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On the Well-Posedness of the Cauchy Problem for a Neutral Differential Equation with Distributed Prehistory

Tamaz Tadumadze*

I. Javakhishvili Tbilisi State University
I. Vekua Institute of Applied Mathematics & Department of Mathematics
2 University St., 0186, Tbilisi, Georgia

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Theorems on the continuous dependence of solutions on perturbations of the initial data and the nonlinear term of the right-hand side are given for the neutral differential equation whose right-hand side is linear with respect to the concentrated prehistory of the phase velocity and nonlinear with respect to the distributed prehistory of the phase coordinates. Under the initial data we understand the collection of initial moment, of delay function and initial functions. Perturbations of the initial data and of the right-hand side of the equation are small in a standard norm and in the integral sense, respectively.

Keywords: Neutral differential equations, Differential equations with distributed prehistory, Well-posedness of the Cauchy problem, Perturbations.

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Let $I=[a,b]$ be a finite interval and let $\mathbb{R}^n$ be the n -dimensional vector space of points $x=(x^1,\dots,x^n)^T$, where T denotes transposition. Suppose that $O\subset\mathbb{R}^n$ is an open set, and E_f is the set of functions $f: I\times O^2\to\mathbb{R}^n$ satisfying the following conditions: for each fixed $(x_1,x_2)\in O^2$ the function $f(\cdot,x_1,x_2):I\to\mathbb{R}^n$ is measurable; for each $f\in E_f$ and compact set $K\subset O$, there exist functions $m_{f,K}(t)\in L_1(I,\mathbb{R}_+)$ and $L_{f,K}(t)\in L_1(I,\mathbb{R}_+), \mathbb{R}_+=[0,\infty)$, such that for almost all $t\in I$

$$\mid f(t,x_1,x_2)\mid\leq m_{f,K}(t)\ \forall(x_1,x_2)\in K^2,$$

$$\mid f(t,x_1,x_2)-f(t,y_1,y_2)\mid\leq L_{f,K}(t)\sum_{i=1}^{2}\mid x_i-y_i\mid$$

$$\forall(x_1,x_2)\in K^2 \text{ and } \forall(y_1,y_2)\in K^2.$$

Two functions $f_1,f_2\in E_f$ are said to be equivalent if for every fixed $(x_1,x_2)\in O\times\mathbb{R}^n$ and for almost all $t\in I$,

$$f_1(t,x_1,x_2)-f_2(t,x_1,x_2)=0.$$

*Corresponding author. Email: tamaz.tadumadze@tsu.ge

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The equivalence classes of functions of the space E_f compose a vector space , which is also denoted by E_f; these classes are also called functions and denoted by f again.

We introduce a topology in E_f using the following basis of neighborhoods of the origin

$$\Big\{V_{K,\delta} : K \subset O \text{ is a compact set and } \delta > 0 \text{ is an arbitrary number}\Big\},$$

where

$$V_{K,\delta} = \Big\{\delta f \in E_f : H(\delta f; K) \le \delta\Big\} \text{ and } H(\delta f; K)$$

$$= \sup\Big\{\Big|\int_{t'}^{t''} \delta f(t, x_1, x_2)dt\Big| : t', t'' \in I, x_i \in K, i = 1, 2\Big\}. \tag{1}$$

Let D be the set of continuous differentiable scalar functions (delay functions) $\tau(t), t \in [a, \infty)$, satisfying the conditions:

$$\tau(t) < t, \dot{\tau}(t) > 0, \inf\{\tau(a) : \tau \in D\} := \hat{\tau} > -\infty.$$

Let Φ_1 denote the set of continuous functions $\varphi : I_1 = [\hat{\tau}, b] \to O$, $\varphi(t)$ is called the initial function of the trajectory; by Φ_2 we denote the set of bounded measurable functions $v : I_1 \to \mathbb{R}^n$ and $v(t)$ is called the initial function of the trajectory derivative .

