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On Some Solutions in the Plane Equilibrium Theory for Solids with Triple-Porosity

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In this paper we consider the 2D linear equilibrium theory of elasticity for triple-porosity/triple-permeability model. We construct the fundamental and singular matrices of solutions to the system of equilibrium equations in terms of elementary functions. Some basic properties of single-layer and double-layer potentials are also established. Representation of regular solution is obtained.

Keywords: Triple-porosity, Fundamental and singular matrices of solutions, Regular solution.

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1. Introduction

The theories of elasticity and thermoelasticity of double porosity are used in many areas of applied science (e.g., in biomechanics, in technology, in biophysics, in solid mechanics, in engineering mechanics, in engineering medicine, engineering geology, in applied and computing mechanics and in applied mathematics) and engineering.

The theory of consolidation with double porosity was first proposed by Aifantis and co-authors in the papers [1-3]. This theory unifies a model proposed by Biot for the consolidation of deformable single porosity media with a model proposed by Barenblatt and co-authors for seepage in undeformable media with two degrees of porosity [4].

Naturally fractured reservoirs are usually characterized using dual-porosity models where all fractures are assumed to have identical properties. However, it is more realistic to assume fractures having different properties. An improvement of the dual-porosity models is to consider the models of triple porosity. Triple-porosity models have been developed as more realistic models to capture reservoir heterogeneity in naturally fractured reservoirs. The triple-porosity/triple-permeability model is applicable to a severely fractured reservoirs with high permeability. The first triple-porosity model was developed by Liu [5,6]. In petroleum literature the first triple porosity model was introduced by Abdassah and Ershaghi [7]. We note that the bone can be consider as a body with triple porosity (where p_1, p_2, p_3

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are correspondingly the pressures in the microporosity, the canalicular-lacunar porosity and the vascular porosity).

The basic results and the historical information on the theory of porous media may be found in [8]. Many analytical solutions for variations on double and triple porosity conceptual models have been developed with different configurations and relationships between fracture and matrix continua (see [9]-[27] and the references cited therein).

In this paper we consider the 2D linear equilibrium theory of elasticity for triple-porosity/triple-permeability model. We construct the fundamental and singular matrices of solutions to the system of equilibrium equations in terms of elementary functions. Some basic properties of single-layer and double-layer potentials are also established. Representation of regular solution is obtained.

2. Basic equations

Let $D^+(D^-)$ be a bounded (respectively, an unbounded) domain in the Euclidean two-dimensional space $\mathbb{R}^2$ bounded by the contour S. Suppose that $S \in C^{1,\beta}$, $0 < \beta \leq 1$, i.e., S is a Lyapunow curve. Let $x = (x_1, x_2)$ is point of space, $\partial_{\mathbf{x}} = \left(\dfrac{\partial}{\partial x_1}, \dfrac{\partial}{\partial x_2}\right)$. Let the domain D be filled with an isotropic triple-porosity material.

The basic homogeneous system of equations for isotropic materials with triple-porosity has the form [1],[10],[11]:

$$\mu\Delta\mathbf{u} + (\lambda+\mu)\text{grad div}\mathbf{u} + \sum_{k=1}^{3}\beta_k \text{grad} p_k = 0, \tag{1}$$

$$\begin{aligned}
&a_1\Delta p_1 + a_{12}(p_2 - p_1) + a_{13}(p_3 - p_1) = 0,\\
&a_2\Delta p_2 + a_{21}(p_1 - p_2) + a_{23}(p_3 - p_2) = 0,\\
&a_3\Delta p_3 + +a_{31}(p_1 - p_3) + a_{32}(p_2 - p_3) = 0,
\end{aligned} \tag{2}$$

where $\mathbf{u} = (u_1, u_2)^\top$ is the displacement vector in a solid, p_j, $j = 1, 2, 3$, are the pressures in the fluid phase, $a_i = \frac{k_i}{\mu'}$ (for the fluid phase, each phase i carries its respectively permeability k_i, a_{ij} is the fluid transfer rate between phase i and phase j. β_j, λ, μ, μ' are constants, Δ is the 2D Laplace operator. Throughout this paper the superscript $\top$ denotes transposition.

From (1),(2) we have

$$\mu\Delta\mathbf{u} + (\lambda+\mu)\text{grad div}\mathbf{u} + \sum_{k=1}^{3}\beta_k \text{grad} p_k = 0, \tag{3}$$

$$\begin{aligned}
&(a_1\Delta - b_1)p_1 + a_{12}p_2 + a_{13}p_3 = 0,\\
&(a_2\Delta - b_2)p_2 + a_{21}p_1 + a_{23}p_3 = 0,\\
&(a_3\Delta - b_3)p_3 + a_{31}p_1 + a_{32}p_2 = 0,
\end{aligned} \tag{4}$$

where

$$b_1 = a_{12} + a_{13}, \quad b_2 = a_{21} + a_{23}, \quad b_3 = a_{31} + a_{32},$$

