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Boundary Value Problems of Elasticity for Semi-ellipse with Non-homogeneous Boundary Conditions at the Segment Between Focuses

Natela Zirakashvili*

I. Vekua Institute of Applied Mathematics of I. Javakhishvili Tbilisi State University
2 University St., 0186, Tbilisi, Georgia

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The paper considers the boundary value problems of elasticity for semi-ellipse, when boundary conditions at the portion of the linear boundary between the focuses are nonzero and outside the focuses are zero. Thus, the continuity conditions for the problem solution are given at the portion of the linear boundary, therefore it is possible to bind the semi-ellipse as a whole ellipse, in which on the section between the focuses the condition of uninterrupted continuation of the problem solution not performed along this part, i.e. we have a crack on which, for example, the tangential stress acts. The problem solution for the cracked ellipse is reduced to the solution of the internal and external problems of elasticity, which are solved quite simply by the method of separation of variables.

Keywords: Boundary value problem elasticity, Separation variables method, Homogeneous isotropic ellipse, Internal linear crack.

AMS Subject Classification: 74B05, 35Q74.

1. Introduction

The paper considers boundary value problems for semi-ellipse $\{0 \leq \xi \leq \xi_1, 0 \leq \eta \leq \pi\}$ (if ξ, η $(0 \leq \xi < \infty,\ 0 \leq \eta < 2\pi)$ are elliptic coordinates and $x,\ y$ are Cartesian coordinates, then $x = c\cosh\xi\cos\eta,\quad y = c\sinh\xi\sin\eta$, where c is the scale factor equaling to 1 in our case), when boundary conditions at the portion of the linear boundary between the focuses are nonzero and outside the focuses are zero. Thus, conditions of uninterrupted continuation of the problem solution (symmetry or anti symmetry) are given at $\eta = 0$ and $\eta = \pi$, therefore it is possible to bind the semi-ellipse as a whole ellipse, in which, for example the tangential stress is given on $\xi = 0$ and the condition of uninterrupted continuation of the problem solution not performed along this part, i.e. we have a crack on which the tangential stress acts.

Many researchers have studied the different problems caused by the cracks existing (made on purpose) or originated in an elliptic body. In [1-3], problems of ultimate equilibrium are solved in closed form for a brittle plate weakened by an elliptic hole with one or two small linear cracks located at the ends of the hole. In [4-6] stress intensity factors are considered for cracks emanating from elliptic holes

*Email: natzira@yahoo.com

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in finite or infinite plates. In [7] the stress intensity factor along the crack front of an elliptical crack in a rotating shaft was studied. In [8] considered medium is composed of an elliptic inclusion and many confocal elliptic layers. The crack is embedded in the elliptic inclusion. The author's earlier works [9-11] deal with the question whether cracks can be helpful in strengthening structures. For example, when building underground structures, tunnels in particular, engineers intentionally make so-called technical openings in the tunnel walls in order to decrease the stress concentration and fortify the walls using various techniques. In [9] the author investigates how the number of cracks and their lengths influence the stress distribution in the tunnel walls, i.e., how the tangential stress concentration on the circular hole contour can be diminished by varying the number of cracks and their lengths. In [10] the author investigates how the tangential stress concentration can be diminished on the contour of an elliptic hole (except the crack ends) by varying the number of cracks and their lengths.

The present work considers the deflected mode of a homogeneous isotropic ellipse when a) the body is weakened with an internal linear crack, in particular, the crack is between focuses F_1 and F_2 (See Fig. 1), which is affected by a tangential stress and which depends on ξ_1 and η, and b) the elliptic body has no cracks. In both cases, tangential stress depending on η acts on the boundary $\xi = \xi_1$.

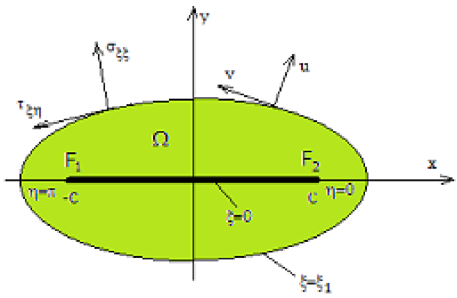


Figure 1. Ellipse with cracks between the focuses F_1 and F_2.

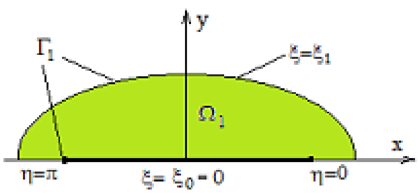


Figure 2. Semi-ellipse

The mathematical model of this problem is built by means of the system of elliptic coordinates ξ, η. The problem is obtained from the relevant problem for the semi-ellipse $\bar{\Omega}_1 = \Omega_1 \bigcup \Gamma_1 = \{\xi_0 = 0 \leq \xi \leq \xi_1, 0 \leq \eta \leq \pi\}$ (See Fig. 2), when at $\eta = 0$ and $\eta = \pi$ the continuity conditions of the problem solution are given and at $\xi = 0$ tangential stress is given, and the continuity condition of the solution is not met along this section. The problem is solved with the method reducing the complex problems of the theory of elasticity to the solution of simple problems [12-14], in particular, to the solution of the internal and external problems of the theory of elasticity simply solved by the method of separation of variables (MSV) [15, 16]. By using MATLAB software, we have obtained the numerical values and drafted 3D and 2D graphs of displacements and distribution of stresses in the body, when there is a) a tangential stress, and b) continuity conditions of the problem solution, symmetry conditions in particular, given at $\xi = 0$.

2. Setting problems in the elliptic coordinates

2.1. *Equilibrium equations, physical law*

Let us describe the equilibrium equations in the elliptic coordinate system ξ, η as follows:

$$\begin{aligned} &\text{a)}\ D_{,\xi} - K_{,\eta} = 0, \qquad &&\text{c)}\ \bar{u}_{,\xi} + \bar{v}_{,\eta} = \tfrac{\kappa-2}{\kappa\mu} h_0^2 D, \\ &\text{b)}\ D_{,\eta} + K_{,\xi} = 0, \qquad &&\text{d)}\ \bar{v}_{,\xi} - \bar{u}_{,\eta} = \tfrac{1}{\mu} h_0^2 K, \end{aligned} \tag{1}$$

where $\kappa = 4\left(1-\nu\right),\ \ \mu = \frac{E}{2(1-\nu)},\ h_0 = \sqrt{\cosh\left(2\xi\right) - \cos\left(2\eta\right)},\ \bar{u} = \frac{2hu}{c^2},\ \bar{v} = \frac{2v}{c^2};\ u$ and v are the components of the displacement vector along $\xi = const$ line normal and tangent, $h_\xi = h_\eta = h = \frac{c}{\sqrt{2}}\sqrt{\cosh\left(2\xi\right) - \cos\left(2\eta\right)}$ are metric coefficients, $\frac{\kappa-2}{\kappa\mu} D$ is the divergence of the displacement vector, $\frac{1}{\mu} K$ is the rotor of the displacement vector, ν is Poisson's ratio and E is the modulus of elasticity.

Hooke's Law will be described as follows:

$$\begin{aligned} \frac{h_0^2}{\mu}\sigma_{\xi\xi} &= \frac{h_0^2}{\mu} D - 2\bar{v}_{,\eta} - \frac{2}{h_0^2}\left[\sinh\left(2\xi\right)\bar{u} - \sin\left(2\eta\right)\bar{v}\right], \\ \frac{h_0^2}{\mu}\sigma_{\eta\eta} &= \frac{h_0^2}{\mu} D - 2\bar{u}_{,\xi} + \frac{2}{h_0^2}\left[\sinh\left(2\xi\right)\bar{u} - \sin\left(2\eta\right)\bar{v}\right], \\ \frac{h_0^2}{\mu}\tau_{\xi\eta} &= \frac{h_0^2}{\mu} K + 2\bar{u}_{,\eta} - \frac{2}{h_0^2}\left[\sin\left(2\eta\right)\bar{u} + \sinh\left(2\xi\right)\bar{v}\right]. \end{aligned} \tag{2}$$

2.2. *Boundary conditions*

Let us set the boundary problem for the semi-ellipse, i.e. let us find the solution of system of equations (1) ($\bar{u}, \bar{v} \in C^2(\Omega)$, $D, K \in C^1(\Omega)$ (Fig. 2) in area $\Omega = \{0 \le \xi < \xi_1, 0 \le \eta < 2\pi\}$ (See Fig. 1), which meets the following boundary conditions:

1) for ellipse with crack

$$\begin{aligned} \eta = 0:&\ \ \bar{v} = 0,\ \ \bar{u}_{,\eta} = 0 \ \ \text{ or }\ \ \bar{u} = 0,\ \ \bar{v}_{,\eta} = 0, \\ \eta = \pi:&\ \ \bar{v} = 0,\ \ \bar{u}_{,\eta} = 0 \ \ \text{ or }\ \ \bar{u} = 0,\ \ \bar{v}_{,\eta} = 0, \\ \xi = 0:&\ \ \frac{h_0^2}{\mu}\sigma_{\xi\xi} = f_1\left(\eta\right),\ \ \frac{h_0^2}{\mu}\tau_{\xi\eta} = f_2\left(\eta\right)\ \ \text{ or }\ \ \bar{u} = \phi_1\left(\eta\right),\ \ \bar{v} = \phi_2\left(\eta\right), \\ \xi = \xi_1:&\ \ \frac{h_0^2}{\mu}\sigma_{\xi\xi} = f_3\left(\eta\right),\ \ \frac{h_0^2}{\mu}\tau_{\xi\eta} = f_4\left(\eta\right)\ \ \text{ or }\ \ \bar{u} = \phi_3\left(\eta\right),\ \ \bar{v} = \phi_4\left(\eta\right); \end{aligned}$$

