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# Explicit Solutions on Same Problems in the Fully Coupled Theory of Elasticity For a Circle with Double Porosity

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The purpose of this paper is to consider the two-dimensional version of the fully coupled theory of elasticity for solids with double porosity the and to solve explicitly some boundary value problems (BVPs) of statics for an elastic circle. The explicit solutions of this BVPs are constructed by means of absolutely and uniformly convergent series. The questions on the uniqueness of a solutions of the problems are investigated.

**Keywords:** Double porosity, Explicit solution, Elastic circle.

**AMS Subject Classification**: 74F10, 74G30, 74G10.

## 1. Introduction

The poroelasticity is an effective and useful model for deformation-driven bone fluid movement in bone tissue. The suggested double porosity model would consider the bone fluid pressure in the vascular porosity and the bone fluid pressure in the lacunar-canalicular porosity.The extensive review of the results in the theory of bone poroelasticity can be found in the survey article of Cowin [1]. A theory of consolidation for elastic materials with double porosity was presented in [2-4], where the physical and mathematical foundations of this theory were considered. In these papers the theory of Aifantis unifies a model proposed by Biot [5] for the consolidation of deformable single porosity media with a model proposed by Barenblatt [6] for seepage in undeformable media with two degrees of porosity. However, Aifantis' quasi-static theory ignored the cross-coupling effect between the volume change of the pores and fissures in the system. The cross-coupled terms were included in the equations of conservation of mass for the pore and fissure fluid and in Darcy's law for solids with double porosity by several authors [7,12]. In [13,14] the fully coupled linear theory of elasticity is considered for solids with double porosity. Four spatial cases of the dynamical equations are considered. The fundamental solutions are constructed by means of elementary functions and the basic properties of the fundamental solutions are established.

Porous media theories play an important role in many branches of engineering, including material science, the petroleum industry, chemical engineering, and

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soil mechanics, as well as biomechanics. The problem of elastic bodies with double porosity was the subject of study for some papers more than fifty years ago. Many authors have investigated the BVPs of the 2-dimensional and 3-dimensional theories of elasticity for materials with double porosity, that are published in a large number of papers (some of these results can be seen in [15-27] and references therein). There the explicit solutions on some BVPs in the form of series and in quadratures are given in a form useful for engineering practice.

The purpose of this paper is to consider the two-dimensional version of the fully coupled theory of elasticity for solids with double porosity and to solve explicitly some BVPs of statics for an elastic circle. The explicit solutions of this BVPs are constructed by means of absolutely and uniformly convergent series. The questions on the uniqueness of solutions of the problems are investigated.

## 2. Basic equations and boundary value problems

Let $D$ be an elastic circle of radius $R$ with boundary $S$, centered at point $O(0,0)$, and let $\bar{D} = D \cup S$. Let us assume that the domain $D$ is filled with an isotropic material with double porosity.

The system of homogeneous equations in the full coupled linear equilibrium theory of elasticity for materials with double porosity can be written as follows [8]

$$\mu\Delta\mathbf{u} + (\lambda+\mu) graddiv\mathbf{u} - grad(\beta_1 p_1 + \beta_2 p_2) = 0, \tag{1}$$

$$\begin{aligned} &(k_1\Delta - \gamma)p_1 + (k_{12}\Delta + \gamma)p_2 = 0,\\ &(k_{21}\Delta + \gamma)p_1 + (k_2\Delta - \gamma)p_2 = 0, \end{aligned} \tag{2}$$

where $\mathbf{u} = \mathbf{u}(u_1, u_2)$ is the displacement vector in a solid, $p_1$ and $p_2$ are the pore and fissure fluid pressures respectively. $\beta_1$ and $\beta_2$ are the effective stress parameters, $\gamma > 0$ is the internal transport coefficient and corresponds to fluid transfer rate with respect to the intensity of flow between the pore and fissures, $\lambda$, $\mu$, are constitutive coefficients, $k_j = \dfrac{\kappa_j}{\mu'}$, $j = 1,2$, $k_{12} = \dfrac{\kappa_{12}}{\mu'}$, $k_{21} = \dfrac{\kappa_{21}}{\mu'}$. $\mu'$ is the fluid viscosity, $\kappa_1$ and $\kappa_2$ are the macroscopic intrinsic permeabilities associated with matrix and fissure porosity, respectively, $\kappa_{12}$ and $\kappa_{21}$ are the cross-coupling permeabilities for fluid flow at the interface between the matrix and fissure phases, $\Delta$ is the Laplace operator.

