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Analytical Solution of Interior Boundary Value Problems of Elasticity for the Domain Bounded by the Parabola

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Abstract. Exact solution of two dimensional problems of elasticity are constructed in the parabolic coordinates in domain bounded by coordinate lines of the parabolic coordinate system. Here we represent internal boundary value problems of elastic equilibrium of the homogeneous isotropic body bounded by coordinate lines of the parabolic coordinate system, when on parabolic border normal or tangential stresses are given. Exact solutions are obtained using the method of separation of variables. Using the MATLAB software numerical results and constructed graphs of the mentioned boundary value problems are obtained.

Keywords: Parabolic coordinates, Separation of variables, Elasticity, Internal boundary value problem, Harmonic function.

AMS Subject Classification: 65N38.

1. Introduction

In order to solve the boundary value problems and boundary-contact problems in the areas with curvilinear border, it is purposeful to consider such tasks in the corresponding curvilinear coordinate system. For example, for a circle and its parts the tasks are considered in the polar coordinate system [1-4], for an ellipse and its parts the tasks are considered in elliptical coordinate system [5-8], for areas with circle with different centers and radiuses the tasks are considered in the bipolar coordinate system [9-11]. The above-mentioned tasks are solved by both analytical and numerical methods. In [12] the boundary value problems are formulated according to the complex potential function, using parabolic coordinate systems.

In the present paper the boundary value problems are considered in a parabolic coordinate system (see appendix A). In the parabolic coordinates are written the equilibrium equation system and Hooke's law, analytical (exact) solution of 2D problems of elasticity are constructed in the domain bounded by coordinate lines of the parabolic coordinate system. Here we represent internal boundary value problems of elastic equilibrium of the homogeneous isotropic body bounded by coordinate lines of the parabolic coordinate system, when on parabolic border normal or tangential stresses are given. Exact solutions are obtained using the

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method of separation of variables. Numerical results and corresponding graphs of above-mentioned problems are presented.

2. Statement of the problems

We consider the homogeneous isotropic elastic body, to which the following area (see fig. 1) corresponds

$$\Omega_1 = \{0 < \xi < \xi_1,\;\; 0 < \eta < \eta_1\} \tag{1}$$

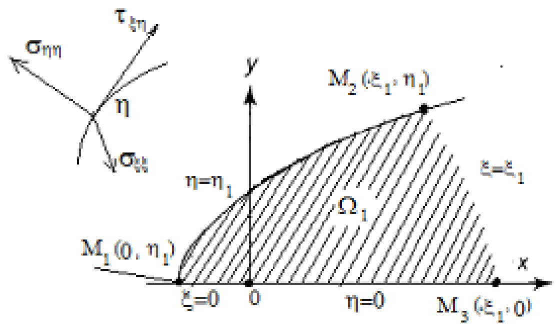

Figure 1. $\Omega_1 = \{0 < \xi < \xi_1,\;\; 0 < \eta < \eta_1\}$ area bounded by parabolic line and line $y = 0$.

In a parabolic coordinate system equilibrium equations with respect to the function D, K, u, v and Hooke's law can be written as [13]

$$\begin{aligned} &\text{a)}\;\; D_{,\xi} - K_{,\eta} = 0, \qquad && \text{c)}\;\; \bar{u}_{,\xi} + \bar{v}_{,\eta} = \frac{æ - 2}{æ\mu} h_0^2 D, \\ &\text{b)}\;\; D_{,\eta} + K_{,\xi} = 0, \qquad && \text{d)}\;\; \bar{v}_{,\xi} - \bar{u}_{,\eta} = \frac{1}{\mu} h_0^2 K, \end{aligned} \tag{2}$$

$$\begin{aligned} &\sigma_{\xi\xi} + \sigma_{\eta\eta} = 2\left(\lambda + \mu\right) h_0^{-2} \left[\left(\bar{u}\right)_{,\xi} + \left(\bar{v}\right)_{,\eta} \right], \\ &\sigma_{\xi\xi} - \sigma_{\eta\eta} = 2\mu \left[\left(h_0^{-2} \bar{u}\right)_{,\xi} - \left(h_0^{-2} \bar{v}\right)_{,\eta} \right], \\ &\tau_{\xi\eta} = \mu h_0^{-2} \left[\left(h_0^{-2} \bar{v}\right)_{,\xi} + \left(h_0^{-2} \bar{u}\right)_{,\eta} \right], \end{aligned} \tag{3}$$

where $\bar{u} = \frac{hu}{c^2}$, $\bar{v} = \frac{hv}{c^2}$; $h_0 = \sqrt{\xi^2 + \eta^2}$, $h = h_\xi = h_\eta = c\sqrt{\xi^2 + \eta^2}$ are Lamé coefficients (see appendix A), u, v are components of the displacement vector at tangents to the coordinate lines η, ξ; $\frac{æ-2}{æ\mu} D$ is the divergence of the displacement vector, $\frac{K}{\mu}$ is the rotor of the displacement vector; $\sigma_{\xi\xi}$, $\sigma_{\eta\eta}$ and $\tau_{\xi\eta} = \tau_{\eta\xi}$ are normal and tangential stresses; subscripts ξ, η denote partial derivatives with respect to the corresponding coordinates; $\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$, $\mu = \frac{E}{2(1-\nu)}$ are elastic Lamé constants; æ $= 4\,(1-\nu)$; ν is the Poisson's ratio and$E$ is the modulus of elasticity.

Boundary conditions can be written as:

$$\text{for } \eta = \eta_1 : \ a)\ \sigma_{\eta\eta} = Q_1(\xi),\ \tau_{\xi\eta} = Q_2(\xi) \quad \text{or} \quad b)\ u = H_1(\xi),\ \ v = H_2(\xi), \tag{4}$$

$$\text{for } \xi = \xi_1 : \ a)\ \ \sigma_{\xi\xi} = F_1(\eta),\ \tau_{\xi\eta} = F_2(\eta) \ \text{or } b)\ u = G_1(\eta),\ v = G_1(\eta), \tag{4'}$$

$$\text{for } \ \xi = 0 : \quad a)\ \ v = 0, \quad \sigma_{\xi\xi} = 0 \quad \text{or} \quad b)\ \ u = 0, \quad \tau_{\xi\eta} = 0, \tag{5}$$

$$\text{for } \ \eta = 0 : \quad a)\ \ u = 0, \quad \sigma_{\eta\eta} = 0, \quad \text{or} \quad b)\ \ v = 0, \quad \tau_{\xi\eta} = 0, \tag{6}$$

where F_i, Q_i $(i = 1, 2)$ with the first derivative, and G_i, H_i with the first and second derivatives can be decomposed into the trigonometric absolute and uniform convergent Fourier series.

Boundary conditions on the linear parts $\xi = 0$ and $\eta = 0$ of consideration area enables us to continue the solutions continuously (symmetrically or anti symmetrically) in the domain, that is the mirror reflection of the consideration area in a relationship $y = 0$ line (see Fig.2).

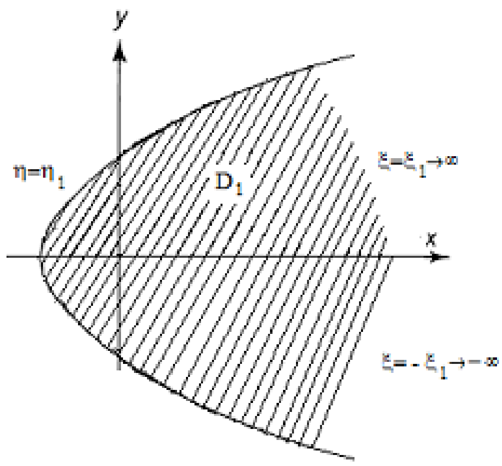

Figure 2. $D_1 = \{-\xi_1 < \xi < \xi_1,\ \ 0 < \eta < \eta_1\}$ area bounded by parabola.

3. Solution of problems

We have to find the solution of tasks (1) (see Figure 1), (2)-(6), which belongs to the $C^2(D_1)$ (see D_1 area on Fig. 2)). The solution is constructed using its general representation by two harmonic functions φ_1 and φ_2 (see Appendix B).

