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On Finite Difference Scheme for One Integro-Differential Model with Source Terms Based on Maxwell System

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Finite difference scheme for one system of nonlinear partial integro-differential equations is studied. Mentioned system is obtained by adding the source terms to the resulting model which is derived after reduction of well-known Maxwell system to the system of nonlinear integro-differential equations. Class of nonlinearity is widened than one has been studied before.

Keywords: System of nonlinear partial differential equations, Source terms, Finite difference scheme, Convergence.

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1. Introduction

In mathematical simulation of many diffusive processes the non-stationary, nonlinear partial differential and integro-differential equations and systems of those equations are obtained very often. Problems connected to the nonlinearity significantly complicates the investigation of such models.

The purpose of this note is to study the finite difference scheme for one diffusion system of nonlinear partial integro-differential equations that is obtained by adding the source terms to the resulting model which is derived after reduction of well-known Maxwell equations [20], describing process of penetration of an electromagnetic field into a substance, to the system of nonlinear integro-differential equations. At first such reduction to the integro-differential model was made in [6] and [7]. Later, a number of scientists studied proposed in the works above integro-differential models for different cases of magnetic field and different kinds of diffusion coefficient (see, for example, [1] - [19], [21] - [25] and references therein).

Let us study the system which is obtained from the above-described integro-

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differential model with source terms in case of the two-component magnetic field:

$$\begin{aligned}
&\frac{\partial U}{\partial t}-\frac{\partial}{\partial x}\left\{\left(1+\int_0^t\left[\left(\frac{\partial U}{\partial x}\right)^2+\left(\frac{\partial V}{\partial x}\right)^2\right]d\tau\right)^p\frac{\partial U}{\partial x}\right\}+|U|^{q-2}U=f_1(x,t),\\
&\frac{\partial V}{\partial t}-\frac{\partial}{\partial x}\left\{\left(1+\int_0^t\left[\left(\frac{\partial U}{\partial x}\right)^2+\left(\frac{\partial V}{\partial x}\right)^2\right]d\tau\right)^p\frac{\partial V}{\partial x}\right\}+|V|^{q-2}V=f_2(x,t),
\end{aligned} \tag{1}$$

where $0<p\le 1$, $q\ge 2$ and f_1, f_2 are given functions.

In the domain $[0,1]\times[0,\infty)$ for the system (1) let us consider the following initial-boundary value problem with the homogeneous Dirichlet boundary conditions:

$$U(0,t)=U(1,t)=V(0,t)=V(1,t)=0, \quad t\ge 0, \tag{2}$$

$$U(x,0)=U_0(x), \quad V(x,0)=V_0(x), \quad x\in[0,1], \tag{3}$$

where $U_0=U_0(x)$ and $V_0=V_0(x)$ are given functions.

Our purpose is to study the convergence of the finite difference scheme for problem (1) - (3).

2. Finite difference scheme and convergence theorem

In the rectangle $Q_T=[0,1]\times[0,T]$, where T is a positive constant, let us construct the finite difference scheme for problem (1) - (3). On Q_T let us introduce a net with mesh points denoted by $(x_i,t_j)=(ih,j\tau)$, where $i=0,1,...,M;\ j=0,1,...,N$ with $h=1/M$, $\tau=T/N$. The initial line is denoted by $j=0$. The discrete approximation at (x_i,t_j) is designed by $(u_i^j,\, v_i^j)$ and the exact solution to the problem (1) - (3) by $(U_i^j,\, V_i^j)$. We will use the following well-known notations [26]:

$$r^j_{x,i}=\frac{r^j_{i+1}-r^j_i}{h}, \quad r^j_{\bar{x},i}=\frac{r^j_i-r^j_{i-1}}{h}, \quad r^j_{t,i}=\frac{r^{j+1}_i-r^j_i}{\tau}.$$