To each element $\mu = (t_0, \tau, \varphi, v, f) \in \Lambda = [a, b) \times D \times \Phi_1 \times \Phi_2 \times E_f$ we assign the neutral differential equation which is the linear and nonlinear with respect to the phase velocity and distributed prehistory on the interval $[\tau(t), t]$, respectively,

$$\dot{x}(t) = A(t)\dot{x}(\sigma(t)) + \int_{\tau(t)}^{t} f(t, x(t), x(s))ds \tag{2}$$

with the initial condition

$$x(t) = \varphi(t), t \in [\hat{\tau}, t_0], \dot{x}(t) = v(t), t \in [\hat{\tau}, t_0). \tag{3}$$

Here $A(t)$ is a continuous $n \times n$ matrix function and $\sigma(t) \in D$ is a fixed delay function in the phase velocity. The symbol $\dot{x}(t)$ on the interval $[\hat{\tau}, t_0)$ is not connected with the derivative of the function $\varphi(t)$.

Definition 1: Let $\mu = (t_0, \tau, \varphi, v, f) \in \Lambda$. A function $x(t) = x(t; \mu) \in O, t \in [\hat{\tau}, t_1], t_1 \in (t_0, b]$, is called a solution of the equation (2) with the initial condition (3) or a solution corresponding to the element μ and defined on the interval $[\hat{\tau}, t_1]$, if it satisfies the condition (3), is absolutely continuous on the interval $[t_0, t_1]$ and satisfies the equation (2) almost everywhere on $[t_0, t_1]$.

To formulate the main results, we introduce the following set:

$$W(K;\alpha)=\Big\{\delta f\in E_f:\exists\, m_{\delta f,K}(t),L_{\delta f,K}(t)\in L_1(I,R_+),$$
$$\int_I [m_{\delta f,K}(t)+L_{\delta f,K}(t)]dt\le\alpha\Big\},$$

where $K\subset O$ is a compact set and $\alpha>0$ is a fixed number not dependent on δf. The set $W(K;\alpha)$ is called the class of perturbations of the right-hand side of equation (2).

Furthermore,

$$B(t_{00};\delta)=\{t_0\in I:\mid t_0-t_{00}\mid<\delta\},\ V(\tau_0;\delta)=\{\tau\in D:\parallel\tau-\tau_0\parallel_I<\delta\},$$
$$V_1(\varphi_0;\delta)=\{\varphi\in\Phi_1:\parallel\varphi-\varphi_0\parallel_{I_1}<\delta\},$$
$$V_2(v_0;\delta)=\{v\in\Phi_2:\parallel v-v_0\parallel_{I_1}<\delta\},$$

where $t_{00}\in[a,b)$ is a fixed point, $\tau_0\in D,\ \varphi_0\in\Phi_1$ and $v_0\in\Phi_2$ are fixed functions, $\delta>0$ is a fixed number; $\parallel\tau\parallel_I=\sup\{|\tau(t)|:\ t\in I\}$.

Theorem 2 : *Let $x_0(t)$ be the solution corresponding to $\mu_0=(t_{00},\tau_0,\varphi_0,v_0,f_0)$ $\in\Lambda$ and defined on $[\hat\tau,t_{10}],t_{10}<b$. Let $K_1\subset O$ be a compact set containing a certain neighborhood of the set $K_0=\varphi_0(I_1)\cup x_0([t_{00},t_{10}])$. Then the following conditions hold:*

(1) there exist numbers $\delta_i>0,i=0,1$ such that to each element

$$\mu=(t_0,\tau,\varphi,v,f_0+\delta f)\in V(\mu_0;K_1,\delta_0,\alpha)=B(t_{00};\delta_0)\times V(\tau_0;\delta_0)$$
$$\times V_1(\varphi_0;\delta_0)\times V_2(v_0;\delta_0)\times\Big[f_0+\Big(W(K_1;\alpha)\cap V_{K_1,\delta_0}\Big)\Big]$$

corresponds solution $x(t;\mu)$ defined on the interval $[\hat\tau,t_{10}+\delta_1]\subset I_1$ and satisfying the condition $x(t;\mu)\in K_1$;

(2) for an arbitrary $\varepsilon>0$ there exists a number $\delta_2=\delta_2(\varepsilon)\in(0,\delta_0)$ such that the following inequality holds for any $\mu\in V(\mu_0;K_1,\delta_2,\alpha)$:

$$\mid x(t;\mu)-x(t;\mu_0)\mid\le\varepsilon\ \forall t\in[\theta,t_{10}+\delta_1],\theta=\max\{t_0,t_{00}\};$$

