Bulletin of TICMI
Vol. 18, No. 1, 2014, 141–145

On the Summability Almost Everywhere by the Methods (c, α) and Abel–Poisson's of Series with Respect to Block–Orthonormal Systems

Givi Nadibaidze *

Department of Mathematics, Tbilisi State University,
Chavchavadze av. 1, 0128, Tbilisi, Georgia

(*Received February 28, 2014; Revised May 29, 2014; Accepted June 25, 2014*)

In the present paper the sufficient conditions on the blocks are established, when the block-orthonormal series are $(c, \alpha), (\alpha > 0)$ and Abel-Poisson's summable almost everywhere and equivalence of the methods (c, α) $(\alpha > 0)$ and Abel-Poisson's are established in certain conditions.

Keywords: Block-orthonormal systems, Abel-Poisson's method.

AMS Subject Classification: 42C10.

Below we shall consider almost everywhere (a.e.) summability by the methods $(c, \alpha), (\alpha > 0)$ and Abel-Poisson's of series with respect to block-orthonormal systems and we shall establish the equivalence in certain conditions of the methods $(c, \alpha), (\alpha > 0)$ and Abel-Poisson's for the summability a.e. of series with respect to block-orthonormal systems.

Definition 1.1: ([1]). Let $\{N_k\}$ be an increasing sequence of natural numbers, $\Delta_k = (N_k, N_{k+1}]$, $(k = 1, 2, ...)$ and let $\{\varphi_n\}$ be a system of functions from $L^2(0, 1)$. The system $\{\varphi_n\}$ will be called a Δ_k-orthonormal system (Δ_k -ONS) if:
1) $\|\varphi_n\|_2 = 1, \quad n = 1, 2, ...;$
2) $(\varphi_i, \varphi_j) = 0$, for $i, j \in \Delta_k, \ i \neq j, \ k \geq 1.$
Let the series

$$\sum_{n=1}^{\infty} a_n \varphi_n (x) \tag{1}$$

be given, where $\{\varphi_n\}$ is a Δ_k -ONS. Below we shall use notations:

$$\sigma_n^{(\alpha)} (x) = \frac{1}{A_n^\alpha} \sum_{k=1}^{n} A_{n-k}^\alpha a_k \varphi_k(x), \tag{2}$$

where $a_0 = 0$ and $A_n^\alpha = \begin{pmatrix} \alpha + n \\ n \end{pmatrix} = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}{n!}$.

*Email: g.nadibaidze@gmail.com

ISSN: 1512-0082 print

Lemma 1.2: *Let the sequence of natural numbers* $\{N_k\}$ *be given and let for the positive nondecreasing sequence* $\{\omega(n)\}$ *the conditions*

$$\min\{k : N_k \geq n\} + n \sum_{k:N_k \geq n} \frac{1}{N_k} = O(\omega(n)), \text{ as } n \to \infty \tag{3}$$

and

$$\sum_{n=1}^{\infty} a_n^2 \omega(n) < \infty \tag{4}$$

be fulfilled. Then for every Δ_k *-ONS* $\{\varphi_n\}$ *we have*

$$\lim_{n\to\infty} \delta_n^{(\alpha)}(x) = 0 \text{ a.e.}, \ \left(\alpha > \frac{1}{2}\right),$$

where

$$\delta_n^{(\alpha)}(x) = \frac{1}{n+1} \sum_{k=1}^{n} \left(\sigma_k^{(\alpha)}(x) - \sigma_k^{(\alpha-1)}(x)\right)^2$$

and $\sigma_k^{(\alpha)}(x)$ *is defined by formula (2).*

Proof: We have

$$\sigma_k^{(\alpha)}(x) - \sigma_k^{(\alpha-1)}(x) = \frac{1}{A_k^{\alpha} A_k^{\alpha-1}} \sum_{j=1}^{k} (A_{k-j}^{\alpha} A_k^{\alpha-1} - A_{k-j}^{\alpha-1} A_k^{\alpha}) a_j \varphi_j(x)$$

$$= \frac{1}{A_k^{\alpha} A_k^{\alpha-1}} \sum_{j=1}^{k} (-\frac{j}{\alpha} A_{k-j}^{\alpha-1} A_k^{\alpha-1}) a_j \varphi_j(x),$$