Introduce the inner products and norms:

$$(r^j,g^j)=h\sum_{i=1}^{M-1}r_i^jg_i^j, \quad (r^j,g^j]=h\sum_{i=1}^{M}r_i^jg_i^j,$$

$$\|r^j\|=(r^j,r^j)^{1/2}, \quad \|r^j]|=(r^j,r^j]^{1/2}.$$

For the problem (1) - (3) let us consider the following finite difference scheme:

$$
\begin{gathered}
u_{t,i}^{j}-\left\{\left(1+\tau\sum_{k=1}^{j+1}\left[(u_{\bar{x},i}^{k})^{2}+(v_{\bar{x},i}^{k})^{2}\right]\right)^{p}u_{\bar{x},i}^{j+1}\right\}_{x}+|u_{i}^{j+1}|^{q-2}u_{i}^{j+1}=f_{1,i}^{j},\\
v_{t,i}^{j}-\left\{\left(1+\tau\sum_{k=1}^{j+1}\left[(u_{\bar{x},i}^{k})^{2}+(v_{\bar{x},i}^{k})^{2}\right]\right)^{p}v_{\bar{x},i}^{j+1}\right\}_{x}+|v_{i}^{j+1}|^{q-2}v_{i}^{j+1}=f_{2,i}^{j},\\
i=1,2,...,M-1;\quad j=0,1,...,N-1,
\end{gathered}
\tag{4}
$$

$$
u_{0}^{j}=u_{M}^{j}=v_{0}^{j}=v_{M}^{j}=0,\quad j=0,1,...,N, \tag{5}
$$

$$
u_{i}^{0}=U_{0,i},\quad v_{i}^{0}=V_{0,i},\ \ i=0,1,...,M. \tag{6}
$$

The following statement takes place.

Theorem 2.1: *If problem (1) - (3) has a sufficiently smooth solution and* $0<p\leq 1$, $q\geq 2$, *then the difference scheme (4) - (6) is stable, its approximation error on problem (1) - (3) is* $O(\tau+h)$ *and its solution converges to the solution of continuous problem (1) - (3) as* $\tau\to 0$, $h\to 0$ *and the following estimates are true:*

$$
\|u^{j}-U^{j}\|\leq C(\tau+h),\quad \|v^{j}-V^{j}\|\leq C(\tau+h).
$$

Here and below C is a positive constant independent from τ and h.

Proof: Let us multiply equations in (4) scalarly by u_{i}^{j+1} and v_{i}^{j+1} respectively. Using the discrete analog of the formula of integration by parts and the relation $(1+S)^{p}\geq 1$ it is not difficult to get the validity of the following inequalities:

$$
\|u^{n}\|^{2}+\sum_{j=1}^{n}\|u_{\bar{x}}^{j}]|^{2}\tau<C,\quad \|v^{n}\|^{2}+\sum_{j=1}^{n}\|v_{\bar{x}}^{j}]|^{2}\tau<C,\quad n=1,2,...,N. \tag{7}
$$

The a priori estimates (7) guarantee the stability of the scheme (4) - (6). Note, that applying the technique as we prove the convergence theorem, it is not difficult to prove the uniqueness of the solution of scheme (4) - (6) too.

Let us introduce the differences $z_{i}^{j}=u_{i}^{j}-U_{i}^{j}$ and $w_{i}^{j}=v_{i}^{j}-V_{i}^{j}$ to get the following relations:

$$
\begin{gathered}
z_{t,i}^{j+1} - \left\{ \left(1 + \tau \sum_{k=1}^{j+1} \left[(u_{\bar{x},i}^k)^2 + (v_{\bar{x},i}^k)^2 \right] \right)^p u_{\bar{x},i}^{j+1} \right. \\
\left. - \left(1 + \tau \sum_{k=1}^{j+1} \left[(U_{\bar{x},i}^k)^2 + (V_{\bar{x},i}^k)^2 \right] \right)^p U_{\bar{x},i}^{j+1} \right\}_x \\
+ |u_i^{j+1}|^{q-2} u_i^{j+1} - |U_i^{j+1}|^{q-2} U_i^{j+1} = -\psi_{1,i}^j, \\
w_{t,i} - \left\{ \left(1 + \tau \sum_{k=1}^{j+1} \left[(u_{\bar{x},i}^k)^2 + (v_{\bar{x},i}^k)^2 \right] \right)^p v_{\bar{x},i}^{j+1} \right. \\
\left. - \left(1 + \tau \sum_{k=1}^{j+1} \left[(U_{\bar{x},i}^k)^2 + (V_{\bar{x},i}^k)^2 \right] \right)^p V_{\bar{x},i}^{j+1} \right\}_x \\
+ |v_i^{j+1}|^{q-2} v_i^{j+1} - |V_i^{j+1}|^{q-2} V_i^{j+1} = -\psi_{2,i}^j,
\end{gathered} \tag{8}
$$