2) for the whole ellipse

$$\begin{aligned} \eta = 0:&\ \ \bar{v} = 0,\ \ \bar{u}_{,\eta} = 0 \ \ \text{ or }\ \ \bar{u} = 0,\ \ \bar{v}_{,\eta} = 0, \\ \eta = \pi:&\ \ \bar{v} = 0,\ \ \bar{u}_{,\eta} = 0 \ \ \text{ or }\ \ \bar{u} = 0,\ \ \bar{v}_{,\eta} = 0, \end{aligned}$$

$$\xi = 0: \quad \bar{u} = 0, \quad \bar{v}_{,\xi} = 0 \ \text{ or } \ \bar{v} = 0, \quad \bar{u}_{,\xi} = 0,$$

$$\xi = \xi_1: \quad \frac{h_0^2}{\mu}\sigma_{\xi\xi} = f_3(\eta), \quad \frac{h_0^2}{\mu}\tau_{\xi\eta} = f_4(\eta) \quad \text{or} \quad \bar{u} = \phi_3(\eta), \quad \bar{v} = \phi_4(\eta).$$

3. Setting and solving concrete problem for ellipse with crack

Let us consider the case when $f_1(\eta) = 0$, $f_2(\eta) = \sin(2\eta)\sinh(2\xi_1)P$, $f_3(\eta) = 0$, $f_2(\eta) = \sin(2\eta)P$ i.e.

$$\eta = 0: \quad \bar{v} = 0, \quad \bar{u}_{,\eta} = 0, \tag{3}$$

$$\eta = \pi: \quad \bar{v} = 0, \quad \bar{u}_{,\eta} = 0, \tag{4}$$

$$\xi = 0: \quad \frac{h_0^2}{\mu}\sigma_{\xi\xi} = 0, \quad \frac{h_0^2}{\mu}\tau_{\xi\eta} = \sin(2\eta)\sinh(2\xi_1)P, \tag{5}$$

$$\xi = \xi_1: \quad \frac{h_0^2}{\mu}\sigma_{\xi\xi} = 0, \quad \frac{h_0^2}{\mu}\tau_{\xi\eta} = \sin(2\eta)P, \tag{6}$$

where P is an arbitrary real number.

The solution of problem (1)-(6) is reduced to the solution of the two following problems:

3.1. *The first problem*

The first problem (external problem) is considered in area $\Omega_2 = \{0 \le \xi < \infty, \ 0 \le \eta \le \pi\}$ given in Fig. 3 with the following boundary conditions:

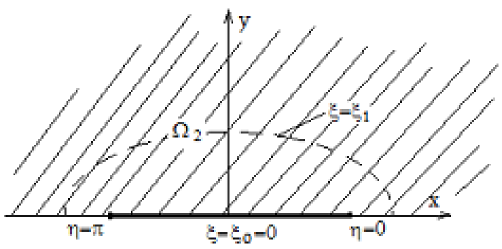


Figure 3. Infinite area with the first problem

$$\eta = 0: \quad \bar{v} = 0, \quad \bar{u}_{,\eta} = 0, \tag{7}$$

$$\eta = \pi: \quad \bar{v} = 0, \quad \bar{u}_{,\eta} = 0, \tag{8}$$

$$\xi = 0: \quad \frac{h_0^2}{\mu}\sigma_{\xi\xi} = 0, \quad \frac{h_0^2}{\mu}\tau_{\xi\eta} = \sin(2\eta)\sinh(2\xi_1)P. \tag{9}$$

The solution of this problem will be described as follows:

$$
\begin{aligned}
\bar{u} &= -\left(\varphi_{1,\eta} - \varphi_{2,\xi}\right)\sinh\xi\sin\eta \\
&\quad + \left[\left(\varphi_{3,\xi} - (\kappa-1)\,\varphi_1\right)\sinh\xi\cos\eta + \left(\varphi_{3,\eta} - (\kappa-1)\,\varphi_2\right)\cosh\xi\sin\eta\right], \\
\bar{v} &= \left(\varphi_{1,\xi} + \varphi_{2,\eta}\right)\sinh\xi\sin\eta(10) \\
&\quad - \left[\left(\varphi_{3,\xi} - (\kappa-1)\,\varphi_1\right)\cosh\xi\sin\eta - \left(\varphi_{3,\eta} - (\kappa-1)\,\varphi_2\right)\sinh\xi\cos\eta\right];
\end{aligned}
\tag{10}
$$

$$
D = \frac{\kappa\mu}{\cosh(2\xi) - \cos(2\eta)}\left[\left(\varphi_{1,\eta} - \varphi_{2,\xi}\right)\cosh\xi\sin\eta - \left(\varphi_{1,\xi} + \varphi_{2,\eta}\right)\sinh\xi\cos\eta\right],
$$

$$
K = \frac{\kappa\mu}{\cosh(2\xi) - \cos(2\eta)}\left[\left(\varphi_{1,\eta} - \varphi_{2,\xi}\right)\sinh\xi\cos\eta + \left(\varphi_{1,\xi} + \varphi_{2,\eta}\right)\cosh\xi\sin\eta\right].
$$

The components of stress tensors are presented with the following formulae:

$$
\begin{aligned}
&\frac{h_0^2}{\mu}\sigma_{\xi\xi} = -\left(\kappa\varphi_{1,\xi} - (\kappa-2)\,\varphi_{2,\eta} + 2\varphi_{3,\eta\eta}\right)\sinh\xi\cos\eta \\
&-\left((\kappa-2)\,\varphi_{1,\eta} + \kappa\varphi_{2,\xi} - 2\varphi_{3,\xi\eta}\right)\cosh\xi\sin\eta - 2\left(\varphi_{1,\xi\eta} + \varphi_{2,\eta\eta}\right)\sinh\xi\sin\eta \\
&-\frac{4\sinh^2\xi}{\cosh(2\xi) - \cos(2\eta)}\left\{\left(\varphi_{1,\xi} + \varphi_{2,\eta}\right)\sinh\xi\cos\eta - \left(\varphi_{1,\eta} - \varphi_{2,\xi}\right)\cosh\xi\sin\eta\right\},
\end{aligned}
$$

$$
\begin{aligned}
&\frac{h_0^2}{\mu}\sigma_{\eta\eta} = \left((\kappa-2)\,\varphi_{1,\xi} - \kappa\varphi_{2,\eta} + 2\varphi_{3,\eta\eta}\right)\sinh\xi\cos\eta \\
&+\left(\kappa\varphi_{1,\eta} + (\kappa-2)\,\varphi_{2,\xi} - 2\varphi_{3,\xi\eta}\right)\cosh\xi\sin\eta + 2\left(\varphi_{1,\xi\eta} + \varphi_{3,\eta\eta}\right)\sinh\xi\sin\eta \\
&+\frac{4\sinh^2\xi}{\cosh(2\xi) - \cos(2\eta)}\left\{\left(\varphi_{1,\xi} + \varphi_{2,\eta}\right)\sinh\xi\cos\eta - \left(\varphi_{1,\eta} - \varphi_{2,\xi}\right)\cosh\xi\sin\eta\right\},
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
&\frac{h_0^2}{\mu}\tau_{\xi\eta} = \left(\kappa\varphi_{1,\xi} - (\kappa-2)\,\varphi_{2,\eta} + 2\varphi_{3,\eta\eta}\right)\cosh\xi\sin\eta \\
&-\left((\kappa-2)\,\varphi_{1,\eta} + \kappa\varphi_{2,\xi} - 2\varphi_{3,\xi\eta}\right)\sinh\xi\cos\eta - 2\left(\varphi_{1,\eta\eta} - \varphi_{2,\xi\eta}\right)\sinh\xi\sin\eta \\
&-\frac{4\sinh^2\xi}{\cosh(2\xi) - \cos(2\eta)}\left\{\left(\varphi_{1,\xi} + \varphi_{2,\eta}\right)\cosh\xi\sin\eta - \left(\varphi_{1,\eta} - \varphi_{2,\xi}\right)\sinh\xi\cos\eta\right\},
\end{aligned}
$$

where φ_i, $i = 1, 2, 3$ are harmonic functions. From boundary conditions (7) and (8), we obtain:

$$\varphi_1=\sum_{n=1}^{\infty}A_{1n}e^{-n\xi}\cos(n\eta),\ \varphi_2=\sum_{n=1}^{\infty}A_{2n}e^{-n\xi}\sin(n\eta),$$
$$\varphi_3=\sum_{n=1}^{\infty}A_{2n}\frac{\kappa-2}{2n}e^{-n\xi}\cos(n\eta). \tag{12}$$