(3) for an arbitrary $\varepsilon>0$ there exists a number $\delta_3=\delta_3(\varepsilon)\in(0,\delta_0)$ such that the following inequality holds for any $\mu\in V(\mu_0;K_1,\delta_3,\alpha)$:

$$\int_{\hat\tau}^{t_{10}+\delta_1}\mid x(t;\mu)-x(t;\mu_0)\mid dt\le\varepsilon.$$

The solution $x(t;\mu_0)$ is the continuation of the solution $x_0(t)$ and to the element $\mu=(t_0,\tau,\varphi,v,f_0+\delta f)\in V(\mu_0;K_1,\delta_0,\alpha)$ corresponds the perturbed differential equation

$$\dot{x}(t) = A(t)\dot{x}(\sigma(t)) + \int_{\tau(t)}^{t} \Big[f_0(t, x(t), x(s)) + \delta f(t, x(t), x(s))\Big] ds$$

with the perturbed initial condition (3).

Now we introduce the set of variations

$$\Im = \Big\{\delta\mu = (\delta t_0, \delta\tau, \delta\varphi, \delta v, \delta f) : \mid \delta t_0 \mid\leq \beta, \parallel \delta\tau \parallel_I \leq \beta, \parallel \delta\varphi \parallel_{I_1} \leq \beta,$$

$$\delta\varphi \in \Phi_1 - \varphi_0, \parallel \delta v \parallel_{I_1} \leq \beta, \delta v \in \Phi_2 - v_0, \delta f = \sum_{i=1}^{k} \lambda_i \delta f_i, \mid \lambda_i \mid\leq \beta, i = \overline{1, k}\Big\},$$

where $\beta > 0$ is a fixed number and $\delta f_i \in E_f - f_0, i = \overline{1, k}$ are fixed functions.

Theorem 3: *Let $x_0(t)$ be the solution corresponding to $\mu_0 \in \Lambda$ and defined on $[\hat{\tau}, t_{10}], t_{i0} \in (a, b), i = 0, 1$. Let $K_1 \subset O$ be a compact set containing a certain neighborhood of the set K_0. Then the following conditions hold:*

(4) there exist numbers $\varepsilon_1 > 0$ and $\delta_1 > 0$ such that for an arbitrary $(\varepsilon, \delta\mu) \in (0, \varepsilon_1) \times \Im$ the element $\mu_0 + \varepsilon\delta\mu \in \Lambda$, and there corresponds the solution $x(t; \mu_0 + \varepsilon\delta\mu)$ defined on the interval $[\hat{\tau}, t_{10} + \delta_1] \subset I_1$. Moreover, $x(t; \mu_0 + \varepsilon\delta\mu) \in K_1$;

(5) the following relations are fulfilled:

$$\lim_{\varepsilon\to 0} \Big[\sup\Big\{ \mid x(t; \mu_0 + \varepsilon\delta\mu) - x(t; \mu_0) \mid : t \in [\theta, t_{10} + \delta_1]\Big\}\Big] = 0$$

and

$$\lim_{\varepsilon\to 0} \int_{\hat{\tau}}^{t_{10}+\delta_1} \mid x(t; \mu_0 + \varepsilon\delta\mu) - x(t; \mu_0) \mid dt = 0$$

uniformly in $\delta\mu \in \Im$, where $\theta = \max\{t_{00}, t_{00} + \varepsilon\delta t_0\}$.

Theorem 3 is a simple corollary of Theorem 2.

Let $U_0 \subset \mathbb{R}^r$ be an open set and let Ω be the set of measurable functions $u(t) \in U_0, t \in I$ satisfying the conditions: $clu(I)$ is a compact set in $\mathbb{R}^r$ and $clu(I) \subset U_0$.

To each element $\varrho = (t_0, \tau, \varphi, v, u) \in \Lambda_1 = [a, b) \times D \times \Phi_1 \times \Phi_2 \times \Omega$ we assign the controlled neutral differential equation with distributed prehistory

$$\dot{x}(t) = A(t)\dot{x}(\sigma(t)) + \int_{\tau(t)}^{t} g(t, x(t), x(s), u(t)) ds \tag{4}$$

with the initial condition (3). Here the function $g(t, x_1, x_2, u)$ is defined on $I \times O^2 \times U_0$ and satisfies the following conditions: for each fixed $(x_1, x_2, u) \in O^2 \times U_0$ the function $g(\cdot, x_1, x_2, u) : I \to \mathbb{R}^n$ is measurable; for each compact set $K \subset O$ and $U \subset U_0$ there exist a functions $m_{K,U}(t), L_{K,U}(t) \in L_1(I, R_+)$ such that for almost all $t \in I$