$$
z_0^j = z_M^j = w_0^j = w_M^j = 0,
$$

$$
z_i^0 = w_i^0 = 0,
$$

where

$$
\psi_{k,i}^j = O(\tau + h), \quad k = 1, 2.
$$

Multiplying the first equation of the system (8) scalarly by $\tau z^{j+1} = \tau \left(z_1^{j+1}, z_2^{j+1}, \ldots, z_{M-1}^{j+1} \right)$ and using again the discrete analogue of the formula of integration by parts we get

$$
\begin{gathered}
\|z^{j+1}\|^2 - (z^{j+1}, z^j) + \tau h \sum_{i=1}^{M} \left\{ \left(1 + \tau \sum_{k=1}^{j+1} \left[(u_{\bar{x},i}^k)^2 + (v_{\bar{x},i}^k)^2 \right] \right)^p u_{\bar{x},i}^{j+1} \right. \\
\left. - \left(1 + \tau \sum_{k=1}^{j+1} \left[(U_{\bar{x},i}^k)^2 + (V_{\bar{x},i}^k)^2 \right] \right)^p U_{\bar{x},i}^{j+1} \right\} z_{\bar{x},i}^{j+1} \\
+ \left(|u_i^{j+1}|^{q-2} u_i^{j+1} - |U_i^{j+1}|^{q-2} U_i^{j+1},\, u^{j+1} - U^{j+1} \right) = -\tau (\psi_1^j, z^{j+1}).
\end{gathered}
$$

Analogously,

$$\|w^{j+1}\|^2 - (w^{j+1}, w^j) + \tau h \sum_{i=1}^{M} \left\{ \left(1 + \tau \sum_{k=1}^{j+1} \left[(u^k_{\bar{x},i})^2 + (v^k_{\bar{x},i})^2 \right] \right)^p v^{j+1}_{\bar{x},i} \right.$$

$$\left. - \left(1 + \tau \sum_{k=1}^{j+1} \left[(U^k_{\bar{x},i})^2 + (V^k_{\bar{x},i})^2 \right] \right)^p V^{j+1}_{\bar{x},i} \right\} w^{j+1}_{\bar{x},i}$$

$$+ \left(|v_i^{j+1}|^{q-2} v_i^{j+1} - |V_i^{j+1}|^{q-2} V_i^{j+1},\, v^{j+1} - V^{j+1} \right) = -\tau(\psi_2^j, w^{j+1}).$$

Adding these two equalities and taking into account monotonicity of the function $g(r) = |r|^{q-2} r$, from these two equalities we have

$$\begin{aligned}
&\|z^{j+1}\|^2 - (z^{j+1}, z^j) + \|w^{j+1}\|^2 - (w^{j+1}, w^j) \\
&+\tau h \sum_{i=1}^{M} \left\{ \left(1 + \tau \sum_{k=1}^{j+1} \left[(u^k_{\bar{x},i})^2 + (v^k_{\bar{x},i})^2 \right] \right)^p u^{j+1}_{\bar{x},i} \right. \\
&\left. - \left(1 + \tau \sum_{k=1}^{j+1} \left[(U^k_{\bar{x},i})^2 + (V^k_{\bar{x},i})^2 \right] \right)^p U^{j+1}_{\bar{x},i} \right\} z^{j+1}_{\bar{x},i} \\
&+\tau h \sum_{i=1}^{M} \left\{ \left(1 + \tau \sum_{k=1}^{j+1} \left[(u^k_{\bar{x},i})^2 + (v^k_{\bar{x},i})^2 \right] \right)^p v^{j+1}_{\bar{x},i} \right. \\
&\left. - \left(1 + \tau \sum_{k=1}^{j+1} \left[(U^k_{\bar{x},i})^2 + (V^k_{\bar{x},i})^2 \right] \right)^p V^{j+1}_{\bar{x},i} \right\} w^{j+1}_{\bar{x},i} \\
&= -\tau(\psi_1^j, z^{j+1}) - \tau(\psi_2^j, w^{j+1}).
\end{aligned} \tag{9}$$

