag{37}$$

where $\bar{r}$ and r are the scalar curvature with respect to the connection D and ∇ respectively.

Let $\bar{C}$ be the conformal curvature tensors on Kenmotsu manifolds with respect to the connections D. Then
$\bar{C}(X,Y)Z = \bar{R}(X,Y)Z - \frac{1}{n-2}[\bar{S}(Y,Z)X - g(X,Z)\bar{Q}Y + g(Y,Z)\bar{Q}X$

$$- \bar{S}(X,Z)Y] + \frac{\bar{r}}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y], \tag{38}$$

where $\bar{Q}$ is the Ricci operator with the connection D on M and

$$\bar{S}(X,Y) = g(\bar{Q}X,Y), \tag{39}$$

$$\bar{S}^2(X,Y) = \bar{S}(\bar{Q}X,Y). \tag{40}$$

Now we shall prove the following theorem.

Theorem 4.1: *Let M be a Kenmotsu manifold with respect to the quarter-symmetric non-metric connection D, then the following relations hold:*

$$\bar{R}(\xi,X)Y = \eta(Y)X - g(X,Y)\xi + \eta(Y)\phi X, \tag{41}$$

$$\eta(\bar{R}(X,Y)Z) = g(X,Z)\eta(Y) - g(Y,Z)\eta(X) + 2g(\phi X,Y)\eta(Z), \tag{42}$$

$$\bar{R}(X,Y)\xi = \eta(X)Y - \eta(Y)X - \eta(Y)\phi X + \eta(X)\phi Y + 2g(\phi X, Y)\xi, \tag{43}$$

$$\bar{S}(X,\xi) = \bar{S}(\xi, X) = -2n\eta(X), \tag{44}$$

$$\bar{Q}X = QX + \phi X, \tag{45}$$

$$\bar{S}^2(X,\xi) = \bar{S}^2(\xi, X) = 4n^2\eta(X), \tag{46}$$

$$\bar{Q}\xi = -2n\xi. \tag{47}$$

Proof: Since M is a Kenmotsu with respect to the quarter-symmetric non-metric connection D,
then replacing $X = \xi$ in (35) and using (10) and (18) we get (41).
Using (10) and (17), from (35) we get (42).
To prove (43), we put $Z = \xi$ in (35) and then we use (19).
Replacing $Y = \xi$ in (36) and using (20) we get (44).
Using (36) and (39) we get (45).
Using (40), (44) and (45) we get (46).
Putting $X = \xi$ in (45) we obtain (47). ◇

5. Kenmotsu manifold with respect to the quarter-symmetric non-metric connection D satisfying the condition $\bar{C}.\bar{S} = 0$.

In this section we shall find out the characterization of Kenmotsu manifold with respect to the quarter-symmetric non-metric connection D satisfying the condition $\bar{C}.\bar{S} = 0$.
We define $\bar{C}.\bar{S} = 0$ on M by

$$(\bar{C}(X,Y).\bar{S})(Z,W) = -\bar{S}(\bar{C}(X,Y)Z, W) - \bar{S}(Z, \bar{C}(X,Y)W), \tag{48}$$

where $X, Y, Z, W \in \chi(M)$.

Theorem 5.1: *Let M be a Kenmotsu manifold with respect to the quarter-symmetric non-metric connection D. If $\bar{C}.\bar{S} = 0$, then*
$\bar{S}^2(X,Y) = -\{\frac{r}{(n-1)} + n - 2\}\bar{S}(X,Y) + 2n\{\frac{r}{n-1} + n + 2\}g(X,Y)]$

$$-2n(2n-1)\eta(X)\eta(Y) - (n-2)\bar{S}(\phi X, Y) \tag{49}$$

Proof: Let us consider M to be a Kenmotsu manifold with respect the quarter-symmetric non-metric connection D satisfying the condition $\bar{C}.\bar{S} = 0$. Then from (48), we get