A vector-function $\mathbf{U(x)} = (u_1, u_2, p_1, p_2)$ defined in the domain $D$ is called regular if it has integrable continuous second derivatives in $D$, and $\mathbf{U(x)}$ itself and its first order derivatives are continuously extendable at every point of the boundary of $D$, i.e., $\mathbf{U(x)} \in C^2(D) \bigcap C^1(\overline{D})$; $\overline{D} = D \bigcup S$, $\mathbf{x} \in D$, $\mathbf{x} = (x_1, x_2)$.

Note that the system (2) would be considered separately. Further we assume that $p_j$ is known, when $\mathbf{x} \in D$. We can write the system (2) as

$$(\Delta + \lambda_1^2)\Delta p_j(\mathbf{x}) = 0, \quad j = 1,2.$$

With the help of this we find the solution of system (2) in the form

$$p_1(\mathbf{x}) = \varphi(\mathbf{x}) + m_1\varphi_1(\mathbf{x}), \quad p_2(\mathbf{x}) = \varphi(\mathbf{x}) + \varphi_1(\mathbf{x}), \tag{3}$$

where

$$\Delta\varphi = 0, \quad (\Delta + \lambda_1^2)\varphi_1 = 0, \quad m_1 = \frac{\gamma - k_{12}\lambda_1^2}{\gamma + k_1\lambda_1^2} = -\frac{k_2 + k_{12}}{k_1 + k_{21}},$$

$$\lambda_1 = i\sqrt{\frac{\gamma k_0}{k_1k_2 - k_{12}k_{21}}} = i\lambda_0, \quad k_0 = k_1 + k_2 + k_{12} + k_{21},$$

$$k_1 > 0, \quad k_2 > 0, \quad \gamma > 0, \quad \mu > 0, \quad \lambda + \mu > 0, \quad k_1k_2 - k_{12}k_{21} > 0.$$

Substitute the expression $\beta_1 p_1 + \beta_2 p_2$ in (1) and search the particular solution of the following nonhomogeneous equation

$$\mu\Delta\mathbf{u} + (\lambda + \mu) graddiv\mathbf{u} = grad[(\beta_1 + \beta_2)\varphi + (m_1\beta_1 + \beta_2)\varphi_1]. \tag{4}$$

It is well-known that a general solution of the last equation is presented in the form

$$\mathbf{u}(\mathbf{x}) = \mathbf{v}(\mathbf{x}) + \mathbf{v}_0(\mathbf{x}), \tag{5}$$

where $\mathbf{v}(\mathbf{x})$ is a general solution of the equation

$$\mu\Delta\mathbf{v} + (\lambda + \mu) graddiv\mathbf{v} = 0. \tag{6}$$

$\mathbf{v}_0(\mathbf{x})$ is a particular solution of the nonhomogeneous equation. We Look for solution $\mathbf{v}_0(\mathbf{x})$ in the form [28]

$$\mathbf{v}_0(\mathbf{x}) = grad\mathbf{F}(\mathbf{x}).$$

Substitute $\mathbf{v}_0$ instead of $\mathbf{u}$ into (4). Now we can find value of the function $F$. And finally, for $textbf v_0$, we obtain

$$\mathbf{v}_0(\mathbf{x}) = \frac{1}{\lambda + 2\mu} grad\left[(\beta_1 + \beta_2)\varphi_0 - \frac{\beta_1 m_1 + \beta_2}{\lambda_1^2}\varphi_1\right], \tag{7}$$

where $\varphi_0$ is a biharmonic function $\Delta\Delta\varphi_0 = 0$ and $\Delta\varphi_0 = \varphi, \quad \Delta\varphi = 0$.

So it remains to study the problem of finding the functions $p_j(\mathbf{x}), \quad j = 1, 2$.

We consider only the interior boundary value problems. The exterior one can be treated quite similarly.

The basic BVPs in the full coupled linear equilibrium theory of elasticity for materials with double porosity are formulated as follows.

The Dirichlet type BVP problem

*Find a regular solution* $\boldsymbol{U}(\boldsymbol{u}, p_1, p_2)$ *to systems (1) and (2) for* $\boldsymbol{x} \in D$ *satisfying the following boundary conditions:*

$$\mathbf{u}(\mathbf{z}) = \mathbf{f}(\mathbf{z}), \quad p_1(\mathbf{z}) = f_3(\mathbf{z}), \quad p_2(\mathbf{z}) = f_4(\mathbf{z}), \quad \mathbf{z} \in S. \tag{8}$$