From formulas (B11)-(B13) we will have:

$$\begin{aligned}\bar{u} &= -\left[\eta\left(\varphi_{1,\eta}-\varphi_{2,\xi}\right)+(æ-1)\,\varphi_1\right]\xi+\left[\frac{\eta_1^2}{\eta}\left(\varphi_{1,\xi}+\varphi_{2,\eta}\right)-(æ-1)\,\varphi_2\right]\eta,\\ \bar{v} &= \left[\frac{\eta_1^2}{\eta}\left(\varphi_{1,\eta}-\varphi_{2,\xi}\right)+(æ-1)\,\varphi_1\right]\eta+\left[\eta\left(\varphi_{1,\xi}+\varphi_{2,\eta}\right)-(æ-1)\,\varphi_2\right]\xi;\end{aligned} \tag{7}$$

$$D=\frac{æ\mu}{h_0^2}\left[\left(\varphi_{1,\eta}-\varphi_{2,\xi}\right)\eta-\left(\varphi_{1,\xi}+\varphi_{2,\eta}\right)\xi\right],$$

$$K=\frac{æ\mu}{h_0^2}\left[\left(\varphi_{1,\eta}-\varphi_{2,\xi}\right)\xi+\left(\varphi_{1,\xi}+\varphi_{2,\eta}\right)\eta\right],$$

where

$$\frac{1}{h^2}\left(\varphi_{i,\xi\xi}+\varphi_{i,\eta\eta}\right)=0,\quad i=1,2. \tag{8}$$

The stress tensor components can be written as:

$$\begin{aligned}\frac{h_0^2}{2\mu}\sigma_{\eta\eta} =& -\left[\frac{\eta_1^2}{\eta}\left(\varphi_{1,\xi\xi}+\varphi_{2,\xi\eta}\right)-\frac{æ}{2}\varphi_{1,\eta}-\frac{æ-2}{2}\varphi_{2,\xi}\right]\eta\\ &+\left[\eta\left(\varphi_{1,\xi\eta}-\varphi_{2,\eta\eta}\right)+\frac{æ-2}{2}\varphi_{1,\xi}-\frac{æ}{2}\varphi_{2,\eta}\right]\xi\\ &-\frac{\eta_1^2-\eta}{\xi^2+\eta^2}\left[\left(\varphi_{1,\eta}-\varphi_{2,\xi}\right)\eta-\left(\varphi_{1,\xi}+\varphi_{2,\eta}\right)\xi\right],\end{aligned}$$

$$\begin{aligned}\frac{h_0^2}{2\mu}\tau_{\xi\eta} =& \left[\frac{\eta_1^2}{\eta}\left(\varphi_{1,\xi\eta}-\varphi_{2,\xi\xi}\right)+\frac{æ-2}{2}\varphi_{1,\xi}-\frac{æ}{2}\varphi_{2,\eta}\right]\eta\\ &+\left[\eta\left(\varphi_{1,\xi\xi}+\varphi_{2,\xi\eta}\right)-\frac{æ}{2}\varphi_{1,\eta}-\frac{æ-2}{2}\varphi_{2,\xi}\right]\xi\\ &-\frac{\eta_1^2-\eta}{\xi^2+\eta^2}\left[\left(\varphi_{1,\eta}-\varphi_{2,\xi}\right)\xi+\left(\varphi_{1,\xi}+\varphi_{2,\eta}\right)\eta\right],\end{aligned} \tag{9}$$

$$\begin{aligned}\frac{h_0^2}{2\mu}\sigma_{\xi\xi} =& \left[\frac{\eta_1^2}{\eta}\left(\varphi_{1,\xi\xi}+\varphi_{2,\xi\eta}\right)-\frac{æ-4}{2}\varphi_{1,\eta}-\frac{æ+2}{2}\varphi_{2,\xi}\right]\eta\\ &-\left[\eta\left(\varphi_{1,\xi\eta}-\varphi_{2,\xi\xi}\right)+\frac{æ+2}{2}\varphi_{1,\xi}-\frac{æ-4}{2}\varphi_{2,\eta}\right]\xi\\ &+\frac{\eta_1^2-\eta}{\xi^2+\eta^2}\left[\left(\varphi_{1,\eta}-\varphi_{2,\xi}\right)\eta-\left(\varphi_{1,\xi}+\varphi_{2,\eta}\right)\xi\right],\end{aligned}$$

From (8) by the separation of variables method we obtain (see Appendix A)

$$\varphi_i=\sum_{n=1}^{\infty}\varphi_{in},\quad i=1,2, \tag{10}$$

where

$$\varphi_{1n} = A_{1n} \cosh(n\eta)\cos(n\xi), \qquad \varphi_{2n} = A_{2n} \sinh(n\eta)\sin(n\xi)$$

or

$$\varphi_{1n} = A_{1n} \sinh(n\eta)\sin(n\xi), \qquad \varphi_{2n} = A_{2n} \cosh(n\eta)\cos(n\xi).$$

For $n = 0$ $\varphi_{10} = A_{10} + a_{02}\xi + +a_{03}\eta + a_{04}\xi\eta$, $\varphi_{20} = A_{20} + b_{02}\xi + b_{03}\eta + b_{04}\xi\eta$, where $A_{10}, a_{02}, \ldots, b_{04}$ are constant coefficients. When $n = 0$ and $0 < \xi < \xi_1$, then the terms ξ, η and $\xi\eta$ will not be contained in φ_{10} and φ_{20}. If the foregoing solutions are presented in expressions of φ_{10} and φ_{20}, then it would be impossible on $\xi = \xi_1$ to satisfy the boundary conditions and $grad\varphi_{i0} = \frac{1}{h}(\varphi_{i0,\xi} + \varphi_{i0,\eta})$ $(i = 1, 2)$ will not be bounded in the point $M(0, 0)$.

Statement 1. Here and in what follows it will be assumed that

1) ξ_1 is a sufficiently large positive number;

2) at $\eta = \eta_1$ given boundary conditions, i.e. displacements or stresses, on the interval $\tilde{\xi}_1 < \xi < \xi_1$ will be equal to zero;

3) when on $\eta = \eta_1$ is given stresses the main vector and main moment will equal to zero.

It is clear that

$$D = \frac{æ}{4}(\sigma_{\xi\xi} + \sigma_{\eta\eta}), \sigma_{\xi\xi} = \frac{4}{æ}D - \sigma_{\eta\eta}.$$

By ultimately opening expressions $\sigma_{\eta\eta}$ and $\tau_{\xi\eta}$ (in details), we can demonstrate that at point $M(0, 0)$, $\sigma_{\eta\eta}$ and $\tau_{\xi\eta}$ (and naturally, $\sigma_{\xi\xi}$, too) are determined, i.e. they are finite.

When at $\eta = \eta_1$ are given and $\bar{v}$, then it is expedient to take instead of them as their equivalent the following expressions:

$$\begin{aligned} &\frac{1}{h_0^2}(\bar{u} \cdot \eta_1 + \bar{v} \cdot \xi) = \eta_1 (\varphi_{1,\xi} + \varphi_{2,\eta}) - (æ - 1)\varphi_2, \\ &\frac{1}{h_0^2}(\bar{u} \cdot \xi - \bar{v} \cdot \eta_1) = \eta_1 (\varphi_{1,\eta} - \varphi_{2,\xi}) + (æ - 1)\varphi_1 \end{aligned} \tag{11}$$

and if at $\eta = \eta_1$ are given $\frac{h_0^2}{2\mu}\sigma_{\eta\eta}$ and $\frac{h_0^2}{2\mu}\sigma_{\xi\eta}$, then instead of them we have to take their equivalent following expressions :

$$\begin{aligned} &\frac{1}{2\mu}(\sigma_{\eta\eta} \cdot \eta_1 - \sigma_{\xi\eta} \cdot \xi) = -\eta_1 (\varphi_{1,\xi\xi} + \varphi_{2,\xi\eta}) - \frac{æ}{2}\varphi_{1,\eta} - \frac{æ - 2}{2}\varphi_{2,\xi}, \\ &\frac{1}{2\mu}(\sigma_{\eta\eta} \cdot \xi + \sigma_{\xi\eta} \cdot \eta_1) = \eta_1 (\varphi_{1,\xi\eta} - \varphi_{2,\xi\xi}) + \frac{æ - 2}{2}\varphi_{1,\xi} - \frac{æ}{2}\varphi_{2,\eta}. \end{aligned} \tag{12}$$

Considering the homogeneous boundary conditions of the concrete problem, we will insert φ_1 and φ_2 functions selected from the (10) in the right sides of the (11) or (12) equation, and we will expand the left sides in Fourier series. In both sides expressions which are with identical combinations of trigonometric functions will equate to each other and will receive the infinite system of linear algebraic equations to unknown coefficients A_{1n} and A_{2n} of harmonic functions, with its main matrix having a block-diagonal form. The dimension of each block is 2×2 and determinant is not equal to zero, but in infinite the determinant of block strive to the finite number different to zero.

It is very easy to establish convergence of (7), (9) functional series on the area $\bar{D}_1 = \{-\xi_1 \le \xi \le \xi_1,\ \ 0 \le \eta \le \eta_1\}$ by construction of the corresponding uniform convergent numerical majorizing series. So we have valid the following

Proposal 1. The functional series corresponding to (7), (9) are absolutely and uniformly convergent series on the area $\bar{D}_1 = \{-\xi_1 \le \xi \le \xi_1,\ \ 0 \le \eta \le \eta_1\}$.

