THE METHOD OF A SMALL PARAMETER FOR THE SHALLOW SHELLS

T. Meunargia

I. Vekua Institute of Applied Mathematics
of I. Javakhishvili Tbilisi State University

Abstract. In the present paper the method of a small parameter is used for the shallow shells of I.Vekua. The small parameter has the form $\varepsilon = \frac{h}{R}$, where h is a semithickness of the shell and R is the characterictic radius of curvature of the midsurface of the shell.

Key words: Shallow shells, small parameter.

MSC 2000: 74K25

1. Introduction

A complete system of three-dimentional equations of elastic bodies of shallow shell type can be written as [1]:

a) equilibrium equations

$$\begin{cases} \nabla_\alpha \sigma^{\alpha\beta} - b_\alpha^\beta \sigma^{\alpha 3} + \dfrac{\partial \sigma^{3\beta}}{\partial x^3} + \Phi^\beta = 0, \\ \nabla_\alpha \sigma^{\alpha 3} + b_{\alpha\beta}\sigma^{\alpha\beta} + \dfrac{\partial \sigma^{33}}{\partial x^3} + \Phi^3 = 0, \quad (\alpha, \beta = 1, 2) \end{cases} \tag{1}$$

where σ^{ij} $(i, j = 1, 2, 3)$ are contravariant components of the stress tensor, $b_{\alpha\beta}(b^{\alpha\beta}, b_\alpha^\beta)$ are covariant (contravariant, mixed) components of the curvature tensor of the midsurface S of the shell Ω, Φ^i $(i = 1, 2, 3)$ are contravariant components of the external force, ∇_α $(\alpha = 1, 2)$ are symbols of covariant derivatives, x^1 and x^2 are curvilinear coordinates of the midsurface, x_3 is thickness coordinate, $-h \leq x_3 \leq h$, and h is a semi-thickness of the shell;

b) Hooke's law

$$\begin{cases} \sigma^{\alpha\beta} = \lambda(\theta - 2HU_3 + \partial_3 U_3)a^{\alpha\beta} + \mu(\nabla^\alpha U^\beta + \nabla^\beta U^\alpha - 2b^{\alpha\beta}U_3) = \sigma^{\beta\alpha}, \\ \sigma^{\alpha 3} = \mu(\partial^3 U^\alpha + \nabla^\alpha U^3 + b_\gamma^\alpha U^\gamma) = \sigma^{3\alpha}, \\ \sigma^{33} = \lambda(\theta - 2HU_3) + (\lambda + 2\mu)\partial_3 U_3, \\ (\theta = \nabla_\gamma U^\gamma, \quad \nabla^\alpha = a^{\alpha\gamma}\nabla_\gamma, \quad \partial^3 = \partial_3 = \dfrac{\partial}{\partial x_3}, \ \ a^{\alpha\beta} = \mathbf{r}^\alpha \mathbf{r}^\beta), \end{cases} \tag{2}$$

where λ and μ are Lamé constants, H is the mean curvature of the surfaces

$$2H = b_1^1 + b_2^2 = b_\alpha^\alpha,$$

$\mathbf{U} = u^\alpha \mathbf{r}_\alpha + U^3 \mathbf{n}$ is the displacement vector, $\mathbf{r}_\alpha(\mathbf{r}^\alpha)(\alpha = 1, 2)$ are covariant (contravariant) base vectors and $\mathbf{n}$ is a normal of the midsurface S;

c) the stress vector $\boldsymbol{\sigma}_{(l)}$ acting on the lateral surface with normal $\mathbf{l}$ has the form [1]

$$\begin{aligned}
&\boldsymbol{\sigma}_{(\mathbf{l})} = \boldsymbol{\sigma}^\alpha l_\alpha == (\sigma^{\alpha\beta}\mathbf{r}_\beta + \sigma^{\alpha 3}\mathbf{n}) l_\alpha = \sigma_{(ll)}\mathbf{l} + \sigma_{(ls)}\mathbf{s} + \sigma_{(ln)}\mathbf{n} \quad \Rightarrow \\
&\sigma_{(ll)} = \sigma^{\alpha\beta} l_\alpha l_\beta, \quad \sigma_{(ls)} = \sigma^{\alpha\beta} l_\alpha s_\beta, \quad \sigma_{(ln)} = \sigma^{\alpha 3} l_\alpha, \\
&(l_\alpha = \mathbf{l}\mathbf{r}_\alpha, \quad s_\alpha = \mathbf{s}\mathbf{r}_\alpha, \quad \mathbf{l} \times \mathbf{s} = \mathbf{n}).
\end{aligned} \tag{3}$$

There exist several methods of reduction of the three-dimensional problems to the two-dimensional problems (Kirchhoff-Love, E. Reissner, K. Friedrichs, A. Green, A. Goldenveizer, I. Vorovich, I. Vekua, etc.)