The Neumann type BVP problem

*Find a regular solution* $\boldsymbol{U}(\boldsymbol{u}, p_1, p_2)$ *to systems (1) and (2) for* $\boldsymbol{x} \in D$ *satisfying the following boundary conditions*

$$\mathbf{P}\left(\frac{\partial}{\partial \mathbf{z}}, \mathbf{n}\right) \mathbf{U}(\mathbf{z}) = \mathbf{f}(\mathbf{z}), \quad \frac{\partial}{\partial n} p_1(\mathbf{z}) = f_3(\mathbf{z}), \quad \frac{\partial}{\partial n} p_2(\mathbf{z}) = f_4(\mathbf{z}), \quad \mathbf{z} \in S, \tag{9}$$

where $\mathbf{f} = (f_1, f_2)$ and $f_j(\mathbf{z})$, $j = 3, 4$, are known functions, $\mathbf{n}(\mathbf{z})$ is the external unit normal vector on $S$ at $\mathbf{z}$ and $\mathbf{P}\left(\frac{\partial}{\partial \mathbf{x}}, \mathbf{n}\right) \mathbf{U}$ is the stress vector in the considered theory

$$\mathbf{P}\left(\frac{\partial}{\partial \mathbf{x}}, \mathbf{n}\right) \mathbf{U} = \mathbf{T}\left(\frac{\partial}{\partial \mathbf{x}}, \mathbf{n}\right) \mathbf{u} - \mathbf{n}(\beta_1 p_1 + \beta_2 p_2), \tag{10}$$

$\mathbf{T}\left(\frac{\partial}{\partial \mathbf{x}}, \mathbf{n}\right) \mathbf{u}$ is the stress vector in the classical theory of elasticity,

$$\mathbf{T}\left(\frac{\partial}{\partial \mathbf{x}}, \mathbf{n}\right) \mathbf{u}(\mathbf{x}) = \mu \frac{\partial}{\partial \mathbf{n}} \mathbf{u}(\mathbf{x}) + \lambda \mathbf{n} div \mathbf{u}(\mathbf{x}) + \mu \sum_{i=1}^{2} n_i(\mathbf{x}) grad u_i(\mathbf{x}).$$

## 3. The uniqueness theorems

In this section we investigate the question of the uniqueness of solutions of the above-mentioned problems.

Let $\mathbf{U}(\mathbf{u}, p_1, p_2)$ be a regular solution of equations (1) and (2) in $D$. Multiply the equation (1) by $\mathbf{u}$, the first equation of (2) by $p_1$ and the second by $p_2$. Integrating over $D$ and summing the results, we arrive at Green's formulas

$$\int_D [E(\mathbf{u}, \mathbf{u}) - (\beta_1 p_1 + \beta_2 p_2) div \mathbf{u}] dx = \int_S \mathbf{u} \mathbf{P}(\partial_{\mathbf{y}}, \mathbf{n}) \mathbf{U} d_y S,$$

$$\int_D \{\gamma(p_1 - p_2)^2 + (k_{12} + k_{21})(grad p_1 \cdot grad p_2)\} dx$$

$$+ \int_D \left\{k_1 (grad p_1)^2 + k_2 (grad p_2)^2\right\} dx = \int_S \mathbf{p} \mathbf{P}^{(1)}(\partial_{\mathbf{y}}, \mathbf{n}) \mathbf{p} d_y S,$$

where

$$E(\mathbf{u},\mathbf{u}) = (\lambda+\mu)(div\mathbf{u})^2 + \mu\left(\frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2}\right)^2 + \mu\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right)^2,$$

$$\mathbf{P}^{(1)}(\partial_{\mathbf{x}},\mathbf{n})\mathbf{p} = \begin{pmatrix} k_1 & k_{12} \\ k_{21} & k_2 \end{pmatrix} \frac{\partial \mathbf{p}}{\partial \mathbf{n}}, \quad \mathbf{p} = (p_1, p_2).$$

Now we prove the following theorems:

**Theorem 3.1:** *The Dirichlet boundary value problem has at most one regular solution in the domain $D$.*

**Proof:** Let the Dirichlet BVP have in the domain $D$ two regular solutions $\mathbf{U}^{(1)}(\mathbf{x})$ and $\mathbf{U}^{(2)}(\mathbf{x})$. Denote $\mathbf{U} = \mathbf{U}^{(1)} - \mathbf{U}^{(2)}$. Evidently the vector $\mathbf{U}$ satisfies equations (1),(2) and the boundary condition $\mathbf{U}(\mathbf{z}) = 0$ on $S$. Note that if $\mathbf{U}$ is a regular solution of equations (1),(2), we have the Green formula and taking into account the fact that the potential energy is positively definite, we conclude that $\mathbf{U}(\mathbf{x}) = C$, for $\mathbf{x} \in D$, where $C = const$. Since $\mathbf{U}(\mathbf{z}) = 0, \quad z \in S$, we have $C = 0$ and $\mathbf{U}(\mathbf{x}) = 0$ at every point $\mathbf{x} \in D$. ☐

**Theorem 3.2:** *Two regular solutions of the Neumann boundary value problem may differ by the vector $(\boldsymbol{u}, p_1, p_2)$, where $\boldsymbol{u}$ is a sum of a rigid displacement vector and $c_1\boldsymbol{x}$, and $p_1 = p_2 = c = const$.*