4. Examples

We will set and solve the concrete internal boundary value problem in stresses. Let us find the solution of equilibrium equation system (2) of the homogeneous isotropic body in the area $\Omega_1 = \{0 < \xi < \xi_1,\ \ 0 < \eta < \eta_1\}$, which satisfies the following boundary conditions:

$$\begin{aligned} &\text{for } \ \xi = 0: \qquad \bar{v} = 0, \quad \bar{u}_{,\xi} = 0, \\ &\text{for } \ \eta = 0: \qquad \bar{u} = 0, \quad \bar{v}_{,\eta} = 0, \end{aligned} \tag{13}$$

$$\text{for } \ \eta = \eta_1: \quad \frac{h_0^2}{2\mu}\sigma_{\eta\eta} = Q_1(\xi)\,, \ \ \frac{h_0^2}{2\mu}\tau_{\xi\eta} = Q_2(\xi) \tag{14}$$

From (11), (13)

$$\varphi_i = \sum_{n=1}^{\infty} \varphi_{in}, \tag{15}$$

where $\varphi_{1n} == A_{1n}\sinh(n\eta)\sin(n\xi)\,, \qquad \varphi_{2n} == A_{2n}\cosh(n\eta)\cos(n\xi)\,.$

By inserting (15) in (7) and (9) we will receive the following expressins for the displacements:

$$\begin{aligned} \bar{u} = \sum_{n=1}^{\infty} \{ &- [n\eta\xi\cosh(n\eta)(A_{1n} + A_{2n}) + (æ - 1)\xi\sinh(n\eta)A_{1n}]\sin(n\xi) \\ &+ \left[n\eta_1^2\sinh(n\eta)(A_{1n} + A_{2n}) - (æ - 1)\eta\cosh(n\eta)A_{2n}\right]\cos(n\xi)\}\,, \\ \bar{v} = \sum_{n=1}^{\infty} \{ &\left[n\eta_1^2\cosh(n\eta)(A_{1n} + A_{2n}) + (æ - 1)\eta\sinh(n\eta)A_{1n}\right]\sin(n\xi) \\ &+ [n\eta\xi\sinh(n\eta)(A_{1n} + A_{2n}) - (æ - 1)\xi\cosh(n\eta)A_{2n}]\cos(n\xi)\}\,, \end{aligned} \tag{16}$$

but for the stresses the following:

$$\begin{aligned}
\frac{h_0^2}{2\mu}\sigma_{\eta\eta} = \sum_{n=1}^{\infty} \Big\{ & \left[n^2\eta_1^2 \sinh(n\eta)(A_{1n}+A_{2n})\right.\\
& \left. +n\eta\cosh(n\eta)\left(\frac{æ}{2}A_{1n}-\frac{æ-2}{2}A_{2n}\right)\right]\sin(n\xi)\\
& +\left[n^2\eta\xi\cosh(n\eta)(A_{1n}+A_{2n})\right.\\
& \left. +n\xi\sinh(n\eta)\left(\frac{æ-2}{2}A_{1n}-\frac{æ}{2}A_{2n}\right)\right]\cos(n\xi)\\
& -\frac{\eta_1^2-\eta^2}{\xi^2+\eta^2}\left[n\eta\cosh(n\eta)(A_{1n}+A_{2n})\sin(n\xi)\right.\\
& \left. -n\xi\sinh(n\eta)(A_{1n}+A_{2n})\cos(n\xi)\right]\Big\},
\end{aligned}$$

$$\begin{aligned}
\frac{h_0^2}{2\mu}\tau_{\xi\eta} = \sum_{n=1}^{\infty} \Big\{ & \left[n^2\eta_1^2 \cosh(n\eta)(A_{1n}+A_{2n})\right.\\
& \left. +n\eta\sinh(n\eta)\left(\frac{æ-2}{2}A_{1n}-\frac{æ}{2}A_{2n}\right)\right]\cos(n\xi)\\
& -\left[n^2\eta\xi\sinh(n\eta)(A_{1n}+A_{2n})\right.\\
& \left. +n\xi\cosh(n\eta)\left(\frac{æ}{2}A_{1n}-\frac{æ-2}{2}A_{2n}\right)\right]\sin(n\xi)\\
& -\frac{\eta_1^2-\eta^2}{\xi^2+\eta^2}\left[n\xi\cosh(n\eta)(A_{1n}+A_{2n})\sin(n\xi)\right.\\
& \left. +n\eta\sinh(n\eta)(A_{1n}+A_{2n})\cos(n\xi)\right]\Big\},
\end{aligned} \tag{17}$$

$$\begin{aligned}
\frac{h_0^2}{2\mu}\sigma_{\xi\xi} = \sum_{n=1}^{\infty} \Big\{ & -\left[n^2\eta_1^2 \sinh(n\eta)(A_{1n}+A_{2n})\right.\\
& \left. +n\eta\cosh(n\eta)\left(\frac{æ-4}{2}A_{1n}-\frac{æ+2}{2}A_{2n}\right)\right]\sin(n\xi)\\
& -\left[n^2\eta\xi\cosh(n\eta)(A_{1n}+A_{2n})\right.\\
& \left. +n\xi\sinh(n\eta)\left(\frac{æ+2}{2}A_{1n}-\frac{æ-4}{2}A_{2n}\right)\right]\cos(n\xi)\\
& +\frac{\eta_1^2-\eta^2}{\xi^2+\eta^2}\left[n\eta\cosh(n\eta)(A_{1n}+A_{2n})\sin(n\xi)\right.\\
& \left. -n\xi\sinh(n\eta)(A_{1n}+A_{2n})\cos(n\xi)\right]\Big\}.
\end{aligned}$$

Example a. We have to solve problem (2), (13), (14), when $Q_1(\xi) = P$ and $Q_2(\xi) = 0$, i.e. at $\eta = \eta_1$ boundary the normal load $\frac{1}{2\mu}\sigma_{\eta\eta} = \frac{P}{h_0^2}$ is given, but tangent stress is equal to zero. From (12), (14), (15) we obtain the following equations:

$$\sum_{n=1}^{\infty}\left[n^2\eta_1\sinh(n\eta_1)(A_{1n}+A_{2n})-n\cosh(n\eta_1)\left(\frac{æ}{2}A_{1n}-\frac{æ-2}{2}A_{2n}\right)\right]\sin(n\xi)$$
$$=\frac{P\eta_1}{\xi^2+\eta_1^2},$$

$$\sum_{n=1}^{\infty}\left[n^2\eta_1\cosh(n\eta_1)(A_{1n}+A_{2n})+n\sinh(n\eta_1)\left(\frac{æ-2}{2}A_{1n}-\frac{æ}{2}A_{2n}\right)\right]\cos(n\xi)$$
$$=\frac{P\xi}{\xi^2+\eta_1^2}.$$

From here we obtained the infinite system of the linear algebraic equations with unknown A_{1n} and A_{2n} coefficients

$$\begin{aligned}&\Big[\Big(n^2\eta_1\sinh(n\eta_1)-n\frac{æ}{2}\cosh(n\eta_1)\Big)A_{1n}\\&\quad+\left(n^2\eta_1\sinh(n\eta_1)+n\frac{æ-2}{2}\cosh(n\eta_1)\right)A_{2n}\Big]=\tilde{F}_{1n},\\&\left[\left(n^2\eta_1\cosh(n\eta_1)+n\frac{æ-2}{2}\sinh(n\eta_1)\right)A_{1n}\right.\\&\quad\left.+\Big(n^2\eta_1\cosh(n\eta_1)-n\frac{æ}{2}\sinh(n\eta_1)\Big)A_{2n}\right]=\tilde{F}_{2n},\qquad n=1,2,\ldots\end{aligned}\tag{18}$$

where $\tilde{F}_{1n}$ and $\tilde{F}_{2n}$ are the coefficients of expansion into Fourier series $f_1(\xi)=\sum_{n=1}^{\infty}\tilde{F}_{1n}\sin(n\xi)$ and $f_2(\xi)=\sum_{n=1}^{\infty}\tilde{F}_{2n}\cos(n\xi)$, respectively, $f_1(\xi)=\dfrac{P\eta_1}{\xi^2+\eta_1^2}$ and $f_2(\xi)=\dfrac{P\xi}{\xi^2+\eta_1^2}$ functions.

As seen, the main matrix of system (18) has a block-diagonal form, dimension of each block is 2×2 . Thus, will be solved two equations, to twoA_{1n} and A_{2n} unknowns. After solving this system, we find A_{1n} and A_{2n} coefficients, and put them into formulas (16) and (17), we get displacements and stresses at any points of the body.

Numerical results obtained for some characteristic points of the body, in particular, $M_1(0,\eta_1)$, $M_2(\xi_1,\eta_1)$, $M_3(\xi_1,0)$ points (see. Fig. 1), when $0.1\le\eta_1\le 3$ for the following data:$\nu=0.3$, $E=2*10^6 kg/cm^2$, $P=-10kg/cm^2$, $0.1\le\eta_1\le 3$, $\xi_1=2*\pi$, $\xi_1=4*\pi$ and $\xi_1=6*\pi$. Numerical calculations and the visual presentation are made by MATLAB's software.

Example b. We solve problem (2), (13), (14), when $Q_1(\xi)=0$ and $Q_2(\xi)=P$, i.e. at $\eta=\eta_1$ is given the tangent stress $\dfrac{1}{2\mu}\tau_{\xi\eta}=\dfrac{P}{h_0^2}$, but the normal stress is

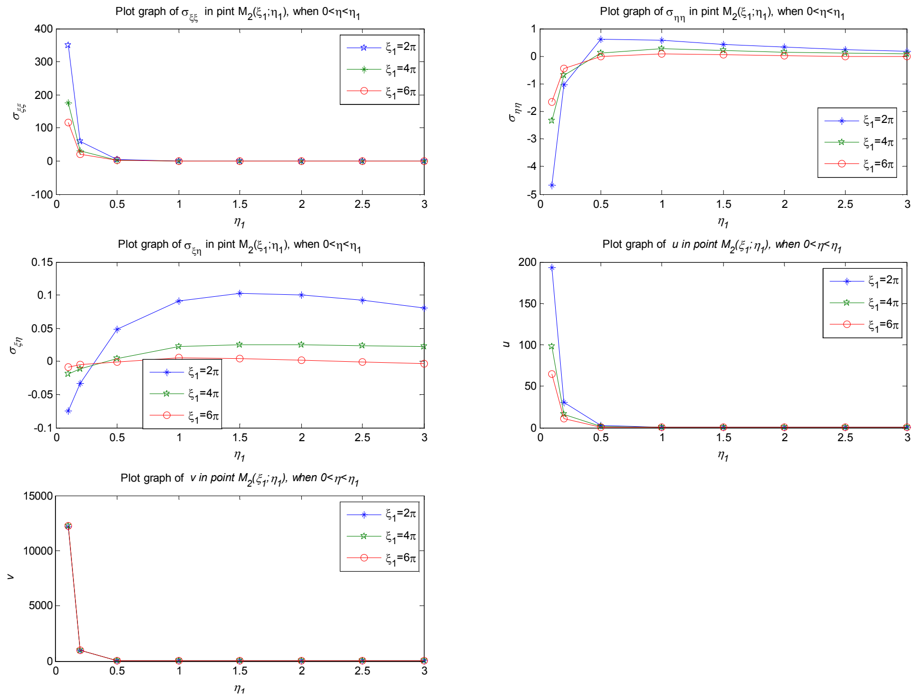

Figure 3. Stresses and displacements in points $M_2\left(\xi_1,\eta_1\right)$ for $\xi_1=2*\pi$, $\xi_1=4*\pi$ and $\xi_1=6*\pi$, when $0.1\leq\eta_1\leq3$.