Equations (3),(4) can be written as

$$A(\partial_{\mathbf{x}})\mathbf{U} = 0 \tag{5}$$

where

$$A(\partial_{\mathbf{x}}) = \| \, A_{pq}(\partial_{\mathbf{x}}) \, \|_{5\cdot 5}, \quad p, q = 1, ..., 5,$$

$$A_{jl}(\partial_{\mathbf{x}}) = \mu \Delta \delta_{jl} + (\lambda + \mu) \frac{\partial^2}{\partial x_j x_l}, \quad A_{j3}(\partial_{\mathbf{x}}) = \beta_1 \frac{\partial}{\partial x_j}$$

$$A_{j4}(\partial x) = \beta_2 \frac{\partial}{\partial x_j}, \quad A_{j5}(\partial x) = \beta_3 \frac{\partial}{\partial x_j}, \quad j, l = 1, 2,$$

$$A_{3j}(\partial_{\mathbf{x}}) = A_{4j}(\partial_{\mathbf{x}}) = A_{5j}(\partial x) = 0, \quad j = 1, 2,$$

$$A_{j+2;j+2}(\partial_{\mathbf{x}}) = a_j \Delta - b_j, \quad A_{34} = a_{12}, \quad A_{35} = a_{13},$$

$$A_{43} = a_{21}, \quad A_{45} = a_{23}, \quad A_{53} = a_{31}, \quad A_{54} = a_{32}, \quad j = 1, 2, 3$$

$\delta_{\alpha\gamma}$ is the Kronecker delta, $\mathbf{U} = (u_1, u_2, p_1, p_2, p_3)^T$.

The matrix $\widetilde{\mathbf{A}}(\partial \mathbf{x}) := \| \, \widetilde{A}_{lj}(\partial \mathbf{x}) \, \|_{5x5} := \mathbf{A}^T(-\partial \mathbf{x})$, where $\widetilde{A}_{lj}(\partial \mathbf{x}) := A_{jl}(-\partial \mathbf{x})$, will be called the associated operator to the differential operator $\mathbf{A}(\partial \mathbf{x})$.

It is easily seen that

$$det A = \mu \mu_0 a_1 a_2 a_3 \Delta^3 (\Delta + \lambda_1^2)(\Delta + \lambda_2^2), \quad \mu_0 = \lambda + 2\mu$$

where λ_j^2 are roots of equation

$$a_1 a_2 a_3 \lambda^2 + [a_2 a_3 b_1 + a_1 a_3 b_2 + a_1 a_2 b_3]\lambda$$

$$+ a_1(b_2 b_3 - a_{23} a_{32}) + a_2(b_1 b_3 - a_{13} a_{31}) + a_3(b_1 b_2 - a_{12} a_{21}) = 0$$

Definition 2.1: A vector-function $\mathbf{U} = (u_1, u_2, p_1, p_2, p_3)$ defined in the domain $D^+(D^-)$ is called regular if $\mathbf{U} \in C^2(D^+) \cap C^1(\overline{D}^+)$. Note that, in the case of the domain D^- the vector additionally satisfies the following conditions at infinity:

$$\mathbf{U}(\mathbf{x}) = O(1), \quad \frac{\partial \mathbf{U}}{\partial x_\alpha} = O(|\mathbf{x}|^{-2}), \quad |\mathbf{x}|^2 = x_1^2 + x_2^2 >> 1, \quad \alpha = 1, 2. \tag{6}$$

3. Matrix of fundamental solutions

Here we apply the method developed in [28] and construct the matrix of fundamental solutions

We introduce the matrix differential operator $\mathbf{B}(\partial_{\mathbf{x}})$ consisting of cofactors of elements of the transposed matrix $\mathbf{A}$ divided on $\mu(\lambda+2\mu)a_1a_2a_3 \neq 0$:

$$\mathbf{B}(\partial_{\mathbf{x}}) :=\| B_{lj}(\partial_{\mathbf{x}}) \|_{5\times 5},$$

where

$$\begin{aligned}
B_{ij} &= \frac{1}{\mu\mu_0}\Delta(\Delta+\lambda_1^2)(\Delta+\lambda_2^2)\left[\delta_{ij}\mu_0\Delta-(\lambda+\mu)\frac{\partial^2}{\partial x_i\partial x_j}\right], \quad i,j=1,2,\\
B_{j3} &= \frac{\Delta k_{13}}{\mu_0a_1a_2a_3}\frac{\partial}{\partial x_j}, \quad B_{j4}=\frac{\Delta k_{14}}{\mu_0a_1a_2a_3}\frac{\partial}{\partial x_j}, \quad B_{j5}=\frac{\Delta k_{15}}{\mu_0a_1a_2a_3}\frac{\partial}{\partial x_j},\\
B_{33} &= \frac{\Delta^2}{a_1a_2a_3}[a_2a_3\Delta^2-(a_2b_3+a_3b_2)\Delta+b_2b_3-a_{23}a_{32}],\\
B_{44} &= \frac{\Delta^2}{a_1a_2a_3}[a_1a_3\Delta^2-(a_1b_3+a_3b_1)\Delta+b_1b_3-a_{13}a_{31}],\\
B_{55} &= \frac{\Delta^2}{a_1a_2a_3}[a_2a_1\Delta^2-(a_2b_1+a_1b_2)\Delta+b_2b_1-a_{12}a_{21}],\\
B_{34} &= -\frac{\Delta^2}{a_1a_2a_3}[a_{12}(a_3\Delta-b_3)-a_{13}a_{32}],\\
B_{43} &= -\frac{\Delta^2}{a_1a_2a_3}[a_{21}(a_3\Delta-b_3)-a_{31}a_{23}],\\
B_{35} &= -\frac{\Delta^2}{a_1a_2a_3}[a_{13}(a_2\Delta-b_2)-a_{12}a_{23}],\\
B_{53} &= -\frac{\Delta^2}{a_1a_2a_3}[a_{31}(a_2\Delta-b_2)-a_{21}a_{32}],\\
B_{45} &= -\frac{\Delta^2}{a_1a_2a_3}[a_{23}(a_1\Delta-b_1)-a_{21}a_{13}],\\
B_{54} &= -\frac{\Delta^2}{a_1a_2a_3}[a_{32}(a_1\Delta-b_1)-a_{12}a_{31}],
\end{aligned}$$