$$| \, g(t, x_1, x_2, u) \, | \leq m_{K,U}(t) \; \forall (x_1, x_2, u) \in K^2 \times U,$$

$$| \, g(t, x_1, x_2, u_1) - g(t, y_1, y_2, u_2) \, | \leq L_{K,U}(t) \Big[\sum_{i=1}^{2} | \, x_i - y_i \, | + | \, u_1 - u_2 \, | \Big]$$

$$\forall (x_1, x_2) \in K^2, \forall (y_1, y_2) \in K^2 \text{ and } (u_1, u_2) \in U^2.$$

Definition 4: Let $\varrho = (t_0, \tau, \varphi, v, u) \in \Lambda_1$. A function $x(t) = x(t; \varrho) \in O, t \in [\hat{\tau}, t_1], t_1 \in (t_0, b]$, is called a solution of equation (4) with the initial condition (3) or a solution corresponding to the element ϱ and defined on the interval $[\hat{\tau}, t_1]$, if it satisfies condition (3), is absolutely continuous on the interval $[t_0, t_1]$ and satisfies equation (4) almost everywhere on $[t_0, t_1]$.

Theorem 5: *Let $x_0(t)$ be the solution corresponding to $\varrho_0 = (t_{00}, \tau_0, \varphi_0, v_0, u_0) \in \Lambda_1$ and defined on $[\hat{\tau}, t_{10}], t_{10} < b$. Let $K_1 \subset O$ be a compact set containing a certain neighborhood of the set $\varphi_0(I_1) \cup x_0([t_{00}, t_{10}])$. Then the following conditions hold:*

(6) there exist numbers $\delta_i > 0, i = 0, 1$ such that to each element

$$\varrho = (t_0, \tau, \varphi, v, u) \in \hat{V}(\varrho_0; \delta_0) = B(t_{00}; \delta_0) \times V(\tau_0; \delta_0) \\ \times V_1(\varphi_0; \delta_0) \times V_2(v_0; \delta_0) \times V_3(u_0; \delta_0)$$

corresponds solution $x(t; \varrho)$ defined on the interval $[\hat{\tau}, t_{10} + \delta_1] \subset I_1$ and satisfying the condition $x(t; \varrho) \in K_1$; here

$$V_3(u_0; \delta_0) = \{ u \in \Omega : \| \, u - u_0 \, \|_I < \delta_0 \};$$

(7) for an arbitrary $\varepsilon > 0$, there exists a number $\delta_2 = \delta_2(\varepsilon) \in (0, \delta_0)$ such that the following inequality holds for any $\varrho \in \hat{V}(\varrho_0; \delta_2)$:

$$| \, x(t; \varrho) - x(t; \varrho_0) \, | \leq \varepsilon \; \forall t \in [\theta, t_{10} + \delta_1], \theta = \max\{t_0, t_{00}\};$$

(8) for an arbitrary $\varepsilon > 0$, there exists a number $\delta_3 = \delta_3(\varepsilon) \in (0, \delta_0)$ such that the following inequality fulfilled for any $\varrho \in \hat{V}(\varrho_0; \delta_3)$:

$$\int_{\hat{\tau}}^{t_{10}+\delta_1} | \, x(t; \varrho) - x(t; \varrho_0) \, | \, dt \leq \varepsilon.$$

We introduce the set of variations

$$\Im_1 = \Big\{ \delta\varrho = (\delta t_0, \delta\tau, \delta\varphi, \delta v, \delta u) : | \, \delta t_0 \, | \leq \beta, \| \, \delta\tau \, \|_I \leq \beta, \| \, \delta\varphi \, \|_{I_1} \leq \beta,$$

$$\delta\varphi \in \Phi_1 - \varphi_0, \| \, \delta v \, \|_{I_1} \leq \beta, \delta v \in \Phi_2 - v_0, \| \, \delta u \, \|_I \leq \beta, \delta u \in V_3 - u_0 \Big\}.$$