equal to zero. From (12), (14), (15) we will receive the following equations:

$$\sum_{n=1}^{\infty}\left[n^2\eta_1\sinh\left(n\eta_1\right)\left(A_{1n}+A_{2n}\right)-n\cosh\left(n\eta_1\right)\left(\frac{æ}{2}A_{1n}-\frac{æ-2}{2}A_{2n}\right)\right]\sin\left(n\xi\right)$$
$$=-\sum_{n=1}^{\infty}\tilde{G}_{1n}\sin\left(n\xi\right),$$

$$\sum_{n=1}^{\infty}\left[n^2\eta_1\cosh\left(n\eta_1\right)\left(A_{1n}+A_{2n}\right)+n\sinh\left(n\eta_1\right)\left(\tfrac{æ-2}{2}A_{1n}-\tfrac{æ}{2}A_{2n}\right)\right]\cos\left(n\xi\right)$$
$$=\sum_{n=1}^{\infty}\tilde{G}_{2n}\cos\left(n\xi\right),$$

where $\tilde{G}_{1n}$ and $\tilde{G}_{2n}$ are the coefficients of expansion into Fourier series $f_1\left(\xi\right)=\sum_{n=1}^{\infty}\tilde{G}_{1n}\sin\left(n\xi\right)$ and $f_2\left(\xi\right)=\sum_{n=1}^{\infty}\tilde{G}_{2n}\cos\left(n\xi\right)$, respectively, $f_1\left(\xi\right)=\dfrac{P\xi}{\left(\xi^2+\eta_1^2\right)}$ and $f_2\left(\xi\right)=\dfrac{P\eta_1}{\left(\xi^2+\eta_1^2\right)}$ functions.

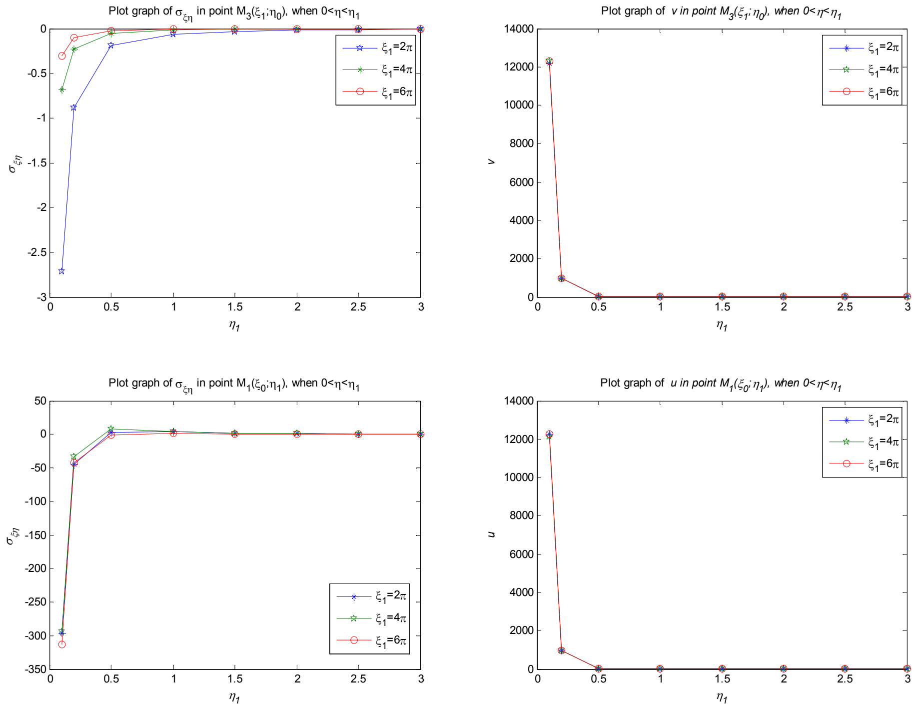

Figure 4. Tangent stress and displacements in points $M_3\left(\xi_1, 0\right)$ and $M_1\left(0, \eta_1\right)$ for $\xi_1 = 2 * \pi$, $\xi_1 = 4 * \pi$ and $\xi_1 = 6 * \pi$, when $0.1 \leq \eta_1 \leq 3$.

$$
\begin{aligned}
&\left[\left(n^2\eta_1 \sinh\left(n\eta_1\right) - n\frac{æ}{2}\cosh\left(n\eta_1\right)\right) A_{1n}\right.\\
&\qquad \left. + \left(n^2\eta_1 \sinh\left(n\eta_1\right) + n\frac{æ-2}{2}\cosh\left(n\eta_1\right)\right) A_{2n}\right] = -\tilde{G}_{1n},\\
&\left[\left(n^2\eta_1 \cosh\left(n\eta_1\right) + n\frac{æ-2}{2}\sinh\left(n\eta_1\right)\right) A_{1n}\right.\\
&\qquad \left. + \left(n^2\eta_1 \cosh\left(n\eta_1\right) - n\frac{æ}{2}\sinh\left(n\eta_1\right)\right) A_{2n}\right] = \tilde{G}_{2n}, \qquad n = 1, 2, \ldots
\end{aligned}
\tag{19}
$$

Here about system (25) we can say the same what has been said about system (24) . For mutual comparison of the results obtained in both problems, the numerical solutions, tables and graphs obtained for the same data that was used in the previous problem.

In points $M_1\left(0, \eta_1\right)$, $(0.1 \leq \eta_1 \leq 3$ and $\xi_1 = 2 * \pi$, $\xi_1 = 4 * \pi$, $\xi_1 = 6 * \pi)$ graphs and tables of values of tangential $\tau_{\xi\eta}$ stresses and normal displacements u are presented, when on the parabolic boundary the normal load (see Fig.4 and Tab.4) and the tangential load is given (see Fig.6 and Tab.8).

In points $M_2\left(\xi_1, \eta_1\right)$, $(0.1 \leq \eta_1 \leq 3$ and $\xi_1 = 2 * \pi$, $\xi_1 = 4 * \pi$, $\xi_1 = 6 * \pi)$ graphs and tables of values of a) normal $\sigma_{\eta\eta}$, tangential $\tau_{\xi\eta}$ and shearing $\sigma_{\xi\xi}$ stresses are presented, when on the parabolic boundary the normal load is given (see Fig. 3 and Tab.1) and when on parabolic boundary tangential load (see Fig. 5 and Tab.5); b) normal u and tangential v displacements, when on the parabolic boundary normal load is given (see Fig.3 and Tab.2) are given and when on the parabolic boundary is given the tangential load (see Fig.5 and Tab.6).

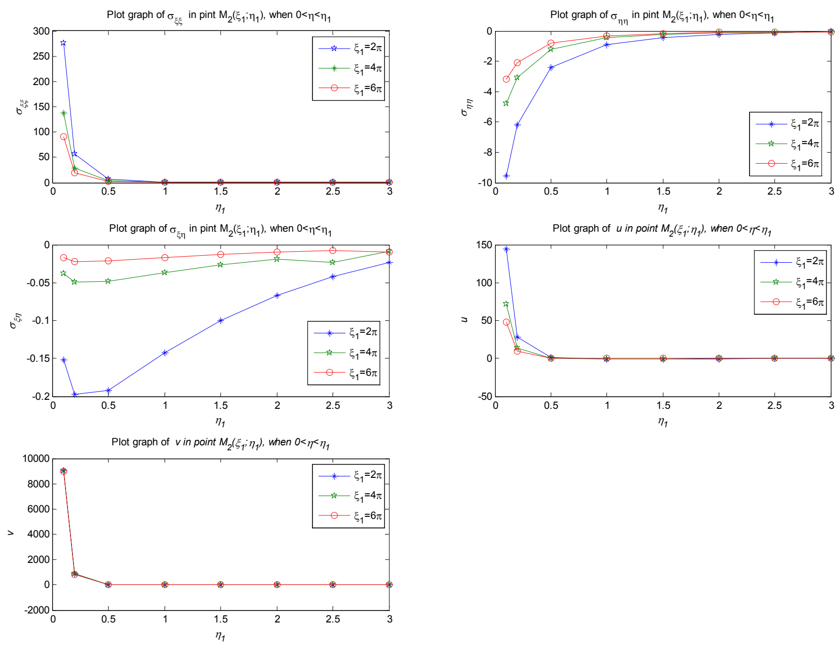

Figure 5. Stresses and displacements in points $M_2\left(\xi_1, \eta_1\right)$ for $\xi_1 = 2 * \pi$, $\xi_1 = 4 * \pi$ and $\xi_1 = 6 * \pi$, when $0.1 \leq \eta_1 \leq 3$.