$$\begin{aligned}
k_{13} &= \beta_1a_2a_3\Delta^2-[\beta_1(a_2b_3+a_3b_2)+\beta_2a_3a_{21}+\beta_3a_2a_{31}]\Delta\\
&\quad+(\beta_1+\beta_2+\beta_3)(b_2b_3-a_{23}a_{32}),\\
k_{14} &= \beta_2a_1a_3\Delta^2-[\beta_2(a_1b_3+a_3b_1)+\beta_1a_3a_{12}+\beta_3a_1a_{23}]\Delta\\
&\quad+(\beta_1+\beta_2+\beta_3)(b_1b_3-a_{13}a_{31}),\\
k_{15} &= \beta_3a_1a_2\Delta^2-[\beta_3(a_1b_2+a_2b_1)+\beta_1a_2a_{13}+\beta_2a_1a_{23}]\Delta\\
&\quad+(\beta_1+\beta_2+\beta_3)(b_1b_2-a_{12}a_{21}), \quad \mu_0:=\lambda+2\mu.
\end{aligned}$$

Substituting the vector $\mathbf{U}(\mathbf{x})=\mathbf{B}(\partial_{\mathbf{x}})\mathbf{\Psi}$ into (5), we get

$$\Delta\Delta\Delta(\Delta+\lambda_1^2)(\Delta+\lambda_2^2)\mathbf{\Psi}=0.$$

The solution of the last equation can be represented as

$$\Delta\Psi = \frac{r^2(\ln r - 1)}{4\lambda_1^2\lambda_2^2} + \frac{1}{\lambda_1^2 - \lambda_2^2}\left[\frac{\varphi_1 - \ln r}{\lambda_1^4} - \frac{\varphi_2 - \ln r}{\lambda_2^4}\right] \tag{7}$$

where

$$\varphi_m = \frac{\pi}{2i} H_0^{(1)}(\lambda_m r),$$

$H_0^{(1)}(\lambda_m r)$ is Hankel's function of the first kind with the index 0

$$H_0^{(1)}(\lambda_m r) = \frac{2i}{\pi}\ln r + \frac{2i}{\pi}\left(\ln\frac{\lambda_m}{2} + C - \frac{i\pi}{2}\right) J_0(\lambda_m r)$$

$$+\frac{2i}{\pi}(J_0(\lambda_m r) - 1)\ln r - \frac{2i}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{(k!)^2}\left(\frac{\lambda_m r}{2}\right)^{2k}\left(\frac{1}{k} + \frac{1}{k-1} + ... + 1\right),$$

$$J_0(\lambda_m r) = \sum_{k=0}^{\infty}\frac{(-1)^k}{(k!)^2}\left(\frac{\lambda_m r}{2}\right)^{2k}, \quad r^2 = (x_1 - y_1)^2 + (x_2 - y_2)^2, \;\; m = 1, 2.$$

As all the components of $\mathbf{B}$ contain the operator Δ, Substituting (7) into $\mathbf{U} = \mathbf{B\Psi}$, we obtain the matrix of fundamental solutions for the equation (5) which we denote by $\mathbf{\Gamma}(\mathbf{x}\text{-}\mathbf{y})$

$$\mathbf{\Gamma}(\mathbf{x}\text{-}\mathbf{y}) = \| \Gamma_{kj}(\mathbf{x}\text{-}\mathbf{y}) \|_{5\times 5},$$

where

$$\Gamma_{\alpha\gamma}(\mathbf{x}\text{-}\mathbf{y}) = \delta_{\alpha\gamma}\frac{\ln r}{\mu} - \frac{\lambda+\mu}{\mu_0\mu}\frac{\partial^2\varphi_0}{\partial x_\alpha \partial x_\gamma}, \quad \varphi_0 = \frac{r^2(\ln r - 1)}{4},$$

$$\Gamma_{j3}(\mathbf{x}\text{-}\mathbf{y}) = -\frac{1}{\mu_0 a_1 a_2 a_3}\frac{\partial k_{13}}{\partial x_j}, \qquad \Gamma_{j4}(\mathbf{x}\text{-}\mathbf{y}) = -\frac{1}{\mu_0 a_1 a_2 a_3}\frac{\partial k_{14}}{\partial x_j},$$

$$\Gamma_{j5}(\mathbf{x}\text{-}\mathbf{y}) = -\frac{1}{\mu_0 a_1 a_2 a_3}\frac{\partial k_{15}}{\partial x_j} \qquad k_{13} = \frac{B_3\varphi_0}{\lambda_1^2\lambda_2^2} + \sum_{k=1}^{2}(-1)^k \alpha_{3k}(\varphi - \varphi_k),$$