2. I. Vekua's Demension Reduction Method

Following I. Vekua we assume the validity of the expansions

$$\begin{aligned}
&(\sigma^{ij}, U^i, \Phi^i) = \sum_{m=0}^{\infty} \left(\overset{(m)}{\sigma}{}^{ij}, \overset{(m)}{U}{}^{i}, \overset{(m)}{\Phi}{}^{i} \right) P_m\left(\frac{x_3}{h}\right) \quad \Rightarrow \\
&\left(\overset{(m)}{\sigma}{}^{ij}, \overset{(m)}{U}{}^{i}, \overset{(m)}{\Phi}{}^{i} \right) = \frac{2m+1}{2h} \int_{-h}^{h} (\sigma^{ij}, U^i, \Phi^i) P_m\left(\frac{x_3}{h}\right) dx_3,
\end{aligned} \tag{4}$$

where P_m are Legendre polynomials of order m.

Substituting the above expansions in relations (1), (2) and (3) having satisfied beforehand the conditions on the face surface $x_3 = \pm h$,

$$\boldsymbol{\sigma}^3(x^1, x^2, \pm h) = \overset{(\pm)}{\boldsymbol{\sigma}}{}^{3},$$

we obtain the following infinite complete system of two-dimensional equations, which in the izometric coordinates

$$ds^2 = \Lambda dz d\bar{z} \quad (z = x^1 + ix^2, \ \Lambda(x^1, x^2) > 0)$$

written in a complex form looks as follows [2]:

a) equilibrium equations:

$$\begin{cases}
\dfrac{h}{\Lambda}\dfrac{\partial}{\partial z}\left(\overset{(m)}{\sigma}{}_{11} - \overset{(m)}{\sigma}{}_{22} + 2i\overset{(m)}{\sigma}{}_{12} \right) + h\dfrac{\partial \overset{(m)}{\sigma}{}^{\alpha}_{\alpha}}{\partial \bar{z}} - \varepsilon H R \overset{(m)}{\sigma}{}_{+} \\
-\varepsilon Q R \overset{(m)}{\sigma}{}_{+} - (2m+1)(\overset{(m-1)}{\sigma}{}_{+} + \overset{(m-3)}{\sigma}{}_{+} + \ldots) + h\overset{(m)}{F}{}_{+} = 0 \\
\\
\dfrac{h}{\Lambda}\left(\dfrac{\partial \overset{(m)}{\sigma}{}_{+}}{\partial z} + \dfrac{\partial \overline{\overset{(m)}{\sigma}{}_{+}}}{\partial \bar{z}} \right) + \varepsilon H R \overset{(m)}{\sigma}{}^{\alpha}_{\alpha} \\
+\varepsilon Re[\overline{Q} R(\overset{(m)}{\sigma}{}^{1}_{1} - \overset{(m)}{\sigma}{}^{2}_{2} + 2i\overset{(m)}{\sigma}{}^{1}_{2})] \\
-(2m+1)(\overset{(m-1)}{\sigma}{}_{33} + \overset{(m-3)}{\sigma}{}_{33} + \ldots) + h\overset{(m)}{F}{}_{+} = 0 \quad (m = 0, 1, \ldots),
\end{cases} \tag{5}$$

where $\varepsilon = \frac{h}{R}$ is a small parameter and R is the characteristic radius of curvature of S.