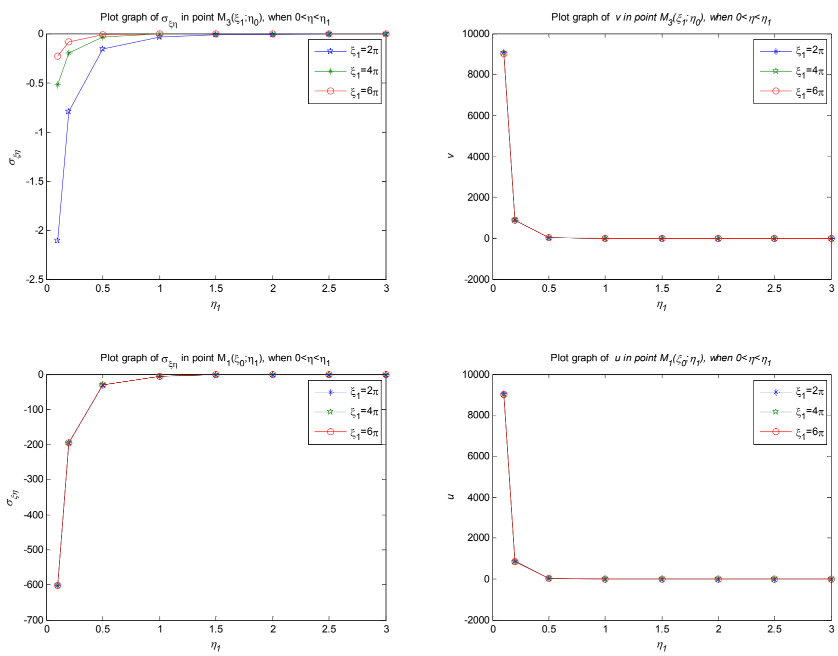

Figure 6. Values of tangent stress and displacements in points $M_3\left(\xi_1, 0\right)$ and $M_1\left(0, \eta_1\right)$ for $\xi_1 = 2 * \pi$, $\xi_1 = 4 * \pi$ and $\xi_1 = 6 * \pi$, when $0.1 \leq \eta_1 \leq 3$.

In points $M_3\left(\xi_1, 0\right)$, ($0.1 \leq \eta_1 \leq 3$ and $\xi_1 = 2 * \pi$, $\xi_1 = 4 * \pi$, $\xi_1 = 6 * \pi$) graphs and tables of values of tangential $\tau_{\xi\eta}$ stresses and tangential v displacements are presented, when on the parabolic boundary normal load is given (see Fig.4 and Tab.3) and when on the parabolic boundary tangential load (see Fig.6 and Tab.7) is given.

The obtained results show that in case of the normal loads on the parabolic boundary the displacements are more than in case of the tangential loads.

5. Conclusion

The main results of this work can be formulated as follows.

(1) The equilibrium equations (2) are written in terms of elliptic coordinates.
(2) The solution of the equilibrium equation (2) is obtained by the method of separation of variables. The solution is constructed using its general representation by two harmonic functions.
(3) In the parabolic coordinates exact solutions of two-dimensional static boundary value problems for the elasticity are constructed for homogeneous isotropic bodies occupying domains bounded by coordinate lines of parabolic coordinates.
(4) To set and solve two concrete internal boundary value problems in stresses.
(5) Numerical values of the components of stress tensor and displacement vector at some points of the body and corresponding graphs are presented.

η_1	$\xi_1 = 2*\pi$			$\xi_1 = 4*\pi$			$\xi_1 = 6*\pi$		
	$\sigma_{\xi\xi}$	$\sigma_{\eta\eta}$	$\tau_{\xi\eta}$	$\sigma_{\xi\xi}$	$\sigma_{\eta\eta}$	$\tau_{\xi\eta}$	$\sigma_{\xi\xi}$	$\sigma_{\eta\eta}$	$\tau_{\xi\eta}$
0.01	$3.48\times10^{+02}$	$-4.68\times10^{+00}$	-7.45×10^{-02}	$1.75\times10^{+02}$	$-2.30\times10^{+00}$	-1.87×10^{-02}	$1.17\times10^{+02}$	$-1.66\times10^{+00}$	-8.81×10^{-03}
0.02	$5.81\times10^{+01}$	$-1.04\times10^{+00}$	3.32×10^{-02}	$2.98\times10^{+01}$	7.08×10^{-01}	1.13×10^{-02}	$1.98\times10^{+01}$	4.48×10^{-01}	-4.75×10^{-03}
0.5	$4.97\times10^{+00}$	6.03×10^{-01}	4.80×10^{-02}	$2.80\times10^{+00}$	1.18×10^{-01}	4.71×10^{-03}	$2.01\times10^{+00}$	-1.22×10^{-02}	-3.24×10^{-04}
1	6.52×10^{-01}	5.75×10^{-01}	9.15×10^{-02}	3.78×10^{-01}	2.81×10^{-01}	2.23×10^{-02}	3.57×10^{-01}	9.74×10^{-02}	5.17×10^{-03}
1.5	1.01×10^{-01}	4.30×10^{-01}	1.03×10^{-01}	8.73×10^{-02}	2.13×10^{-01}	2.54×10^{-02}	1.58×10^{-01}	5.36×10^{-02}	4.27×10^{-03}
2	3.58×10^{-02}	3.16×10^{-01}	1.01×10^{-01}	9.10×10^{-03}	1.60×10^{-01}	2.55×10^{-02}	1.02×10^{-01}	2.00×10^{-02}	2.12×10^{-03}
2.5	-6.86×10^{-02}	2.32×10^{-01}	9.22×10^{-02}	-1.48×10^{-02}	1.21×10^{-01}	2.42×10^{-02}	8.27×10^{-02}	-4.38×10^{-03}	-5.81×10^{-04}
3	-6.89×10^{-02}	1.70×10^{-01}	8.10×10^{-02}	-2.09×10^{-02}	9.28×10^{-02}	2.21×10^{-02}	7.60×10^{-02}	-2.18×10^{-02}	-3.47×10^{-03}

Table 1. Values of stresses at point $M_2(\xi_1,\eta_1)$, when $0.1 \le \eta_1 \le 3$

η_1	$\xi_1 = 2*\pi$		$\xi_1 = 4*\pi$		$\xi_1 = 6*\pi$	
	u	v	u	v	u	v
0.01	$1.931722\times10^{+02}$	$1.213736\times10^{+04}$	$9.748764\times10^{+01}$	$1.225066\times10^{+04}$	$6.511376\times10^{+001}$	$1.227366\times10^{+004}$
0.02	$3.070168\times10^{+01}$	$9.645219\times10^{+02}$	$1.557706\times10^{+01}$	$9.787358\times10^{+02}$	$1.041636\times10^{+001}$	$9.817192\times10^{+002}$
0.5	$2.449950\times10^{+00}$	$3.078698\times10^{+01}$	$1.253830\times10^{+00}$	$3.151218\times10^{+01}$	8.340506×10^{-001}	$3.144297\times10^{+001}$
1	7.756216×10^{-01}	$4.873374\times10^{+00}$	3.965540×10^{-01}	$4.983244\times10^{+00}$	2.568356×10^{-001}	$4.841236\times10^{+000}$
1.5	6.356776×10^{-01}	$2.662720\times10^{+00}$	3.216085×10^{-01}	$2.694301\times10^{+00}$	2.016807×10^{-001}	$2.534394\times10^{+000}$
2	5.864609×10^{-01}	$1.842421\times10^{+00}$	2.974889×10^{-01}	$1.869178\times10^{+00}$	1.811615×10^{-001}	$1.707407\times10^{+000}$
2.5	5.405192×10^{-01}	$1.358473\times10^{+00}$	2.777550×10^{-01}	$1.396149\times10^{+00}$	1.640132×10^{-001}	$1.236631\times10^{+000}$
3	4.876470×10^{-01}	$1.021326\times10^{+00}$	2.559660×10^{-01}	$1.072188\times10^{+00}$	1.458306×10^{-001}	9.162808×10^{-001}

Table 2. Values of displacements at point $M_2(\xi_1,\eta_1)$, when $0.1 \le \eta_1 \le 3$

η_1	$\xi_1 = 2*\pi$		$\xi_1 = 4*\pi$		$\xi_1 = 6*\pi$	
	$\tau_{\xi\eta}$	v	$\tau_{\xi\eta}$	v	$\tau_{\xi\eta}$	v
0.01	$-2.719203\times10^{+00}$	$1.217236\times10^{+04}$	-6.853097×10^{-01}	$1.228477\times10^{+04}$	-3.054502×10^{-001}	$1.230772\times10^{+004}$
0.02	-8.869845×10^{-01}	$9.759616\times10^{+02}$	-2.252623×10^{-01}	$9.901484\times10^{+02}$	-1.002611×10^{-001}	$9.930414\times10^{+002}$
0.5	-1.900663×10^{-01}	$3.234437\times10^{+01}$	-4.901431×10^{-02}	$3.332241\times10^{+01}$	-2.207415×10^{-002}	$3.351348\times10^{+001}$
1	-6.460882×10^{-02}	$4.330191\times10^{+00}$	-1.714687×10^{-02}	$4.490987\times10^{+00}$	-7.713641×10^{-003}	$4.524553\times10^{+000}$
1.5	-3.308990×10^{-02}	1.636474×10^{-00}	-9.078479×10^{-03}	$1.704337\times10^{+00}$	-4.112702×10^{-003}	$1.718664\times10^{+000}$
2	-1.820946×10^{-02}	7.383573×10^{-01}	-5.189954×10^{-03}	7.772619×10^{-01}	-2.370961×10^{-003}	7.855813×10^{-001}
2.5	-1.014523×10^{-02}	3.478495×10^{-01}	-3.078695×10^{-03}	3.723484×10^{-01}	-1.392777×10^{-003}	3.776920×10^{-001}
3	-5.667714×10^{-03}	1.664251×10^{-01}	-1.767139×10^{-03}	1.821680×10^{-01}	8.245882×10^{-004}	1.856983×10^{-001}