$$\bar{S}(\bar{C}(X,Y)Z, W) + \bar{S}(Z, \bar{C}(X,Y)W) = 0, \tag{50}$$

where $X, Y, Z, W \in \chi(M)$. Now putting $X = \xi$ in (50), we get

$$\bar{S}(\bar{C}(\xi, X)Y, Z) + \bar{S}(Y, \bar{C}(\xi, X)Z) = 0. \tag{51}$$

Using (41) and (44) we have
$\bar{S}(\bar{C}(\xi, X)Y, Z) = [\frac{\bar{r}}{(n-1)(n-2)} - \frac{(n+2)}{n-2}][2n\eta(Z)g(X, Y) + \eta(Y)\bar{S}(X, Z)]$

$$+ \eta(Y)\bar{S}(\phi X, Z) + \frac{1}{n-2}[2n\eta(Z)\bar{S}(X, Y) + \bar{S}^2(X, Z)\eta(Y)] \tag{52}$$

and
$\bar{S}(\bar{C}(\xi, X)Y, Z) = [\frac{\bar{r}}{(n-1)(n-2)} - \frac{(n+2)}{n-2}][2n\eta(Y)g(X,) + \eta(Z)\bar{S}(X, Y)]$

$$+ \eta(Z)\bar{S}(\phi X, Y) + \frac{1}{n-2}[2n\eta(Y)\bar{S}(X, Z) + \bar{S}^2(X, Y)\eta(Z)]. \tag{53}$$

Using (52) and (53) in (51) we get
$2n[\frac{r}{(n-1)(n-2)} - \frac{(n+2)}{n-2}]\{\eta(Z)g(X, Y) + \eta(Y)g(X, Z)\} + \eta(Z)\bar{S}(\phi X, Y)$

$+[\frac{r}{(n-1)(n-2)} + 1]\{\eta(Z)\bar{S}(X, Y) + \eta(Y)\bar{S}(X, Z)\} + \eta(Y)\bar{S}(\phi X, Z)$

$$+\frac{1}{n-2}[\eta(Z)\bar{S}^2(X, Y) + \bar{S}^2(X, Z)\eta(Y)], \tag{54}$$

Replacing $Z = \xi$ in (54) and using (44) and (46) we get
$\bar{S}^2(X, Y) = -\{\frac{r}{(n-1)} + n - 2\}\bar{S}(X, Y) + 2n\{\frac{r}{n-1} + n + 2\}g(X, Y)]$

$$-2n(2n-1)\eta(X)\eta(Y) - (n-2)\bar{S}(\phi X, Y).$$ ◇

A Kenmotsu manifold M with the quarter-symmetric non-metric connection D is said to be $\eta-$Einstein if its Ricci tensor $\bar{S}$ is of the form

$$\bar{S}(X, Y) = Ag(X, Y) + B\eta(X)\eta(Y), \tag{55}$$

where A and B are smooth functions on M.
Now putting $X = Y = e_i$, $i = 1, 2, ..., 2n+1$ in (55) and taking summation for $1 \le i \le n$ we get,

$$A(2n + 1) + B = r. \tag{56}$$

Again replacing $X = Y = \xi$ in (55) we have

$$A + B = -2n. \tag{57}$$

Solving (56) and (57) we obtain
$A = \frac{r}{2n} + 1$ and $B = -[\frac{r}{2n} + 2n + 1]$.

Thus the Ricci tensor of an $\eta-$Einstein manifold with the quarter-symmetric non-metric connection D is given by

$$\bar{S}(X,Y) = [\frac{r}{2n}+1]g(X,Y) - [\frac{r}{2n}+2n+1]\eta(X)\eta(Y). \tag{58}$$

6. $\eta-$Einstein Kenmotsu manifold with respect to the quarter-symmetric non-metric connection D satisfying the condition $\bar{C}.\bar{S} = 0$.

Theorem 6.1: *Let M be an $\eta-$Einstein Kenmotsu manifold with the restriction $U = Y = \xi$ in $\chi(M)$. Then $\bar{C}.\bar{S} = 0$ iff*

$g(X,Z) = \eta(X)\eta(Z)$, where $X, Z \in \chi(M)$.