Then

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x^1} - i\frac{\partial}{\partial x^2}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x^1} + i\frac{\partial}{\partial x^2}\right),$$

$$\overset{(m)}{\sigma}_{+} = \overset{(m)}{\sigma}_{1} + i\overset{(m)}{\sigma}_{2}, \quad \overset{(m)}{F}_{+} = \overset{(m)}{F}_{1} + i\overset{(m)}{F}_{2},$$

$$2Q = b_1^1 - b_2^2 + 2ib_1^2, \quad \overset{(m)}{F}_{i} = \overset{(m)}{\Phi}_{i} + \frac{2m+1}{2h}[\overset{(+)}{\sigma}_{3i} - (-1)^m \overset{(-)}{\sigma}_{3i}].$$

b) Hooke's law

$$\begin{cases} h(\overset{(m)}{\sigma}_{11} - \overset{(m)}{\sigma}_{22} + 2i\overset{(m)}{\sigma}_{12}) = 4\mu\Lambda\left(h\frac{\partial}{\partial \bar{z}}\frac{1}{\Lambda}\overset{(m)}{U}_{+} - \varepsilon Q R \overset{(m)}{U}_{3}\right), \\ h\overset{(m)}{\sigma}{}_{\alpha}^{\alpha} = 2(\lambda+\mu)\left(h\overset{(m)}{\theta} - 2H\varepsilon R\overset{(m)}{U}_{3}\right) + 2\lambda \overset{(m)}{U}{}'_{3}, \\ h\overset{(m)}{\sigma}_{+} = \mu\left[2h\frac{\partial \overset{(m)}{U}_{3}}{\partial \bar{z}} + \varepsilon R(H\overset{(m)}{U}_{+} + Q\overline{\overset{(m)}{U}_{+}}) + \overset{(m)}{U}{}'_{+}\right], \\ h\overset{(m)}{\sigma}_{33} = \lambda(h\overset{(m)}{\theta} - 2H\varepsilon R\overset{(m)}{U}_{3}) + (\lambda+2\mu)\overset{(m)}{U}{}'_{3}, \end{cases} \tag{6}$$

where

$$\overset{(m)}{U}_{+} = \overset{(m)}{U}_{1} + i\overset{(m)}{U}_{2}, \quad \overset{(m)}{U}_{\alpha} = \overset{(m)}{\mathbf{U}}\mathbf{r}_{\alpha}, \ (\alpha = 1,2),$$

$$\overset{(m)}{\theta} = \frac{1}{\Lambda}\left(\frac{\partial \overset{(m)}{U}_{+}}{\partial z} + \frac{\partial \overline{\overset{(m)}{U}_{+}}}{\partial \bar{z}}\right), \quad \overset{(m)}{\mathbf{U}}{}' = (2m+1)\left(\overset{(m+1)}{\mathbf{U}} + \overset{(m+3)}{\mathbf{U}} + ...\right);$$

c) the boundary conditions for stress tensor on the lateral contour Γ:

$$\overset{(m)}{\sigma}_{(ll)} = \overset{(m)}{f}_{1}, \quad \overset{(m)}{\sigma}_{(ls)} = \overset{(m)}{f}_{2}, \quad \overset{(m)}{\sigma}_{(ln)} = \overset{(m)}{f}_{3}, \Rightarrow$$

$$\begin{cases} \overset{(m)}{\sigma}_{(ll)} + i\overset{(m)}{\sigma}_{(ls)} = \frac{1}{2}\left[\overset{(m)}{\sigma}{}_{\alpha}^{\alpha} - (\overset{(m)}{\sigma}_{11} - \overset{(m)}{\sigma}_{22} + 2i\overset{(m)}{\sigma}_{12})\left(\frac{d\bar{z}}{ds}\right)^2\right] \\ = \overset{(m)}{f}_{1} + i\overset{(m)}{f}_{2}, \\ \overset{(m)}{\sigma}_{(ln)} = -Im\left(\overset{(m)}{\sigma}_{+}\frac{d\bar{z}}{ds}\right) = \overset{(m)}{f}_{3}, \end{cases} \tag{7}$$

$$(m = 0, 1, ...).$$

Substituting (6) in (5) and using the formula

$$4h^2\frac{1}{\Lambda}\frac{\partial}{\partial z}\Lambda\frac{\partial}{\partial \bar{z}}\frac{1}{\Lambda}\overset{(m)}{U}_{+} = 4h^2\frac{\partial}{\partial \bar{z}}\left(\frac{1}{\Lambda}\frac{\partial \overset{(m)}{U}_{+}}{\partial z}\right) + 2\varepsilon^2 K R^2 \overset{(m)}{U}_{+},$$

where K is the Gaussian curvature of the midsurface S, we obtain a system of differential equations in terms of the displacement vector components:

$$
\left[\begin{aligned}
&4\mu h^2\frac{\partial}{\partial\bar z}\left(\frac{1}{\Lambda}\frac{\partial\overset{(m)}{U}_+}{\partial z}\right)+2(\lambda+\mu)h^2\frac{\partial\overset{(m)}{\theta}}{\partial\bar z}+2\lambda h\frac{\partial\overset{(m)}{U}{}'_3}{\partial\bar z}\\
&-(2m+1)\mu\left[\left(2h\frac{\partial\overset{(m-1)}{U}_3}{\partial\bar z}+\overset{(m-1)}{U}{}'_+\right)+\left(2h\frac{\partial\overset{(m-3)}{U}_+}{\partial\bar z}+\overset{(m-3)}{U}{}'_+\right)+\dots\right]\\
&-\varepsilon R\Bigg\{\frac{4\mu h}{\Lambda}\frac{\partial\Lambda Q\overset{(m)}{U}_3}{\partial z}+4(\lambda+\mu)h\frac{\partial H\overset{(m)}{U}_3}{\partial\bar z}\\
&+\mu\left[H\left(2h\frac{\partial\overset{(m)}{U}_3}{\partial\bar z}+\overset{(m)}{U}{}'_+\right)+Q\left(2h\frac{\partial\overset{(m)}{U}_3}{\partial\bar z}+\overline{\overset{(m)}{U}{}'_+}\right)\right]\\
&-(2m+1)\mu\left[\left(H\overset{(m-1)}{U}_++Q\overline{\overset{(m-1)}{U}_+}\right)+\dots\right]\Bigg\}\\
&+\varepsilon^2R^2\mu\left[(2K-H^2-Q\bar Q)\overset{(m)}{U}_+-2HQ\overline{\overset{(m)}{U}_+}\right]+h^2\overset{(m)}{F}_+=0,
\end{aligned}\right. \tag{8_1}
$$

$$
\left[\begin{aligned}
&\mu\left(h^2\nabla^2\overset{(m)}{U}_3+h\overset{(m)}{\theta}{}'\right)-(2m+1)\Bigg[\left(\lambda h\overset{(m-1)}{\theta}+(\lambda+2\mu)\overset{(m-1)}{U}{}'_3\right)\\
&+\left(\lambda h\overset{(m-3)}{\theta}+(\lambda+2\mu)\overset{(m-3)}{U}{}'_3\right)+\dots\Bigg]\\
&+\varepsilon R\Bigg\{\frac{2\mu h}{\Lambda}Re\frac{\partial(H\overset{(m)}{U}_++Q\overline{\overset{(m)}{U}_+})}{\partial z}\\
&+H\left[2(\lambda+\mu)h\overset{(m)}{\theta}+2\lambda\overset{(m)}{U}{}'_3\right]+4\mu hRe\left[\overline{Q}\frac{\partial}{\partial\bar z}\left(\frac{1}{\Lambda}\overset{(m)}{U}_+\right)\right]\\
&+2(2m+1)\lambda H\left(\overset{(m-1)}{U}_3+\overset{(m-3)}{U}_3+\dots\right)\Bigg\}\\
&-4\varepsilon^2R^2\left[(\lambda+\mu)H^2+\mu Q\overline{Q}\right]\overset{(m)}{U}_3h^2\overset{(m)}{F}_3=0,
\end{aligned}\right. \tag{8_2}
$$

$$\left(m=0,1,...;\quad\nabla^2=\frac{4}{\Lambda}\frac{\partial^2}{\partial z\partial\bar z}\right).$$

The passage to finite systems is performed by considering a finite expansion (4), where $m=0,1,...,N$.

3. Approximation of order $N = 0$

By introducing the following notation

$$\overset{(0)}{U}_i = U_i, \quad (\overset{(0)}{U}{}'_i = 0)$$

$$\overset{(0)}{\sigma}_{ij} = T_{ij}, \quad \overset{(0)}{F}_i = X_i,$$

we obtain:

a) equilibrium equations

$$\begin{cases} \frac{h}{\Lambda}\frac{\partial}{\partial z}(T_{11} - T_{22} + 2iT_{12}) + h\frac{\partial}{\partial \bar{z}}T^{\alpha}_{\alpha} - \varepsilon R(HT_+ + Q\overline{T}_+) + hX_+ = 0, \\ \frac{h}{\Lambda}\left(\frac{\partial T_+}{\partial z} + \frac{\partial \overline{T_+}}{\partial \bar{z}}\right) + \varepsilon R[HT^{\alpha}_{\alpha} + Re(\overline{Q}(T^1_1 - T^2_2 + 2iT^2_1))] + hX_3 = 0; \end{cases} \tag{9}$$