Note that,

$$\left\{ \left(1 + \tau \sum_{k=1}^{j+1} \left[(u^k_{\bar{x},i})^2 + (v^k_{\bar{x},i})^2 \right] \right)^p u^{j+1}_{\bar{x},i} \right.$$

$$\left. - \left(1 + \tau \sum_{k=1}^{j+1} \left[(U^k_{\bar{x},i})^2 + (V^k_{\bar{x},i})^2 \right] \right)^p U^{j+1}_{\bar{x},i} \right\} \left(u^{j+1}_{\bar{x},i} - U^{j+1}_{\bar{x},i} \right)$$

$$+\left\{\left(1+\tau\sum_{k=1}^{j+1}\left[(u_{\bar{x},i}^{k})^2+(v_{\bar{x},i}^{k})^2\right]\right)^p v_{\bar{x},i}^{j+1}\right.$$

$$\left.-\left(1+\tau\sum_{k=1}^{j+1}\left[(U_{\bar{x},i}^{k})^2+(V_{\bar{x},i}^{k})^2\right]\right)^p V_{\bar{x},i}^{j+1}\right\}\left(v_{\bar{x},i}^{j+1}-V_{\bar{x},i}^{j+1}\right)$$

$$=\int_0^1\frac{d}{d\mu}\left(1+\tau\sum_{k=1}^{j+1}\left\{\left[U_{\bar{x},i}^{k}+\mu(u_{\bar{x},i}^{k}-U_{\bar{x},i}^{k})\right]^2+\left[V_{\bar{x},i}^{k}+\mu(v_{\bar{x},i}^{k}-V_{\bar{x},i}^{k})\right]^2\right\}\right)^p$$

$$\times\left[U_{\bar{x},i}^{j+1}+\mu(u_{\bar{x},i}^{j+1}-U_{\bar{x},i}^{j+1})\right]d\mu\left(u_{\bar{x},i}^{j+1}-U_{\bar{x},i}^{j+1}\right)$$

$$+\int_0^1\frac{d}{d\mu}\left(1+\tau\sum_{k=1}^{j+1}\left\{\left[U_{\bar{x},i}^{k}+\mu(u_{\bar{x},i}^{k}-U_{\bar{x},i}^{k})\right]^2+\left[V_{\bar{x},i}^{k}+\mu(v_{\bar{x},i}^{k}-V_{\bar{x},i}^{k})\right]^2\right\}\right)^p$$

$$\times\left[V_{\bar{x},i}^{j+1}+\mu(v_{\bar{x},i}^{j+1}-V_{\bar{x},i}^{j+1})\right]d\mu\left(v_{\bar{x},i}^{j+1}-V_{\bar{x},i}^{j+1}\right)$$

$$=2p\int_0^1\left(1+\tau\sum_{k=1}^{j+1}\left\{\left[U_{\bar{x},i}^{k}+\mu(u_{\bar{x},i}^{k}-U_{\bar{x},i}^{k})\right]^2+\left[V_{\bar{x},i}^{k}+\mu(v_{\bar{x},i}^{k}-V_{\bar{x},i}^{k})\right]^2\right\}\right)^{p-1}$$

$$\times\tau\sum_{k=1}^{j+1}\left\{\left[U_{\bar{x},i}^{k}+\mu(u_{\bar{x},i}^{k}-U_{\bar{x},i}^{k})\right]\left(u_{\bar{x},i}^{k}-U_{\bar{x},i}^{k}\right)\right.$$

$$\left.+\left[V_{\bar{x},i}^{k}+\mu(v_{\bar{x},i}^{k}-V_{\bar{x},i}^{k})\right]\left(v_{\bar{x},i}^{k}-V_{\bar{x},i}^{k}\right)\right\}$$

$$\times\left[U_{\bar{x},i}^{j+1}+\mu(u_{\bar{x},i}^{j+1}-U_{\bar{x},i}^{j+1})\right]d\mu\left(u_{\bar{x},i}^{j+1}-U_{\bar{x},i}^{j+1}\right)$$

$$+\int_0^1\left(1+\tau\sum_{k=1}^{j+1}\left\{\left[U_{\bar{x},i}^{k}+\mu(u_{\bar{x},i}^{k}-U_{\bar{x},i}^{k})\right]^2+\left[V_{\bar{x},i}^{k}+\mu(v_{\bar{x},i}^{k}-V_{\bar{x},i}^{k})\right]^2\right\}\right)^p$$