$$\alpha_{3k}=\frac{1}{\lambda_2^2-\lambda_1^2}\left[\beta_1 a_2 a_3+\frac{A_3}{\lambda_k^2}+\frac{B_3}{\lambda_k^4}\right],\quad \varphi=\ln r,$$

$$B_3=(\beta_1+\beta_2+\beta_3)(b_2b_3-a_{23}a_{32}),\quad A_3=\beta_1(a_2b_3+a_3b_2)+\beta_2a_3a_{21}+\beta_3a_2a_{31},$$

$$k_{14}=\frac{B_1\varphi_0}{\lambda_1^2\lambda_2^2}+\sum_{k=1}^{2}(-1)^k\alpha_{4k}(\varphi-\varphi_k),\qquad \alpha_{4k}=\frac{1}{\lambda_2^2-\lambda_1^2}\left[\beta_2 a_1 a_3+\frac{A_1}{\lambda_k^2}+\frac{B_1}{\lambda_k^4}\right],$$

$$B_1=(\beta_1+\beta_2+\beta_3)(b_1b_3-a_{13}a_{31}),\quad A_1=\beta_2(a_1b_3+a_3b_2)+\beta_1a_3a_{12}+\beta_3a_1a_{32},$$

$$k_{15}=\frac{B_2\varphi_0}{\lambda_1^2\lambda_2^2}+\sum_{k=1}^{2}(-1)^k\alpha_{5k}(\varphi-\varphi_k),\qquad \alpha_{5k}=\frac{1}{\lambda_2^2-\lambda_1^2}\left[\beta_3 a_2 a_1+\frac{A_2}{\lambda_k^2}+\frac{B_2}{\lambda_k^4}\right],$$

$$B_2=(\beta_1+\beta_2+\beta_3)(b_2b_1-a_{12}a_{21}),\quad A_2=\beta_3(a_1b_2+a_2b_1)+\beta_1a_1a_{23}+\beta_2a_2a_{13},$$

$$\Gamma_{33}(\mathbf{x}\text{-}\mathbf{y})=\frac{b_2b_3-a_{23}a_{32}}{\lambda_1^2\lambda_2^2a_1a_2a_3}\varphi+\sum_{k=1}^{2}(-1)^k\gamma_{1k}\varphi_k,$$

$$\gamma_{1k}=\frac{1}{(\lambda_2^2-\lambda_1^2)a_1a_2a_3}\left[a_2a_3\lambda_k^2+\frac{b_2b_3-a_{23}a_{32}}{\lambda_k^2}+a_2b_3+a_3b_2\right],$$

$$\Gamma_{44}(\mathbf{x}\text{-}\mathbf{y})=\frac{b_1b_3-a_{13}a_{31}}{\lambda_1^2\lambda_2^2a_1a_2a_3}\varphi+\sum_{k=1}^{2}(-1)^k\gamma_{2k}\varphi_k,$$

$$\gamma_{2k}=\frac{1}{(\lambda_2^2-\lambda_1^2)a_1a_2a_3}\left[a_1a_3\lambda_k^2+\frac{b_1b_3-a_{13}a_{31}}{\lambda_k^2}+a_1b_3+a_3b_1\right],$$

$$\Gamma_{55}(\mathbf{x}\text{-}\mathbf{y})=\frac{b_1b_2-a_{12}a_{21}}{\lambda_1^2\lambda_2^2a_1a_2a_3}\varphi+\sum_{k=1}^{2}(-1)^k\gamma_{3k}\varphi_k,$$

$$\gamma_{3k}=\frac{1}{(\lambda_2^2-\lambda_1^2)a_1a_2a_3}\left[a_1a_2\lambda_k^2+\frac{b_1b_2-a_{12}a_{21}}{\lambda_k^2}+a_1b_2+a_2b_1\right],$$

$$\Gamma_{34}(\mathbf{x}\text{-}\mathbf{y})=-\frac{1}{a_1a_2a_3}\left[a_3a_{12}\frac{\varphi_1-\varphi_2}{\lambda_2^2-\lambda_1^2}-\frac{b_1b_3-a_{13}a_{31}}{\lambda_2^2-\lambda_1^2}\left(\frac{\varphi-\varphi_1}{\lambda_1^2}-\frac{\varphi-\varphi_2}{\lambda_2^2}\right)\right],$$

$$\Gamma_{35}(\mathbf{x}\text{-}\mathbf{y})=-\frac{1}{a_1a_2a_3}\left[a_2a_{13}\frac{\varphi_1-\varphi_2}{\lambda_2^2-\lambda_1^2}-\frac{b_1b_2-a_{12}a_{21}}{\lambda_2^2-\lambda_1^2}\left(\frac{\varphi-\varphi_1}{\lambda_1^2}-\frac{\varphi-\varphi_2}{\lambda_2^2}\right)\right],$$

$$\Gamma_{43}(\mathbf{x}\text{-}\mathbf{y})=-\frac{1}{a_1a_2a_3}\left[a_3a_{21}\frac{\varphi_1-\varphi_2}{\lambda_2^2-\lambda_1^2}-\frac{b_2b_3-a_{23}a_{32}}{\lambda_2^2-\lambda_1^2}\left(\frac{\varphi-\varphi_1}{\lambda_1^2}-\frac{\varphi-\varphi_2}{\lambda_2^2}\right)\right],$$