**Proof:** Let

$$\mathbf{P}(\partial_{\mathbf{z}},\mathbf{n})\mathbf{U}(\mathbf{z}) = 0, \quad \frac{\partial}{\partial \mathbf{n}} p_1(\mathbf{z}) = 0, \quad \frac{\partial}{\partial \mathbf{n}} p_2(\mathbf{z}) = 0, \quad \mathbf{z} \in S.$$

Then applying Green's Formulas to a regular solution and taking into account the positive definiteness of the potential energy we have

$$u_1 = \alpha_1 - \varepsilon x_2 + c_1 x_1, \;\; u_2 = \alpha_2 + \varepsilon x_1 + c_1 x_2, \;\; p_1 = p_2 = c, \quad for \quad \mathbf{x} \in D,$$

where

$$c_1 = \frac{c(\beta_1 + \beta_2)}{2(\lambda+\mu)}$$

*and* $\varepsilon, \;\; \alpha_1, \;\; \alpha_2, \;\; c$ are arbitrary constants. ☐

## 4. Explicit solution of the Dirichlet BVP

A solution of the system (2) with boundary conditions $p_1(\mathbf{z}) = f_3(\mathbf{z}), \quad p_2(\mathbf{z}) = f_4(\mathbf{z}), \quad \mathbf{z} \in S$ is sought in the form (3), where the functions $\varphi$ and $\varphi_1$ are unknown in the circle $D$. On the basis of boundary conditions we reformulate the problem in question as follows

$$\varphi(\mathbf{z}) = h(\mathbf{z}), \quad \varphi_1(\mathbf{z}) = h_1(\mathbf{z}), \quad \mathbf{z} \in S, \tag{11}$$

where

$$
\begin{aligned}
h &= \frac{1}{k_0}[(k_1 + k_{21})f_3 + (k_2 + k_{12})f_4],\\
h_1 &= \frac{1}{k_0}(k_1 + k_{21})(f_4 - f_3), \quad k_0 \neq 0.
\end{aligned}
\tag{12}
$$

The functions $h(z)$ and $h_1$ in (12) can be represented in Fourier series. Obviously the function $\varphi$ is solution of the equation $\Delta\varphi = 0$ and it is represented in the form of the following series [30]

$$
\varphi(\mathbf{x}) = \sum_{k=0}^{\infty} \left(\frac{\rho}{R}\right)^k (\mathbf{Y}_k \cdot \boldsymbol{\nu}_k(\psi)), \tag{13}
$$

where

$$
\mathbf{x} = (\rho, \psi), \quad \rho^2 = x_1^2 + x_2^2, \quad \mathbf{Y}_k = (A_k, B_k),
$$

$$
\boldsymbol{\nu}_k = (\cos k\psi, \sin k\psi), \quad \mathbf{Y}_0 = (A_0, 0), \quad A_0 = \frac{1}{2\pi}\int_0^{2\pi} h(\theta)d\theta,
$$

$$
A_k = \frac{1}{\pi}\int_0^{2\pi} h(\theta)\cos k\theta d\theta, \quad B_k = \frac{1}{\pi}\int_0^{2\pi} h(\theta)\sin k\theta d\theta.
$$

The metaharmonic function $\varphi_1(\mathbf{x})$ in the circle $D$ can be written as follows [29]

$$
\varphi_1(\mathbf{x}) = I_0(\lambda_0\rho)C_0 + \sum_{k=1}^{\infty} I_k(\lambda_0\rho)(\mathbf{Z}_k \cdot \boldsymbol{\nu}_k(\psi)) \tag{14}
$$

where $I_k(\lambda_0\rho)$ is the Bessel function of an imaginary argument, $\mathbf{Z}_k = (C_k, D_k)$, $C_0$, $C_k$, $D_k$ are the unknown quantities. Keeping in mind (14) and boundary conditions (11) we obtain the values of $C_k$ and $D_k$

$$
C_0 = \frac{1}{2\pi I_0(\lambda_0 R)}\int_0^{2\pi} h_1(\theta)d\theta, \qquad C_k = \frac{1}{\pi I_k(\lambda_0 R)}\int_0^{2\pi} h_1(\theta)\cos k\theta d\theta, \tag{15}
$$

$$
D_k = \frac{1}{\pi I_k(\lambda_0 R)}\int_0^{2\pi} h_1(\theta)\sin k\theta d\theta.
$$

If we substitute the values of $\varphi$ and $\varphi_1$ into (3), we find the functions $p_1(\mathbf{x})$ and $p_2(\mathbf{x})$ in $D$.