It is purposeful to substitute boundary conditions (9) with the following equivalent conditions:

$$-\frac{2}{\kappa\mu}\left(\sigma_{\xi\xi}\sinh\xi_0\cos\eta-\tau_{\xi\eta}\cosh\xi_0\sin\eta\right)=\varphi_{1,\xi},$$
$$\frac{\kappa-2}{2}\varphi_{1,\eta}-\frac{1}{\mu}\left(\sigma_{\xi\xi}\cosh\xi_0\sin\eta+\tau_{\xi\eta}\sinh\xi_0\cos\eta\right)=\varphi_{2,\xi},$$

i.e. conditions (9) will be described as follows:

$$\kappa\varphi_{1,\xi}=P\sinh(2\xi_1)\sin\eta\sin(2\eta),$$
$$(\kappa-2)\,\varphi_{1,\mu}-2\varphi_{2,\xi}=0. \tag{13}$$

From (12), (13), we obtain the following system of equations:

$$\sum_{n=1}^{\infty}-\kappa ne^{-n\xi}A_{1n}\cos(n\eta)=P\sinh(2\xi_1)\sin\eta\sin(2\eta),$$
$$\sum_{n=1}^{\infty}\left[-(\kappa-2)\,nA_{1n}+2nA_{2n}\right]e^{-n\xi}\sin(n\eta)=0,$$

that is:

$$\sum_{n=1}^{\infty}-\kappa ne^{-n\xi}A_{1n}\cos(n\eta)=\frac{1}{2}P\sinh(2\xi_1)\left[\cos\eta-\cos(3\eta)\right],$$
$$\sum_{n=1}^{\infty}\left[-(\kappa-2)\,nA_{1n}+2nA_{2n}\right]e^{-n\xi}\sin(n\eta)=0. \tag{14}$$

From (14), the following system is obtained:

$$\kappa e^{-\xi}A_{11}=-\frac{1}{2}P\sinh(2\xi_1),\qquad((\kappa-2)\,A_{11}-2A_{21})\,e^{-\xi}=0,$$
$$3\kappa e^{-3\xi}A_{13}=\frac{1}{2}P\sinh(2\xi_1),\qquad((\kappa-2)\,A_{13}-2A_{23})\,e^{-3\xi}=0. \tag{15}$$

The solution of system (15) is as follows:

$$
\begin{aligned}
A_{11} &= -\frac{P}{2\kappa} e^{\xi} \sinh\left(2\xi_1\right), \qquad A_{21} = -\frac{\kappa - 2}{4\kappa} P e^{\xi} \sinh\left(2\xi_1\right), \\
A_{13} &= \frac{P}{6\kappa} e^{3\xi} \sinh\left(2\xi_1\right), \qquad A_{23} = \frac{\kappa - 2}{12\kappa} P e^{3\xi} \sinh\left(2\xi_1\right).
\end{aligned} \tag{16}
$$

By substituting the expressions (12) in expressions (10) and (11), one obtains the components of the displacement vectors and stress tensor of form:

$$
\begin{aligned}
\bar{u} &= e^{-\xi}\left\{\left(A_{11} - A_{21}\right)\sinh\xi\sin^2\eta - \left[\left(\kappa - 1\right)A_{11} - \frac{\kappa - 2}{2}A_{21}\right]\sinh\xi\cos^2\eta \right. \\
&\left. -\frac{3\kappa - 4}{2}A_{21}\cosh\xi\sin^2\eta\right\} + e^{-3\xi}\left\{3\left(A_{13} - A_{23}\right)\sinh\xi\sin\eta\sin\left(3\eta\right)\right. \\
&-\left[\left(\kappa - 1\right)A_{13} - \frac{\kappa - 2}{2}A_{23}\right]\sinh\xi\cos\eta\cos\left(3\eta\right) \\
&\left. -\frac{3\kappa - 4}{2}A_{23}\cosh\xi\sin\eta\sin\left(3\eta\right)\right\}, \\
\bar{v} &= e^{-\xi}\left\{-\left(A_{11} - A_{21}\right)\sinh\xi\sin\eta\cos\eta + \left[\left(\kappa - 1\right)A_{11}\cosh\xi + \right.\right. \\
&\left.\left.\left(\frac{\kappa - 2}{2}\cosh\xi - \frac{3\kappa - 4}{2}\sinh\xi\right)A_{21}\right]\sin\eta\cos\eta\right\} \\
&+e^{-3\xi}\left\{-3\left(A_{13} - A_{23}\right)\sinh\xi\sin\eta\cos\left(3\eta\right) + \left[\left(\kappa - 1\right)A_{13} + \frac{\kappa - 2}{2}A_{23}\right]\right. \\
&\left.\times\ \cosh\xi\sin\eta\cos\eta - \frac{3\kappa - 4}{2}A_{23}\sinh\xi\cos\eta\sin\left(3\eta\right)\right\}.
\end{aligned} \tag{17}
$$

$$
\begin{aligned}
\frac{h_0^2}{\mu}\sigma_{\xi\xi} &= e^{-\xi}\left\{\left[\kappa A_{11} + 2\left(\kappa - 2\right)A_{21}\right]\sinh\xi\cos^2\eta\right. \\
&+\left[\left(\kappa - 2\right)A_{11} + 2\left(\kappa - 1\right)A_{21}\right]\cosh\xi\sin^2\eta - 2\left(A_{11} - A_{21}\right)\sin\xi\sin^2\eta \\
&\left.+\frac{4\sinh^2\xi\left(A_{11} - A_{21}\right)}{\cosh\left(2\xi\right) - \cos\left(2\eta\right)}\left[\sinh\xi\cos^2\eta - \cosh\xi\sin^2\eta\right]\right\} \\
&+3e^{-3\xi}\left\{\left[\kappa A_{13} + 2\left(\kappa - 2\right)A_{23}\right]\sinh\xi\cos\eta\cos\left(3\eta\right) + \left[\left(\kappa - 2\right)A_{13}\right.\right. \\
&\left.+2\left(\kappa - 1\right)A_{23}\right]\cosh\xi\sin\eta\sin\left(3\eta\right) - 6\left(\mathrm{A}_{13} - A_{23}\right)\sin\xi\sin\eta\sin\left(3\eta\right) \\
&\left.+\frac{4\sinh^2\xi\left(A_{13} - A_{23}\right)}{\cosh\left(2\xi\right) - \cos\left(2\eta\right)}\left[\sinh\xi\cos\eta\cos\left(3\eta\right) - \cosh\xi\sin\eta\sin\left(3\eta\right)\right]\right\},
\end{aligned}
$$

$$\frac{h_0^2}{\mu}\sigma_{\eta\eta} = e^{-\xi}\left\{\left[-\left(\kappa-2\right)A_{11} - 2\left(\kappa-1\right)A_{21}\right]\sinh\xi\cos^2\eta\right.$$
$$+\left[\kappa A_{11} - 2\left(\kappa-2\right)A_{21}\right]\cosh\xi\sin^2\eta + 2\left(\mathrm{A}_{11} - A_{21}\right)\sin\xi\sin^2\eta$$
$$\left.-\frac{4\sinh^2\xi\left(A_{11}-A_{21}\right)}{\cosh\left(2\xi\right)-\cos\left(2\eta\right)}\left[\sinh\xi\cos^2\eta - \cosh\xi\sin^2\eta\right]\right\}$$
$$+3e^{-3\xi}\left\{\left[-\left(\kappa-2\right)A_{13} - 2\left(\kappa-1\right)A_{23}\right]\sinh\xi\cos\eta\cos\left(3\eta\right) - \left[\kappa A_{13}\right.\right.$$
$$\left.+2\left(\kappa-2\right)A_{23}\right]\cosh\xi\sin\eta\sin\left(3\eta\right) + 6\left(A_{13}-A_{23}\right)\sinh\xi\sin\eta\sin\left(3\eta\right)$$
$$\left.-\frac{4\sinh^2\xi\left(A_{13}-A_{23}\right)}{\cosh\left(2\xi\right)-\cos\left(2\eta\right)}\left[\sinh\xi\cos\eta\cos\left(3\eta\right) - \cosh\xi\sin\eta\sin\left(3\eta\right)\right]\right\},$$
$$\frac{h_0^2}{\mu}\tau_{\xi\eta} = e^{-\xi}\left\{-\left[\kappa A_{11} + 2\left(\kappa-2\right)A_{21}\right]\cosh\xi\sin\eta\cos\eta\right.$$
$$+\left[\kappa A_{11} + 2\left(\kappa-1\right)A_{21}\right]\sinh\xi\sin\eta\cos\eta$$
$$\left.+\frac{4\sinh^2\xi\left(A_{11}-A_{21}\right)}{\cosh\left(2\xi\right)-\cos\left(2\eta\right)}\left[\cosh\xi\sin\eta\cos\eta + \sinh\xi\sin\eta\cos\eta\right]\right\}$$
$$+3e^{-3\xi}\left\{-\left[\kappa A_{13} + 2\left(\kappa-2\right)A_{23}\right]\cosh\xi\sin\eta\cos\left(3\eta\right)\right.$$
$$+\left[\kappa A_{13} + \left(\kappa-2\right)A_{23}\right]\sinh\xi\sin\left(3\eta\right)\cos\eta + 6\left(A_{13}-A_{23}\right)\sinh\xi\sin\eta\cos\left(3\eta\right)$$
$$\left.+\frac{4\sinh^2\xi\left(A_{13}-A_{23}\right)}{\cosh\left(2\xi\right)-\cos\left(2\eta\right)}\left[\cosh\xi\sin\eta\cos\left(3\eta\right) + \sinh\xi\sin\left(3\eta\right)\cos\eta\right]\right\}. \tag{18}$$

From (16), (17), (18), the numerical values of the displacement vector and stress tensor components are obtained at any point of the body.