$$\Gamma_{45}(\mathbf{x}\text{-}\mathbf{y})=-\frac{1}{a_1a_2a_3}\left[a_1a_{23}\frac{\varphi_1-\varphi_2}{\lambda_2^2-\lambda_1^2}-\frac{b_1b_2-a_{12}a_{21}}{\lambda_2^2-\lambda_1^2}\left(\frac{\varphi-\varphi_1}{\lambda_1^2}-\frac{\varphi-\varphi_2}{\lambda_2^2}\right)\right],$$

$$\Gamma_{53}(\mathbf{x}\text{-}\mathbf{y})=-\frac{1}{a_1a_2a_3}\left[a_2a_{31}\frac{\varphi_1-\varphi_2}{\lambda_2^2-\lambda_1^2}-\frac{b_2b_3-a_{23}a_{32}}{\lambda_2^2-\lambda_1^2}\left(\frac{\varphi-\varphi_1}{\lambda_1^2}-\frac{\varphi-\varphi_2}{\lambda_2^2}\right)\right],$$

$$\Gamma_{54}(\mathbf{x}\text{-}\mathbf{y}) = -\frac{1}{a_1a_2a_3}\left[a_1a_{32}\frac{\varphi_1-\varphi_2}{\lambda_2^2-\lambda_1^2} - \frac{b_1b_3-a_{13}a_{31}}{\lambda_2^2-\lambda_1^2}\left(\frac{\varphi-\varphi_1}{\lambda_1^2} - \frac{\varphi-\varphi_2}{\lambda_2^2}\right)\right],$$

$$\Gamma_{31} \equiv \Gamma_{32} \equiv \Gamma_{41} \equiv \Gamma_{42} \equiv \Gamma_{51} \equiv \Gamma_{52} \equiv 0, \quad \varphi_k = H_0^{(1)}(\lambda_k r), \quad \varphi = \ln r, \quad k = 1,2,$$

The following assertion holds

Theorem 3.1 : *The elements of the matrix $\mathbf{\Gamma}(\boldsymbol{x}\text{-}\boldsymbol{y})$ has a logarithmic singularity as $\boldsymbol{x} \to \boldsymbol{y}$ and each column of the matrix $\mathbf{\Gamma}(\boldsymbol{x}\text{-}\boldsymbol{y})$, considered as a vector, is a solution of system (5) at every point $\boldsymbol{x}$ if $\boldsymbol{x} \neq \boldsymbol{y}$.*

Let us consider the matrix $\widetilde{\mathbf{\Gamma}}(\mathbf{x}) := \mathbf{\Gamma}^\top(\text{-}\mathbf{x})$. The following basic properties of $\widetilde{\mathbf{\Gamma}}(\mathbf{x})$ may be easily verified.

Theorem 3.2 : *Each column of the matrix $\widetilde{\mathbf{\Gamma}}(\boldsymbol{x}\text{-}\boldsymbol{y})$, considered as a vector, satisfies the associated system $\widetilde{\boldsymbol{A}}(\partial_{\boldsymbol{x}})\widetilde{\mathbf{\Gamma}}(\boldsymbol{x}\text{-}\boldsymbol{y}) = 0$, at every point $\boldsymbol{x}$ if $\boldsymbol{x} \neq \boldsymbol{y}$ and the elements of the matrix $\widetilde{\mathbf{\Gamma}}(\boldsymbol{x}\text{-}\boldsymbol{y})$ have a logarithmic singularity as $\boldsymbol{x} \to \boldsymbol{y}$, where $\widetilde{\boldsymbol{A}}(\partial_{\boldsymbol{x}}) = \boldsymbol{A}^\top(-\partial_{\boldsymbol{x}})$.*

4. Matrix of singular solutions

Using the matrix of fundamental solutions, we construct the so-called singular matrices of solutions by means of elementary functions.

Let $\mathbf{n}(\mathbf{z}) = (n_1, n_2)$ denote the exterior (with respect to D^+) unite normal vector of S at the point $\mathbf{z} = (z_1, z_2) \in S$. Denoting the stress vector by $\mathbf{P}(\partial_{\mathbf{z}}, \mathbf{n})\mathbf{U}$ we have

$$\mathbf{P}(\partial_{\mathbf{x}}, \mathbf{n})\mathbf{U} = \mathbf{T}(\partial_{\mathbf{x}}, \mathbf{n})\mathbf{u} + \mathbf{n}\sum_{k=1}^{3}\beta_k p_k,$$

where $\mathbf{T}(\partial_{\mathbf{x}}, \mathbf{n})$ is the stress vector in the classical theory of elasticity

$$\boldsymbol{T}(\partial\mathbf{x}, \mathbf{n})\mathbf{u} = \begin{pmatrix} \mu\dfrac{\partial}{\partial n} + (\lambda+\mu)n_1\dfrac{\partial}{\partial x_1} & (\lambda+\mu)n_1\dfrac{\partial}{\partial x_2} + \mu\dfrac{\partial}{\partial s} \\ (\lambda+\mu)n_2\dfrac{\partial}{\partial x_1} - \mu\dfrac{\partial}{\partial s} & \mu\dfrac{\partial}{\partial n} + (\lambda+\mu)n_2\dfrac{\partial}{\partial x_2} \end{pmatrix}\mathbf{u},$$

$$\frac{\partial}{\partial \mathbf{n}} = n_1\frac{\partial}{\partial x_1} + n_2\frac{\partial}{\partial x_2},$$

$$\frac{\partial}{\partial s} = n_2\frac{\partial}{\partial x_1} - n_1\frac{\partial}{\partial x_2}.$$