$$\times \left(u_{\bar{x},i}^{j+1} - U_{\bar{x},i}^{j+1}\right) d\mu \left(u_{\bar{x},i}^{j+1} - U_{\bar{x},i}^{j+1}\right)$$

$$+2p \int_0^1 \left(1 + \tau \sum_{k=1}^{j+1} \left\{ \left[U_{\bar{x},i}^k + \mu(u_{\bar{x},i}^k - U_{\bar{x},i}^k)\right]^2 + \left[V_{\bar{x},i}^k + \mu(v_{\bar{x},i}^k - V_{\bar{x},i}^k)\right]^2 \right\}\right)^{p-1}$$

$$\times \tau \sum_{k=1}^{j+1} \left\{ \left[U_{\bar{x},i}^k + \mu(u_{\bar{x},i}^k - U_{\bar{x},i}^k)\right] \left(u_{\bar{x},i}^k - U_{\bar{x},i}^k\right)\right.$$

$$\left. + \left[V_{\bar{x},i}^k + \mu(v_{\bar{x},i}^k - V_{\bar{x},i}^k)\right] \left(v_{\bar{x},i}^k - V_{\bar{x},i}^k\right) \right\}$$

$$\times \left[V_{\bar{x},i}^{j+1} + \mu(v_{\bar{x},i}^{j+1} - V_{\bar{x},i}^{j+1})\right] d\mu \left(v_{\bar{x},i}^{j+1} - V_{\bar{x},i}^{j+1}\right)$$

$$+ \int_0^1 \left(1 + \tau \sum_{k=1}^{j+1} \left\{ \left[U_{\bar{x},i}^k + \mu(u_{\bar{x},i}^k - U_{\bar{x},i}^k)\right]^2 + \left[V_{\bar{x},i}^k + \mu(v_{\bar{x},i}^k - V_{\bar{x},i}^k)\right]^2 \right\}\right)^{p}$$

$$\times \left(v_{\bar{x},i}^{j+1} - V_{\bar{x},i}^{j+1}\right) d\mu \left(v_{\bar{x},i}^{j+1} - V_{\bar{x},i}^{j+1}\right)$$

$$= 2p \int_0^1 \left(1 + \tau \sum_{k=1}^{j+1} \left\{ \left[U_{\bar{x},i}^k + \mu(u_{\bar{x},i}^k - U_{\bar{x},i}^k)\right]^2 + \left[V_{\bar{x},i}^k + \mu(v_{\bar{x},i}^k - V_{\bar{x},i}^k)\right]^2 \right\}\right)^{p-1}$$

$$\times \tau \sum_{k=1}^{j+1} \left\{ \left[U_{\bar{x},i}^k + \mu(u_{\bar{x},i}^k - U_{\bar{x},i}^k)\right] \left(u_{\bar{x},i}^k - U_{\bar{x},i}^k\right) + \left[V_{\bar{x},i}^k + \mu(v_{\bar{x},i}^k - V_{\bar{x},i}^k)\right] \left(v_{\bar{x},i}^k - V_{\bar{x},i}^k\right) \right\}$$

$$\times \left\{ \left[U_{\bar{x},i}^{j+1} + \mu(u_{\bar{x},i}^{j+1} - U_{\bar{x},i}^{j+1})\right] \left(u_{\bar{x},i}^{j+1} - U_{\bar{x},i}^{j+1}\right)\right.$$

$$\left. + \left[V_{\bar{x},i}^{j+1} + \mu(v_{\bar{x},i}^{j+1} - V_{\bar{x},i}^{j+1})\right] d\mu \left(v_{\bar{x},i}^{j+1} - V_{\bar{x},i}^{j+1}\right) \right\} d\mu$$

$$+ \int_0^1 \left(1 + \tau \sum_{k=1}^{j+1} \left\{ \left[U_{\bar{x},i}^k + \mu(u_{\bar{x},i}^k - U_{\bar{x},i}^k)\right]^2 + \left[V_{\bar{x},i}^k + \mu(v_{\bar{x},i}^k - V_{\bar{x},i}^k)\right]^2 \right\}\right)^{p}$$

