SOME HARDY SPACES ESTIMATES FOR MULTILINEAR LITTLEWOOD-PALEY S-OPERATORS

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Abstract In this paper, we prove the boundedness for the multilinear Littlewood-Paley S-operators on certain Hardy and Herz-Hardy spaces.

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1. Introduction

Let ψ be a function on R^n which satisfies the following properties:
(1) $\int \psi(x)dx = 0$,
(2) $|\psi(x)| \le C(1+|x|)^{-(n+1)}$,
(3) $|\psi(x+y)-\psi(x)| \le C|y|(1+|x|)^{-(n+2)}$ when $2|y| < |x|$;
Let m be a positive integer and let A be a function on R^n. We denote $\Gamma(x) = \{(y,t) \in R^{n+1}_+ : |x-y| < t\}$. The multilinear Littlewood-Paley operator is defined by

$$S^A_\psi(f)(x) = \left[\int\limits_{\Gamma(x)} |F^A_t(f)(x,y)|^2 \frac{dydt}{t^{n+1}}\right]^{1/2},$$

where

$$F^A_t(f)(x,y) = \int\limits_{R^n} \frac{f(z)\psi_t(y-z)}{|x-z|^m} R_{m+1}(A;x,z)dz,$$

$$R_{m+1}(A;x,y) = A(x) - \sum_{|\alpha|\le m} \frac{1}{\alpha!} D^\alpha A(y)(x-y)^\alpha,$$

and $\psi_t(x) = t^{-n}\psi(x/t)$ for $t > 0$. We also define that

$$S_\psi(f)(x) = \left(\int\limits_{\Gamma(x)} |f * \psi_t(y)|^2 \frac{dydt}{t^{n+1}}\right)^{1/2},$$

which is the Littlewood-Paley operator (see [15]).

It is well known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [2-6]). The main purpose of this paper is to consider the continuity of the multilinear Littlewood-Paley operators on certain Hardy and Herz-Hardy spaces. Let us first introduce some definitions (see [7] [8] [9] [10] [11] [12]).

Definition 1. Let A be a function on R^n and let m be a positive integer and $0 < p \leq 1$. A bounded measurable function a on R^n is said to be a $(p, D^m A)$ atom if

i) $\text{supp} a \subset B = B(x_0, r)$,

ii) $||a||_{L^\infty} \leq |B|^{-1/p}$,

iii) $\int a(y)dy = \int a(y)D^\alpha A(y)dy = 0, |\alpha| = m$;

A temperate distribution f is said to belong to $H^p_{D^m A}(R^n)$, if, in the Schwartz distributional sense, it can be written as

$$f(x) = \sum_{j=0}^{\infty} \lambda_j a_j(x),$$

where a_j's are $(p, D^m A)$ atoms, $\lambda_j \in C$ and $\sum_{j=0}^{\infty} |\lambda_j|^p < \infty$. Moreover, $||f||_{H^p_{D^m A}} \sim \left(\sum_{j=0}^{\infty} |\lambda_j|^p\right)^{1/p}$.

Let $B_k = \{x \in R^n : |x| \leq 2^k\}, C_k = B_k \setminus B_{k-1}, k \in Z$, $m_k(\lambda, f) = |\{x \in C_k : |f(x)| > \lambda\}|$; for $k \in N$, let $\tilde{m}_k(\lambda, f) = m_k(\lambda, f)$ and $\tilde{m}_0(\lambda, f) = |\{x \in B_0 : |f(x)| > \lambda\}|$.

Definition 2. Let $0 < p$, $q < \infty$, $\alpha \in R$.

(1) The homogeneous Herz space is defined by

$$\dot{K}^{\alpha,p}_q = \{f \in L^q_{loc}(R^n \setminus \{0\}) : ||f||_{\dot{K}^{\alpha,p}_q(R^n)} < \infty\},$$

where

$$||f||_{\dot{K}^{\alpha,p}_q} = \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}||f\chi_k||^p_{L^q}\right]^{1/p};$$

(2) The nonhomogeneous Herz space is defined by

$$K^{\alpha,p}_q(R^n) = \{f \in L^q_{loc}(R^n) : ||f||_{K^{\alpha,p}_q(R^n)} < \infty\},$$

where

$$||f||_{K^{\alpha,p}_q} = \left[\sum_{k=1}^{\infty} 2^{k\alpha p}||f\chi_k||^p_{L^q} + ||f\chi_{B_0}||^p_{L^q}\right]^{1/p}.$$

Definition 3. Let m be a positive integer and let A be a function on R^n, $\alpha \in R$, $0 < p < \infty$, $1 < q \leq \infty$, A function $a(x)$ on R^n is called a central $(\alpha, q, D^m A)$-atom (or a central $(a, q, D^m A)$-atom of restrict type), if