We introduce the following notations

$$\boldsymbol{R}(\partial\mathbf{x},\mathbf{n}) = \begin{pmatrix} T_{11}(\partial x,n) & T_{12}(\partial x,n) & \beta_1 n_1 & \beta_2 n_1 & \beta_3 n_1 \\ T_{21}(\partial x,n) & T_{22}(\partial x,n) & \beta_1 n_2 & \beta_2 n_2 & \beta_3 n_2 \\ 0 & 0 & a_1\dfrac{\partial}{\partial n} & 0 & 0 \\ 0 & 0 & 0 & a_2\dfrac{\partial}{\partial n} & 0 \\ 0 & 0 & 0 & 0 & a_3\dfrac{\partial}{\partial n} \end{pmatrix},$$

$$\widetilde{\boldsymbol{R}}(\partial\mathbf{x},\mathbf{n}) = \begin{pmatrix} T_{11}(\partial x,n) & T_{12}(\partial x,n) & 0 & 0 & 0 \\ T_{21}(\partial x,n) & T_{22}(\partial x,n) & 0 & 0 & 0 \\ 0 & 0 & a_1\dfrac{\partial}{\partial n} & 0 & 0 \\ 0 & 0 & 0 & a_2\dfrac{\partial}{\partial n} & 0 \\ 0 & 0 & 0 & 0 & a_3\dfrac{\partial}{\partial n} \end{pmatrix},$$

where

$$T_{kj}(\partial\mathbf{x},\mathbf{n}) = \mu\delta_{kj}\frac{\partial}{\partial n} + \lambda n_k\frac{\partial}{\partial x_j} + \mu n_j\frac{\partial}{\partial x_k}, \quad k,j,=1,2.$$

Applying the operator $\mathbf{R}(\partial_{\mathbf{x}},\mathbf{n})$ to the matrix $\boldsymbol{\Gamma}(\mathbf{x}\text{-}\mathbf{y})$, we obtain

$$\mathbf{R}(\partial_{\mathbf{x}},\mathbf{n})\boldsymbol{\Gamma}(\mathbf{x}\text{-}\mathbf{y}) :=\| M_{kj} \|_{5\times 5}, \tag{8}$$

where

$$\begin{aligned} M_{11} &= \delta_{kj}\frac{\partial\ln r}{\partial n} - 2\frac{\lambda+\mu}{\mu_0}\frac{\partial}{\partial s}\frac{\partial^2\varphi_0}{\partial x_1\partial x_2}, \quad M_{12} = \frac{\partial}{\partial s}\ln r - 2\frac{\lambda+\mu}{\mu_0}\frac{\partial}{\partial s}\frac{\partial^2\varphi_0}{\partial x_2^2}, \\ M_{21} &= -\frac{\partial}{\partial s}\ln r + 2\frac{\lambda+\mu}{\mu_0}\frac{\partial}{\partial s}\frac{\partial^2\varphi_0}{\partial x_1^2}, \quad M_{22} = \delta_{kj}\frac{\partial\ln r}{\partial n} + 2\frac{\lambda+\mu}{\mu_0}\frac{\partial}{\partial s}\frac{\partial^2\varphi_0}{\partial x_1\partial x_2}, \end{aligned} \tag{9}$$

$$M_{13}=-\frac{2\mu}{\mu_0 a_1 a_2 a_3}\frac{\partial}{\partial s}\frac{\partial}{\partial x_2}k_{13},\quad M_{23}=\frac{2\mu}{\mu_0 a_1 a_2 a_3}\frac{\partial}{\partial s}\frac{\partial}{\partial x_1}k_{13},$$

$$M_{14}=-\frac{2\mu}{\mu_0 a_1 a_2 a_3}\frac{\partial}{\partial s}\frac{\partial}{\partial x_2}k_{14},\quad M_{24}=\frac{2\mu}{\mu_0 a_1 a_2 a_3}\frac{\partial}{\partial s}\frac{\partial}{\partial x_1}k_{14},$$

$$M_{15}=-\frac{2\mu}{\mu_0 a_1 a_2 a_3}\frac{\partial}{\partial s}\frac{\partial}{\partial x_2}k_{15},\quad M_{25}=\frac{2\mu}{\mu_0 a_1 a_2 a_3}\frac{\partial}{\partial s}\frac{\partial}{\partial x_1}k_{15},$$

$$M_{33}=a_1\frac{\partial}{\partial n}\Gamma_{33}\quad M_{34}=a_1\frac{\partial}{\partial n}\Gamma_{34}\quad M_{35}=a_1\frac{\partial}{\partial n}\Gamma_{35},$$

$$M_{43}=a_2\frac{\partial}{\partial n}\Gamma_{43},\quad M_{44}=a_2\frac{\partial}{\partial n}\Gamma_{44},\quad M_{45}=a_2\frac{\partial}{\partial n}\Gamma_{45},$$