$$\times\left[\left(u_{\bar{x},i}^{j+1}-U_{\bar{x},i}^{j+1}\right)^2+\left(v_{\bar{x},i}^{j+1}-V_{\bar{x},i}^{j+1}\right)^2\right]d\mu$$

$$=2p\int_0^1\left(1+\tau\sum_{k=1}^{j+1}\left\{\left[U_{\bar{x},i}^k+\mu(u_{\bar{x},i}^k-U_{\bar{x},i}^k)\right]^2\right.\right.$$

$$\left.\left.+\left[V_{\bar{x},i}^k+\mu(v_{\bar{x},i}^k-V_{\bar{x},i}^k)\right]^2\right\}\right)^{p-1}\xi_i^{j+1}(\mu)\xi_{t,\,i}^j(\mu)\,d\mu$$

$$+\int_0^1\left(1+\tau\sum_{k=1}^{j+1}\left\{\left[U_{\bar{x},i}^k+\mu(u_{\bar{x},i}^k-U_{\bar{x},i}^k)\right]^2+\left[V_{\bar{x},i}^k+\mu(v_{\bar{x},i}^k-V_{\bar{x},i}^k)\right]^2\right\}\right)^p$$

$$\times\left[\left(u_{\bar{x},i}^{j+1}-U_{\bar{x},i}^{j+1}\right)^2+\left(v_{\bar{x},i}^{j+1}-V_{\bar{x},i}^{j+1}\right)^2\right]d\mu,$$

where

$$\xi_i^{j+1}(\mu)=\tau\sum_{k=1}^{j+1}\left\{\left[U_{\bar{x},i}^k+\mu(u_{\bar{x},i}^k-U_{\bar{x},i}^k)\right]\left(u_{\bar{x},i}^k-U_{\bar{x},i}^k\right)\right.$$

$$\left.+\left[V_{\bar{x},i}^k+\mu(v_{\bar{x},i}^k-V_{\bar{x},i}^k)\right]\left(v_{\bar{x},i}^k-V_{\bar{x},i}^k\right)\right\},$$

$$\xi_i^0(\mu)=0,$$

and therefore,

$$\xi_{t,\,i}^j(\mu)=\left[U_{\bar{x},i}^{j+1}+\mu(u_{\bar{x},i}^{j+1}-U_{\bar{x},i}^{j+1})\right]\left(u_{\bar{x},i}^{j+1}-U_{\bar{x},i}^{j+1}\right)$$

$$+\left[V_{\bar{x},i}^{j+1}+\mu(v_{\bar{x},i}^{j+1}-V_{\bar{x},i}^{j+1})\right]\left(v_{\bar{x},i}^{j+1}-V_{\bar{x},i}^{j+1}\right).$$

Introducing the following notation

$$s_i^{j+1}(\mu)=\tau\sum_{k=1}^{j+1}\left\{\left[U_{\bar{x},i}^k+\mu(u_{\bar{x},i}^k-U_{\bar{x},i}^k)\right]^2+\left[V_{\bar{x},i}^k+\mu(v_{\bar{x},i}^k-V_{\bar{x},i}^k)\right]^2\right\},$$

from the previous equality we have

$$\left\{\left(1+\tau\sum_{k=1}^{j+1}\left[(u_{\bar{x},i}^{k})^2+(v_{\bar{x},i}^{k})^2\right]\right)^p u_{\bar{x},i}^{j+1}\right.$$

$$\left.-\left(1+\tau\sum_{k=1}^{j+1}\left[(U_{\bar{x},i}^{k})^2+(V_{\bar{x},i}^{k})^2\right]\right)^p U_{\bar{x},i}^{j+1}\right\}\left(u_{\bar{x},i}^{j+1}-U_{\bar{x},i}^{j+1}\right)$$

$$+\left\{\left(1+\tau\sum_{k=1}^{j+1}\left[(u_{\bar{x},i}^{k})^2+(v_{\bar{x},i}^{k})^2\right]\right)^p v_{\bar{x},i}^{j+1}\right.$$

$$\left.-\left(1+\tau\sum_{k=1}^{j+1}\left[(U_{\bar{x},i}^{k})^2+(V_{\bar{x},i}^{k})^2\right]\right)^p V_{\bar{x},i}^{j+1}\right\}\left(v_{\bar{x},i}^{j+1}-V_{\bar{x},i}^{j+1}\right)$$

$$=2p\int\limits_0^1\left(1+s_i^{j+1}(\mu)\right)^{p-1}\xi_i^{j+1}\xi_{t,\,i}^{j}\,d\mu$$

$$+\int\limits_0^1\left(1+s_i^{j+1}(\mu)\right)^p\left[\left(u_{\bar{x},i}^{j+1}-U_{\bar{x},i}^{j+1}\right)^2+\left(v_{\bar{x},i}^{j+1}-V_{\bar{x},i}^{j+1}\right)^2\right]d\mu.$$