Proof: Let M be an $\eta-$Einstein Kenmotsu manifold with respect to the quarter-symmetric non-metric connection D satisfying $\bar{C}.\bar{S} = 0$. Using (48) in (58), we get

$\eta(\bar{C}(X,Y)Z)\eta(W) + \eta(\bar{C}(X,Y)W)\eta(Z) = 0.$

Using (38), (42), (44) and (58) in the above equation we obtain
$4g(\phi U,X)\eta(Y)\eta(Z) = \frac{n+1}{n-2}\{\frac{r}{2n(n-1)+1}\}[g(U,Y)\eta(X)\eta(Z)$

$$+g(U,Z)\eta(X)\eta(Y) - g(X,Y)\eta(U)\eta(Z) - g(X,Z)\eta(Y)\eta(U)]. \tag{59}$$

Putting $U = Y = \xi$ in (59) we get
$g(X,Z) = \eta(X)\eta(Z)$.
Conversely,
$\bar{C}.\bar{S} = 4g(\phi U,X)\eta(Y)\eta(Z) - \frac{n+1}{n-2}\{\frac{r}{2n(n-1)+1}\}[g(U,Y)\eta(X)\eta(Z)$
$+g(U,Z)\eta(X)\eta(Y) - g(X,Y)\eta(U)\eta(Z) - g(X,Z)\eta(Y)\eta(U)]$.
Putting $U = Y = \xi$ in the above equation we get
$\bar{C}.\bar{S} = -g(X,Z) + \eta(X)\eta(Z)$.
Thus $\bar{C}.\bar{S} = 0$. ◇

7. Ricci pseudosymmetric Kenmotsu manifolds with quarter-symmetric non-metric connection D

Theorem 7.1: *A Ricci pseudosymmetric Kenmotsu manifold M with quarter-symmetric non-metric connection D with restriction $Y = W = \xi \in \chi(M)$ and $L_{\bar{S}} = -1$ is an $\eta-$Einstein manifold.*

Proof: Kenmotsu manifold M with quarter-symmetric non-metric connection D is called a Ricci pseudosymmetric Kenmotsu manifold if

$$(\bar{R}(X,Y).\bar{S})(Z,W) = L_{\bar{S}}[((X \wedge Y).\bar{S})(Z,W)], \tag{60}$$

or,
$\bar{S}(\bar{R}(X,Y)Z,W) + \bar{S}(Z,\bar{R}(X,Y)W)$

$$= L_{\bar{S}}[\bar{S}((X \wedge Y)Z,W) + \bar{S}(Z,(X \wedge Y)W)]. \tag{61}$$

Putting $Y = W = \xi$, in (61) and using (2), (41) and (44), we have

$$[L_{\bar{S}} + 1][\bar{S}(Z,X) + 2ng(Z,X)] = -\bar{S}(Z,\phi X). \tag{62}$$

Then for $L_{\bar{S}} = -1$, (62) becomes
$\bar{S}(Z,\phi X) = 0$.
Then (36) implies that M is an $\eta-$Einstein manifold. ◇

Corollary 7.1: If M is a Ricci semi-symmetric α-Sasakian manifold with quarter-symmetric non-metric connection D with restriction $Y = W = \xi$, then
$\bar{S}(Z,X) + 2ng(Z,X) + \bar{S}(Z,\phi X) = 0$.

Proof: Sine M is a Ricci semi-symmetric Kenmotsu manifold with quarter-symmetric non-metric connection D, then $L_{\bar{C}} = 0$. Putting $L_{\bar{C}} = 0$ in (62) we get
$\bar{S}(Z,X) + 2ng(Z,X) + \bar{S}(Z,\phi X) = 0$. ◇

8. Pseudosymmetric Kenmotsu manifold and Weyl- pseudosymmetric Kenmotsu manifold with quarter-symmetric non-metric connection

In the present section we shall give the definition of pseudosymmetric Kenmotsu manifold and Weyl-pseudosymmetric Kenmotsu manifold with quarter-symmetric non-metric connection and discuss some of there properties.