3.2. *The second problem*

The second problem (internal problem) is considered on the area $\Omega_3 = \{0 \le \xi \le \xi_1,\ \ 0 \le \eta \le \pi\}$ given in Fig. 4 with the following boundary conditions:

$$\eta = 0: \quad \bar{v} = 0, \quad \bar{u}_{,\eta} = 0, \tag{19}$$
$$\eta = \pi: \quad \bar{v} = 0, \quad \bar{u}_{,\eta} = 0, \tag{20}$$
$$\xi = 0: \quad \bar{u} = 0, \ \bar{v}_{,\xi} = 0, \tag{21}$$
$$\xi = \xi_1: \quad \frac{2}{\mu}\sigma_{\xi\xi} = 0 - \tilde{g}_{11}\left(\eta\right), \quad \frac{2}{\mu}\tau_{\xi\eta} = \sin\left(2\eta\right)P - \tilde{g}_{12}\left(\eta\right). \tag{22}$$

Here $\tilde{g}_{11}(\eta) = \frac{2}{\mu}\sigma_{\xi\xi}\Big|_{\xi=\xi_1}$, $\tilde{g}_{12}(\eta) = \frac{2}{\mu}\tau_{\xi\eta}\Big|_{\xi=\xi_1}$, where $\sigma_{\xi\xi}$ and $\tau_{\xi\eta}$ are the solution of **the first problem** (1), (2), (7)-(9). Continuity conditions of the problem solution, symmetry conditi

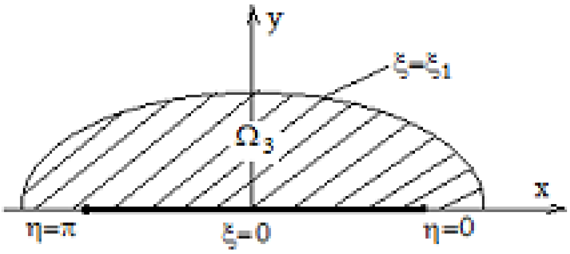


Figure 4. Area with the second problem.

The solution of this problem is presented with harmonic functions φ_1 and φ_2:

$$\begin{aligned} \bar{u} &= \left[\sinh^2\xi_1\cosh\xi\left(\varphi_{1,\eta}+\varphi_{2,\xi}\right)+(\kappa-1)\sinh\xi\varphi_2\right]\cos\eta \\ &\quad -\left[\cosh^2\xi_1\sinh\xi\left(\varphi_{1,\xi}-\varphi_{2,\eta}\right)-(\kappa-1)\cosh\xi\varphi_1\right]\sin\eta, \\ \bar{v} &= -\left[\cosh^2\xi_1\sinh\xi\left(\varphi_{1,\eta}+\varphi_{2,\xi}\right)+(\kappa-1)\cosh\xi\varphi_2\right]\sin\eta \\ &\quad -\left[\sinh^2\xi_1\cosh\xi\left(\varphi_{1,\xi}-\varphi_{2,\eta}\right)-(\kappa-1)\sinh\xi\varphi_1\right]\cos\eta. \end{aligned} \tag{23}$$

$$D = \frac{\kappa\mu}{\cosh(2\xi)-\cos(2\eta)}\left[\left(\varphi_{1,\xi}-\varphi_{2,\eta}\right)\cosh\xi\sin\eta+\left(\varphi_{1,\eta}+\varphi_{2,\xi}\right)\sinh\xi\cos\eta\right],$$

$$K = \frac{\kappa\mu}{\cosh(2\xi)-\cos(2\eta)}\left[\left(\varphi_{1,\xi}-\varphi_{2,\eta}\right)\sinh\xi\cos\eta-\left(\varphi_{1,\eta}+\varphi_{2,\xi}\right)\cosh\xi\sin\eta\right].$$

By substituting the expressions (23) for the displacement vector and the expressions for the divergence D and the rotor component K in Hooke's law (2), one obtains the components of the stress tensor of form:

$$\frac{h_0^2}{\mu}\sigma_{\xi\xi} = \left[2\sinh^2\xi_1\left(\varphi_{1,\xi\eta}-\varphi_{2,\eta\eta}\right)\cosh\xi-\left((\kappa-2)\varphi_{1,\eta}-\kappa\varphi_{2,\xi}\right)\sinh\xi\right]\cos\eta$$

$$+\left[2\cosh^2\xi_1\left(\varphi_{1,\eta\eta}+\varphi_{2,\xi\eta}\right)\sinh\xi+\left(\kappa\varphi_{1,\xi}+(\kappa-2)\varphi_{2,\eta}\right)\cosh\xi\right]\sin\eta$$

$$-\frac{4\sinh(\xi_1+\xi)\sinh(\xi_1-\xi)}{\cosh(2\xi)-\cos(2\eta)}\left\{\left(\varphi_{1,\xi}-\varphi_{2,\eta}\right)\cosh\xi\sin\eta+\left(\varphi_{1,\eta}+\varphi_{2,\xi}\right)\sinh\xi\cos\eta\right\},$$

$$\frac{h_0^2}{\mu}\sigma_{\eta\eta} = -\left[2\sinh^2\xi_1\left(\varphi_{1,\xi\eta}-\varphi_{2,\eta\eta}\right)\cosh\xi-\left(\kappa\varphi_{1,\eta}-(\kappa-2)\varphi_{2,\xi}\right)\sinh\xi\right]\cos\eta$$

$$-\left[2\cosh^2\xi_1\left(\varphi_{1,\eta\eta}+\varphi_{2,\xi\eta}\right)\sinh\xi+\left((\kappa-2)\varphi_{1,\xi}+\kappa\varphi_{2,\eta}\right)\cosh\xi\right]\sin\eta$$

$$+\frac{4\sinh(\xi_1+\xi)\sinh(\xi_1-\xi)}{\cosh(2\xi)-\cos(2\eta)}\left\{\left(\varphi_{1,\xi}-\varphi_{2,\eta}\right)\cosh\xi\sin\eta+\left(\varphi_{1,\eta}+\varphi_{2,\xi}\right)\sinh\xi\cos\eta\right\},$$

$$\frac{h_0^2}{\mu}\tau_{\xi\eta} = -\left[2\cosh^2\xi_1\left(\varphi_{1,\xi\eta}-\varphi_{2,\eta\eta}\right)\sinh\xi - \left(\left(\kappa-2\right)\frac{\partial\varphi_1}{\partial\eta}-\kappa\frac{\partial\varphi_2}{\partial\xi}\right)\cosh\xi\right]\sin\eta$$

$$+\left[2\sinh^2\xi_1\cosh\xi\left(\varphi_{1,\eta\eta}+\varphi_{2,\xi\eta}\right)+\left(\kappa\frac{\partial\varphi_1}{\partial\xi}+\left(\kappa-2\right)\frac{\partial\varphi_2}{\partial\eta}\right)\sinh\xi\right]\cos\eta \tag{24}$$

$$+\frac{4\sinh\left(\xi_1+\xi\right)\sinh\left(\xi_1-\xi\right)}{\cosh\left(2\xi\right)-\cos\left(2\eta\right)}\left\{\left(\varphi_{1,\xi}-\varphi_{2,\eta}\right)\sinh\xi\cos\eta-\left(\varphi_{1,\eta}+\varphi_{2,\xi}\right)\cosh\xi\sin\eta\right\}.$$

By employing the MSV and taking into account the boundary conditions (19), (20), (21) we can write harmonic functions φ_1, φ_2 in the following form:

$$\varphi_1=\sum_{n=1}^{\infty}B_{1n}\frac{\sinh\left(n\xi\right)}{\cosh\left(n\xi_1\right)}\sin\left(n\eta\right), \qquad \varphi_2=\sum_{n=1}^{\infty}B_{2n}\frac{\cosh\left(n\xi\right)}{\cosh\left(n\xi_1\right)}\cos\left(n\eta\right). \tag{25}$$