b) Hooke's law

$$\begin{aligned} &h(T_{11} - T_{22} + 2iT_{12}) = 4\mu\Lambda\left[h\frac{\partial}{\partial \bar{z}}\left(\frac{1}{\Lambda}u_+\right) - \varepsilon QRu_3\right], \\ &hT^{\alpha}_{\alpha} = h(T^1_1 + T^2_2) = 2(\lambda + \mu)(h\theta - 2H\varepsilon Ru_3), \\ &hT_+ = \mu(2h\frac{\partial u_3}{\partial \bar{z}} + \varepsilon R(Hu_+ + Q\overline{u_+})], \\ &hT_{33} = \lambda(R\theta - 2H\varepsilon Ru_3), \quad \theta = \frac{1}{\Lambda}\left(\frac{\partial u_+}{\partial z} + \frac{\partial \overline{u_+}}{\partial \bar{z}}\right); \end{aligned} \tag{10}$$

c) equilibrium equations in terms of the displacement vector components

$$\left[\begin{aligned} &4\mu h^2\frac{\partial}{\partial \bar{z}}\left(\frac{1}{\Lambda}\frac{\partial u_+}{\partial z}\right) + 2(\lambda + \mu)h^2\frac{\partial \theta}{\partial \bar{z}} - \varepsilon R\left[\frac{4\mu h}{\Lambda}\frac{\partial \Lambda Q u_3}{\partial z}\right. \\ &\left.+4(\lambda + \mu)h\frac{\partial H u_3}{\partial \bar{z}} + 2\mu h\left(H\frac{\partial u_3}{\partial \bar{z}} + Q\frac{\partial u_3}{\partial z}\right)\right] \\ &+\varepsilon^2 R^2\mu\left[(2K - H^2 - Q\overline{Q})u_+ - 2HQ\overline{u_+}\right] + h^2X_+ = 0, \\ \\ &\mu h^2\nabla^2 u_3 + \varepsilon R\left\{2\mu h Re\left[\frac{1}{\Lambda}\frac{\partial(Hu_+ + Q\overline{u_+})}{\partial z} + 2\overline{Q}\frac{\partial}{\partial \bar{z}}\left(\frac{1}{\Lambda}u_+\right)\right]\right. \\ &\left.+ 2(\lambda + 2\mu)hH\theta\right\} - 4R^2\varepsilon^2\left[(\lambda + \mu)H^2 + \mu Q\overline{Q}\right]u_3 + h^2X_3 = 0. \end{aligned}\right. \tag{11}$$

To determine the components of the displacement vector and the stress tensor we shall use expansions with respect to the small parameter ε:

$$u_i = \overset{(0)}{u}_i + \varepsilon\overset{(1)}{u}_i + \varepsilon^2\overset{(2)}{u}_i + \ldots$$

$$T_{ij} = \overset{(0)}{T}_{ij} + \varepsilon\overset{(1)}{T}_{ij} + \varepsilon^2\overset{(2)}{T}_{ij} + \ldots$$

$$X_i = \overset{(0)}{X}_i + \varepsilon\overset{(1)}{X}_i + \varepsilon^2\overset{(2)}{X}_i + \ldots$$

and then we equate to zero the factors of ε^n. These equations may be written in the form:

a)

$$\left\{\begin{array}{l} \frac{h}{\Lambda}\frac{\partial}{\partial z}\left(\overset{(n)}{T}_{11}-\overset{(n)}{T}_{22}+2i\overset{(n)}{T}_{12}\right)+h\frac{\partial}{\partial \bar{z}}\overset{(n)}{T}{}_{\alpha}^{\alpha} \\ =R(H\overset{(n-1)}{T}_{+}+Q\overline{\overset{(n-1)}{T}_{+}})-h\overset{(n)}{X}_{+}, \\ \frac{h}{\Lambda}\left(\frac{\partial\overset{(n)}{T}_{+}}{\partial z}+\frac{\partial\overline{\overset{(n)}{T}_{+}}}{\partial \bar{z}}\right)=-R\left[H\overset{(n-1)}{T}{}_{\alpha}^{\alpha}\right. \\ \left.+Re\left(\overline{Q}(\overset{(n-1)}{T}{}_{1}^{1}-\overset{(n-1)}{T}{}_{2}^{2}+2i\overset{(n-1)}{T}{}_{2}^{1}\right)\right]-h\overset{(n)}{X}_{3}=0; \end{array}\right. \tag{12}$$