Definition 8.1: A Kenmotsu manifold M with quarter-symmetric non-metric connection D is said to be pseudosymmetric Kenmotsu manifold with quarter-symmetric non-metric connection if the curvature tensor$\bar{R}$ of M with respect to D satisfies the conditions

$$(\bar{R}(X,Y) \circ \bar{R})(U,V,W) = L_{\bar{R}}[((X \wedge Y) \circ \bar{R})(U,V,W)], \tag{63}$$

where $(\bar{R}(X,Y) \circ \bar{R})(U,V,W) = \bar{R}(X,Y)(\bar{R}(U,V)W)$

$$-\bar{R}(\bar{R}(X,Y)U,V)W - \bar{R}(U,\bar{R}(X,Y)V)W - \bar{R}(U,V)(R(X,Y)W), \tag{64}$$

and $((X \wedge Y) \circ \bar{R})(U,V,W) = (X \wedge Y)(\bar{R}(U,V)W)$

$$-\bar{R}((X \wedge Y)U,V)W - \bar{R}(U,(X \wedge Y)V)W - \bar{R}(U,V)((X \wedge Y)W). \tag{65}$$

Definition 8.2: A Kenmotsu manifold M with quarter-symmetric non-metric connection D is said to be Weyl-pseudosymmetric Kenmotsu manifold with quarter-symmetric non-metric connection if the curvature tensor$\bar{R}$ of M with respect to D satisfies the conditions

$$(\bar{R}(X,Y) \circ \bar{C})(U,V,W) = L_{\bar{C}}[((X \wedge Y) \circ \bar{C})(U,V,W)], \tag{66}$$

where $(\bar{R}(X,Y) \circ \bar{C})(U,V,W) = \bar{R}(X,Y)(\bar{C}(U,V)W)$

$$-\bar{C}(\bar{R}(X,Y)U,V)W - \bar{C}(U,\bar{R}(X,Y)V)W - \bar{C}(U,V)(R(X,Y)W), \tag{67}$$

and $((X \wedge Y) \circ \bar{C})(U,V,W) = (X \wedge Y)(\bar{C}(U,V)W)$

$$-\bar{C}((X \wedge Y)U,V)W - \bar{C}(U,(X \wedge Y)V)W - \bar{C}(U,V)((X \wedge Y)W). \qquad (68)$$

Theorem 8.1: *Let M be a Kenmotsu manifold. If M is Weyl-pseudosymmetric with the connection D then M is either conformally flat and $\eta-$Einstein manifold or $L_{\bar{C}} = -1$.*

Proof: Let M be a Weyl-pseudosymmetric Kenmotsu manifold and $X,Y,U,V,W \in \chi(M)$. Then using (67) and (68) in (66), we have
$\bar{R}(X,Y)(\bar{C}(U,V)W) - \bar{C}(\bar{R}(X,Y)U,V)W$
$-\bar{C}(U,\bar{R}(X,Y)V)W - \bar{C}(U,V)(R(X,Y)W)$
$= L_{\bar{C}}[(X \wedge Y)(\bar{C}(U,V)W) - \bar{C}((X \wedge Y)U,V)W$

$$-\bar{C}(U,(X \wedge Y)V)W - \bar{C}(U,V)((X \wedge Y)W)]. \qquad (69)$$

Replacing X with ξ in (69) we obtain
$\bar{R}(\xi,Y)(\bar{C}(U,V)W) - \bar{C}(\bar{R}(\xi,Y)U,V)W$
$-\bar{C}(U,\bar{R}(\xi,Y)V)W - \bar{C}(U,V)(R(\xi,Y)W)$
$= L_{\bar{C}}[(\xi \wedge Y)(\bar{C}(U,V)W) - \bar{C}((\xi \wedge Y)U,V)W$