Theorem 6: *Let $x_0(t)$ be the solution corresponding to $\varrho_0 \in \Lambda_1$ and defined* *on* $[\hat{\tau}, t_{10}], t_{i0} \in (a,b), i = 0,1$. *Let* $K_1 \subset O$ *be a compact set containing a certain neighborhood of the set* K_0. *Then the following conditions hold:*

(9) there exist numbers $\varepsilon_1 > 0$ *and* $\delta_1 > 0$ *such that for an arbitrary* $(\varepsilon, \delta\varrho) \in (0, \varepsilon_1) \times \Im_1$ *the element* $\varrho_0 + \varepsilon\delta\varrho \in \Lambda_1$, *and there corresponds the solution* $x(t; \varrho_0 + \varepsilon\delta\varrho)$ *defined on the interval* $[\hat{\tau}, t_{10} + \delta_1] \subset I_1$. *Moreover,* $x(t; \varrho_0 + \varepsilon\delta\varrho) \in K_1$;

(10) the following relations are fulfilled:

$$\lim_{\varepsilon \to 0} \left[\sup \left\{ \mid x(t; \varrho_0 + \varepsilon\delta\varrho) - x(t; \varrho_0) \mid : t \in [\theta, t_{10} + \delta_1] \right\} \right] = 0$$

and

$$\lim_{\varepsilon \to 0} \int_{\hat{\tau}}^{t_{10}+\delta_1} \mid x(t; \varrho_0 + \varepsilon\delta\varrho) - x(t; \varrho_0) \mid dt = 0$$

uniformly in $\delta\varrho \in \Im_1$, *where* $\theta = \max\{t_{00}, t_{00} + \varepsilon\delta t_0\}$.

Theorem 6 is a simple corollary of Theorem 5.

Some comments. In Theorem 2 perturbations of the right-hand side of equation (2) are small in the integral sense (see (1)). Theorems 2,3,5,6 and similar theorems play an important role in the theory of optimal control, in proving variation formulas of solution, in the sensitivity analysis of equations. For the first time, theorem on the continuous dependence of the solution when perturbations of the right-hand side are small in the integral sense has been given in [1] for the ordinary differential equation. Theorem 2 is an analog of the theorems proved in [2,3]. Finally, we note that theorems analogous to Theorem 2 for various classes of neutral equations and equations with distributed prehistory are proved in [4-9].

References

[1] M.A. Krasnosel'skii, S.G. Krein, *On the averaging principle in nonlinear mechanics* (in Russian), Usp. Mat. Nauk., **10**, 3 (1955), 147-152

[2] R.V. Gamkrelidze, *Principles of optimal control theory*, Plenum Press, New York-London, 1978

[3] R.V. Gamkrelidze, G.L. Kharatishvili, *Extremal problems in linear topological spaces* (in Russian), Izv. Akad. Nauk SSSR, Ser. Mat., **33**, 4 (1969), 781-839

[4] G. Kharatishvili, T. Tadumadze, N. Gorgodze, *Continuous dependence and differentiability of solution with respect to initial data and right-hand side for differential equations with deviating argument*, Mem. Diff. Equ. Math. Phys, **19** (2000), 3-105

[5] T. Tadumadze, N. Gorgodze, I. Ramishvili, *On the well-posedness of the Cauchy problem for quasi-linear differential equations of neutral type*, J. Math. Sci. (N.Y.), **151**, 6 (2008), 3611-3630

[6] N. Gorgodze, I. Ramishvili, T. Tadumadze, *Continuous dependence of solution of a neutral functional differential equation on the right-hand side and initial data considering perturbations of variable delays*, Georgian Math. J., **23**, 4 (2016), 519-535

[7] T. Tadumadze, N. Gorgodze, *Variation formulas of a solution and initial data optimization problems for quasi-linear neutral functional differential equations with discontinuous initial condition*, Mem. Diff. Equ. Math. Phys, **63** (2014), 1-77

[8] T. Tadumadze, F. Dvalishvili, *Continuous dependence of the solution of the differential equation with distributed delay on the initial data and right-hand side*, Sem. I. Vekua Inst. Appl. Math., Rep.,**26-27** (2000-2001), 15-30

[9] P. Dvalishvili, T. Tadumadze, *On the well-posedness of the Cauchy problem for differential equations with distributed prehistory considering delay function perturbations*, Transactions of A. Razmadze Mathematical Institute, **170** (2016), 7-18.