b) Hooke's law

$$\left\{\begin{array}{l} h\left(\overset{(n)}{T}_{11}-\overset{(n)}{T}_{22}+2i\overset{(n)}{T}_{12}\right)=4\mu\Lambda\left[h\frac{\partial}{\partial \bar{z}}\left(\frac{1}{\Lambda}\overset{(n)}{u}_{+}\right)-QR\overset{(n-1)}{u}_{3}\right], \\ h\overset{(n)}{T}{}_{\alpha}^{\alpha}=2(\lambda+\mu)\left[h\overset{(n)}{\theta}-2HR\overset{(n-1)}{u}_{3}\right], \\ \left(\overset{(n)}{\theta}=\frac{1}{\Lambda}\left(\frac{\partial\overset{(n)}{u}_{+}}{\partial z}+\frac{\partial\overline{\overset{(n)}{u}_{+}}}{\partial \bar{z}}\right)\right) \\ h\overset{(n)}{T}_{+}=\mu\left[2h\frac{\partial\overset{(n)}{u}_{3}}{\partial \bar{z}}+R(H\overset{(n-1)}{u}_{+}+Q\overline{\overset{(n-1)}{u}_{+}})\right], \\ h\overset{(n)}{T}_{33}=\lambda\left(h\overset{(n)}{\theta}-2HR\overset{(n-1)}{u}_{3}\right); \end{array}\right. \tag{13}$$

c) equilibrium equations in terms of the displacement vector components:

$$\left\{\begin{array}{l} 4\mu h^2\frac{\partial}{\partial \bar{z}}\left(\frac{1}{\Lambda}\frac{\partial\overset{(n)}{u}_{+}}{\partial z}\right)+2(\lambda+\mu)h^2\frac{\partial\overset{(n)}{\theta}}{\partial \bar{z}}=\overset{(n)}{A}_{+}(z,\bar{z}), \\ \mu h^2\nabla^2\overset{(n)}{u}_{3}=\overset{(n)}{A}_{3}(z,\bar{z}), \end{array}\right. \tag{14}$$

where

$$\overset{(n)}{A}_{+}=-h^2\overset{(n)}{X}_{+}-R\left[2\mu h\left(\frac{2}{\Lambda}\frac{\partial\Lambda Q\overset{(n-1)}{u}_{3}}{\partial z}+H\frac{\partial\overset{(n-1)}{u}_{3}}{\partial \bar{z}}+Q\frac{\partial\overset{(n-1)}{u}_{3}}{\partial z}\right)\right.$$

$$\left.+4(\lambda+\mu)h\frac{\partial H\overset{(n-1)}{u}_{3}}{\partial \bar{z}}\right]-\mu R^2\left[\left(2K-H^2-Q\overline{Q}\right)\overset{(n-2)}{u}_{+}-2HQ\overline{\overset{(n-2)}{u}_{+}}\right],$$

$$\overset{(n)}{A}_3 = -h^2 \overset{(n)}{X}_3 - R \left\{ 2\mu h Re \left[\frac{1}{\Lambda} \frac{\partial (H \overset{(n-1)}{u}_+ + Q \overline{\overset{(n-1)}{u}_+})}{\partial z} + 2\overline{Q} \frac{\partial}{\partial \bar{z}} \left(\frac{1}{\Lambda} \overset{(n-1)}{u}_+ \right) \right] \right.$$
$$\left. + 2(\lambda+\mu) h H \overset{(n-1)}{\theta} \right\} + 4R^2 \left[(\lambda+\mu) H^2 + 2\mu Q \overline{Q} \right] \overset{(n-2)}{u}_3;$$

d) for the boundary conditions (7), we obtain

$$\begin{cases} T_{(ll)} + i T_{(ls)} = \dfrac{1}{2} \left[T^\alpha_\alpha - (T_{11} - T_{22} + 2iT_{12}) \left(\dfrac{d\bar{z}}{ds} \right)^2 \right] = f_1 + i f_2, \\ T_{(ln)} = -Im \left(T_+ \dfrac{d\bar{z}}{ds} \right) = f_3, \\ (\overset{0}{f}_i = f_i), \end{cases} \Rightarrow$$

