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A Note on Vilenkin-Fejér Means on the Martingale Hardy Spaces H_p

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The main aim of this note is to derive necessary and sufficient conditions for the convergence of Fejér means in terms of the modulus of continuity of the Hardy spaces H_p, $(0 < p \leq 1)$.

Keywords: Vilenkin system, Vilenkin-Fejér means, Martingale Hardy space.

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1. Introduction and preliminary results

Let $\mathbb{P}_+$ denote the set of the positive integers and $\mathbb{P} := \mathbb{P}_+ \cup \{0\}$.

Let $m := (m_0, m_1, \dots)$ denote a sequence of the positive integers not less than 2.

Denote by

$$Z_{m_k} := \{0, 1, \dots, m_k - 1\}$$

the additive group of integers modulo m_k.

Define the group G_m as the complete direct product of the group Z_{m_j} with the product of the discrete topologies of Z_{m_j}'s.

The direct product μ of the measures

$$\mu_k(\{j\}) := 1/m_k, (j \in Z_{m_k})$$

is the Haar measure on G_m with $\mu(G_m) = 1$.

In this paper we consider bounded Vilenkin groups only, which are defined by the condition $\sup_n m_n < \infty$.

The elements of G_m are represented by sequences

$$x := (x_0, x_1, \dots, x_k, \dots) \qquad (x_k \in Z_{m_k}).$$

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It is easy to give a base for the neighbourhoods of G_m :

$$I_0(x) := G_m, \quad I_n(x) := \{y \in G_m \mid y_0 = x_0, \ldots, y_{n-1} = x_{n-1}\} \ (x \in G_m, \ n \in \mathbb{P}).$$

Denote $I_n := I_n(0)$ and $\overline{I_n} := G_m \setminus I_n$, for $n \in \mathbb{P}$. Let $e_n := (0, \ldots, 0, x_n = 1, 0, \ldots) \in G_m$, $(n \in \mathbb{P})$.

The norm (or quasi-norm) of the space $L_p(G_m)$ is defined by

$$\|f\|_p := \left(\int_{G_m} |f(x)|^p \, d\mu(x) \right)^{1/p} \qquad (0 < p < \infty).$$

The space $weak - L_p(G_m)$ consists of all measurable functions f, for which

$$\|f\|^p_{weak - L_p} := \sup_{\lambda > 0} \lambda^p \mu(f > \lambda) < +\infty.$$

If we define the so-called generalized number system based on m in the following way:

$$M_0 := 1, \qquad M_{k+1} := m_k M_k \qquad (k \in \mathbb{P}),$$

then every $n \in \mathbb{P}$ can be uniquely expressed as $n = \sum_{j=0}^{\infty} n_j M_j$, where $n_j \in Z_{m_j}$ $(j \in \mathbb{P})$ and only a finite number of n_j's differ from zero. Let $|n| := \max\{j \in \mathbb{P}; n_j \neq 0\}$.

Next, we define the complex valued function $r_k(x) : G_m \to \mathbb{C}$, called the generalized Rademacher functions in the following way:

$$r_k(x) := \exp(2\pi \imath x_k / m_k) \qquad \left(\imath^2 = -1, \ x \in G_m, \ k \in \mathbb{P}\right).$$

Moreover, the Vilenkin system $\psi := (\psi_n : n \in \mathbb{P})$ on G_m is defined as follows:

$$\psi_n := \prod_{k=0}^{\infty} r_k^{n_k}(x) \qquad (n \in \mathbb{P}).$$

In particular, we call this system the Walsh-Paley one when $m \equiv 2$. It is known that the Vilenkin system is orthonormal and complete in $L_2(G_m)$ (see e.g. [1, 15]).

Hence we can introduce analogues of the usual definitions in Fourier-analysis. If $f \in L_1(G_m)$ we can define the Fourier coefficients, the partial sums of the Fourier series, the Fejér means, the Dirichlet and Fejér kernels with respect to the Vilenkin system in the usual manner:

$$\widehat{f}(n) := \int_{G_m} f \overline{\psi}_n d\mu, \quad S_n f := \sum_{k=0}^{n-1} \widehat{f}(k) \psi_k, \quad \sigma_n f := \frac{1}{n} \sum_{k=1}^{n} S_k f,$$

$$D_n := \sum_{k=0}^{n-1} \psi_k, \qquad K_n := \frac{1}{n} \sum_{k=1}^{n} D_k \quad (n \in \mathbb{P}_+).$$

Recall that

$$D_{M_n}(x)=\begin{cases} M_n, \text{ if } x\in I_n, \\ 0, \quad \text{ if } \ x\notin I_n. \end{cases} \tag{1}$$