$$M_{53}=a_3\frac{\partial}{\partial n}\Gamma_{53},\quad M_{54}=a_3\frac{\partial}{\partial n}\Gamma_{54},\quad M_{55}=a_3\frac{\partial}{\partial n}\Gamma_{55},$$

$$M_{31}=M_{32}=M_{41}=M_{42}=M_{51}=M_{52}=0.$$

Let $[\mathbf{R}(\partial_{\mathbf{y}},\mathbf{n})\mathbf{\Gamma}(\mathbf{y}\text{-}\mathbf{x})]^\top$ be the matrix which we get from $[\mathbf{R}(\partial_{\mathbf{x}},\mathbf{n})\mathbf{\Gamma}(\mathbf{x}\text{-}\mathbf{y})]$ by transposition of the columns and rows and the variables $\mathbf{x}$ and $\mathbf{y}$ (analogously $\left[\tilde{\boldsymbol{R}}(\partial\mathbf{y},\mathbf{n})\tilde{\mathbf{\Gamma}}(\mathbf{y}\text{-}\mathbf{x})\right]^\top$).

Let us introduce the single-layer and double-layer potentials.

The vector-functions defined by the equalities

$$\mathbf{V}(\mathbf{x};\boldsymbol{g}=\frac{1}{\pi}\int\limits_S \mathbf{\Gamma}(\mathbf{x}-\boldsymbol{y})\mathbf{g}(\mathbf{y})d_{\mathbf{y}}S,$$

$$\widetilde{\boldsymbol{V}}(\mathbf{x};\boldsymbol{g})=\frac{1}{\pi}\int\limits_S \mathbf{\Gamma}^T(\mathbf{y}-\boldsymbol{x})\mathbf{g}(\mathbf{y})d_{\mathbf{y}}S$$

will be called single layer potentials, while the vector-functions defined by the equalities

$$\mathbf{W}(\mathbf{x};\boldsymbol{h})=\frac{1}{\pi}\int\limits_S \left[\boldsymbol{R}(\partial_{\mathbf{y}},\mathbf{n})\mathbf{\Gamma}(\mathbf{y}-\mathbf{x})\right]^\top\mathbf{h}(\mathbf{y})d_{\mathbf{y}}S,$$

$$\widetilde{\boldsymbol{W}}(\mathbf{x};\mathbf{h})=\frac{1}{\pi}\int\limits_S \left[\widetilde{\boldsymbol{R}}(\partial_{\mathbf{y}},\mathbf{n})\mathbf{\Gamma}^T(\mathbf{y}-\mathbf{x})\right]^\top\mathbf{h}(\mathbf{y})d_{\mathbf{y}}S$$

will be called double layer potentials. Here $\mathbf{g}$ and $\mathbf{h}$ are the continuous (or Hölder continuous) vectors and S is a closed Lyapunov curve.

The following theorems hold true.

Theorem 4.1: *The vectors $\widetilde{\boldsymbol{V}}(\boldsymbol{x};\boldsymbol{g})$ and $\boldsymbol{W}(\boldsymbol{x};\boldsymbol{h}$ are solutions of the system*

$\widetilde{\boldsymbol{A}}(\partial_{\boldsymbol{x}})\boldsymbol{U} = 0$ *at any point* $\boldsymbol{x} \notin S$.*The vectors* $\boldsymbol{V}(\boldsymbol{x};\boldsymbol{g})$ *and* $\widetilde{\boldsymbol{W}}(\boldsymbol{x};\boldsymbol{h})$ *are solutions of the system* $\boldsymbol{A}(\partial_{\boldsymbol{x}})\boldsymbol{U} = 0$ *at any point* $\boldsymbol{x} \notin S$. *The elements of the matrices* $[\boldsymbol{R}(\partial_{\boldsymbol{y}}, \boldsymbol{n})\boldsymbol{\Gamma}(\boldsymbol{y} - \boldsymbol{x})]^T$ *and* $\left[\widetilde{\boldsymbol{R}}(\partial_{\boldsymbol{y}}, \boldsymbol{n})\boldsymbol{\Gamma}^T(\boldsymbol{x} - \boldsymbol{y})\right]^T$ *contain a singular part integrable in the sense of the Cauchy principal value.*

Theorem 4.2: *If* $S \in C^{1,\eta}(S)$, $\boldsymbol{g}, \boldsymbol{h} \in C^{0,\delta}(S)$, $\quad 0 < \delta < \eta \leq 1$, *then the vectors* $\boldsymbol{W}(\boldsymbol{x},\boldsymbol{h})$, $\boldsymbol{V}(\boldsymbol{x},\boldsymbol{g})$, $\widetilde{\boldsymbol{W}}(\boldsymbol{x},\boldsymbol{h})$ *and* $\widetilde{\boldsymbol{V}}(\boldsymbol{x},\boldsymbol{g})$ *are regular vector-functions in* $D^+(D^-)$, *and when the point* $\boldsymbol{x}$ *tends to any point* $\boldsymbol{z}$ *of the boundary* S *from inside or from outside we have the following formulas:*