$$\begin{cases} h \left[\dfrac{\lambda+\mu}{2\mu} \overset{(n)}{\theta} - \Lambda \dfrac{\partial}{\partial \bar{z}} \left(\dfrac{1}{\Lambda} \overset{(n)}{u}_+ \right) \left(\dfrac{d\bar{z}}{ds} \right)^2 \right] = \dfrac{h}{2\mu} \overset{(n)}{f}_+ \\ +R \left[\dfrac{\lambda+\mu}{\mu} H - \Lambda Q \left(\dfrac{d\bar{z}}{ds} \right)^2 \right] \overset{(n-1)}{u}_3, \\ 2h Im \left(\dfrac{\partial \overset{(n)}{u}_3}{\partial \bar{z}} \dfrac{d\bar{z}}{ds} \right) = -\dfrac{h}{\mu} \overset{(n)}{f}_3 - R Im \left[\left(H \overset{(n-1)}{u}_+ + Q \overline{\overset{(n-1)}{u}_+} \right) \left(\dfrac{d\bar{z}}{ds} \right) \right], \\ (f_+ = f_1 + i f_2). \end{cases} \tag{15}$$

We may now write the general solution of the system of the equations (14) in an explicit form [2]

$$\begin{aligned} \overset{(n)}{u}_+ &= -æ \int_S\int \frac{\overset{(n)}{\varphi}{}'(\zeta)\, dS}{\overline{\zeta} - \bar{z}} + \left(\frac{1}{\pi} \int_S\int \frac{dS}{\overline{\zeta} - \bar{z}} \right) \overline{\overset{(n)}{\varphi}{}'(z)} - \overline{\overset{(n)}{\psi}(z)} \\ &+ \frac{1}{\pi} \int_S\int \frac{\overset{(n)}{B}_+(\zeta, \overline{\zeta}) dS}{\overline{\zeta} - \bar{z}}, \\ \overset{(n)}{u}_3 &= f(z) + \overline{f(z)} + \frac{2}{\pi} \int_S\int B_3(\zeta, \overline{\zeta}) \ln|\zeta - z| dS, \\ &(\zeta = \xi + i\eta \in G, \quad dS = \Lambda(\zeta, \overline{\zeta}) d\xi d\eta) \end{aligned} \tag{16}$$

where

$$\overset{(n)}{B}_+(z, \overline{z}) = -\frac{\lambda + 3\mu}{8\mu(\lambda+2\mu)h^2} \frac{1}{\pi} \int_G\int \left(æ \frac{\overset{(n)}{A}_+(\zeta, \overline{\zeta})}{\zeta - z} - \frac{\overline{\overset{(n)}{A}_+(\zeta, \overline{\zeta})}}{\overline{\zeta} - \overline{z}} \right) d\xi d\eta,$$

$$\overset{(n)}{B}_3(z,\overline{z}) = \frac{1}{4\mu h^2}\overset{(n)}{A}_3(z,\overline{z}), \quad \left(æ = \frac{\lambda+3\mu}{\lambda+\mu}\right).$$

In this way the general solution of the system (14) is expressed by three arbitrary analytic functions $f(z)$, $\varphi(z)$ and $\psi(z)$ of z. Accordingly, it ensures the satisfication of three arbitrary given physical or kinematic conditions.

4. Approximation of Order $N = 1$

Consider, now, the system of equations (5), (6) and (8) for constructing approximation of order $N = 1$. In this case we have

$$U_i = \overset{0}{U}_i + P_1\left(\frac{x_3}{h}\right)\overset{(1)}{U}_i, \quad \sigma_{ij} = \overset{(0)}{\sigma}_{ij} + P_1\left(\frac{x_3}{h}\right)\overset{(1)}{\sigma}_{ij},$$

$$F_i = \overset{(0)}{F}_i + P_1\left(\frac{x_3}{h}\right)\overset{(1)}{F}_i, \quad \overset{(0)}{U}{}'_i = \overset{(1)}{U}_i, \quad \overset{(1)}{U}{}'_i = 0.$$