The σ-algebra generated by the intervals $\{I_n(x) : x\in G_m\}$ will be denoted by $\digamma_n$ $(n\in\mathbb{P})$. Denote by $f=\left(f^{(n)}, n\in\mathbb{P}\right)$ the martingale with respect to $\digamma_n$ $(n\in\mathbb{P})$ (for details see e.g. [16]). The maximal function of the martingale f is defined by

$$f^*(x)=\sup_{n\in\mathbb{P}}\left|f^{(n)}(x)\right|.$$

In the case $f\in L_1(G_m)$, the maximal functions can also be given by

$$f^*(x)=\sup_{n\in\mathbb{P}}\frac{1}{|I_n(x)|}\left|\int_{I_n(x)}f(u)\,d\mu(u)\right|$$

For $0<p<\infty$ the Hardy martingale spaces $H_p(G_m)$ consist of all martingales for which

$$\|f\|_{H_p}:=\|f^*\|_p<\infty. \tag{2}$$

If $f\in L_1(G_m)$, then it is easy to show that the sequence $(S_{M_n}(f) : n\in\mathbb{P})$ is a martingale. If $f=\left(f^{(n)}, n\in\mathbb{P}\right)$ is a martingale, then the Vilenkin-Fourier coefficients must be defined in a slightly different manner:

$$\widehat{f}(i):=\lim_{k\to\infty}\int_{G_m}f^{(k)}(x)\,\overline{\psi}_i(x)\,d\mu(x).$$

The Vilenkin-Fourier coefficients of $f\in L_1(G_m)$ are the same as those of the martingale $(S_{M_n}(f) : n\in\mathbb{P})$ obtained from f.

For the martingale f we consider the following maximal operators:

$$\sigma^*f:=\sup_{n\in\mathbb{P}}|\sigma_n f|,\quad \sigma^{\#}f:=\sup_{n\in\mathbb{P}}|\sigma_{M_n}f|,\quad \widetilde{\sigma}_p^*:=\sup_{n\in\mathbb{P}_+}\frac{|\sigma_n|}{n^{1/p-2}\log^{2[1/2+p]}(n+1)},$$

where $0<p\leq 1/2$ and $[1/2+p]$ denotes the integer part of $1/2+p$.

A weak type-$(1,1)$ inequality for the maximal operator of Fejér means σ^* can be found in Schipp [8] for Walsh series and in Pl, Simon [7] for bounded Vilenkin series. Fujji [3] and Simon [10] verified that σ^* is bounded from H_1 to L_1.

Weisz [17] generalized this result and proved the following:

Theorem W1 (Weisz): *The maximal operator σ^* is bounded from the martingale space H_p to the space L_p for $p>1/2$.*

Simon [9] gave a counterexample, which shows that boundedness does not hold for $0<p<1/2$. The counterexample for $p=1/2$ is due to Goginava [4], (see also [2]). Weisz [18] proved that σ^* is bounded from the Hardy space $H_{1/2}$ to the space $L_{weak-1/2}$.

In [12] and [13] it was proved that the maximal operators $\widetilde{\sigma}_p^*$ with respect to Vilenkin systems, where $0 < p \leq 1/2$ and $[1/2+p]$ denotes the integer part of $1/2+p$, is bounded from the Hardy space H_p to the space L_p. Moreover, we showed that the order of deviant behaviour of the n-th Fejér means was given exactly. As a corollary it was pointed out that

$$\|\sigma_n f\|_p \leq c_p n^{1/p-2} \log^{2[1/2+p]} n \, \|f\|_{H_p}, \quad (n = 2, 3, ...) . \tag{3}$$

Weisz [19] also proved that the following is true:

Theorem W2 (Weisz): *The maximal operator $\sigma^{\#} f$ is bounded from the martingale Hardy space $H_p(G_m)$ to the space $L_p(G_m)$ for $p > 0$.*

Moreover, he also considered the norm convergence of Fejér means of Vilenkin-Fourier series and proved the following:

Theorem W3 (Weisz): *Let $k \in \mathbb{P}$. Then*

$$\|\sigma_k f\|_{H_p} \leq c_p \|f\|_{H_p}, \quad (f \in H_p, \quad p > 1/2)$$

and

$$\|\sigma_{M_k} f\|_{H_p} \leq c_p \|f\|_{H_p}, \quad (f \in H_p, \quad p > 1/2) .$$

For the Walsh system Goginava [6] proved a very unexpected fact:

Theorem G1 (Goginava): *Let $0 < p \leq 1$. Then there exists a martingale $f \in H_p$, such that*

$$\sup_{n\in\mathbb{P}} \||\sigma_{M_k} f|\|_{H_p} = +\infty, \quad (0 < p \leq 1/2) .$$

In [11] (see also [5]) it was proved that there exists a martingale $f \in H_p$, such that

$$\sup_{n\in\mathbb{P}} \|\sigma_n f\|_{H_p} = +\infty, \quad (0 < p \leq 1/2) .$$

In [14] it was proved that the following statements are true:

Theorem T2 (Tephnadze): *a) Let $0 < p \leq 1/2$, $f \in H_p$, $M_N < n \leq M_{N+1}$ and*

$$\omega_{H_p}(1/M_N, f) = o\left(1/M_N^{1/p-2} N^{2[1/2+p]}\right), \ \textit{as} \ \ N \to \infty. \tag{4}$$

Then

$$\|\sigma_n f - f\|_p \to 0, \ \textit{when} \ n \to \infty.$$

b) Let $0 < p < 1/2$ and $M_N < n \leq M_{N+1}$. Then there exists a martingale

$f \in H_p(G_m)$, for which

$$\omega_{H_p}\left(1/M_N, f\right) = O\left(1/M_N^{1/p-2}\right), \quad as \quad N \to \infty \tag{5}$$

and

$$\|\sigma_n f - f\|_{L_{p,\infty}} \nrightarrow 0, \ \ as \ \ n \to \infty.$$

c) Let $M_N < n \leq M_{N+1}$. Then there exists a martingale $f \in H_{1/2}(G_m)$, for which

$$\omega_{H_{1/2}}\left(1/M_N, f\right) = O\left(1/N^2\right), \quad as \quad N \to \infty \tag{6}$$

and

$$\|\sigma_n f - f\|_{1/2} \nrightarrow 0, \ \ as \ \ n \to \infty.$$

In this paper we will show that Theorem W3 of Weisz are simple corollary of Theorems W1 and W2. It is very important, because we do not have definition of conjugate transform of martingales, with the same properties as Walsh series. Moreover we will improve inequality (3) and show that

$$\|\sigma_n f\|_{H_p} \leq c_p n^{1/p-2} \log^{2[1/2+p]} n \, \|f\|_{H_p}, \quad (n = 2, 3, ...).$$

On the other hand, it gives chance to generalize Theorem T2 and derive necessary and sufficient conditions for the convergence of Fejér means in terms of the modulus of continuity of the Hardy spaces H_p, $(0 < p \leq 1)$. We will also generalize Theorem G1 for the bounded Vilenkin system.

2. The main result

Theorem 2.1 : *a) Let $f \in H_p$, where $1/2 < p \leq 1$. Then*

$$\|\sigma_n f\|_{H_p} \leq c_p \|f\|_{H_p}, \ \ (n \in \mathbb{P}).$$

b) Let $f \in H_p$, where $0 < p \leq 1/2$. Then

$$\|\sigma_n f\|_{H_p} \leq c_p n^{1/p-2} \log^{2[1/2+p]} n \, \|f\|_{H_p}, \quad (n \in \mathbb{P}).$$

c) Let $f \in H_p$, where $p > 0$. Then

$$\|\sigma_{M_n} f\|_{H_p} \leq c_p \|f\|_{H_p}, \quad (n \in \mathbb{P}).$$

d) Let $p > 1/2$ and $f \in H_p$. Then

$$\|\sigma_n f - f\|_{H_p} \to 0, \text{ when } n \to \infty.$$

e) Let $0<p\leq 1/2$, $f\in H_p$, $M_N<n\leq M_{N+1}$ *and*

$$\omega_{H_p}\left(1/M_N,f\right)=o\left(1/M_N^{1/p-2}N^{2[1/2+p]}\right),\ \text{as}\ \ N\to\infty. \tag{7}$$

Then

$$\|\sigma_n f-f\|_{H_p}\to 0,\ \text{when}\ n\to\infty.$$

Proof: Let $f\in H_p$, $p>1$ and $M_N\leq n<M_{N+1}$. Then

$$E\sigma_n f:=\left(S_{M_k}\sigma_n f:k\geq 0\right)=\left(\frac{M_0}{n}\sigma_{M_0}f,...,\frac{M_N}{n}\sigma_{M_N}f,\ \sigma_n f\right) \tag{8}$$

and

$$\left(E\sigma_n f\right)^*\leq\sup_{0\leq k\leq N}\left|\frac{M_k}{n}\sigma_{M_k}f\right|+\left|\sigma_n f\right|\leq\sigma^{\#}f+\left|\sigma_n f\right|.$$