$$-\bar{C}(U,(\xi \wedge Y)V)W - \bar{C}(U,V)((\xi \wedge Y)W)]. \qquad (70)$$

Using (2), (41) in (70) and taking the inner product of (70) with ξ, we get
$-\bar{C}(U,V,W,Y) + \eta(\bar{C}(U,V)W)\eta(Y) - g(Y,U)\eta(\bar{C}(\xi,V)W)$
$+\eta(U)\eta(\bar{C}(Y,V)W) - g(Y,V)\eta(\bar{C}(U,\xi)W) + \eta(V)\eta(\bar{C}(U,Y)W)$
$+\eta(W)\eta(\bar{C}(U,V)Y) + \eta(U)\eta(\bar{C}(\phi Y,V)W) + \eta(V)\eta(\bar{C}(U,\phi Y)W)$
$+\eta(W)\eta(\bar{C}(U,V)\phi Y) - g(Y,W)\eta(\bar{C}(U,V)\xi)$
$= L_{\bar{C}}[\bar{C}(U,V,W,Y) - \eta(Y)\eta(\bar{C}(U,V)W) + g(Y,U)\eta(\bar{C}(\xi,V)W) - \eta(U)\eta(\bar{C}(Y,V)W + g(Y,V)\eta(\bar{C}(U,\xi)W) - \eta(V)\eta(\bar{C}(U,Y)W) - \eta(W)\eta(\bar{C}(U,V)Y) + g(Y,W)\eta(\bar{C}(U,V)\xi)]$.
Then putting $Y = U = \xi$, we get

$$[L_{\bar{C}} + 1]\eta(\bar{C}(\xi,V)W) = 0. \qquad (71)$$

Now (71) gives either $\eta(\bar{C}(\xi,V)W) = 0$ or $L_{\bar{C}} = -1$.

Now $L_{\bar{C}} \neq -1$, then $\eta(\bar{C}(\xi,V)W) = 0$, and we have that M is conformally flat which gives
$\bar{S}(V,W) = Ag(V,W) + B\eta(V)\eta(W)$,
where $A = n + 2 + \frac{\bar{r}}{n-1}$
and $B = -[3n + 2 + \frac{\bar{r}}{n-1}]$.
This shows that M is an $\eta-$Einstein manifold.

If $\eta(\bar{C}(\xi,V)W) \neq 0$, then we have $L_{\bar{C}} = -1$. ◇

Theorem 8.2: *Let M be a Kenmotsu manifold. If M is pseudosymmetric then either M is a spece of constant curvature and $g(X,Y) = \eta(X)\eta(Y)$ or*

$L_{\bar{R}} = -1$, *for* $X, Y \in \chi(M)$.

Proof: Let M be a pseudosymmetric Kenmotsu manifold and $X, Y, U, V, W \in \chi(M)$. Then using (64) and (65) in (63), we have

$$\bar{R}(X,Y)(\bar{R}(U,V)W) - \bar{R}(\bar{R}(X,Y)U,V)W$$
$$-\bar{R}(U,\bar{R}(X,Y)V)W - \bar{R}(U,V)(R(X,Y)W)$$
$$= L_{\bar{R}}[(X \wedge Y)(\bar{R}(U,V)W) - \bar{R}((X \wedge Y)U,V)W$$
$$-\bar{R}(U,(X \wedge Y)V)W - \bar{R}(U,V)((X \wedge Y)W)]. \quad (72)$$

Replacing X with ξ in (72) we obtain

$$\bar{R}(\xi,Y)(\bar{R}(U,V)W) - \bar{R}(\bar{R}(\xi,Y)U,V)W$$
$$-\bar{R}(U,\bar{R}(\xi,Y)V)W - \bar{R}(U,V)(R(\xi,Y)W)$$
$$= L_{\bar{R}}[(\xi \wedge Y)(\bar{R}(U,V)W) - \bar{R}((\xi \wedge Y)U,V)W$$
$$-\bar{R}(U,(\xi \wedge Y)V)W - \bar{R}(U,V)((\xi \wedge Y)W)]. \quad (73)$$