Table 3. Values of tangent stress and displacement at point $M_3(\xi_1, 0)$, when $0.1 \le \eta_1 \le 3$

η_1	$\xi_1 = 2*\pi$		$\xi_1 = 4*\pi$		$\xi_1 = 6*\pi$	
	$\tau_{\xi\eta}$	u	$\tau_{\xi\eta}$	u	$\tau_{\xi\eta}$	u
0.01	$-2.959080\times10^{+02}$	$1.225105\times10^{+04}$	$-2.942147\times10^{+02}$	$1.213890\times10^{+04}$	$-3.130625\times10^{+002}$	$1.227383\times10^{+004}$
0.02	$-4.449785\times10^{+01}$	$9.788597\times10^{+02}$	$-3.280970\times10^{+01}$	$9.650104\times10^{+02}$	$-4.223036\times10^{+001}$	$9.817745\times10^{+002}$
0.5	$2.978055\times10^{+00}$	$3.153712\times10^{+01}$	$7.631027\times10^{+00}$	$3.088431\times10^{+01}$	-4.602453×10^{-001}	$3.145403\times10^{+001}$
1	$3.549046\times10^{+00}$	$4.998998\times10^{+00}$	$3.704994\times10^{+00}$	$4.934710\times10^{+00}$	$1.840741\times10^{+000}$	$4.848044\times10^{+000}$
1.5	$1.811397\times10^{+00}$	$2.713428\times10^{+00}$	$1.904996\times10^{+00}$	$2.737547\times10^{+00}$	6.784791×10^{-001}	$2.542406\times10^{+000}$
2	$1.033931\times10^{+00}$	$1.892703\times10^{+00}$	$1.094364\times10^{+00}$	$1.933508\times10^{+00}$	1.904068×10^{-001}	$1.716991\times10^{+000}$
2.5	6.349922×10^{-01}	$1.423510\times10^{+00}$	6.746135×10^{-01}	$1.462057\times10^{+00}$	-3.359406×10^{-002}	$1.247460\times10^{+000}$
3	4.106701×10^{-01}	$1.102318\times10^{+00}$	4.362036×10^{-01}	$1.131771\times10^{+00}$	-1.403026×10^{-001}	9.278131×10^{-001}

Table 4. Values of tangent tress and normal displacement at point $M_1(0, \eta_1)$, when $0.1 \le \eta_1 \le 3$

η_1	$\xi_1 = 2*\pi$			$\xi_1 = 4*\pi$			$\xi_1 = 6*\pi$		
	$\sigma_{\xi\xi}$	$\sigma_{\eta\eta}$	$\tau_{\xi\eta}$	$\sigma_{\xi\xi}$	$\sigma_{\eta\eta}$	$\tau_{\xi\eta}$	$\sigma_{\xi\xi}$	$\sigma_{\eta\eta}$	$\tau_{\xi\eta}$
0.01	$2.76\times10^{+02}$	$-9.56\times10^{+00}$	-1.52×10^{-01}	$1.37\times10^{+02}$	$-4.79\times10^{+00}$	-3.81×10^{-02}	$9.13\times10^{+01}$	$-3.19\times10^{+00}$	-1.69×10^{-02}
0.02	$5.69\times10^{+01}$	$-6.21\times10^{+00}$	-1.98×10^{-01}	$2.81\times10^{+01}$	$-3.11\times10^{+00}$	-4.94×10^{-02}	$1.87\times10^{+01}$	$-2.07\times10^{+00}$	-2.20×10^{-02}
0.5	$6.67\times10^{+00}$	$-2.42\times10^{+00}$	-1.92×10^{-01}	$3.20\times10^{+00}$	$-1.22\times10^{+00}$	-4.84×10^{-02}	$2.11\times10^{+00}$	-8.17×10^{-01}	-2.17×10^{-02}
1	1.45×10^{-01}	-8.93×10^{-01}	-1.42×10^{-01}	6.72×10^{-01}	-4.59×10^{-01}	-3.66×10^{-02}	4.51×10^{-01}	-3.19×10^{-01}	-1.69×10^{-02}
1.5	6.25×10^{-01}	-4.18×10^{-01}	-9.99×10^{-02}	2.91×10^{-01}	-2.24×10^{-01}	-2.68×10^{-02}	2.08×10^{-01}	-1.68×10^{-01}	-1.34×10^{-02}
2	3.29×10^{-01}	-2.09×10^{-01}	-6.66×10^{-02}	1.58×10^{-01}	-1.20×10^{-01}	-1.91×10^{-02}	1.13×10^{-01}	-9.25×10^{-02}	-9.81×10^{-03}
2.5	1.85×10^{-01}	-1.05×10^{-01}	-4.16×10^{-02}	1.46×10^{-01}	-1.17×10^{-01}	-2.32×10^{-02}	7.45×10^{-02}	-5.94×10^{-02}	-7.88×10^{-03}
3	1.07×10^{-001}	-4.92×10^{-02}	-2.35×10^{-02}	5.83×10^{-02}	-3.75×10^{-02}	-8.96×10^{-03}	7.31×10^{-02}	-5.98×10^{-02}	-9.52×10^{-03}

Table 5. Values of stresses at point $M_2(\xi_1, \eta_1)$, when $0.1 \le \eta_1 \le 3$

η_1	$\xi_1 = 2*\pi$		$\xi_1 = 4*\pi$		$\xi_1 = 6*\pi$	
	u	v	u	v	u	v
0.01	$1.442944\times10^{+002}$	$9.066284\times10^{+003}$	$7.175766\times10^{+001}$	$9.017334\times10^{+003}$	$4.774305\times10^{+001}$	$8.999352\times10^{+003}$
0.02	$2.784333\times10^{+001}$	$8.747241\times10^{+002}$	$1.373038\times10^{+001}$	$8.627051\times10^{+002}$	$9.106392\times10^{+001}$	$8.582572\times10^{+002}$
0.5	$1.246640\times10^{+000}$	$1.566574\times10^{+001}$	5.495786×10^{-001}	$1.381242\times10^{+001}$	3.479472×10^{-001}	$1.311730\times10^{+001}$
1	-6.007224×10^{-001}	$-3.774450\times10^{+00}$	-3.384055×10^{-001}	$-4.252529\times10^{+000}$	-2.358731×10^{-001}	$-4.446103\times10^{+000}$
1.5	-5.277845×10^{-001}	$-2.210778\times10^{+00}$	-2.943083×10^{-001}	$-2.465592\times10^{+000}$	-2.060385×10^{-001}	$-2.589156\times10^{+000}$
2	-3.750689×10^{-001}	$-1.178317\times10^{+00}$	-2.174606×10^{-001}	$-1.366346\times10^{+000}$	-1.522572×10^{-001}	$-1.434990\times10^{+000}$
2.5	-2.441181×10^{-001}	-6.135357×10^{-01}	-1.674292×10^{-001}	-8.415912×10^{-001}	-1.106320×10^{-001}	-8.341453×10^{-001}
3	-1.423399×10^{-001}	-2.981160×10^{-01}	-1.038323×10^{-001}	-4.349317×10^{-001}	-8.968710×10^{-002}	-5.635207×10^{-001}

Table 6. Values of displacements at point $M_2(\xi_1, \eta_1)$, when $0.1 \le \eta_1 \le 3$

η_1	$\xi_1 = 2*\pi$		$\xi_1 = 4*\pi$		$\xi_1 = 6*\pi$	
	$\tau_{\xi\eta}$	v	$\tau_{\xi\eta}$	v	$\tau_{\xi\eta}$	v
0.01	$-2.108357\times10^{+000}$	$9.095177\times10^{+003}$	-5.237980×10^{-001}	$9.045212\times10^{+003}$	-2.324449×10^{-001}	$9.027032\times10^{+003}$
0.02	-7.962980×10^{-001}	$8.881000\times10^{+002}$	-1.958982×10^{-001}	$8.756040\times10^{+002}$	-8.666218×10^{-002}	$8.710514\times10^{+002}$
0.5	-1.591929×10^{-001}	$2.082162\times10^{+001}$	-3.712390×10^{-002}	$1.882728\times10^{+001}$	-1.608475×10^{-002}	$1.809816\times10^{+001}$
1	-3.436416×10^{-002}	$-1.114244\times10^{+000}$	-6.691920×10^{-003}	$-1.596807\times10^{+00}$	-2.656536×10^{-003}	$-1.773543\times10^{+000}$
1.5	-1.442399×10^{-002}	-5.521852×10^{-001}	-2.323125×10^{-003}	-7.484135×10^{-001}	-8.044633×10^{-004}	-8.209315×10^{-001}
2	-8.326926×10^{-003}	-1.602579×10^{-001}	-1.253521×10^{-003}	-2.548025×10^{-001}	-3.991349×10^{-004}	-2.904809×10^{-001}
2.5	-5.270993×10^{-003}	-2.685660×10^{-002}	-8.035682×10^{-004}	-7.544702×10^{-002}	-2.507020×10^{-004}	-9.440888×10^{-002}
3	-3.372185×10^{-003}	9.063354×10^{-003}	-5.345054×10^{-004}	-1.650107×10^{-002}	-1.673819×10^{-004}	-2.694384×10^{-002}