Therefore

$$\int_0^1 \left(\sigma_n^{(\alpha)}(x) - \sigma_n^{(\alpha-1)}(x)\right)^2 dx = \int_0^1 \left(\frac{1}{A_n^{\alpha} A_n^{\alpha-1}} \sum_{j=1}^{n} (-\frac{j}{\alpha} A_{n-j}^{\alpha-1} A_n^{\alpha-1}) a_j \varphi_j(x)\right)^2 dx$$

$$= \frac{1}{(A_n^{\alpha} A_n^{\alpha-1})^2} \int_0^1 (\sum_{i=0}^{k(n)-1} \sum_{j=N_i+1}^{N_{i+1}} (-\frac{j}{\alpha} A_{n-j}^{\alpha-1} A_n^{\alpha-1}) a_j \varphi_j(x)$$

$$+\sum_{j=N_{k(n)}+1}^{n}(-\frac{j}{\alpha}A_{n-j}^{\alpha-1}A_n^{\alpha-1})a_j\varphi_j(x))^2dx$$

$$\leq \frac{2}{\left(A_n^{\alpha}\right)^2\alpha^2}\left(k(n)\sum_{j=1}^{N_{k(n)}}j^2\left(A_{n-j}^{\alpha-1}\right)^2a_j^2+\sum_{j=N_{k(n)}+1}^{n}j^2\left(A_{n-j}^{\alpha-1}\right)^2a_j^2\right).$$

Then for the $\alpha>\frac{1}{2}$ we have

$$\int_0^1\delta_{2^m}^{(\alpha)}(x)dx\leq\frac{2}{\alpha^2(2^m+1)}\sum_{n=1}^{2^m}\frac{k(n)}{(A_n^{\alpha})^2}\sum_{j=1}^{N_{k(n)}}j^2\left(A_{n-j}^{\alpha-1}\right)^2a_j^2$$

$$+\frac{2}{\alpha^2(2^m+1)}\sum_{n=1}^{2^m}\frac{1}{(A_n^{\alpha})^2}\sum_{j=N_{k(n)}+1}^{n}j^2\left(A_{n-j}^{\alpha-1}\right)^2a_j^2\leq\frac{ck(2^m)}{2^m}\sum_{j=1}^{N_{k(2^m)}}ja_j^2+\frac{c}{2^m}\sum_{j=1}^{2^m}ja_j^2.$$

Hence

$$\sum_{m=1}^{\infty}\int_0^1\delta_{2^m}^{(\alpha)}(x)dx\leq c\left(\sum_{i=1}^{\infty}\sum_{\log_2 N_i<m\leq\log_2 N_{i+1}}\frac{k(2^m)}{2^m}\sum_{j=1}^{N_{k(2^m)}}ja_j^2+\sum_{j=1}^{\infty}ja_j^2\sum_{2^m\geq j}\frac{1}{2^m}\right)$$

$$\leq c\left(\sum_{j=1}^{\infty}ja_j^2[(k(j)+1)\sum_{m\geq\log_2 N_{k(j)+1}}\frac{1}{2^m}+\sum_{l=2}^{\infty}\sum_{m\geq\log_2 N_{k(j)+l}}\frac{1}{2^m}]+\sum_{j=1}^{\infty}a_j^2\right)$$

$$\leq c\left(\sum_{j=1}^{\infty}a_j^2\left(\min\{k:N_k\geq j\}+j\sum_{k:N_k\geq j}\frac{1}{N_k}\right)+\sum_{j=1}^{\infty}a_j^2\right).$$

Then by conditions (3),(4) we obtain

$$\sum_{m=1}^{\infty}\delta_{2^m}^{(\alpha)}(x)<\infty\quad a.e.$$

Hence Levi's theorem implies

$$\lim_{m\to\infty}\delta_{2^m}^{(\alpha)}(x)=0\quad a.e.$$

Now if $2^m < n < 2^{m+1}$, then

$$0 \le \delta_n^{(\alpha)}(x) \le 2\delta_{2^{m+1}}^{(\alpha)}(x).$$

Therefore

$$\lim_{n\to\infty} \delta_n^{(\alpha)}(x) = 0 \ \ a.e.$$

□

Lemma 1.3: *Let the sequence of natural numbers $\{N_k\}$ be given, $\{\varphi_n\}$ is an arbitrary Δ_k -ONS and let for the positive nondecreasing sequence $\{\omega(n)\}$ the conditions (3),(4) be fulfilled. If corresponding series (1) is summable a.e. on $(0,1)$ to the function $S(x)$ by the method $(c,\alpha),(\alpha > 1/2)$, then a.e. on $(0,1)$ we have*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left| S(x) - \sigma_k^{\alpha-1}(x) \right| = 0$$

and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left| S(x) - \sigma_k^{\alpha-1}(x) \right|^2 = 0,$$

where $\sigma_k^{(\alpha)}(x)$ is defined by formula (2).