1) $\text{supp} a \subset B(0, r)$ for some $r > 0$ (or for some $r \geq 1$),

2) $||a||_{L^q} \leq |B(0, r)|^{-\alpha/n}$,

3) $\int a(x)dx = \int a(x)D^\beta A(x)dx = 0$, $|\beta| = m$;
A temperate distribution f is said to belong to $H\dot{K}^{\alpha,p}_{q,D^mA}(R^n)$ (or $HK^{\alpha,p}_{q,D^mA}(R^n)$), if it can be written as $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$ (or $f = \sum_{j=0}^{\infty} \lambda_j a_j$) in the $S'(R^n)$ sense, where a_j is a central (α, q, D^mA)-atom (or a central (α, q, D^mA)-atom of restrict type) supported on $B(0, 2^j)$ and $\sum_{j=-\infty}^{\infty} |\lambda_j|^p < \infty$ (or $\sum_{j=0}^{\infty} |\lambda_j|^p < \infty$), moreover, $||f||_{H\dot{K}^{\alpha,p}_{q,D^mA}}$ (or $||f||_{HK^{\alpha,p}_{q,D^mA}}$) $\sim \left(\sum_j |\lambda_j|^p\right)^{1/p}$.

Now we can state our main theorems.

Theorem 1. *Let $1 \geq p > n/(n+1)$, $D^\beta A \in BMO(R^n)$ for $|\beta| = m$. Then S^A_ψ is bounded from $H^p_{D^mA}(R^n)$ to $L^p(R^n)$.*

Theorem 2. *Let $0 < p < \infty$, $1 < q < \infty$, $n(1-1/q) \leq \alpha < n(1-1/q)+1$ and $D^\beta A \in BMO(R^n)$ for $|\beta| = m$. Then S^A_ψ is bounded from $H\dot{K}^{\alpha,p}_{q,D^mA}$ (R^n) to $\dot{K}^{\alpha,p}_q(R^n)$.*

Theorem 3. *Let $D^\beta A \in BMO(R^n)$ for $|\beta| = m$ and $0 < p \leq 1 \leq q < \infty$, $\alpha = n(1-1/q)+1$. Then, for any $\lambda > 0$ and $f \in HK^{\alpha,p}_{q,D^mA}(R^n)$, we have*

$$\left[\sum_{k=0}^{\infty} 2^{k\alpha p} \tilde{m}_k(\lambda, S^A_\psi(f))^{p/q}\right]^{1/p}$$
$$\leq C\lambda^{-1}||f||_{HK^{\alpha,p}_{q,D^mA}(R^n)} \left(1 + \log^+(\lambda^{-1}||f||_{HK^{\alpha,p}_{q,D^mA}(R^n)})\right).$$

Remark. Note that when $m = 0$, S^A_ψ is just the commutator of Littlewood-Paley operator (see [1]). Theorem 1-3 are an extension of the results in [1].

2. Proof of Theorems

We begin with some preliminary lemmas.

Lemma 1.[4] *Let A be a function on R^n and $D^\alpha A \in L^q(R^n)$ for $|\alpha| = m$ and some $q > n$. Then*

$$|R_m(A; x, y)| \leq C|x-y|^m \sum_{|\alpha|=m} \left(\frac{1}{|\tilde{Q}(x,y)|} \int_{\tilde{Q}(x,y)} |D^\alpha A(z)|^q dz\right)^{1/q},$$

where $\tilde{Q}(x, y)$ is the cube centered at x and having side length $5\sqrt{n}|x-y|$.

Lemma 2. *Let $1 < p < \infty$ and $D^\alpha A \in L^r(R^n)$, $|\alpha| = m$, $1 < r \leq \infty$, $1/q = 1/p + 1/r$. Then S^A_ψ is bounded from $L^p(R^n)$ to $L^q(R^n)$, that is*

$$||S^A_\psi(f)||_{L^q} \leq C \sum_{|\alpha|=m} ||D^\alpha A||_{L^r} ||f||_{L^p}.$$