By combining (2) and (3) we get

$$\begin{aligned}\|\sigma_n f\|_{H_p}\ :&=\|(E\sigma_n f)^*\|_p\leq\left\|\sigma^{\#}f\right\|_p+\|\sigma_n f\|_p \\ &\leq c_p\|f\|_{H_p},\quad(1/2<p\leq 1)\end{aligned} \tag{9}$$

and

$$\begin{aligned}\|\sigma_n f\|_{H_p}\ :&=\|(\sigma_n f)^*\|_p\leq\left\|\sup_{k\in_+}|\sigma_{M_k}f|\right\|_p+\|\sigma_n f\|_p \\ &\leq c_p\left(n^{1/p-2}\log^{2[1/2+p]}n\right)\|f\|_{H_p},\quad(0<p\leq 1/2).\end{aligned} \tag{10}$$

On the other hand, if $n=M_N$, for some $n\in\mathbb{P}$, by using (8), we obtain that

$$\left(E\sigma_{M_N}f\right)^*\leq\sup_{0\leq k\leq N}\left|\frac{M_k}{n}\sigma_{M_k}f\right|\leq\sup_{k\in\mathbb{N}_+}\left|\sigma_{M_k}f\right|=:\sigma^{\#}f$$

and

$$\begin{aligned}\|\sigma_{M_N}f\|_{H_p}\ :&=\|(E\sigma_{M_N}f)^*\|_p\leq\left\|\sigma^{\#}f\right\|_p \\ &\leq c_p\|f\|_{H_p},\quad(p>0).\end{aligned} \tag{11}$$

It is easy to show that (see [14])

$$\sigma_n S_{M_N}f-S_{M_N}f=\frac{M_N}{n}S_{M_N}\left(\sigma_{M_N}f-f\right). \tag{12}$$

Hence, according to (12), we have

$$\begin{aligned}
&\|\sigma_n f - f\|_{H_p} \\
&\leq c_p \|\sigma_n f - \sigma_n S_{M_N} f\|_{H_p} + c_p \|\sigma_n S_{M_N} f - S_{M_N} f\|_{H_p} + c_p \|S_{M_N} f - f\|_{H_p} \\
&= c_p \|\sigma_n (S_{M_N} f - f)\|_{H_p} + c_p \|S_{M_N} f - f\|_{H_p} + \frac{c_p M_N}{n} \|S_{M_N} \sigma_{M_N} f - S_{M_N} f\|_{H_p} \\
&: = III + IV + V.
\end{aligned}$$

For IV we have that

$$IV = c_p \omega_{H_p} (1/M_n, f) \to 0, \quad \text{as} \quad n \to \infty, \quad (p > 0).$$

Since

$$\|S_{M_n} f\|_{H_p} \leq c_p \|f\|_{H_p}, \quad p > 0 \tag{13}$$

we obtain

$$V \leq \|S_{M_N} (\sigma_{M_N} f - f)\|_{H_p} \leq \|\sigma_{M_N} f - f\|_{H_p} \to 0, \quad \text{as} \quad n \to \infty.$$

Let $1/2 < p \leq 1$. Then, by using (9) we obtain

$$III \leq c_p \|S_{M_N} f - f\|_{H_p} \leq c_p \omega_{H_p} (1/M_N, f) \to \infty, \quad \text{as} \quad n \to \infty.$$

On the other hand, for $0 < p \leq 1/2$ we can apply (10) and under condition (7) we get

$$III \leq c_p \left(n^{1/p-2} \log^{2[1/2+p]} n \right) \omega_{H_p} (1/M_N, f) \to 0, \quad \text{as} \quad n \to \infty.$$

The proof is complete. □

Theorem 2.2: *Let $0 < p \leq 1$. Then the operator $|\sigma_{M_n} f|$ is not bounded from the martingale Hardy space $H_p(G_m)$ to the martingale Hardy space $H_p(G_m)$.*

Proof: Let

$$f_A = D_{M_{A+1}} - D_{M_A}.$$

It is evident that

$$\widehat{f}_A(i) = \begin{cases} 1, \text{ if } i = M_A, ..., M_{A+1} - 1, \\ 0, \text{otherwise.} \end{cases}$$

Then we can write

$$S_i f_A = \begin{cases} D_i - D_{M_A}, & \text{if } i = M_A, ..., M_{A+1} - 1, \\ f_A, & \text{if } i \geq M_{A+1}, \\ 0, & \text{otherwise.} \end{cases} \tag{14}$$