A solution $\mathbf{v}(\mathbf{x}) = (v_1, v_2)$ of homogeneous equation (6) is sought in the form [17]

$$\begin{aligned} v_1(\mathbf{x}) &= \frac{\partial}{\partial x_1}[\Phi_1 + \Phi_2] - \frac{\partial \Phi_3}{\partial x_2}, \\ v_2(\mathbf{x}) &= \frac{\partial}{\partial x_2}[\Phi_1 + \Phi_2] + \frac{\partial \Phi_3}{\partial x_1}, \end{aligned} \tag{16}$$

where $\Phi_1$, $\Phi_2$ and $\Phi_3$ are scalar functions,

$$\begin{aligned} &\Delta\Phi_1 = 0, \quad \Delta\Delta\Phi_2 = 0, \quad \Delta\Delta\Phi_3 = 0, \\ &(\lambda + 2\mu)\frac{\partial}{\partial x_1}\Delta\Phi_2 - \mu\frac{\partial}{\partial x_2}\Delta\Phi_3 = 0, \\ &(\lambda + 2\mu)\frac{\partial}{\partial x_2}\Delta\Phi_2 + \mu\frac{\partial}{\partial x_1}\Delta\Phi_3 = 0. \end{aligned} \tag{17}$$

Taking into account (5) and boundary conditions (8), we can write

$$\mathbf{v}(\mathbf{z}) = \mathbf{\Psi}(\mathbf{z}), \quad z \in S, \tag{18}$$

where $\mathbf{\Psi}(\mathbf{z}) = \mathbf{f}(\mathbf{z}) - \mathbf{v}_0(\mathbf{z})$ is the known vector; $\varphi(z)$ and $\varphi_1(z)$ are defined by equalities (11). On the basis of equation $\Delta\varphi_0 = \varphi$ the function $\varphi_0$ is represented in the following form

$$\varphi_0(x) = \frac{R^2}{4}\sum_{k=0}^{\infty}\frac{1}{k+1}\left(\frac{\rho}{R}\right)^{k+2}(\mathbf{Y}_k \cdot \boldsymbol{\nu}_k(\psi)), \tag{19}$$

where $\mathbf{Y}_k$ is defined by (13).

In view of (17) we can represent the harmonic function $\Phi_1$, biharmonic functions $\Phi_2$ and $\Phi_3$ in the form

$$\begin{aligned} \Phi_1 &= \sum_{k=0}^{\infty}\left(\frac{\rho}{R}\right)^{k}(\mathbf{X}_{k1} \cdot \boldsymbol{\nu}_k(\psi)), \\ \Phi_2 &= \sum_{k=0}^{\infty}\left(\frac{\rho}{R}\right)^{k+2}(\mathbf{X}_{k2} \cdot \boldsymbol{\nu}_k(\psi)), \\ \Phi_3 &= \frac{\lambda + 2\mu}{\mu}\sum_{k=0}^{\infty}\left(\frac{\rho}{R}\right)^{k+2}(\mathbf{X}_{k2} \cdot \mathbf{s}_k(\psi)), \end{aligned} \tag{20}$$

where $\mathbf{X}_{ki} = (X_{ki1}, X_{ki2})$, $k = 1, 2$ are the unknown two-component vectors, $\boldsymbol{\nu}_k = (\cos k\psi, \sin k\psi)$, $\mathbf{s}_k = (-\sin k\psi, \cos k\psi)$.

Using the formulas

$$\frac{\partial}{\partial x_1} = n_1\frac{\partial}{\partial \rho} - \frac{n_2}{\rho}\frac{\partial}{\partial \psi}, \quad \frac{\partial}{\partial x_2} = n_2\frac{\partial}{\partial \rho} + \frac{n_1}{\rho}\frac{\partial}{\partial \psi},$$

let us rewrite the boundary conditions (18) in the form

$$v_n(\mathbf{z}) = \Psi_n(\mathbf{z}), \quad v_s(\mathbf{z}) = \Psi_s(\mathbf{z}), \quad \mathbf{z} \in S, \tag{21}$$

where $v_n$ and $\Psi_n(\mathbf{z})$ are the normal components of the vectors $\boldsymbol{v} = (v_1, v_2)$ and $\Psi = (\Psi_1, \Psi_2)$ respectively; $v_s$ and $\Psi_s(\mathbf{z})$ are the tangent components of the vectors $\boldsymbol{v} = (v_1, v_2)$ and $\Psi = (\Psi_1, \Psi_2)$ respectively. Substituting the equalities (16),(20) into (21), we get

$$\begin{aligned}
&v_n = \frac{\partial}{\partial\rho}(\Phi_1 + \Phi_2) - \frac{1}{\rho}\frac{\partial}{\partial\psi}\Phi_3,\\
&v_s = \frac{1}{\rho}\frac{\partial}{\partial\psi}(\Phi_1 + \Phi_2) + \frac{\partial}{\partial\rho}\Phi_3,\\
&\Psi_n = n_1\Psi_1 + n_2\Psi_2, \quad \Psi_s = -n_2\Psi_1 + n_1\Psi_2,\\
&\mathbf{n} = (n_1, n_2), \quad \mathbf{s} = (-n_2, n_1), \quad n_1 = \frac{x_1}{\rho}, \quad n_2 = \frac{x_2}{\rho}.
\end{aligned} \tag{22}$$