Proof. By Minkowski inequality and the condition of ψ, we have

$$S^A_\psi(f)(x) \leq \int_{R^n} \frac{|f(z)|\ |R_{m+1}(A; x, z)|}{|x-z|^m} \left(\int_{\Gamma(x)} |\psi_t(y-z)|^2 \frac{dydt}{t^{1+n}}\right)^{1/2} dz$$

$$\leq C\int_{R^n}\frac{|f(z)||R_{m+1}(A;x,z)|}{|x-z|^m}\left(\int_0^\infty\int_{|x-y|\leq t}\frac{t^{-2n}}{(1+|y-z|/t)^{2n+4}}\frac{dydt}{t^{1+n}}\right)^{1/2}dz$$

$$\leq C\int_{R^n}\frac{|f(z)||R_{m+1}(A;x,z)|}{|x-z|^m}\left(\int_0^\infty\int_{|x-y|\leq t}\frac{t^{-2n}}{(1+|y-z|/t)^{2n+4}}\frac{dydt}{t^{n+1}}\right)^{1/2}dz,$$

$$\leq C\int_{R^n}\frac{|f(z)||R_{m+1}(A;x,z)|}{|x-z|^m}\left(\int_0^\infty\int_{|x-y|\leq t}\frac{2^{2n+4}\cdot t^{1-n}}{(2t+|y-z|)^{2n+2}}dydt\right)^{1/2}dz,$$

noting that $2t+|y-z|\geq 2t+|x-z|-|x-y|\geq t+|x-z|$ when $|x-y|\leq t$ and

$$\int_0^\infty\frac{tdt}{(t+|x-z|)^{2n+2}}=C|x-z|^{-2n},$$

we obtain

$$\begin{aligned}S_\psi^A(f)(x) &\leq C\int_{R^n}\frac{|f(z)|}{|x-z|^m}|R_{m+1}(A;x,z)|\left(\int_0^\infty\frac{tdt}{(t+|x-z|)^{2n+2}}\right)^{1/2}dz\\ &= C\int_{R^n}\frac{|f(z)|}{|x-z|^{m+n}}|R_{m+1}(A;x,z)|dz,\end{aligned}$$

thus, the lemma follows from [5] and [6].

Now let us turn to the proof of Theorems in this paper.

Proof of Theorem 1. It suffices to show that there exists a constant $c>0$ such that for every (p,D^mA) atom a,

$$||S_\psi^A(a)||_{L^p}\leq C.$$

Let a be a (p,D^mA) atom supported on a ball $B=B(x_0,r)$. We write

$$\int_{R^n}[S_\psi^A(a)(x)]^pdx=\int_{|x-x_0|\leq 2r}[S_\psi^A(a)(x)]^pdx+\int_{|x-x_0|>2r}[S_\psi^A(a)(x)]^pdx=I+II.$$

For I, taking $q>1$, by Hölder's inequality and the L^q-boundedness of S_ψ^A (see Lemma 2), we see that

$$I\leq C||S_\psi^A(a)||_{L^q}^p\cdot|B(x_0,2r)|^{1-p/q}\leq C||a||_{L^q}^p|B|^{1-p/q}\leq C.$$

To obtain the estimate of II, we need to estimate $S_\psi^A(a)(x)$ for $x\in(2B)^c$. Let $\tilde{B}=5\sqrt{n}B$ and $\tilde{A}(x)=A(x)-\sum_{|\alpha|=m}\frac{1}{\alpha!}(D^\alpha A)_{\tilde{B}}\cdot x^\alpha$, where $(A)_B$ are the mean values of A on B. Then $R_m(A;x,y)=R_m(\tilde{A};x,y)$. We write, by the vanishing moment of a,

$$\begin{aligned}
F_t^A(a)(x,y) &= \int_B \left[\frac{\psi_t(y-z)}{|x-z|^m}-\frac{\psi_t(y-x_0)}{|y-x_0|^m}\right] R_m(\tilde{A};x,z)a(z)dz \\
&+ \int_B \frac{\psi_t(y-x_0)}{|y-x_0|^m}[R_m(\tilde{A};x,z)-R_m(\tilde{A};x,x_0)]a(y)dy \\
&- \sum_{|\alpha|=m}\frac{1}{\alpha!}\int_B \frac{\psi_t(y-z)(x-z)^\alpha}{|x-z|^m}(D^\alpha A(z)-(D^\alpha A)_B)a(z)dz,
\end{aligned}$$

thus,

$$\begin{aligned}
&S_\psi^A(a)(x) \\
\le &\left[\int_{\Gamma(x)}\left(\int_B\left|\frac{\psi_t(y-z)}{|x-z|^m}-\frac{\psi_t(y-x_0)}{|x-x_0|^m}\right||R_m(\tilde{A};x,z)||a(z)|dz\right)^2\frac{dydt}{t^{n+1}}\right]^{1/2} \\
+ &\left[\int_{\Gamma(x)}\left(\int_B\frac{|\psi_t(y-x_0)|}{|x-x_0|^m}|R_m(\tilde{A};x,z)-R_m(\tilde{A};x,x_0)||a(z)|dz\right)^2\frac{dydt}{t^{n+1}}\right]^{1/2} \\
+ &\left[\int_{\Gamma(x)}\left|\sum_{|\alpha|=m}\frac{1}{\alpha!}\int_B\frac{\psi_t(y-z)(x-z)^\alpha}{|x-z|^m}(D^\alpha A(z)-(D^\alpha A)_B)a(z)dz\right|^2\frac{dydt}{t^{n+1}}\right]^{1/2} \\
\equiv &\ II_1+II_2+II_3.
\end{aligned}$$