Using (2), (41) in (70) and taking the inner product of (73) with ξ, we get

$$-\bar{R}(U,V,W,Y) + \eta(\bar{R}(U,V)W)\eta(Y) - g(Y,U)\eta(\bar{R}(\xi,V)W)$$
$$+\eta(U)\eta(\bar{R}(Y,V)W) - g(Y,V)\eta(\bar{R}(U,\xi)W) + \eta(V)\eta(\bar{R}(U,Y)W)$$
$$+\eta(W)\eta(\bar{R}(U,V)Y) + \eta(U)\eta(\bar{R}(\phi Y,V)W) + \eta(V)\eta(\bar{R}(U,\phi Y)W)$$
$$+\eta(W)\eta(\bar{R}(U,V)\phi Y) - g(Y,W)\eta(\bar{R}(U,V)\xi)$$
$$= L_{\bar{R}}[\bar{R}(U,V,W,Y) - \eta(Y)\eta(\bar{R}(U,V)W) + g(Y,U)\eta(\bar{R}(\xi,V)W)$$
$$-\eta(U)\eta(\bar{R}(Y,V)W + g(Y,V)\eta(\bar{R}(U,\xi)W) - \eta(V)\eta(\bar{R}(U,Y)W)$$
$$-\eta(W)\eta(\bar{R}(U,V)Y) + g(Y,W)\eta(\bar{R}(U,V)\xi)].$$

Then putting $Y = U = \xi$, we get

$$[L_{\bar{C}} + 1]\eta(\bar{R}(\xi,V)W) = 0. \quad (74)$$

Now (71) gives either $\eta(\bar{R}(\xi,V)W) = 0$ or $L_{\bar{R}} = -1$.

Now $L_{\bar{R}} \neq -1$, then $\eta(\bar{R}(\xi,V)W) = 0$, and we have that M is a space of constant curvature and $\eta(\bar{R}(\xi,V)W) = 0$ gives
$g(V,W) = \eta(V)\eta(W)$.

If $\eta(\bar{R}(\xi,V)W) \neq 0$, then we have $L_{\bar{R}} = -1$. ◇

9. Example of pseudosymmetric Kenmotsu manifold with quarter-symmetric non-metric connection *D*

Let us consider the three dimensional manifold $M = \{(x_1, x_2, x_3) \in R^3 : x_1, x_2, x_3 \in R\}$, where (x_1, x_2, x_3) are the standard coordinates of R^3. We consider the vector fields

$$e_1 = x_1\frac{\partial}{\partial x_3},\ e_2 = x_1\frac{\partial}{\partial x_2} \text{ and } e_3 = -x_1\frac{\partial}{\partial x_1}.$$

Clearly, $\{e_1, e_2, e_3\}$ is a set of linearly independent vectors for each point of M and hence a basis of M. The non-metric g is defined by

$g(e_1, e_2) = g(e_2, e_3) = g(e_1, e_3) = 0,$
$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1.$

Let η be the 1−form defined by $\eta(Z) = g(Z, e_3)$, for any $Z \in \chi(M)$ and the $(1,1)-$ tensor field ϕ is defined by

$$\phi e_1 = e_2,\ \phi e_2 = -e_1,\ \phi e_3 = 0.$$

From the linearity of ϕ and g, we have

$\eta(e_3) = 1,$
$\phi^2(X) = -X + \eta(X)e_3$ and
$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$, for any $X \in \chi(M)$.

Then for $e_3 = \xi$, the structure (ϕ, ξ, η, g) defines an almost contact metric structure on M.

Let ∇ be the Levi-Civita connection with respect to the metric g. Then we have

$$[e_1, e_2] = 0,\ [e_1, e_3] = e_1,\ [e_2, e_3] = e_2.$$

Koszul's formula is defined by

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Y g(Z, X) - Zg(X, Y) - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]).$$

Then from the above formula we can calculate the following,

$$\nabla_{e_1} e_1 = -e_3, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = e_1,$$
$$\nabla_{e_2} e_1 = 0, \quad \nabla_{e_2} e_2 = -e_3, \quad \nabla_{e_2} e_3 = e_2,$$
$$\nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = 0.$$

Hence the structure (ϕ, ξ, η, g) is a Kenmotsu manifold. [8]

Using (22), we find D, the quarter-symmetric non-metric connection on M

$$D_{e_1} e_1 = -e_3, \quad D_{e_1} e_2 = 0, \quad D_{e_1} e_3 = e_1 + e_2,$$
$$D_{e_2} e_1 = 0, \quad D_{e_2} e_2 = -e_3, \quad D_{e_2} e_3 = e_2 - e_1,$$
$$D_{e_3} e_1 = 0, \quad D_{e_3} e_2 = 0, \quad D_{e_3} e_3 = 0.$$