Lemma 1.3 is possible to prove by standard method using Lemma 1.2 (see [2, proof [5.8.2]]).

Lemma 1.4: *Let the sequence of natural numbers $\{N_k\}$ be given, $\{\varphi_n\}$ is an arbitrary Δ_k -ONS and for the positive nondecreasing sequence $\{\omega(n)\}$ the conditions (3), (4) are fulfilled. If the corresponding series (1) is summable a.e. on $(0,1)$ by the Poisson's method, then the series (1) is summable a.e. on $(0,1)$ by the all methods$(c,\alpha),(\alpha > 0)$.*

Lemma 1.4 is possible to prove by standard method using Lemma 1.3 (see [2, proof [5.8.4]]).

Theorem 1.5: *Let the sequence of natural numbers $\{N_k\}$ be given, $\{\varphi_n\}$ is an arbitrary Δ_k -ONS and for the positive nondecreasing sequence $\{\omega(n)\}$ the conditions (3), (4) are fulfilled. Then for the corresponding series (1) all methods $(c,\alpha),(\alpha > 0)$ and Abel-Poisson's method are equivalent.*

Proof: Let conditions (3), (4) be fulfilled. Then we have

$$\min\{k : N_k \ge n\} + n^2 \sum_{k:N_k\ge n} \frac{1}{N_k^2} \le \min\{k : N_k \ge n\} + n^2 \sum_{k:N_k\ge n} \frac{1}{nN_k}$$

$$= \min\{k : N_k \ge n\} + n \sum_{k:N_k\ge n} \frac{1}{N_k} = O\left(\omega(n)\right) \ \text{ as } \ n \to \infty. \tag{$*$}$$

Therefore using [3, Lemma 1] we have

$$\sum_{n=1}^{\infty} n \left(\sigma_n (x) - \sigma_{n-1} (x)\right)^2 < \infty \ \ a.e. \tag{5}$$

Let the corresponding series (1) be summable a.e. by Abel-Poisson's method. Then by mentioned method is summable a.e. series

$$\sum_{n=1}^{\infty} \left(\sigma_n (x) - \sigma_{n-1} (x)\right) \tag{6}$$

Hence by (5) we obtain that series (6) is summable a.e. by the method $(c, 1)$. Then by Lemma 1.4 we finished proof of Theorem 1.5. □

Theorem 1.6: *Let the sequence of natural numbers*$\{N_k\}$ *be given,* $\{\varphi_n\}$ *is an arbitrary* Δ_k *-ONS and for the positive nondecreasing sequence* $\{\omega(n)\}$ *the conditions (3),(4) are fulfilled.Then for corresponding series (1) to be summable a.e. by* $(c, \alpha), (\alpha > 0)$ *and Abel-Poisson's methods it is necessary and sufficient that the subsequence of partial sums* $\{S_{2^n}\}$ *of (1) be convergent a.e.*

Proof: Theorem 1.6 will be proved using Theorem 1.5, estimate (*) and [3, Theorem 2]. □

Finally, using Theorem 1.6 and method of proof [3, Theorem 3] we have

Corollary 1.7: *Let the sequence of natural numbers*$\{N_k\}$ *be given,* $\{\varphi_n\}$*is an arbitrary* Δ_k *-ONS and for the sequence* $\omega(n) = (\log_2 \log_2 n)^2$ *the conditions (3),(4) are fulfilled. Then corresponding series (1) is summable a.e. on* $(0, 1)$ *by all methods* $(c, \alpha), (\alpha > 0)$ *and Abel-Poisson's method.*

References

[1] V.F. Gaposhkin, *Series of block-orthonormal and block-independent systems* (Russian), Izv. VUZ, ser. Mathem., **34**, 5 (1990), 13-20

[2] Kaczmarz S., Steinhaus H., *Theorie der Orthogonaleihen*, New York, 1951

[3] G. Nadibaidze, *On the convergence and summability of series with respect to block- orthonormal systems*, Georgian Math. J., **2**, 5 (1995), 517-534