By Lemma 1, for $z\in B$ and $x\in 2^{k+1}B\setminus 2^kB$, we know

$$|R_m(\tilde{A};x,z)|\le C|x-z|^m\sum_{|\alpha|=m}|D^\alpha A(x)-(D^\alpha A)_{2^kB}|;$$

By the condition of ψ and Minkowski's inequality, similar to the proof of Lemma 2, noting that $|x-z|\sim|x-x_0|$ for $z\in B$ and $x\in R^n\setminus B$, we obtain

$$\begin{aligned}
&II_1\le \\
&\left[\int_{\Gamma(x)}\left(\int_B\left(\left|\frac{1}{|x-z|^m}-\frac{1}{|x-x_0|^m}\right||\psi_t(y-x_0)|+\frac{|\psi_t(y-z)-\psi_t(y-x_0)|}{|x-z|^m}\right)\right.\right. \\
&\left.\left.|R_m(\tilde{A};x,z)||a(z)|dz\right)^2\frac{dydt}{t^{n+1}}\right]^{1/2}
\end{aligned}$$

$$\leq C\int_B \frac{|x_0-z|}{|x-x_0|^{m+1}}\left(\int_{\Gamma(x)} |\psi_t(y-x_0)|^2\frac{dydt}{t^{n+1}}\right)^{1/2} |R_m(\tilde{A};x,z)||a(z)|dz$$

$$+ C\int_B \frac{1}{|x-x_0|^{m}}\left(\int_{\Gamma(x)} |\psi_t(y-z)-\psi_t(y-x_0)|^2\frac{dydt}{t^{n+1}}\right)^{1/2}$$

$$\times|R_m(\tilde{A};x,z)||a(z)|dz$$

$$\leq C\frac{|B|^{1/n-1/p}}{|x-x_0|^{m+1}}\left(\int_{\Gamma(x)} \frac{t^{1-n}dydt}{(2t+|y-x_0|)^{2n+2}}\right)^{1/2}\left(\int_B |R_m(\tilde{A};x,z)|dz\right)$$

$$+C\frac{|B|^{1/n-1/p}}{|x-x_0|^{m}}\left(\int_{\Gamma(x)} \frac{t^{1-n}dydt}{(2t+|y-x_0|)^{2n+4}}\right)^{1/2}\left(\int_B |R_m(\tilde{A};x,z)|dz\right)$$

$$\leq C|x-x_0|^{-(m+n+1)}|B|^{1/n-1/p}\left(\int_B |R_m(A;x,z)|dz\right)$$

$$\leq Ck|x-x_0|^{-n-1}|B|^{1/n-1/p+1}\sum_{|\alpha|=m}|D^\alpha A(x)-(D^\alpha A)_{2^{k+1}B}|;$$

On the other hand, by the formula (see [4]):

$$R_m(\tilde{A};x,z)-R_m(\tilde{A};x,x_0)=\sum_{|\beta|<m}\frac{1}{\beta!}R_{m-|\beta|}(D^\beta\tilde{A};z,x_0)(x-x_0)^\beta$$

and Lemma 1, we get

$$|R_m(\tilde{A};x,z)-R_m(\tilde{A};x,x_0)|\leq C\sum_{|\beta|<m}\sum_{|\alpha|=m}|x_0-z|^{m-|\beta|}|x-x_0|^{|\beta|}||D^\alpha A||_{BMO},$$

so that

$$II_2\leq C\int_B |x-x_0|^{-(n+m)}\sum_{|\beta|<m}\left|R_{m-|\beta|}(D^\beta\tilde{A};z,x_0)\right||x-x_0|^{|\beta|}|a(z)|dz$$

$$\leq C\int_B |x-x_0|^{-(n+m)}\sum_{|\beta|<m}|x_0-z|^{m-|\beta|}|x-x_0|^{|\beta|}\sum_{|\alpha|=m}||D^\alpha A||_{BMO}|a(z)|dz$$

$$\leq C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\int_B\frac{|x_0-z|}{|x-x_0|^{n+1}}|a(z)|dz$$

$$\leq C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}|x-x_0|^{-n-1}|B|^{1/n-1/p+1};$$