Table 7. Values of tangent stress and displacement at point $M_3(\xi_1,0)$, when $0.1 \le \eta_1 \le 3$

η_1	$\xi_1 = 2*\pi$		$\xi_1 = 4*\pi$		$\xi_1 = 6*\pi$	
	$\tau_{\xi\eta}$	u	$\tau_{\xi\eta}$	u	$\tau_{\xi\eta}$	u
0.01	$-6.009727\times10^{+002}$	$9.067432\times10^{+003}$	$-6.011787\times10^{+002}$	$9.017619\times10^{+003}$	$-6.014531\times10^{+002}$	$8.999479\times10^{+003}$
0.02	$-1.952012\times10^{+002}$	$8.751671\times10^{+002}$	$-1.952601\times10^{+002}$	$8.628143\times10^{+002}$	$-1.954875\times10^{+002}$	$8.583055\times10^{+002}$
0.5	$-3.055434\times10^{+001}$	$1.571526\times10^{+001}$	$-3.061269\times10^{+001}$	$1.382335\times10^{+001}$	$-3.083989\times10^{+001}$	$1.312191\times10^{+001}$
1	$-5.753816\times10^{+000}$	$-3.821955\times10^{+000}$	$-5.810127\times10^{+000}$	$-4.265972\times10^{+000}$	$-6.036614\times10^{+000}$	$-4.452356\times10^{+000}$
1.5	$-1.852409\times10^{+000}$	$-2.272905\times10^{+000}$	$-1.905542\times10^{+000}$	$-2.483095\times10^{+000}$	$-2.130849\times10^{+000}$	$-2.597341\times10^{+000}$
2	-7.244901×10^{-001}	$-1.236568\times10^{+000}$	-7.735399×10^{-001}	$-1.383542\times10^{+000}$	-8.814041×10^{-001}	$-1.443045\times10^{+000}$
2.5	-3.044763×10^{-001}	-6.603179×10^{-001}	-6.097619×10^{-001}	-8.580841×10^{-001}	-4.556970×10^{-001}	-8.414498×10^{-001}
3	-1.266048×10^{-001}	-3.303540×10^{-001}	-1.660994×10^{-001}	-4.471540×10^{-001}	-3.852802×10^{-001}	-5.706131×10^{-001}

Table 8. Values of tangent tress and normal displacement at point $M_1(0,\eta_1)$, when $0.1 \le \eta_1 \le 3$

6. Appendices

Appendix A.

Some basic formulas in parabolic coordinates

In orthogonal parabolic coordinate system $\xi, \eta(-\infty < \xi < \infty,\ \ 0 \leq \eta < \infty)$ [l4,15] we have

$$h_\xi = h_\eta = h = c\sqrt{\xi^2 + \eta^2}, \qquad x = \frac{c}{2}\left(\xi^2 - \eta^2\right), \quad y = c\xi\eta,$$

where $h_\xi,\ \ h_\eta$ are Lame's coefficients of the system of parabolic coordinates, c is a scale coefficient,$x,\ \ y$ are Cartesian coordinates.

The coordinate axes are parabolas

$$y^2 = -2c\xi_0^2\left(x - \frac{c\xi_0^2}{2}\right), \qquad \xi_0 = const,$$

$$y^2 = -2c\eta_0^2\left(x + \frac{c\eta_0^2}{2}\right), \qquad \eta_0 = const.$$

Laplace's equation $\Delta f = 0$, where $f = f\left(\xi, \eta\right)$, in the parabolic coordinates has the form

$$\frac{1}{c^2\left(\xi^2 + \eta^2\right)}\left(\frac{\partial^2 f}{\partial \xi^2} + \frac{\partial^2 f}{\partial \eta^2}\right) = 0.$$

We have to find solution of the equation in the following form

$$f = X\left(\xi\right) \cdot E\left(\eta\right),$$

Then by separation of variables we will receive.

$$\frac{1}{c^2\left(\xi^2 + \eta^2\right)}\left[\frac{X''}{X} + \frac{E'}{E}\right] = 0.$$

From here

$$X'' + mX = 0,$$
$$E'' - mE = 0,$$

where m is any constant, their solutions are [16]

$$X = C_1 \cos\left(m\xi\right) + C_2 \sin\left(m\xi\right),$$

$$E = C_3 e^{m\eta} + C_4 e^{-m\eta} = C_{_3}^* \cosh\left(m\eta\right) + C_{_4}^* \sinh\left(m\eta\right).$$

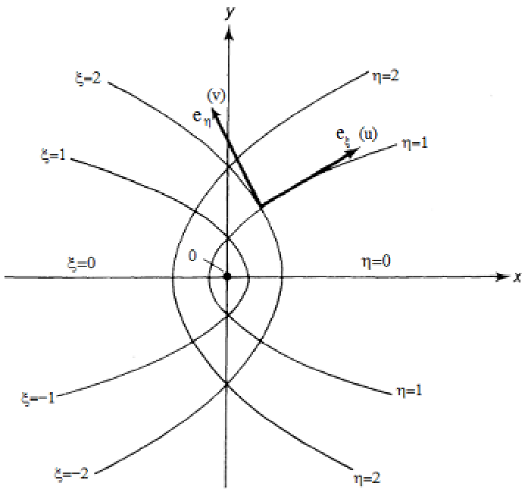

Figure 7. Parabolic coordinate system.

So

$$f(\xi,\eta) = \left(C_3 e^{m\eta} + C_4 e^{-m\eta}\right)\left(C_1\cos(m\xi) + C_2\sin(m\xi)\right)$$

or

$$f(\xi,\eta) = \left(C_3^* \cosh(m\eta) + C_4^* \sinh(m\eta)\right)\left(C_1\cos(m\xi) + C_2\sin(m\xi)\right).$$

e_ξ, e_η are unit vectors .

Appendix B. Solution of system of partial differential equations

We solve of system of partial differential equations (2)
We have introduce φ_1 harmonic function and if we take

$$\text{a)}\quad D = \frac{æ\mu}{h_0^2}\left(\frac{\partial\varphi_1}{\partial\eta}\eta - \frac{\partial\varphi_1}{\partial\xi}\xi\right),$$

$$\text{b)}\quad K = \frac{æ\mu}{h_0^2}\left(\frac{\partial\varphi_1}{\partial\eta}\xi + \frac{\partial\chi\varphi_1}{\partial\xi}\eta\right), \tag{B1}$$

then (2a) and (2b) equations will be satisfied identically, while (2c) and (2d) equations will receive the following lform:

$$\text{a)}\quad \frac{\partial\bar{u}}{\partial\xi} + \frac{\partial\bar{v}}{\partial\eta} = (æ - 2)\left(\frac{\partial\varphi_1}{\partial\eta}\eta - \frac{\partial\varphi_1}{\partial\xi}\xi\right),$$

$$\text{b)}\quad \frac{\partial v}{\partial\xi} - \frac{\partial\bar{u}}{\partial\eta} = æ\left(\frac{\partial\varphi_1}{\partial\eta}\xi + \frac{\partial\varphi_1}{\partial\xi}\eta\right), \tag{B2}$$

$$\text{a)} \quad \frac{\partial \bar{u}}{\partial \xi} + \frac{\partial \bar{v}}{\partial \eta} = (æ - 2)\left(\frac{\partial \varphi_1}{\partial \eta}\eta - \frac{\partial \varphi_1}{\partial \xi}\xi\right),$$

$$\text{b)} \quad \frac{\partial}{\partial \xi}\left(v - æ\varphi_1\eta\right) = \frac{\partial}{\partial \eta}\left(\bar{u} + æ\varphi_1\xi\right), \tag{B3}$$

From (B3b) imply that exists such type harmonic function ϕ, for which fulfil the following

$$\bar{u} = \frac{\partial \phi}{\partial \xi} - æ\varphi_1\xi, \qquad \bar{v} = \frac{\partial \phi}{\partial \eta} + æ\varphi_1\eta. \tag{B4}$$

Considering (B4), from the equation (B3a) will be obtain following

$$h^2\Delta\phi = \frac{\partial^2 \phi}{\partial \xi^2} + \frac{\partial^2 \phi}{\partial \eta^2} = æ\varphi_1 + æ\frac{\partial \varphi_1}{\partial \xi}\xi - æ\varphi_1 - æ\frac{\partial \varphi_1}{\partial \eta}\eta$$
$$+ (æ - 2)\left(\frac{\partial \varphi_1}{\partial \eta}\eta - \frac{\partial \varphi_1}{\partial \xi}\xi\right) = 2\left(\frac{\partial \varphi_1}{\partial \xi}\xi - \frac{\partial \varphi_1}{\partial \eta}\eta\right) \tag{B5}$$

General solution of the system (B2) can be write in the form $\bar{u} = \psi_1, \ \ \bar{v} = \psi_2$, where

$$\frac{\partial \psi_1}{\partial \xi} + \frac{\partial \psi_2}{\partial \eta} = 0, \qquad \frac{\partial \psi_2}{\partial \xi} - \frac{\partial \psi_1}{\partial \eta} = 0.$$

The full solution of the equation system (B2) is written in following form

$$\bar{u} = \frac{\partial \phi}{\partial \xi} - æ\varphi_1\xi + \psi_1, \qquad \bar{v} = \frac{\partial \phi}{\partial \eta} + æ\varphi_1\eta + \psi_2 \tag{B6}$$

where ϕ is the partial solution of the (B5).