Let us expand the functions $\Psi_n$ *and* $\Psi_s$ in Fourier series, those Fourier coefficients are $\gamma_k$ and $\delta_k$, respectively

$$\boldsymbol{\gamma}_0 = (\gamma_{01}, 0), \quad \boldsymbol{\gamma}_k = (\gamma_{k1}, \gamma_{k2}), \quad \boldsymbol{\delta}_0 = (\delta_{01}, 0), \quad \boldsymbol{\delta}_k = (\delta_{k1}, \delta_{k2}),$$

$$\gamma_{01} = \frac{1}{2\pi}\int_0^{2\pi} \Psi_n(\theta) d\theta, \quad \delta_{01} = \frac{1}{2\pi}\int_0^{2\pi} \Psi_s(\theta) d\theta,$$

$$\begin{aligned}
&\gamma_{k1} = \frac{1}{\pi}\int_0^{2\pi} \Psi_n(\theta)\cos k\theta d\theta, \quad \delta_{k1} = \frac{1}{\pi}\int_0^{2\pi} \Psi_s(\theta)\cos k\theta d\theta,\\
&\gamma_{k2} = \frac{1}{\pi}\int_0^{2\pi} \Psi_s(\theta)\sin k\theta d\theta, \quad \delta_{k2} = \frac{1}{\pi}\int_0^{2\pi} \Psi_n(\theta)\sin k\theta d\theta.
\end{aligned} \tag{23}$$

If we substitute (22) into (21), then passing to limit as $\rho \longrightarrow R$, for determining the unknown values we obtain the following system of algebraic equations

$$\begin{aligned}
&\frac{2}{R}X_{01i} = \frac{\gamma_{0i}}{2}, \quad \frac{2(\lambda + 2\mu)}{\mu R}X_{02i} = \frac{\delta_{0i}}{2},\\
&\frac{k}{R}X_{k1i} + \frac{k(\lambda + 3\mu)}{\mu R}X_{k2i} = \gamma_{ki}, \quad i = 1, 2, \quad k = 1, 2, ...,\\
&\frac{k}{R}X_{k1i} + \frac{k(\lambda + 3\mu) + 2(\lambda + 2\mu)}{\mu R}X_{k2i} = \delta_{ki}.
\end{aligned} \tag{24}$$

From (24) we find

$$X_{01i} = \frac{\gamma_{0i} R}{4}, \quad X_{02i} = \frac{\delta_{0i} R \mu}{4(\lambda + 2\mu)},$$

$$X_{k1i} = \frac{\gamma_{ki} R}{k} - \frac{(\delta_{ki} - \gamma_{ki}) R}{2k(\lambda + \mu)} [(\lambda + 3\mu)k + 2\mu],$$

$$X_{k2i} = \mu \frac{(\delta_{ki} - \gamma_{ki}) R}{2(\lambda + \mu)}, \quad i = 1, 2, \quad k = 1, 2, ...,$$

Thus the solution of the Dirichlet boundary problem is represented by the sum (5) in which $\mathbf{v}(\mathbf{x})$ is defined by means of formula (16), $\mathbf{v}_0(\mathbf{x})$ by formula (7), $\varphi_0(\mathbf{x})$ by formula (19) and $\varphi_1(\mathbf{x})$ by formulas (14) and (5).

It can be proved that if the functions $\mathbf{f}$ and $f_j, \quad j = 3, 4$ satisfy the following conditions on $S$

$$\mathbf{f} \in C^3(S), \quad f_j \in C^3(S), \quad j = 3, 4,$$

then the resulting series are absolutely and uniformly convergent.

## 5. Explicit solution of the Neumann BVP

We sought the solution of the Neumann BVP in the form (3), where the functions $\varphi$ and $\varphi_1$ are unknown in the circle $D$. Taking into account formulas (3) and (9), the boundary conditions can be rewritten as

$$\frac{\partial \varphi(\mathbf{z})}{\partial R} = h(\mathbf{z}), \quad \frac{\partial \varphi_1(\mathbf{z})}{\partial R} = h_1(\mathbf{z}), \quad \mathbf{z} \in S. \tag{25}$$

$h(\mathbf{z})$ and $h_1(\mathbf{z})$ are given by (12), where $f_3 = \dfrac{\partial p_1}{\partial R}$, $f_4 = \dfrac{\partial p_2}{\partial R}$, $\displaystyle\int_S h(y) d_y S = 0$.