For II_3 , we write

$$\begin{aligned} & \int\limits_B \frac{\psi_t(y-z)(x-z)^\alpha}{|x-y|^m}(D^\alpha A(z)-(D^\alpha A)_B)a(z)dz \\ = & \int\limits_B \left[\frac{\psi_t(y-z)(x-z)^\alpha}{|x-z|^m} - \frac{\psi_t(y-x_0)(y-x_0)^\alpha}{|x-x_0|^m}\right] \\ \times & [D^\alpha A(z)-(D^\alpha A)_B]a(z)dz. \end{aligned}$$

Similar to the estimate of II_1, we obtain

$$\begin{aligned} II_3 &\leq C\sum_{|\alpha|=m}|x-x_0|^{-(n+1)}\int\limits_B |x_0-z||D^\alpha A(z)-(D^\alpha A)_B||a(z)|dz \\ &\leq C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}|B|^{1/n-1/p+1}|x-x_0|^{-n-1}. \end{aligned}$$

Therefore, recall that $p > n/(n+1)$,

$$\begin{aligned} II &\leq \sum_{k=1}^{\infty}\int\limits_{2^{k+1}B\backslash 2^kB}[S_\psi^A(a)(x)]^p dx \\ &\leq C\sum_{k=1}^{\infty}\int\limits_{2^{k+1}B\backslash 2^kB} k^p|x-x_0|^{-p(n+1)}|B|^{p(1+1/n-1/p)} \\ &\quad \times\left(\sum_{|\alpha|=m}|D^\alpha A(x)-(D^\alpha A)_{2^{k+1}B}|\right)^p dx \\ &\quad +C\left(\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\right)^p\sum_{k=1}^{\infty}\int\limits_{2^{k+1}B\backslash 2^kB}|x-x_0|^{-p(n+1)}|B|^{p(1+1/n-1/p)}dx \\ &\leq C\left(\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\right)^p\sum_{k=1}^{\infty}k^p 2^{k(n-p-pn)} \leq C\left(\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\right)^p, \end{aligned}$$

which together with the estimate for I yields the desired result. This finishes the proof of Theorem 1.

Proof of Theorem 2. Let $f \in H\dot{K}^{\alpha,p}_{q,D^mA}(R^n)$ and $f(x) = \sum_{j=-\infty}^{\infty}\lambda_j a_j(x)$ be the atomic decomposition for f as in Definition 3. We write

$$||S_\psi^A(f)||_{\dot{K}_q^{\alpha,p}(R^n)} \leq C\left[\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=-\infty}^{k-3}|\lambda_j|\ ||S_\psi^A(a_j)\chi_k||_{L^q}\right)^p\right]^{1/p}$$

$$+C\left[\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=k-2}^{\infty}|\lambda_j|\ ||S_\psi^A(a_j)\chi_k||_{L^q}\right)^p\right]^{1/p}=I+II.$$

For II, by the boundedness of S_ψ^A on $L^q(R^n)$ (see Lemma 2), we have

$$\begin{aligned}II&\le C\left[\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=k-2}^{\infty}|\lambda_j|\ ||a_j||_{L^q}\right)^p\right]^{1/p}\\&\le C\left[\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=k-2}^{\infty}|\lambda_j|2^{-j\alpha}\right)^p\right]^{1/p}\end{aligned}$$

$$\begin{aligned}&\le C\begin{cases}\left[\sum_{k=-\infty}^{\infty}2^{k\alpha p}\sum_{j=k-2}^{\infty}|\lambda_j|^p2^{-j\alpha p}\right]^{1/p}, & 0<p\le 1\\ \left[\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=k-2}^{\infty}|\lambda_j|^p2^{-j\alpha p/2}\right)\left(\sum_{j=k-2}^{\infty}2^{-j\alpha p'/2}\right)^{p/p'}\right]^{1/p}, & p>1\end{cases}\\&\le C\begin{cases}\left[\sum_{j=-\infty}^{\infty}|\lambda_j|^p\left(\sum_{k=-\infty}^{j+2}2^{(k-j)\alpha p}\right)|\lambda_j|^p2^{-j\alpha p}\right]^{1/p}, & 0<p\le 1\\ \left[\sum_{j=-\infty}^{\infty}|\lambda_j|^p\left(\sum_{k=-\infty}^{j+2}2^{(k-j)\alpha p/2}\right)\right]^{1/p}, & p>1\end{cases}\\&\le C\left(\sum_{j=-\infty}^{\infty}|\lambda_j|^p\right)^{1/p}\le C||f||_{H\dot{K}_{q,D^mA}^{\alpha,p}}.\end{aligned}$$