$$[\boldsymbol{W}(\boldsymbol{z},\boldsymbol{h})]^{\pm} = \mp\boldsymbol{h}(\boldsymbol{z}) + \frac{1}{\pi}\int\limits_S [\boldsymbol{R}(\partial_{\boldsymbol{y}}, \boldsymbol{n})\boldsymbol{\Gamma}(\boldsymbol{y} - \boldsymbol{z})]^T \boldsymbol{h}(\boldsymbol{y})d_{\boldsymbol{y}}S,$$

$$[\widetilde{\boldsymbol{W}}(\boldsymbol{z},\boldsymbol{h})]^{\pm} = \mp\boldsymbol{h}(\boldsymbol{z}) + \frac{1}{\pi}\int\limits_S \left[\widetilde{\boldsymbol{R}}(\partial_{\boldsymbol{y}}, \boldsymbol{n})\boldsymbol{\Gamma}^T(\boldsymbol{y} - \boldsymbol{z}))\right]^T \boldsymbol{h}(\boldsymbol{y})d_{\boldsymbol{y}}S,$$

$$[\boldsymbol{R}(\partial_{\boldsymbol{z}}, \boldsymbol{n})\,\boldsymbol{V}(\boldsymbol{z},\boldsymbol{g})]^{\pm} = \pm\boldsymbol{g}(\boldsymbol{z}) + \frac{1}{\pi}\int\limits_S \boldsymbol{R}(\partial_{\boldsymbol{z}}, \boldsymbol{n})\boldsymbol{\Gamma}(\boldsymbol{z} - \boldsymbol{y})\boldsymbol{g}(\boldsymbol{y})d_{\boldsymbol{y}}S,$$

$$[\widetilde{\boldsymbol{R}}(\partial_{\boldsymbol{z}}, \boldsymbol{n})\widetilde{\boldsymbol{V}}(\boldsymbol{z},\boldsymbol{g})]^{\pm} = \pm\boldsymbol{g}(\boldsymbol{z}) + \frac{1}{\pi}\int\limits_S \widetilde{\boldsymbol{R}}(\partial_{\boldsymbol{z}}, \boldsymbol{n})\boldsymbol{\Gamma}^T(\boldsymbol{y} - \boldsymbol{z})\boldsymbol{g}(\boldsymbol{y})d_{\boldsymbol{y}}S.$$

Here the integrals are understood in the principal value sense.

5. A representation of regular solutions

In this section we represent the general solution of Eqs.(1)-(4) by means of harmonic, biharmonic and metaharmonic functions.

It is evident that we can investigate separately Eqs. (4), which contain only p_j. Solving the system 4, we get

$$p_1 = \varphi + \sum_{k=1}^{2} A_k\varphi_k, \quad p_2 = \varphi + \sum_{k=1}^{2} B_k\varphi_k,$$

$$p_3 = \varphi + \sum_{k=1}^{2}\varphi_k,$$

where

$$\Delta\varphi = 0, \quad (\Delta + \lambda_k^2)\varphi_k = 0, \quad k = 1, 2,$$

$$A_k = \frac{a_{13}(a_2\lambda_k^2 + b_2) + a_{12}a_{23}}{(a_1\lambda_k^2 + b_1)(a_2\lambda_k^2 + b_2) - a_{12}a_{21}},$$

$$B_k = \frac{a_{23}(a_1\lambda_k^2 + b_1) + a_{13}a_{12}}{(a_1\lambda_k^2 + b_1)(a_2\lambda_k^2 + b_2) - a_{12}a_{21}},$$

$$A_k a_{31} + B_k a_{32} = b_3 + a_3\lambda_k^2 \quad k = 1, 2.$$

Assuming that p_j are known functions, we can rewrite Eq. (3) in the following form

$$\mu\Delta\mathbf{u} + (\lambda+\mu)\mathrm{graddiv}\mathbf{u} + \mathrm{grad}\left[\sum_{k=1}^{3}\beta_k\varphi + \sum_{k=1}^{2}(\beta_1 A_k + \beta_2 B_k + \beta_3)\varphi_k)\right] = 0, \quad (10)$$

Clearly, the general solution of equation (10) can be represented in the form

$$\mathbf{u} = \mathbf{v} + \mathbf{u}_0$$

where

$$\mu\Delta\mathbf{v} + (\lambda+\mu)\mathrm{grad\ div}\mathbf{v} = 0$$

and $\mathbf{u}_0$ is a particular solution of equation (10)

$$\mathbf{u}_0 = -\frac{1}{\mu_0}\mathrm{grad}\left[\sum_{k=1}^{3}\beta_k\varphi_0 - \sum_{k=1}^{2}\frac{1}{\lambda_k^2}(\beta_1 A_k + \beta_2 B_k + \beta_3)\varphi_k)\right].$$

Here φ_0, satisfies the equation

$$\Delta\varphi_0 = \varphi, \quad \Delta\varphi = 0.$$

6. Conclusions

The main results of this work can be formulated as follows:

1. The fundamental and singular matrices of solutions of the system of equations of motion in the equilibrium theory of elasticity for triple porosity materials are constructed explicitly.
2. Some basic properties for single and double layer potentials are established.
3. Representation of regular solution is obtained.

Remark 1 : By using the above-mentioned method, it is possible to construct explicitly the fundamental and singular matrices of solutions of the systems of equations in the modern linear theories of elasticity, thermoelasticity and poroelasticity for materials with microstructures and for elastic materials with multiple porosity, etc.

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