It is purposeful to substitute boundary conditions (22) with the following equivalent conditions:

$$\begin{aligned}
&\frac{2}{\mu}\left(\cosh\xi_1\sin\eta\sigma_{\xi\xi}+\sinh\xi_1\cos\eta\tau_{\xi\eta}\right)\\
&=\sinh\left(2\xi_1\right)\left(\varphi_{1,\eta\eta}+\varphi_{2,\xi\eta}\right)+\kappa\varphi_{1,\xi}+\left(\kappa-2\right)\varphi_{2,\eta},\\
&\frac{2}{\mu}\left(\sinh\xi_1\cos\eta\sigma_{\xi\xi}-\cosh\xi_1\sin\eta\tau_{\xi\eta}\right)\\
&=\sinh\left(2\xi_1\right)\left(\varphi_{1,\xi\eta}-\varphi_{2,\eta\eta}\right)-\left(\kappa-2\right)\varphi_{1,\eta}+\kappa\varphi_{2,\xi},
\end{aligned}$$

i.e. conditions (22) will be described as follows:

$$\begin{aligned}
&\sinh\left(2\xi_1\right)\left(\varphi_{1,\eta\eta}+\varphi_{2,\xi\eta}\right)+\kappa\varphi_{1,\xi}+\left(\kappa-2\right)\varphi_{2,\eta}\\
&=\left(-\cosh\xi_1\sin\eta\tilde{g}_{11}+\sinh\xi_1\cos\eta\left(\sin\left(2\eta\right)P-\tilde{g}_{12}\right)\right),\\
&\sinh\left(2\xi_1\right)\left(\varphi_{1,\xi\eta}-\varphi_{2,\eta\eta}\right)-\left(\kappa-2\right)\varphi_{1,\eta}+\kappa\varphi_{2,\xi}\\
&=\left(-\sinh\xi_1\cos\eta\tilde{g}_{11}-\cosh\xi_1\sin\eta\left(\sin\left(2\eta\right)P-\tilde{g}_{12}\right)\right),
\end{aligned} \tag{26}$$

By inserting (25) in (26), we obtain:

$$\sum_{n=1}^{\infty}\left[-n^2\sinh(2\xi_1)\frac{\sinh(n\xi)}{\cosh(n\xi_1)}(B_{1n}+B_{2n})\right.$$
$$\left.+n\frac{\cosh(n\xi)}{\cosh(n\xi_1)}(\kappa B_{1n}-(\kappa-2)B_{2n})\right]\sin(n\eta)=\sum_{n=1}^{\infty}\tilde{F}_{1n}\sin(n\eta),$$
$$\sum_{n=1}^{\infty}\left[n^2\sinh(2\xi_1)\frac{\cosh(n\xi)}{\cosh(n\xi_1)}(B_{1n}+B_{2n})\right. \tag{27}$$
$$\left.-n\frac{\sinh(n\xi)}{\cosh(n\xi_1)}((\kappa-2)B_{1n}-\kappa B_{2n})\right]\cos(n\eta)=\sum_{n=1}^{\infty}\tilde{F}_{2n}\cos(n\eta),$$

where $\tilde{F}_{1n}=\frac{2}{\pi}\int_0^{\pi}F_1(\eta)\sin(n\eta)\,d\eta$ and $\tilde{F}_{2n}=\frac{2}{\pi}\int_0^{\pi}F_2(\eta)\cos(n\eta)\,d\eta$ are the coefficients of expansion of functions:

$$F_1(\eta)=-\cosh\xi_1\sin\eta\tilde{g}_{11}+\sinh\xi_1\cos\eta\,(P\sin(2\eta)-\tilde{g}_{11}(\eta))$$

and

$$F_2=-\sinh\xi_1\cos\eta\tilde{g}_{11}-\cosh\xi_1\sin\eta\,(P\sin(2\eta)-\tilde{g}_{12}(\eta))$$

into Fourier series ($F_1(\eta)$ - according to sinuses and $F_2(\eta)$ - according to cosines), respectively.

After equating the expressions at the same trigonometric functions in both sides of equations in (27), we obtain an infinite system of linear algebraic equations to unknown quantities B_{1n} and B_{2n}.

$$\left[-n^2\sinh(2\xi_1)\frac{\sinh(n\xi)}{\cosh(n\xi_1)}+n\kappa\frac{\cosh(n\xi)}{\cosh(n\xi_1)}\right]B_{1n}-\left[n^2\sinh(2\xi_1)\frac{\sinh(n\xi)}{\cosh(n\xi_1)}\right.$$
$$\left.+n(\kappa-2)\frac{\cosh(n\xi)}{\cosh(n\xi_1)}\right]B_{2n}=\tilde{F}_{1n},$$
$$\left[n^2\sinh(2\xi_1)\frac{\cosh(n\xi)}{\cosh(n\xi_1)}-n(\kappa-2)\frac{\sinh(n\xi)}{\cosh(n\xi_1)}\right]B_{1n} \tag{28}$$
$$+\left[n^2\sinh(2\xi_1)\frac{\cosh(n\xi)}{\cosh(n\xi_1)}+n\kappa\frac{\cosh(n\xi)}{\cosh(n\xi_1)}\right]B_{2n}=\tilde{F}_{2n},\qquad n=1,2,\ldots.$$

As one can see, the leading matrix of system (28) is a block-diagonal one (See Fig. 5).

The dimension of each block $D_i,\ \ i=1,2,\ldots$ is 2×2 and $\det D_i\neq 0$, while when $i\to\infty$, $\det D_i\to M$, where $M\neq 0$ is the finite number.

Let us find B_{1n}, B_{2n} from (28), and we will obtain the following expressions of

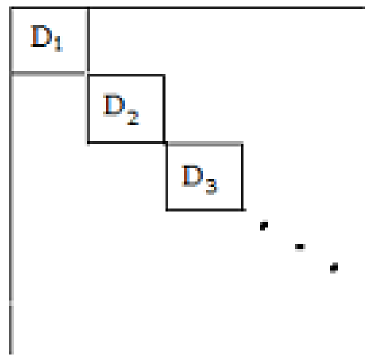


Figure 5. Image of the leading matrix

the displacement vector and stress tensor components from (23)-(25):

$$\bar{u} = \sum_{n=1}^{\infty} \left\{ \left[n \sinh^2 \xi_1 \cosh \xi \frac{\sinh (n\xi)}{\cosh (n\xi_1)} (B_{1n} + B_{2n}) \right.\right.$$
$$\left. + (\kappa - 1) \sinh \xi \frac{\cosh (n\xi)}{\cosh (n\xi_1)} B_{2n} \right] \cos \eta \cos (n\eta) - \left[n \cosh^2 \xi_1 \sinh \xi \frac{\cosh (n\xi)}{\cosh (n\xi_1)} \right.$$
$$\left.\left. \times (B_{1n} + B_{2n}) - (\kappa - 1) \cosh \xi \frac{\sinh (n\xi)}{\cosh (n\xi_1)} B_{1n} \right] \sin \eta \sin (n\eta) \right\},$$
$$\bar{v} = \sum_{n=1}^{\infty} \left\{ - \left[n \cosh^2 \xi_1 \sinh \xi \frac{\sinh (n\xi)}{\cosh (n\xi_1)} (B_{1n} + B_{2n}) \right.\right. \tag{29}$$
$$\left. + (\kappa - 1) \cosh \xi \frac{\cosh (n\xi)}{\cosh (n\xi_1)} B_{2n} \right] \sin \eta \cos (n\eta)$$
$$- \left[n \sinh^2 \xi_1 \cosh \xi \frac{\cosh (n\xi)}{\cosh (n\xi_1)} (B_{1n} + B_{2n}) \right.$$
$$\left.\left. - (\kappa - 1) \sinh \xi \frac{\sinh (n\xi)}{\cosh (n\xi_1)} B_{1n} \right] \cos \eta \sin (n\eta) \right\}.$$

$$\frac{h_0^2}{\mu} \sigma_{\xi\xi} = \sum_{n=1}^{\infty} \left\{ \left[2n^2 \sinh^2 \xi_1 \cosh \xi \frac{\cosh (n\xi)}{\cosh (n\xi_1)} (B_{1n} + B_{2n}) \right.\right.$$
$$-n \sinh \xi \frac{\sinh (n\xi)}{\cosh (n\xi_1)} ((\kappa - 2) B_{1n} - \kappa B_{2n})$$
$$\left. -n \frac{4 \sinh (\xi_1 + \xi) \sinh (\xi_1 - \xi)}{\cosh (2\xi) - \cos (2\eta)} \frac{\sinh \xi \sinh (n\xi)}{\cosh (n\xi_1)} (B_{1n} + B_{2n}) \right] \cos \eta \cos (n\eta)$$

$$+\left[-2n^2\cosh^2\xi_1\sinh\xi\frac{\sinh(n\xi)}{\cosh(n\xi_1)}(B_{1n}+B_{2n})\right.$$

$$+n\cosh\xi\frac{\cosh(n\xi)}{\cosh(n\xi_1)}(\kappa B_{1n}-(\kappa-2)B_{2n})$$

$$\left.-n\frac{4\sinh(\xi_1+\xi)\sinh(\xi_1-\xi)}{\cosh(2\xi)-\cos(2\eta)}\frac{\cosh\xi\cosh(n\xi)}{\cosh(n\xi_1)}(B_{1n}+B_{2n})\right]\sin\eta\sin(n\eta)\Bigg\},$$

$$\frac{h_0^2}{\mu}\sigma_{\eta\eta}=\sum_{n=1}^{\infty}\Bigg\{\left[-2n^2\sinh^2\xi_1\cosh\xi\frac{\cosh(n\xi)}{\cosh(n\xi_1)}(B_{1n}+B_{2n})\right.$$

$$+n\sinh\xi\frac{\sinh(n\xi)}{\cosh(n\xi_1)}(\kappa B_{1n}-(\kappa-2)B_{2n})$$

$$\left.+n\frac{4\sinh(\xi_1+\xi)\sinh(\xi_1-\xi)}{\cosh(2\xi)-\cos(2\eta)}\frac{\sinh\xi\sinh(n\xi)}{\cosh(n\xi_1)}(B_{1n}+B_{2n})\right]\cos\eta\cos(n\eta)$$