For I, similar to the proof of Theorem 1, we have, for $x\in C_k$, $j\le k-3$,

$$\begin{aligned}S_\psi^A(a_j)(x)&\le C|x-x_0|^{-n-m-1}|B_j|^{1/n}\left(\int_{B_j}|a_j(y)||R_m(\tilde{A};x,y)|dy\right)\\&\quad+C\sum_{|\beta|=m}||D^\beta A||_{BMO}(k-j)|x-x_0|^{-n-1}|B_j|^{1/n}\int_{B_j}|a(y)|dy\\&\le C2^{-k(n+1)}2^{j(1+n(1-1/q)-\alpha)}\left(\sum_{|\beta|=m}|D^\beta A(x)-(D^\beta A)_{B_k}|\right)\\&\quad+C\sum_{|\beta|=m}||D^\beta A||_{BMO}(k-j)2^{-k(n+1)}2^{j(1+n(1-1/q)-\alpha)},\end{aligned}$$

thus,

$$
\begin{aligned}
I &\leq C\Big[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\Big(\sum_{j=-\infty}^{k-3} |\lambda_j| 2^{-k(n+1)+j(1+n(1-1/q)-\alpha)} \\
&\quad \sum_{|\beta|=m} \left(\int_{B_k} |D^\beta A(x) - (D^\beta A)_{B_k}|^q dx\right)^{1/q} \Big)^p\Big]^{1/p} \\
&\quad + C\Big[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\Big(\sum_{j=-\infty}^{k-3} |\lambda_j|(k-j) 2^{-k(n+1)+j(1+n(1-1/q)-\alpha)} 2^{kn/q} \\
&\quad \sum_{|\beta|=m} ||D^\beta A||_{BMO}\Big)^p\Big]^{1/p} = I_1 + I_2;
\end{aligned}
$$

To estimate I_1 and I_2, we consider two cases.
Case 1 $0 < p \leq 1$.

$$
\begin{aligned}
&I_1 \leq \\
&C\left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \sum_{j=-\infty}^{k-3} |\lambda_j|^p 2^{[-k(n+1)+j(1+n(1-1/q)-\alpha)]p} 2^{knp/q} \Big(\sum_{|\beta|=m} ||D^\beta A||_{BMO}\Big)^p\right]^{1/p}
\end{aligned}
$$

$$
\begin{aligned}
&= C\sum_{|\beta|=m} ||D^\beta A||_{BMO} \left[\sum_{j=-\infty}^{\infty} |\lambda_j|^p \sum_{k=j+3}^{\infty} 2^{(j-k)(1+n(1-1/q)-\alpha)p}\right]^{1/p} \\
&\leq C\sum_{|\beta|=m} ||D^\beta A||_{BMO} \left(\sum_{j=-\infty}^{\infty} |\lambda_j|^p\right)^{1/p} \leq C||f||_{H\dot{K}^{\alpha,p}_{q,D^m A}},
\end{aligned}
$$

$$
\begin{aligned}
I_2 &\leq C\sum_{|\beta|=m} ||D^\beta A||_{BMO} \left[\sum_{j=-\infty}^{\infty} |\lambda_j|^p \sum_{k=j+3}^{\infty} (k-j)^p 2^{(j-k)(1+n(1-1/q)-\alpha)p}\right]^{1/p} \\
&\leq C\sum_{|\beta|=m} ||D^\beta A||_{BMO} \left(\sum_{j=-\infty}^{\infty} |\lambda_j|^p\right)^{1/p} \leq C||f||_{H\dot{K}^{\alpha,p}_{q,D^m A}};
\end{aligned}
$$

Case 2 $p > 1$. By Hölder's inequality, we deduce

$$
I_1 \leq C\sum_{|\beta|=m} ||D^\beta A||_{BMO}\Big[\sum_{j=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-3} |\lambda_j|^p 2^{(j-k)p(1(1-1/q)-\alpha)/2}\right)
$$

$$
\begin{aligned}
&\times \left(\sum_{j=-\infty}^{k-3} 2^{(j-k)p'(1+n(1-1/q)-\alpha)/2}\right)^{p/p'}\Bigg]^{1/p} \\
&\le C\left(\sum_{j=-\infty}^{\infty}|\lambda_j|^p\right)^{1/p} \le C||f||_{H\dot{K}^{\alpha,p}_{q,D^mA}},
\end{aligned}
$$

$$
\begin{aligned}
I_2 &\le C\sum_{|\beta|=m}||D^\beta A||_{BMO}\left[\sum_{j=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-3}|\lambda_j|^p(k-j)^p 2^{(j-k)p(1+n(1-1/q)-\alpha)/2}\right)\right. \\
&\times \left.\left(\sum_{j=-\infty}^{k-3} 2^{(j-k)p'(1+n(1-1/q)-\alpha)/2}\right)^{p/p'}\right]^{1/p} \\
&\le C\sum_{|\beta|=m}||D^\beta A||_{BMO}\left[\sum_{j=-\infty}^{\infty}|\lambda_j|^p\sum_{k=j+3}^{\infty}(k-j)^p 2^{(j-k)p(1+n(1-1/q)-\alpha)/2}\right]^{1/p} \\
&\le C\left(\sum_{j=-\infty}^{\infty}|\lambda_j|^p\right)^{1/p} \le C||f||_{H\dot{K}^{\alpha,p}_{q,D^mA}}.
\end{aligned}
$$