Thus, for the unknown harmonic function $\varphi$ we obtain the Neumann problem, the solution that is represented in the form of series [30]

$$\varphi(\mathbf{x}) = c_0 + \sum_{k=1}^{\infty} \frac{R}{k} \left(\frac{\rho}{R}\right)^k (\mathbf{Y}_k \cdot \boldsymbol{\nu}_k(\psi)), \tag{26}$$

where $c_0$ is an arbitrary constant; $\quad \mathbf{Y}_k = (A_k, B_k)$,

$$A_k = \frac{1}{\pi} \int_0^{2\pi} h(\theta) \cos k\theta d\theta, \quad B_k = \frac{1}{\pi} \int_0^{2\pi} h(\theta) \sin k\theta d\theta.$$

The metaharmonic function $\varphi_1(\mathbf{x})$ in the circle $D$ can be written as (14), where $\mathbf{Z}_k = (C_k, D_k)$; $\; C_0, \; C_k, \; D_k$ are the unknown quantities. Keeping in mind (12) and boundary conditions (25), we obtain the values of $C_0$, $C_k$ and $D_k$

$$C_0 = \frac{1}{2\pi\lambda_0 I_0'(\lambda_0 R)} \int\limits_0^{2\pi} h_1(\theta)d\theta, \qquad C_k = \frac{1}{\pi\lambda_0 I_k'(\lambda_0 R)} \int\limits_0^{2\pi} h_1(\theta)\cos k\theta d\theta, \tag{27}$$

$$D_k = \frac{1}{\pi\lambda_0 I_k'(\lambda_0 R)} \int\limits_0^{2\pi} h_1(\theta)\sin k\theta d\theta,$$

where

$$I_k'(\xi) = \frac{\partial I_k(\xi)}{\partial \xi}, \quad \frac{\partial I_k(\lambda_0\rho)}{\partial\rho} = \lambda_0 I_k'(\lambda_0\rho) \quad I_k'(\lambda_0 R) \neq 0, \quad k = 0, 1, 2, ....$$

Considering equality (5) and (10), the boundary condition (9), for $\mathbf{T}(\partial_\mathbf{z}, \mathbf{n})\mathbf{v}(\mathbf{z})$ can be rewritten as

$$\mathbf{T}(\partial_\mathbf{z}, \mathbf{n})\mathbf{v}(\mathbf{z})(\mathbf{z}) = \mathbf{\Omega}(\mathbf{z}), \quad \mathbf{z} \in S, \tag{28}$$

where

$$\mathbf{\Omega}(\ \mathbf{z}) = \mathbf{f}(\mathbf{z}) + \mathbf{n}(\mathbf{z})[a\varphi(\mathbf{z}) + b\varphi_1(\mathbf{z})] - \mathbf{T}(\partial_\mathbf{z}, \mathbf{n})\mathbf{v}_0(\mathbf{z})$$

is the known vector, $\mathbf{\Omega} = (\Omega_1, \Omega_2)$; $\varphi$ is defined by (26) and $\varphi_1$ - formulas (14) and (27); $\quad a = \beta_1 + \beta_2, \quad b = m_1\beta_1 + \beta_2$.

Let us rewrite the boundary conditions (28) in the form

$$[\mathbf{T}(\partial_\mathbf{z}, \mathbf{n})\mathbf{v}(\mathbf{z})]_n = \Omega_n(\mathbf{z}), \quad [\mathbf{T}(\partial_\mathbf{z}, \mathbf{n})\mathbf{v}(\mathbf{z})]_s = \Omega_s(\mathbf{z}), \tag{29}$$

where $[\mathbf{T}(\partial_\mathbf{z}, \mathbf{n})\mathbf{v}(\mathbf{z})]_n$ and $\Omega_n(\mathbf{z})$ are the normal components of the vectors $\mathbf{T}(\partial_\mathbf{z}, \mathbf{n})textbffv$ and $\mathbf{\Omega}(\mathbf{z})$ respectively; $[\mathbf{T}(\partial_\mathbf{z}, \mathbf{n})\mathbf{v}(\mathbf{z})]_s$ and $\Omega_s(\mathbf{z})$ are the tangent components of the vectors $\mathbf{T}(\partial_\mathbf{z}, \mathbf{n})\mathbf{v}(\mathbf{z}))$ and $\mathbf{\Omega}(\mathbf{z})$ respectively;