$$+\left[2n^2\cosh^2\xi_1\sinh\xi\frac{\sinh(n\xi)}{\cosh(n\xi_1)}(B_{1n}+B_{2n})\right.$$

$$-n\cosh\xi\frac{\cosh(n\xi)}{\cosh(n\xi_1)}((\kappa-2)B_{1n}-\kappa B_{2n})$$

$$\left.+n\frac{4\sinh(\xi_1+\xi)\sinh(\xi_1-\xi)}{\cosh(2\xi)-\cos(2\eta)}\frac{\cosh\xi\cosh(n\xi)}{\cosh(n\xi_1)}(B_{1n}+B_{2n})\right]\sin\eta\sin(n\eta)\Bigg\},$$

$$\frac{h_0^2}{\mu}\tau_{\xi\eta}=\sum_{n=1}^{\infty}\Bigg\{-\left[2n^2\cosh^2\xi_1\sinh\xi\frac{\cosh(n\xi)}{\cosh(n\xi_1)}(B_{1n}+B_{2n})\right.$$

$$-n\cosh\xi\frac{\sinh(n\xi)}{\cosh(n\xi_1)}((\kappa-2)B_{1n}-\kappa B_{2n})$$

$$\left.-n\frac{4\sinh(\xi_1+\xi)\sinh(\xi_1-\xi)}{\cosh(2\xi)-\cos(2\eta)}\frac{\cosh\xi\sinh(n\xi)}{\cosh(n\xi_1)}(B_{1n}+B_{2n})\right]\sin\eta\cos(n\eta)$$

$$+\left[-2n^2\sinh^2\xi_1\cosh\xi\frac{\sinh(n\xi)}{\cosh(n\xi_1)}(B_{1n}+B_{2n})\right.$$

$$+n\sinh\xi\frac{\cosh(n\xi)}{\cosh(n\xi_1)}(\kappa B_{1n}-(\kappa-2)B_{2n}) \tag{30}$$

$$\left.+n\frac{4\sinh(\xi_1+\xi)\sinh(\xi_1-\xi)}{\cosh(2\xi)-\cos(2\eta)}\frac{\sinh\xi\cosh(n\xi)}{\cosh(n\xi_1)}(B_{1n}+B_{2n})\right]\cos\eta\sin(n\eta)\Bigg\}.$$

From (29), (30), we obtain the numerical values of the displacement vector and stress tensor components at any point of the body, where B_{1n}, B_{2n} is the solution of system (28).

Note. To obtain the solution of infinite system (28), $n = 1, \dots, 15$.

The sum of the solution of the first problem and solution of the second problem is the solution of problems (1)-(6) (i.e. tensions and displacements at any point of the body).

4. Setting and solving concrete problem for a whole ellipse

Let us set the boundary value problem for a semi-ellipse when the continuity conditions (symmetry conditions) of the solution are given at the linear boundary which means that it is possible to bound the semi-ellipse into a whole ellipse. So, let us find the solution of system of equations (1) in the area $\Omega_1 = \{0 < \xi < \xi_1, 0 < \eta < \pi\}$ (Fig. 2), which meets the following boundary conditions:

$$\eta = 0: \quad \bar{v} = 0, \ \ \bar{u}_{,\eta} = 0, \tag{31}$$

$$\eta = \pi: \quad \bar{v} = 0, \ \ \bar{u}_{,\eta} = 0, \tag{32}$$

$$\xi = 0: \quad \bar{u} = 0, \ \ \bar{v}_{,\xi} = 0, \tag{33}$$

$$\xi = \xi_1: \quad \frac{h_0^2}{\mu}\sigma_{\xi\xi} = 0, \ \ \frac{h_0^2}{\mu}\tau_{\xi\eta} = \sin(2\eta)\, P. \tag{34}$$

The components of the displacement vector and stress tensor are given by harmonic functions φ_1 and φ_2, and are presented by formulae (23), (24).

By employing the MSV and taking into account the boundary conditions (31), (32), (33), we can write functions φ_1 and φ_2 in the following form:

$$\varphi_1 = \sum_{n=1}^{\infty} B_{1n} \frac{\sinh(n\xi)}{\cosh(n\xi_1)} \sin(n\eta), \qquad \varphi_2 = \sum_{n=1}^{\infty} B_{2n} \frac{\cosh(n\xi)}{\cosh(n\xi_1)} \cos(n\eta). \tag{35}$$

Like with the second problem of the previous paragraph, we will here substitute boundary conditions (34) with the equivalent:

$$\begin{aligned} &\sinh(2\xi_1)(\varphi_{1,\eta\eta} + \varphi_{2,\xi\eta}) + \kappa\varphi_{1,\xi} + (\kappa-2)\,\varphi_{2,\eta} = \sinh\xi_1 \cos\eta \sin(2\eta)\, P, \\ &\sinh(2\xi_1)(\varphi_{1,\xi\eta} - \varphi_{2,\eta\eta}) - (\kappa-2)\,\varphi_{1,\eta} + \kappa\varphi_{2,\xi} = -\cosh\xi_1 \sin\eta \sin(2\eta)\, P. \end{aligned} \tag{36}$$

By inserting (35) in (36), we obtain:

$$\begin{aligned} &\sum_{n=1}^{\infty} \Big[-n^2 \sinh(2\xi_1) \frac{\sinh(n\xi)}{\cosh(n\xi_1)} (B_{1n} + B_{2n}) \\ &\qquad + n \tfrac{\cosh(n\xi)}{\cosh(n\xi_1)} (\kappa B_{1n} - (\kappa-2)\, B_{2n}) \Big] \sin(n\eta) = \sinh\xi_1 \cos\eta \sin(2\eta)\, P, \\ &\textstyle\sum_{n=1}^{\infty} \Big[n^2 \sinh(2\xi_1) \tfrac{\cosh(n\xi)}{\cosh(n\xi_1)} (B_{1n} + B_{2n}) \\ &\qquad - n \tfrac{\sinh(n\xi)}{\cosh(n\xi_1)} ((\kappa-2)\, B_{1n} - \kappa B_{2n}) \Big] \cos(n\eta) = -\cosh\xi_1 \sin\eta \sin(2\eta)\, P. \end{aligned} \tag{37}$$

After equating the expressions at the same trigonometric functions in both sides of equations (37), we obtain the following infinite system of linear algebraic equations to unknown quantities B_{1n} and B_{2n}.

$$\begin{gathered}
\left[-\sinh\left(2\xi_1\right)\frac{\sinh(\xi)}{\cosh(\xi_1)}+\kappa\frac{\cosh(\xi)}{\cosh(\xi_1)}\right]B_{11}-\left[\sinh\left(2\xi_1\right)\frac{\sinh(\xi)}{\cosh(\xi_1)}\right.\\
\left.+\left(\kappa-2\right)\frac{\cosh(\xi)}{\cosh(\xi_1)}\right]B_{21}=\frac{P}{2}\sinh\left(\xi_1\right),\\
\left[\sinh\left(2\xi_1\right)\frac{\cosh(\xi)}{\cosh(\xi_1)}-\left(\kappa-2\right)\frac{\sinh(\xi)}{\cosh(\xi_1)}\right]B_{11}+\left[\sinh\left(2\xi_1\right)\frac{\cosh(\xi)}{\cosh(\xi_1)}\right.\\
\left.+\kappa\frac{\cosh(\xi)}{\cosh(\xi_1)}\right]B_{21}=-\frac{P}{2}\cosh\left(\xi_1\right),\\
3\left[-3\sinh\left(2\xi_1\right)\frac{\sinh(3\xi)}{\cosh(3\xi_1)}+\kappa\frac{\cosh(3\xi)}{\cosh(3\xi_1)}\right]B_{13}-3\left[3\sinh\left(2\xi_1\right)\frac{\sinh(3\xi)}{\cosh(3\xi_1)}\right.\\
\left.+\left(\kappa-2\right)\frac{\cosh(3\xi)}{\cosh(3\xi_1)}\right]B_{23}=\frac{P}{2}\sinh\left(\xi_1\right),\\
3\left[3\sinh\left(2\xi_1\right)\frac{\cosh(3\xi)}{\cosh(3\xi_1)}-\left(\kappa-2\right)\frac{\sinh(3\xi)}{\cosh(3\xi_1)}\right]B_{13}+3\left[3\sinh\left(2\xi_1\right)\frac{\cosh(3\xi)}{\cosh(3\xi_1)}\right.\\
\left.+\kappa\frac{\cosh(3\xi)}{\cosh(3\xi_1)}\right]B_{23}=\frac{P}{2}\cosh\left(\xi_1\right),
\end{gathered}\tag{38}$$

The values of displacements and stresses are calculated formulae (29), (30) at any point to be considered, where B_{1n}, B_{2n}, $n=1,3$ are the solution of system (38).

5. Numerical examples and discussion

By using MATLAB software, the numerical values of the displacement vector and stress tensor components were obtained at the points of a) the ellipse weakened with the crack and b) the ellipse without cracks (whole ellipse), and the 3D (See Fig. 6-9) and 2D (See Fig. 10-17) graphs of displacements and stresses distribution respectively.