This finishes the proof of Theorem 2.

Remark. Theorem 2 also holds for nonhomogeneous Herz-type space.

Proof of Theorem 3. Let $f \in HK^{\alpha,p}_{q,D^mA}(R^n)$ and let $f(x) = \sum_{j=0}^{\infty}\lambda_j a_j(x)$ be the atomic decomposition for f as in Definition 3. We write

$$
\begin{aligned}
&\left[\sum_{k=0}^{\infty}2^{k\alpha p}\tilde{m}_k(\lambda, S^A_\psi(f))^{p/q}\right]^{1/p} \le C\left[\sum_{k=0}^{3}2^{k\alpha p}\tilde{m}_k(\lambda, S^A_\psi(f))^{p/q}\right]^{1/p} \\
&+C\left[\sum_{k=4}^{\infty}2^{k\alpha p}\tilde{m}_k\left(\lambda/2, \sum_{j=0}^{k-3}|\lambda_j|S^A_\psi(a_j)\right)^{p/q}\right]^{1/p} \\
&+C\left[\sum_{k=4}^{\infty}2^{k\alpha p}\tilde{m}_k\left(\lambda/2, S^A_\psi\left(\sum_{j=k-2}^{\infty}\lambda_j a_j\right)\right)^{p/q}\right]^{1/p} \equiv I_1+I_2+I_3.
\end{aligned}
$$

For I_1, I_3, by the weak (q,q) type boundedness of S^A_ψ and $0 < p \le 1$, we have

$$
\begin{aligned}
I_1 &\le C\lambda^{-1}\left[\sum_{k=0}^{3}2^{k\alpha p}||f||^p_{L^q}\right]^{1/p} \le C\lambda^{-1}\left(\sum_{j=0}^{\infty}|\lambda_j|^p||a_j||^p_{L^q}\right)^{1/p} \\
&\le C\lambda^{-1}\left(\sum_{j=0}^{\infty}|\lambda_j|^p\cdot 2^{-j\alpha p}\right)^{1/p}
\end{aligned}
$$

$$\leq \ C\lambda^{-1}\left(\sum_{j=0}^{\infty}|\lambda_j|^p\right)^{1/p} \leq C\lambda^{-1}||f||_{HK^{\alpha,p}_{q,D^mA}},$$

$$\begin{aligned} I_3 \ &\leq \ C\lambda^{-1}\left[\sum_{k=4}^{\infty}2^{k\alpha p}\left|\left|\sum_{j=k-2}^{\infty}\lambda_j a_j\right|\right|_{L^q}^p\right]^{1/p} \\ &\leq \ C\lambda^{-1}\left[\sum_{k=4}^{\infty}2^{k\alpha p}\sum_{j=k-2}^{\infty}|\lambda_j|^p 2^{-j\alpha p}\right]^{1/p} \\ &\leq \ C\lambda^{-1}\left[\sum_{j=0}^{\infty}|\lambda_j|^p\sum_{k=0}^{j+2}2^{(k-j)\alpha p}\right]^{1/p} \leq C\lambda^{-1}\left(\sum_{j=0}^{\infty}|\lambda_j|^p\right)^{1/p} \\ &\leq \ C\lambda^{-1}||f||_{HK^{\alpha,p}_{q,D^mA}}. \end{aligned}$$

For I_2, by the same argument as the proof of Theorem 1 and 2, we have

$$S^A_\psi(a_j)(x) \leq C2^{-k(n+1)}\left(\sum_{|\beta|=m}|D^\beta A(x)-(D^\beta A)_{B_k}| + k\sum_{|\beta|=m}||D^\beta A||_{BMO}\right),$$

therefore,

$$\begin{aligned} I_2 \leq C &\left[\sum_{k=4}^{\infty}2^{k\alpha p}\tilde{m}_k\left(\lambda/4, C2^{-k(n+1)}\sum_{|\beta|=m}|D^\beta A(x)-(D^\beta A)_{B_k}|\sum_{j=0}^{\infty}|\lambda_j|\right)^{p/q}\right]^{1/p} \\ &+C\left[\sum_{k=4}^{\infty}2^{k\alpha p}\tilde{m}_k\left(\lambda/4, C2^{-k(n+\varepsilon)}k\sum_{|\beta|=m}||D^\beta A||_{BMO}\sum_{j=0}^{\infty}|\lambda_j|\right)^{p/q}\right]^{1/p} \\ &\equiv I_2^{(1)} + I_2^{(2)}. \end{aligned}$$