After substituting this equality in (9) we get

$$\begin{gathered}\|z^{j+1}\|^2-(z^{j+1},z^j)+\|w^{j+1}\|^2-(w^{j+1},w^j)\\+2\tau hp\sum_{i=1}^{M}\int\limits_0^1\left(1+s_i^{j+1}(\mu)\right)^{p-1}\xi_i^{j+1}\xi_{t,\,i}^{j}\,d\mu\\+\tau h\sum_{i=1}^{M}\int\limits_0^1\left(1+s_i^{j+1}(\mu)\right)^p\left[\left(u_{\bar{x},i}^{j+1}-U_{\bar{x},i}^{j+1}\right)^2+\left(v_{\bar{x},i}^{j+1}-V_{\bar{x},i}^{j+1}\right)^2\right]d\mu\\=-\tau(\psi_1^j,z^{j+1})-\tau(\psi_2^j,w^{j+1}).\end{gathered}\tag{10}$$

Tacking into account restriction $p>0$ and relation $s_i^{j+1}(\mu)\geq 0$,

$$(r^{j+1},r^j)=\frac{1}{2}\|r^{j+1}\|^2+\frac{1}{2}\|r^j\|^2-\frac{1}{2}\|r^{j+1}-r^j\|^2,$$

$$\tau\,\xi_i^{j+1}\xi_{t,\,i}^{j}=\frac{1}{2}\left(\xi_i^{j+1}\right)^2-\frac{1}{2}\left(\xi_i^{j}\right)^2+\frac{\tau^2}{2}\left(\xi_{t,\,i}^{j}\right)^2$$

from (10) we have

$$\begin{gathered}
\|z^{j+1}\|^2-\frac{1}{2}\|z^{j+1}\|^2-\frac{1}{2}\|z^j\|^2+\frac{1}{2}\|z^{j+1}-z^j\|^2\\
+\|w^{j+1}\|^2-\frac{1}{2}\|w^{j+1}\|^2-\frac{1}{2}\|w^j\|^2+\frac{1}{2}\|w^{j+1}-w^j\|^2\\
+hp\sum_{i=1}^{M}\int\limits_0^1\left(1+s_i^{j+1}(\mu)\right)^{p-1}\left[\left(\xi_i^{j+1}\right)^2-\left(\xi_i^j\right)^2\right]d\mu\\
+\tau^2hp\sum_{i=1}^{M}\int\limits_0^1\left(1+s_i^{j+1}(\mu)\right)^{p-1}\left(\xi_{t,\,i}^j\right)^2d\mu\\
+\tau h\sum_{i=1}^{M}\left[\left(u_{\bar{x},i}^{j+1}-U_{\bar{x},i}^{j+1}\right)^2+\left(v_{\bar{x},i}^{j+1}-V_{\bar{x},i}^{j+1}\right)^2\right]\\
\le-\tau(\psi_1^j,z^{j+1})-\tau(\psi_2^j,w^{j+1}).
\end{gathered}\tag{11}$$

From (11) we arrive at

$$\begin{gathered}
\frac{1}{2}\|z^{j+1}\|^2-\frac{1}{2}\|z^j\|^2+\frac{\tau^2}{2}\|z_t^j\|^2\\
+\frac{1}{2}\|w^{j+1}\|^2-\frac{1}{2}\|w^j\|^2+\frac{\tau^2}{2}\|w_t^j\|^2\\
+hp\sum_{i=1}^{M}\int\limits_0^1\left(1+s_i^{j+1}(\mu)\right)^{p-1}\left[\left(\xi_i^{j+1}\right)^2-\left(\xi_i^j\right)^2\right]d\mu\\
+\tau\left(\|z_{\bar{x}}^{j+1}]|^2+\|w_{\bar{x}}^{j+1}]|^2\right)\le\frac{\tau}{2}\left(\|\psi_1^j\|^2+\|\psi_2^j\|^2\right)+\frac{\tau}{2}\left(\|z^{j+1}\|^2+\|w^{j+1}\|^2\right).
\end{gathered}\tag{12}$$