Numerical values are obtained for the following data:$\nu=0.3$, $E=10^6 kg/cm^2$, $P=-10kg/cm^2$, $\xi_0=0$, $\xi_1=1$.

Fig. 6 shows 3D graph of distribution of the stress tensor components in semi-ellipse and Fig. 8 shows 3D graph of distribution of the components of a displacement vector, when variable tangential stress is given at $\xi=0$ (i.e. on the section between the focuses) and normal stress equals to 0, while Fig. 7 and Fig. 9 show the relevant graphs of the same components when there are symmetry conditions given on the section between the focuses.

Fig. 10 and Fig. 11 show the graphs of components of the stress tensor $\sigma_{\xi\xi}$, $\sigma_{\eta\eta}$ and $\tau_{\xi\eta}$, when $\eta=\frac{\pi}{4}$ or $\eta=\frac{\pi}{2}$ or $\eta=\frac{3\pi}{4}$ and $0\le\xi\le\xi_1=1$ (See hyperbolic lines in Fig. 18), in particular, when there are tangential stresses (Fig. 10) and symmetry conditions (Fig. 11) given on the section between the focuses. As expected, in both cases (we mean the boundary conditions on the section between the focuses), the graphs show that the values of stress $\sigma_{\xi\xi}$on $\eta=\frac{\pi}{4}$ and $\eta=\frac{3\pi}{4}$ are the same, and

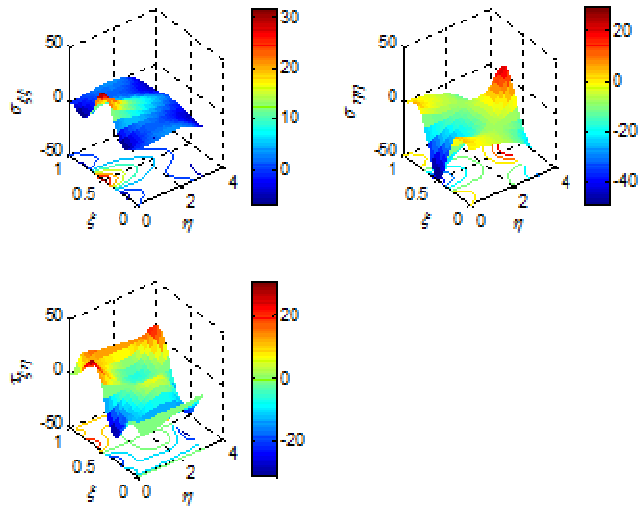


Figure 6. Distribution of $\sigma_{\xi\xi}, \sigma_{\eta\eta}$ and $\tau_{\xi\eta}$ in semi-ellipse when tangential stress is given at $\xi = 0$

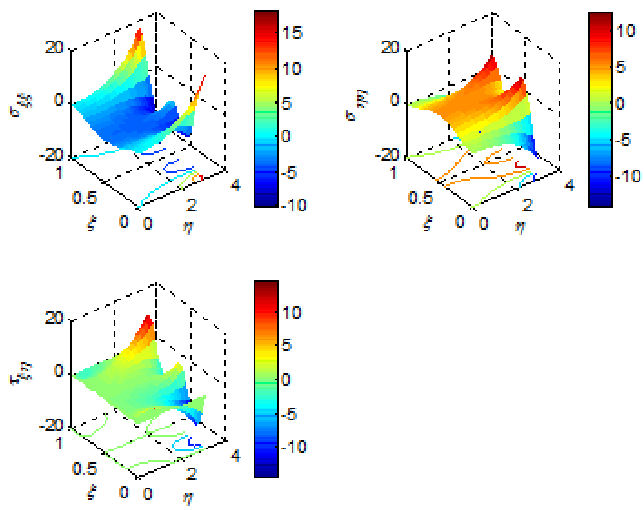


Figure 7. Distribution of $\sigma_{\xi\xi}, \sigma_{\eta\eta}$ and $\tau_{\xi\eta}$ in semi-ellipse when symmetry conditions are given at $\xi = 0$

the values of $\sigma_{\eta\eta}$ are also the same, while the values of$\tau_{\xi\eta}$ differ with the sign only and are very little (almost 0) at $\eta = \frac{\pi}{2}$.

Fig. 12 and 13 show the graphs of u and v, the components of a displacement vector when $\eta = \frac{\pi}{4}$ or $\eta = \frac{\pi}{2}$ or $\eta = \frac{3\pi}{4}$ and $0 \leq \xi \leq \xi_1 = 1$ (See Fig. 18). The graphs show that the values of displacement u are the same at $\eta = \frac{\pi}{4}$ and $\eta = \frac{3\pi}{4}$ and the values of displacement v differ with the sign only and equal to 0 at $\eta = \frac{\pi}{2}$ as it was expected.

Fig. 14 and 15 show the graphs of $\sigma_{\xi\xi}$, $\sigma_{\eta\eta}$ and $\tau_{\xi\eta}$in both cases when $\xi = \frac{\xi_1 - \xi_0}{2} = \frac{1}{2}$ and $0 \leq \eta \leq \pi$ (See elliptic line in Fig. 18) , while Fig. 16 and 17 show the graphs of u and v for the same values of ξ and η.

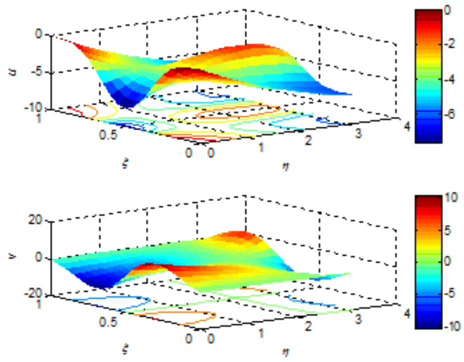


Figure 8. Distribution of u and v in semi-ellipse when tangential stress is given at $\xi = 0$

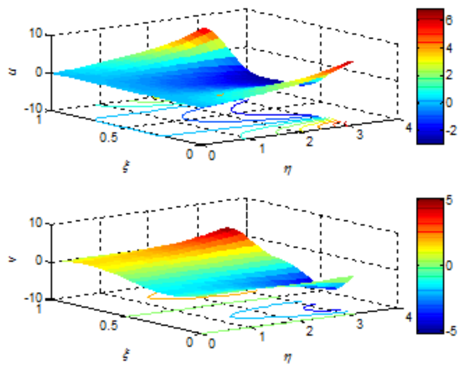


Figure 9. Distribution of u and v in semi-ellipse when symmetry conditions are given at $\xi = 0$

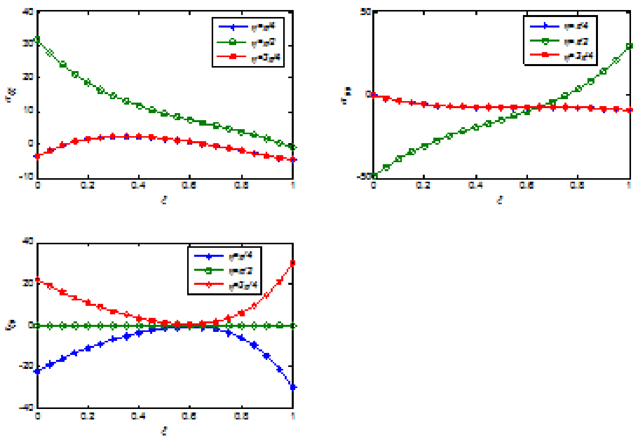


Figure 10. The components of the stress tensor obtained for fixed $\eta = \frac{\pi}{4}, \eta = \frac{\pi}{2}$ and $\eta = \frac{3\pi}{4}$ and when ξ changes $(0 \le \xi \le \xi_1 = 1)$ when tangential stress is given at $\xi = 0$

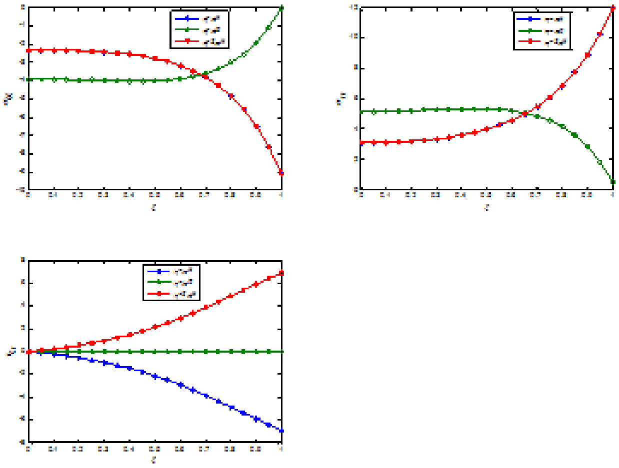


Figure 11. The components of the stress tensor obtained for fixed $\eta = \frac{\pi}{4}, \eta = \frac{\pi}{2}$ and $\eta = \frac{3\pi}{4}$ and when ξ changes $(0 \le \xi \le \xi_1 = 1)$ when symmetry conditions are given at $\xi = 0$