For $I_2^{(1)}$, by using John-Nirenberg inequality (see [15]), we gain

$$\begin{aligned} I_2^{(1)} \ &\leq \ C\left[\sum_{k=4}^{\infty}2^{k\alpha p}\left(\exp\left(-\frac{C2^{k(n+1)}\lambda}{\sum_{|\beta|=m}||D^\beta A||_{BMO}\sum_{j=0}^{\infty}|\lambda_j|}\right)2^{kn}\right)^{p/q}\right]^{1/p} \\ &\leq \ C\left[\sum_{k=0}^{\infty}2^{k(n+1)p}\exp\left(-\frac{C\lambda 2^{k(n+1)}}{\sum_{|\beta|=m}||D^\beta A||_{BMO}\sum_{j=0}^{\infty}|\lambda_j|}\right)\right]^{1/p} \\ &\leq \ C\left[\int_0^{\infty}x^{p-1}\exp\left(-\frac{c\lambda x}{\sum_{|\beta|=m}||D^\beta A||_{BMO}\sum_{j=0}^{\infty}|\lambda_j|}\right)dx\right]^{1/p} \end{aligned}$$

$$
\begin{aligned}
&= C\lambda^{-1}||D^{\beta}A||_{BMO}\sum_{j=0}^{\infty}|\lambda_j|\left(\int_0^{\infty}t^{p-1}e^{-t}dt\right)^{1/p}\\
&\le C\lambda^{-1}\left(\sum_{j=0}^{\infty}|\lambda_j|^p\right)^{1/p}\\
&\le C\lambda^{-1}||f||_{HK^{\alpha,p}_{q,D^mA}};
\end{aligned}
$$

For $I_2^{(2)}$, by using the following fact: If there exists $u > 1$, such that $2^x/x \le u$ for $x \ge 3$, then $2^x \le cu\log^+ u$. We have, if

$$
\left|\left\{x\in C_k : C2^{-k(n+1)}k\sum_{|\beta|=m}||D^{\beta}A||_{BMO}\sum_{j=0}^{\infty}|\lambda_j| > \lambda/4\right\}\right| \ne 0,
$$

then

$$
1 < 2^{k(n+1)/k(n+1)} < C\lambda^{-1}\sum_{|\beta|=m}||D^{\beta}A||_{BMO}\sum_{j=0}^{\infty}|\lambda_j|,
$$

thus,

$$
2^{k(n+1)} \le C\lambda^{-1}\sum_{j=0}^{\infty}|\lambda_j|\log^+\left(\lambda^{-1}\sum_{j=0}^{\infty}|\lambda_j|\right).
$$

Let K_λ be the maximal integer k which satisfies this estimate, then

$$
\begin{aligned}
I_2^{(2)} &\le C\left(\sum_{k=4}^{K_\lambda}2^{k\alpha p}2^{knp/q}\right)^{1/p} \le C2^{K_\lambda(n+1)}\\
&\le C\lambda^{-1}\sum_{j=0}^{\infty}|\lambda_j|\log^+\left(\lambda^{-1}\sum_{j=0}^{\infty}|\lambda_j|\right)\\
&\le C\lambda^{-1}\left(\sum_{j=0}^{\infty}|\lambda_j|^p\right)^{1/p}\log^+\left(\lambda^{-1}\left(\sum_{j=0}^{\infty}|\lambda_j|^p\right)^{1/p}\right)\\
&\le C\lambda^{-1}||f||_{HK^{\alpha,p}_{q,D^mA}}\log^+\left(\lambda^{-1}||f||_{HK^{\alpha,p}_{q,D^mA}}\right).
\end{aligned}
$$

Now, summing up the above estimates, we have

$$
\begin{aligned}
&\left[\sum_{k=0}^{\infty}2^{k\alpha p}\tilde{m}_k(\lambda, S_\psi^A(f))^{p/q}\right]^{1/p}\\
&\le C\lambda^{-1}||f||_{H^{\alpha,p}_{q,D^mA}}\left(1+\log^+\left(\lambda^{-1}||f||_{HK^{\alpha,p}_{q,D^mA}}\right)\right).
\end{aligned}
$$

This completes the proof of Theorem 3.

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