If we take $æ = const$, then

$$\phi = \frac{\xi^2 - \eta^2}{2}\varphi_1$$

and (B6) formula will receive the following form:

$$\bar{u} = \frac{\xi^2 - \eta^2}{2}\frac{\partial \varphi_1}{\partial \xi} - (æ - 1)\,\varphi_1\xi + \psi_1, \qquad \bar{v} = \frac{\xi^2 - \eta^2}{2}\frac{\partial \varphi_1}{\partial \eta} + (æ - 1)\,\varphi_1\eta + \psi_2.$$

From here

$$\bar{u} = \left(\frac{\xi^2 - \eta^2}{2}\frac{\partial \varphi_1}{\partial \xi} + \xi\eta\frac{\partial \varphi_1}{\partial \eta}\right) - \xi\eta\frac{\partial \varphi_1}{\partial \eta} - (æ - 1)\,\varphi_1\xi + \psi_1,$$

$$\bar{v} = \left(\frac{\xi^2 - \eta^2}{2}\frac{\partial \varphi_1}{\partial \eta} - \xi\eta\frac{\partial \varphi_1}{\partial \xi}\right) + \xi\eta\frac{\partial \varphi_1}{\partial \xi} + (æ - 1)\,\varphi_1\eta + \psi_2.$$

Without loss of generality the expression in brackets can be taken to be zero, because we already have in $\bar{u}$ and $\bar{v}$ of the solutions Laplacian (we mean ψ_1 and ψ_2). Therefore, the solutions of system (2) are given in the following form:

$$\text{a)}\quad h_0^2 D = æ\mu\left(\frac{\partial\varphi_1}{\partial\eta}\eta - \frac{\partial\varphi_1}{\partial\xi}\xi\right),$$

$$\text{b)}\quad h_0^2 K = æ\mu\left(\frac{\partial\varphi_1}{\partial\eta}\xi + \frac{\partial\varphi_1}{\partial\xi}\eta\right), \tag{B7}$$

$$\text{c)}\quad \bar{u} = -\xi\eta\frac{\partial\varphi_1}{\partial\eta} - (æ-1)\,\varphi_1\xi + \psi_1,$$

$$\text{d)}\quad \bar{v} = \xi\eta\frac{\partial\varphi_1}{\partial\xi} + (æ-1)\,\varphi_1\eta + \psi_2.$$

Now we have to write down three versions of ψ_1 and ψ_2 function representation. In the first version

$$\begin{aligned}\psi_1 &= \frac{\partial\bar{\varphi}_1}{\partial\eta} + \frac{\partial\tilde{\varphi}_1}{\partial\eta} + \frac{\partial\varphi_2}{\partial\eta},\\ \psi_2 &= \frac{\partial\bar{\varphi}_1}{\partial\xi} + \frac{\partial\tilde{\varphi}_1}{\partial\xi} + \frac{\partial\varphi_2}{\partial\xi},\end{aligned} \tag{B8}$$

$\bar{\varphi}_1, \tilde{\varphi}_1, \varphi_2$ are harmonic functions, in addition, $\bar{\varphi}_1, \tilde{\varphi}_1$ are selected so that at $\eta = \alpha$, where $\alpha = \eta_1$ or $\alpha = \eta_2$, satisfy the following equations

$$-\xi\eta\frac{\partial\varphi_1}{\partial\eta} - (æ-1)\,\varphi_1\xi + \frac{\partial\bar{\varphi}_1}{\partial\eta} + \frac{\partial\tilde{\varphi}_1}{\partial\eta} = 0,$$

$$\xi\eta\frac{\partial\varphi_1}{\partial\xi} + (æ-1)\,\varphi_1\eta + \frac{\partial\bar{\varphi}_1}{\partial\xi} + \frac{\partial\tilde{\varphi}_1}{\partial\xi} = 0.$$

In the second version

$$\begin{aligned}\psi_1 &= -\alpha\left(\frac{\xi^2-(\eta-\alpha)^2}{2}\frac{\partial\varphi_1}{\partial\xi} + \xi\eta\frac{\partial\varphi_1}{\partial\eta}\right) + \frac{\xi^2-\eta^2}{2}\frac{\partial\varphi_2}{\partial\xi} + \xi\eta\frac{\partial\varphi_2}{\partial\eta},\\ \psi_2 &= \alpha\left(\xi\eta\frac{\partial\varphi_1}{\partial\xi} - \frac{\xi^2-(\eta-\alpha)^2}{2}\frac{\partial\varphi_1}{\partial\eta}\right) + \frac{\xi^2-\eta^2}{2}\frac{\partial\varphi_2}{\partial\eta} - \xi\eta\frac{\partial\varphi_2}{\partial\xi},\end{aligned} \tag{B9}$$

where φ_2 is the harmonic function.

In the third version

$$\begin{aligned}\psi_1 &= -\alpha^2\left(\frac{\xi^2-\eta^2}{2}\frac{\partial\varphi_1}{\partial\xi} + \xi\eta\frac{\partial\varphi_1}{\partial\eta}\right) + \frac{\xi^2-\eta^2}{2}\frac{\partial\varphi_2}{\partial\xi} + \xi\eta\frac{\partial\varphi_2}{\partial\eta},\\ \psi_2 &= \alpha^2\left(\xi\eta\frac{\partial\varphi_1}{\partial\xi} - \frac{\xi^2-\eta^2}{2}\frac{\partial\varphi_1}{\partial\eta}\right) + \frac{\xi^2-\eta^2}{2}\frac{\partial\varphi_2}{\partial\eta} - \xi\eta\frac{\partial\varphi_2}{\partial\xi}\end{aligned} \tag{B10}$$

Inserting (B8) in (B7c,d), we will get

$$\text{a)}\ \ \bar{u} = -\xi\eta\frac{\partial\varphi_1}{\partial\eta} - (æ-1)\,\varphi_1\xi + \frac{\partial\bar{\varphi}_1}{\partial\eta} + \frac{\partial\tilde{\varphi}_1}{\partial\eta} + \frac{\partial\varphi_2}{\partial\eta},$$

$$\text{b)}\ \ \bar{v} = \xi\eta\frac{\partial\varphi_1}{\partial\xi} + (æ-1)\,\varphi_1\eta + \frac{\partial\bar{\varphi}_1}{\partial\xi} + \frac{\partial\tilde{\varphi}_1}{\partial\xi} + \frac{\partial\varphi_2}{\partial\xi}. \qquad \text{(B11)}$$

Inserting (B9) in (B7c,d), we will have

$$\text{a)}\ \ \bar{u} = -\alpha\left(\frac{\xi^2-(\eta-\alpha)^2}{2}\frac{\partial\varphi_1}{\partial\xi} + \xi\eta\frac{\partial\varphi_1}{\partial\eta}\right) - \xi\eta\frac{\partial\varphi_1}{\partial\eta} - (æ-1)\,\varphi_1\xi + \frac{\xi^2-\eta^2}{2}\frac{\partial\varphi_2}{\partial\xi} + \xi\eta\frac{\partial\varphi_2}{\partial\eta},$$

$$\text{b)}\ \ \bar{v} = \alpha\left(\xi\eta\frac{\partial\varphi_1}{\partial\xi} - \frac{\xi^2-(\eta-\alpha)^2}{2}\frac{\partial\varphi_1}{\partial\eta}\right) + \xi\eta\frac{\partial\varphi_1}{\partial\xi} + (æ-1)\,\varphi_1\eta + \frac{\xi^2-\eta^2}{2}\frac{\partial\varphi_2}{\partial\eta} - \xi\eta\frac{\partial\varphi_2}{\partial\xi}. \qquad \text{(B12)}$$

Inserting (B10) in (B7c,d), we will get

$$\text{a)}\ \ \bar{u} = -\alpha^2\left(\frac{\xi^2-\eta^2}{2}\frac{\partial\varphi_1}{\partial\xi} + \xi\eta\frac{\partial\varphi_1}{\partial\eta}\right) - \xi\eta\frac{\partial\varphi_1}{\partial\eta} - (æ-1)\,\varphi_1\xi + \frac{\xi^2-\eta^2}{2}\frac{\partial\varphi_2}{\partial\xi} + \xi\eta\frac{\partial\varphi_2}{\partial\eta},$$

$$\text{b)}\ \ \bar{v} = \alpha^2\left(\xi\eta\frac{\partial\varphi_1}{\partial\xi} - \frac{\xi^2-\eta^2}{2}\frac{\partial\varphi_1}{\partial\eta}\right) + \xi\eta\frac{\partial\varphi_1}{\partial\xi} + (æ-1)\,\varphi_1\eta + \frac{\xi^2-\eta^2}{2}\frac{\partial\varphi_2}{\partial\eta} - \xi\eta\frac{\partial\varphi_2}{\partial\xi}. \qquad \text{(B13)}$$

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