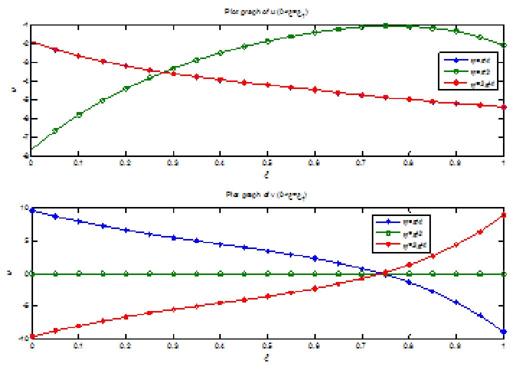


Figure 12. Components of a displacement vector obtained for fixed$\eta = \frac{\pi}{4}, \eta = \frac{\pi}{2}$ and $\eta = \frac{3\pi}{4}$ and when ξchanges $(0 \le \xi \le \xi_1 = 1)$ when tangential stress is given at $\xi = 0$

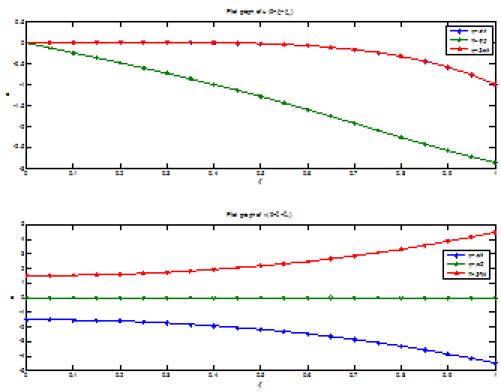
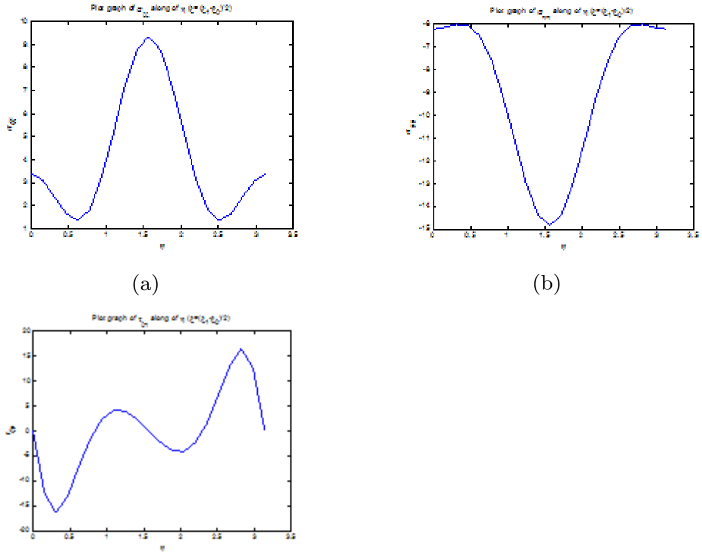
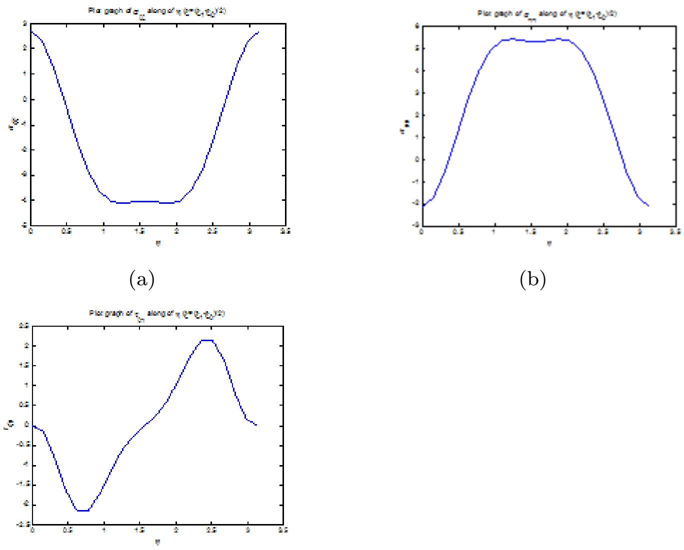


Figure 13. Components of a displacement vector obtained for fixed $\eta = \frac{\pi}{4}, \eta = \frac{\pi}{2}$ and $\eta = \frac{3\pi}{4}$ and when ξchanges $(0 \le \xi \le \xi_1 = 1)$ when symmetry conditions are given at $\xi = 0$



(a) (b) (c)

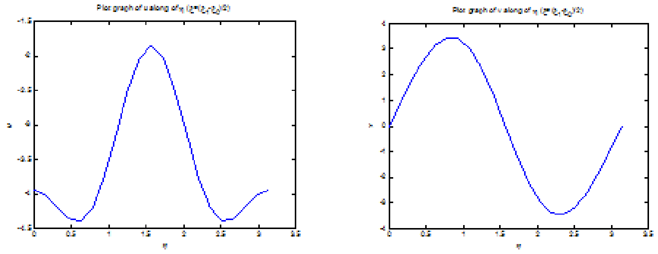
Figure 14. Components of stress tensor obtained for fixed $\xi = \frac{\xi_1 - \xi_0}{2} = \frac{1}{2}$ and when η changes $(0 \le \eta \le \pi)$ when tangential stress is given at $\xi = 0$.



(a) (b) (c)

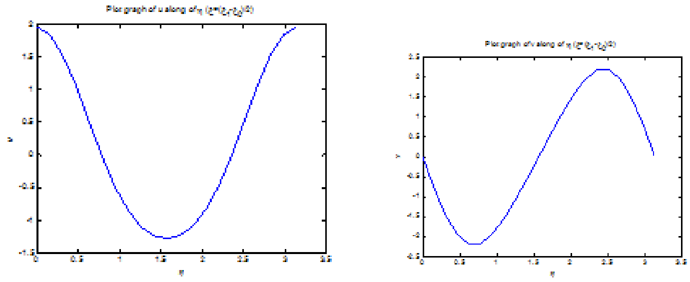
Figure 15. Components of stress tenzor obtained for fixed $\xi = \frac{\xi_1 - \xi_0}{2} = \frac{1}{2}$ and when η changes $(0 \le \eta \le \pi)$, when symmetry conditions are given at $\xi = 0$.

It is known that in the problem, there is a line of symmetry if the elastic properties of the material, the geometric configuration of the boundary and loading conditions are symmetric with respect to this line. The elastic properties of a homogeneous isotropic body are the same at all points and in all directions; so it remains only to follow the last two requirements. The presence of the line of symmetry leads to two physical effects. First, there are no normal (relative to line)



(a) (b)

Figure 16. Components of a displacement vector obtained for fixed $\xi = \frac{\xi_1 - \xi_0}{2} = \frac{1}{2}$ and when η changes $(0 \leq \eta \leq \pi)$ when tangential stress is given at $\xi = 0$.



(a) (b)

Figure 17. Components of a displacement vector obtained for fixed $\xi = \frac{\xi_1 - \xi_0}{2} = \frac{1}{2}$ and when η changes $(0 \leq \eta \leq \pi.)$ when symmetry conditions are given at $\xi = 0$.

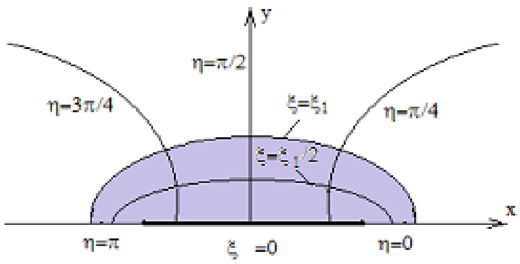


Figure 18. Semi-ellipse with elliptic and hyperbolic lines

displacements on it, and, second, there are no tangential stresses acting along it. It is obvious that in problems (1)-(6), there is a line of symmetry$x = 0$, i.e. axes$y(\eta = \pi/2)$. Therefore, tangential stresses $\tau_{\xi\eta}$ equal to zero for all values of ξat $\eta = \pi/2$ (see Fig.10 and Fig. 11). Also, normal displacements$v$, relative to line symmetry $\eta = \pi/2$, equal to zero for all values of ξ at $\eta = \pi/2$ (see Fig 12 and Fig. 13).

It should also be noted that the absolute values of normal and tangential displacements and stresses when tangential stress acts on the section between the focuses, are higher than in terms of symmetry conditions along the same section. This can be explained by the fact that the excitation applied to the internal section of the ellipse causes the weakening of the body and increase in the normal and tangential displacements and stresses in it.

6. Conclusions

The principal outcomes of the present paper can be summarized as follows:

(1) Mathematical modeling of the boundary value problem of the theory of elasticity for the elliptic body with an internal crack by setting a problem in the elliptic coordinate system.
(2) Reduction of the solution of the set problem to the solutions of the relevant, internal and external problems, which can be solved analytically quite simply by the method of separation of variables.
(3) Obtaining the numerical values of the components of stress tensor and displacement vector at the points of the ellipse and the ellipse weakened by an internal crack.
(4) Visualization and discussion of the obtained results.

The calculations and graphs were made by using MATLAB software.

As the bodies of an elliptic shape are common in practice, e.g. in building, mechanical engineering, biology, medicine, etc., the study of the deformed mode of such bodies is topical and consequently, in my opinion, setting the problems considered in the article and the method of their solution is interesting from the practical point of view.

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