Using (23), the torson tensor $\bar{T}$, with respect to quarter-symmetric non-metric connection D as follows:

$\bar{T}(e_i, e_i) = 0,\ \forall i = 1, 2, 3$
$\bar{T}(e_1, e_2) = 0,\ \bar{T}(e_1, e_3) = e_2,\ \bar{T}(e_2, e_3) = -e_1.$

Also $(D_{e_1} g)(e_2, e_3) = -1$, $(D_{e_2} g)(e_3, e_1) = 1$
and $(D_{e_3} g)(e_1, e_2) = 0$.
Thus M is a 3-dimensional Kenmotsu manifold with quarter-symmetric non-metric connection D.

Now we calculate curvature tensor $\bar{R}$ and Ricci tensors $\bar{S}$ as follows:

$$\bar{R}(e_1, e_2)e_3 = 0, \quad \bar{R}(e_1, e_3)e_3 = -(e_1 + e_2),$$
$$\bar{R}(e_3, e_2)e_2 = -e_3, \quad \bar{R}(e_3, e_1)e_1 = -e_3,$$
$$\bar{R}(e_2, e_1)e_1 = e_1 - e_2, \quad \bar{R}(e_2, e_3)e_3 = e_1 - e_2,$$
$$\bar{R}(e_1, e_2)e_2 = -(e_1 + e_2).$$

From the definition of $\bar{S}$, $\bar{S}(X, Y) = \Sigma_i g(\bar{R}(e_i, X)Y, e_i)$, $i = 1, 2, 3$, we get
$\bar{S}(e_1, e_1) = \bar{S}(e_2, e_2) = \bar{S}(e_3, e_3) = -2$, $\bar{S}(e_1, e_2) = 1$,

$\bar{S}(e_1, e_3) = \bar{S}(e_2, e_3) = 0.$

Again using (2) we get
$(e_1, e_2)e_3 = 0, \qquad (e_i \wedge e_i)e_j = 0, \ \forall i, j = 1, 2, 3,$
$(e_1 \wedge e_2)e_2 = (e_1 \wedge e_3)e_3 = e_1, \qquad (e_2 \wedge e_1)e_1 = (e_2 \wedge e_3)e_3 = e_2,$
$(e_3 \wedge e_2)e_2 = (e_3 \wedge e_1)e_1 = e_3.$

Now $\bar{R}(e_1, e_2)(\bar{R}(e_3, e_1)e_2) = 0, \qquad \bar{R}(\bar{R}(e_1, e_2)e_3, e_1)e_2 = 0,$
$\bar{R}(e_3, \bar{R}(e_1, e_2)e_1)e_2 = -e_3,$
$(\bar{R}(e_3, e_1)(\bar{R}(e_1, e_2)e_2) = e_3.$
Then $(\bar{R}(e_1, e_2).\bar{R})(e_3, e_1, e_2) = 0.$

Again $(e_1 \wedge e_2)(\bar{R}(e_3, e_1)e_2) = 0, \qquad \bar{R}((e_1 \wedge e_2)e_3, e_1)e_2 = 0,$
$\bar{R}(e_3, (e_1 \wedge e_2)e_1)e_2 = e_3,$
$\bar{R}(e_3, e_1)((e_1 \wedge e_2)e_2) = -e_3.$
Then $((e_1, e_2).\bar{R})(e_3, e_1, e_2) = 0.$

Thus

$$(\bar{R}(e_1, e_2).\bar{R})(e_3, e_1, e_2) = L_{\bar{R}}[((e_1, e_2).\bar{R})(e_3, e_1, e_2)],$$

for any function $L_{\bar{R}} \in C^\infty(M)$.
Similarly, we can show any combination of e_1, e_2 and e_3 (60).
Hence M is a pseudosymmetric Kenmotsu manifold with quarter-symmetric non-metric connection.

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