Using discrete analogue of Poincaré inequality [26]

$$\|r^{j+1}\|^2\le\|r_{\bar{x}}^{j+1}]|^2,$$

from (12) we get

$$\begin{gathered}
\|z^{j+1}\|^2-\|z^j\|^2+\tau^2\|z_t^j\|^2+\|w^{j+1}\|^2-\|w^j\|^2+\tau^2\|w_t^j\|^2\\
+2hp\sum_{i=1}^{M}\int\limits_0^1\left(1+s_i^{j+1}(\mu)\right)^{p-1}\left[\left(\xi_i^{j+1}\right)^2-\left(\xi_i^j\right)^2\right]d\mu\\
+\tau\left(\|z_{\bar{x}}^{j+1}]|^2+\|w_{\bar{x}}^{j+1}]|^2\right)\le\tau\left(\|\psi_1^j\|^2+\|\psi_2^j\|^2\right).
\end{gathered}\tag{13}$$

Summing (13) from $j=0$ to $j=n-1$ we arrive at

$$\|z^n\|^2+\tau^2\sum_{j=0}^{n-1}\|z_t^j\|^2+\|w^n\|^2+\tau^2\sum_{j=0}^{n-1}\|w_t^j\|^2$$

$$+2hp\sum_{j=0}^{n-1}\sum_{i=1}^{M}\int_0^1\left(1+s_i^{j+1}(\mu)\right)^{p-1}\left[\left(\xi_i^{j+1}\right)^2-\left(\xi_i^j\right)^2\right]d\mu \tag{14}$$

$$+\tau\sum_{j=0}^{n-1}\left(\|z_{\bar{x}}^{j+1}]|^2+\|w_{\bar{x}}^{j+1}]|^2\right)\le\tau\sum_{j=0}^{n-1}\left(\|\psi_1^j\|^2+\|\psi_2^j\|^2\right).$$

Note, that since $s_i^{j+1}(\mu)\ge s_i^j(\mu)$ and $p\le 1$, for the second line of the formula (14) we have

$$\sum_{j=0}^{n-1}\left(1+s_i^{j+1}(\mu)\right)^{p-1}\left[\left(\xi_i^{j+1}\right)^2-\left(\xi_i^j\right)^2\right]$$

$$=\left(1+s_i^1(\mu)\right)^{p-1}\left(\xi_i^1\right)^2-\left(1+s_i^1(\mu)\right)^{p-1}\left(\xi_i^0\right)^2$$

$$+\left(1+s_i^2(\mu)\right)^{p-1}\left(\xi_i^2\right)^2-\left(1+s_i^2(\mu)\right)^{p-1}\left(\xi_i^1\right)^2$$

$$+\cdots+\left(1+s_i^n(\mu)\right)^{p-1}\left(\xi_i^n\right)^2-\left(1+s_i^n(\mu)\right)^{p-1}\left(\xi_i^{n-1}\right)^2$$

$$=\left(1+s_i^n(\mu)\right)^{p-1}\left(\xi_i^n\right)^2+\sum_{j=1}^{n-1}\left[\left(1+s_i^j(\mu)\right)^{p-1}-\left(1+s_i^{j+1}(\mu)\right)^{p-1}\right]\left(\xi_i^j\right)^2\ge 0.$$

Tacking into account the last relation and (14) one can deduce

$$\|z^n\|^2+\|w^n\|^2+\tau^2\sum_{j=0}^{n-1}\|z_t^j\|^2+\tau^2\sum_{j=0}^{n-1}\|w_t^j\|^2$$

$$+\tau\sum_{j=0}^{n-1}\left(\|z_{\bar{x}}^{j+1}]|^2+\|w_{\bar{x}}^{j+1}]|^2\right)\le\tau\sum_{j=0}^{n-1}\left(\|\psi_1^j\|^2+\|\psi_2^j\|^2\right). \tag{15}$$

From (15) we get validity of the theorem. □

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