$$\begin{aligned}
&[\mathbf{T}(\partial_\mathbf{z}, \mathbf{n})\mathbf{v}(\mathbf{z})]_n = (\lambda + \mu)\left[\frac{\partial v_n(\mathbf{z})}{\partial\rho}\right]_{\rho=R} + \frac{\lambda}{R}\frac{\partial v_s(\mathbf{z})}{\partial\psi}, \\
&[\mathbf{T}(\partial_\mathbf{z}, \mathbf{n})\mathbf{v}(\mathbf{z})]_s = \mu\left[\frac{\partial v_s(\mathbf{z})}{\partial\rho}\right]_{\rho=R} + \frac{\mu}{R}\frac{\partial v_n(\mathbf{z})}{\partial\psi}; \qquad (30) \\
&\Omega_n(\mathbf{z}) = f_n(\mathbf{z}) + a\varphi(\mathbf{z}) + b\varphi_1(\mathbf{z}) - [\mathbf{T}(\partial_\mathbf{z}, \mathbf{n})\mathbf{v}_0(\mathbf{z})]_n, \\
&\Omega_s(\mathbf{z}) = f_s(\mathbf{z}) - [\mathbf{T}(\partial_\mathbf{z}, \mathbf{n})\mathbf{v}_0(\mathbf{z})]_s, \quad \mathbf{z} \in S.
\end{aligned}$$

$\mathbf{v}_0$ is defined by means of formula (7), where function $\varphi_0(x)$ is the solution of equation $\Delta\varphi_0 = \varphi$ and is represented in the form [17]

$$\varphi_0(\mathbf{x}) = m_0 + \frac{R^3}{4}\sum_{k=1}^{\infty}\frac{1}{k(k+1)}\Big(\frac{\rho}{R}\Big)^{k+2}(\mathbf{Y}_k \cdot \boldsymbol{\nu}_k(\psi)),$$

$\mathbf{Y}_k = (A_k, B_k)$, $A_k$ and $B_k$ are defined by (26); $m_0$ is an arbitrary constant.

Let us expand the functions $\Omega_n$ *and* $\Omega_s$ in Fourier series, those Fourier coefficients are $\boldsymbol{\gamma}_0 = (\gamma_{01}, 0)$, $\boldsymbol{\gamma}_k = (\gamma_{k1}, \gamma_{k2})$ and $\boldsymbol{\delta}_0 = (\delta_{01}, 0)$, $\boldsymbol{\delta}_k = (\delta_{k1}, \delta_{k2})$, respectively.

Taking into account the formulas (22),(20) and (30), then passing to limit as $\rho \longrightarrow R$, for determining the unknown values we obtain the following system of algebraic equations.

According to uniqueness theorem, we assume that the determinant of the system is not zero. The solution of the system has

$$X_{01i} = \frac{\gamma_{0i}R^2}{4(\lambda+2\mu)}, \quad X_{02i} = \frac{\delta_{0i}R^2}{4(\lambda+2\mu)},$$

$$X_{k1i} = \frac{R^2}{a_3}\delta_{ki} - \frac{a_4R^2}{a_2a_3 - a_1a_4}(\mu\gamma_{ki} - a_1\delta_{ki}),$$

$$X_{k2i} = \frac{a_3R^2}{a_2a_3 - a_1a_4}(\mu\gamma_{ki} - a_1\delta_{ki}),$$

where

$$a_1 = \mu k[2(\lambda+\mu)k - (\lambda+2\mu)],$$

$$a_2 = 2(\lambda+\mu)(\lambda+3\mu)k^2 + (\lambda+2\mu)[(3\lambda+5\mu)k + 2\mu],$$

$$a_3 = \mu k(2k-1), \quad a_4 = (\lambda+3\mu)k(2k+3) + 2(\lambda+2\mu).$$

We assume that the functions $\mathbf{f}$ and $f_j$, $j = 3, 4$ satisfies the following conditions on $S$

$$\mathbf{f} \in C^2(S), \quad f_j \in C^2(S), \quad j = 3, 4.$$

Under these conditions the resulting series are absolutely and uniformly convergent.

## 6. Conclusions

1. The main purpose of this work has been to present some explicit solutions of BVPs in the fully coupled theory of elasticity for solids with double porosity. Solutions of the considered boundary value problems are obtained in the form of absolutely and uniformly convergent series that is useful to obtain numerical solutions of the boundary value problems. The Green's formulas are obtained.The uniqueness theorems of the BVPs are proved. The solutions are sought by means

of harmonic, biharmonic and metaharmonic functions, which properties are well known in mathematical physics.

2. The obtained results may be of practical use in micro and nanomechanics, mechanics of materials, engineering mechanics, engineering medicine, biomechanics, engineering geology, geomechanics, applied and computing mechanics, in the applied mathematics.

3. Using the above mentioned method gives an opportunity to research the wide class of problems for the systems of equations in the modern linear theories of elasticity, thermoelasticity and poroelasticity for materials with microstructures and construct explicitly the solutions of basic BVPs for a circle, sphere and etc., in a complete version.

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