From (1) we get (c.f. [12] and [13])

$$\|f_A\|_{H_p} = \left\|\sup_{n\in\mathbb{P}} S_{M_n}(f_A)\right\|_p = \|f_A\|_p \le M_A^{1-1/p}. \tag{15}$$

Let $x \in I_{A+1}$. Applying (14), we obtain that

$$\sigma_{M_{A+1}} f_A(x) = \frac{1}{M_{A+1}} \sum_{j=0}^{M_{A+1}} S_j f_A(x) = \frac{1}{M_{A+1}} \sum_{j=M_A+1}^{M_{A+1}} S_j f_A(x) \tag{16}$$

$$= \frac{1}{M_{A+1}} \sum_{j=M_A}^{M_{A+1}} \left(D_j(x) - D_{M_A}(x)\right) = \frac{1}{M_{A+1}} \sum_{j=M_A}^{M_{A+1}} (j - M_A)$$

$$= \frac{1}{M_{A+1}} \sum_{j=0}^{(m_A-1)M_A} j \ge cM_A.$$

By using (16), we find that

$$S_{M_N}\left(\left|\sigma_{M_{A+1}} f_A\right|; x\right) = \int_{G_m} \left|\sigma_{M_{A+1}} f_A(t)\right| D_{M_N}(x-t)\, d\mu(t) \tag{17}$$

$$\ge \int_{I_{A+1}} \left|\sigma_{M_{A+1}} f_A(x)\right| D_{M_N}(x-t)\, d\mu(t)$$

$$\ge cM_A \int_{I_{A+1}} D_{M_N}(x-t)\, d\mu(t).$$

According to (17), we have that

$$S_{M_N}\left(\left|\sigma_{M_{A+1}} f_A\right|; x\right) \ge cD_{M_N}(x), \quad N = 0, 1, ..., A,$$

and

$$\sup_N S_{M_N}\left(\left|\sigma_{M_{A+1}} f_A\right|; x\right) \ge \sup_{1\le N<A} S_{M_N}\left(\left|\sigma_{M_{A+1}} f_A\right|; x\right) \ge c \sup_{1\le N<A} D_{M_N}(x).$$

Let $x \in I_N \backslash I_{N+1}$, for some $s = 0, 1, ..., A$. Then, from (1) it follows that

$$\sup_{N\in\mathbb{P}} S_{M_N}\left(\left|\sigma_{M_{A+1}} f_A\right|; x\right) \ge cM_N.$$

Let $0 < p < 1$. Then

$$\begin{aligned} & \left\|\left|\sigma_{M_{A+1}} f_A\right|\right\|_{H_p}^p \qquad (18) \\ = & \left\| \sup_{1 \leq N < A-1} S_{M_N} \left(\left|\sigma_{M_{A+1}} f_A\right|; x\right) \right\|_p^p \\ \geq & \int_{G_m} \left(\sup_{1 \leq N < A-1} S_{M_N} \left(\left|\sigma_{M_{A+1}} f_A\right|; x\right) \right)^p d\mu(x) \\ \geq & \sum_{s=1}^{A} \int_{I_N \setminus I_{N+1}} \left(\sup_{1 \leq N < A-1} S_{M_N} \left(\left|\sigma_{M_{A+1}} f_A\right|; x\right) \right)^p d\mu(x) \\ \geq & c \sum_{s=1}^{A} \frac{M_s^p}{M_s} = c_p > 0. \end{aligned}$$

Let $p = 1$. Then we obtain

$$\left\|\left|\sigma_{M_{A+1}} f_A\right|\right\|_{H_1} \geq cA. \qquad (19)$$

By combining (15), (18) and (19) we can conclude that

$$\frac{\left\|\left|\sigma_{M_{A+1}} f_A\right|\right\|_{H_p}}{\|f_A\|_{H_p}} \geq \frac{c_p}{M_A^{1-1/p}} \to \infty, \quad \text{as} \quad A \to \infty, \quad 0 < p < 1$$

and

$$\frac{\left\|\left|\sigma_{M_{A+1}} f_A\right|\right\|_{H_1}}{\|f_A\|_{H_1}} \geq cA \to \infty, \quad \text{as} \quad A \to \infty.$$

The proof is complete. □

As the consequence of our result we have the following negative result:

Corollary 2.3: *Let $0 < p \leq 1$. Then the maximal operator $\sigma^{\#} f$ is not bounded from the martingale Hardy space $H_p(G_m)$ to the martingale Hardy space $H_p(G_m)$.*

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