By introducing the following notation

$$\overset{(0)}{U}_i = u_i, \quad \overset{(1)}{U}_i = v_i, \quad \overset{(0)}{\sigma}_{ij} = T_{ij}, \quad \overset{(1)}{\sigma}_{ij} = S_{ij}, \quad \overset{(0)}{F}_i = X_i, \quad \overset{(1)}{F}_i = Y_i,$$

we obtain:

a) equilibrium equations:

$$\begin{cases} \frac{h}{\Lambda}\frac{\partial}{\partial z}(T_{11} - T_{22} + 2iT_{12}) + h\frac{\partial}{\partial \overline{z}}T^\alpha_\alpha - \varepsilon R(HT_+ + Q\overline{T_+}) + hX_+ = 0, \\ \frac{h}{\Lambda}\left(\frac{\partial T_+}{\partial z} + \frac{\partial \overline{T_+}}{\partial \overline{z}}\right) + \varepsilon R\left[HT^\alpha_\alpha + Re(\overline{Q}(T^1_1 - T^2_2 + 2iT^1_2))\right] + hX_3 = 0, \\ \frac{h}{\Lambda}\frac{\partial}{\partial z}(S_{11} - S_{22} + 2iS_{12}) + h\frac{\partial}{\partial \overline{z}}S^\alpha_\alpha - \varepsilon R(HS_+ + Q\overline{S_+}) - 3T_+ + hY_+ = 0, \\ \frac{h}{\Lambda}\left(\frac{\partial S_+}{\partial z} + \frac{\partial \overline{S_+}}{\partial \overline{z}}\right) + \varepsilon R\left[HS^\alpha_\alpha + Re(\overline{Q}(S^1_1 - S^2_2 + 2iS^1_2))\right] - 3T_{33} + hY_3 = 0; \end{cases} \tag{17}$$

b) Hooke's law

$$\begin{cases} \left[\begin{array}{l} h(T_{11} - T_{22} + 2iT_{12}) = 4\mu\Lambda\left[h\frac{\partial}{\partial \overline{z}}\left(\frac{1}{\Lambda}u_+\right) - \varepsilon RQu_3\right], \\ hT^\alpha_\alpha = h(T^1_1 + T^2_2) = 2(\lambda+\mu)(h\theta - 2H\varepsilon Ru_3) + 2\lambda v_3, \\ hT_+ = \mu\left[2h\frac{\partial u_3}{\partial \overline{z}} + \varepsilon R(Hu_+ + Q\overline{u_+}) + v_+\right], \\ hT_{33} = \lambda(h\theta - 2H\varepsilon Ru_3) + (\lambda+2\mu)v_3, \end{array}\right. \\ \left[\begin{array}{l} h(S_{11} - S_{22} + 2iS_{12}) = 4\mu\Lambda\left[h\frac{\partial}{\partial \overline{z}}\left(\frac{1}{\Lambda}v_+\right) - \varepsilon RQv_3\right], \\ hS^\alpha_\alpha = h(S^1_1 + S^2_2) = 2(\lambda+\mu)(h\rho - 2H\varepsilon Rv_3), \\ hS_+ = h(S_{13} + iS_{23}) = \mu\left[2h\frac{\partial v_3}{\partial \overline{z}} + \varepsilon R(Hv_+ + Q\overline{v_+})\right], \\ hS_{33} = \lambda(h\rho - 2H\varepsilon Rv_3), \end{array}\right. \end{cases} \tag{18}$$

where

$$\theta = \frac{1}{\Lambda}\left(\frac{\partial u_+}{\partial z} + \frac{\partial \overline{u_+}}{\partial \bar z}\right), \quad \rho = \frac{1}{\Lambda}\left(\frac{\partial v_+}{\partial z} + \frac{\partial \overline{v_+}}{\partial \bar z}\right), \quad (u_+ = u_1 + iu_2, \quad v_+ = v_1 + iv_2).$$

c) equilibrium equations in terms of the displacement vector components:

$$\left[\begin{array}{l} 4\mu h^2 \frac{\partial}{\partial \bar z}\left(\frac{1}{\Lambda}\frac{\partial u_+}{\partial z}\right) + 2(\lambda+\mu)h^2\frac{\partial\theta}{\partial \bar z} + 2\lambda h \frac{\partial v_3}{\partial \bar z} - \varepsilon R\left[\frac{4\mu h}{R}\frac{\partial \Lambda Q u_3}{\partial z}\right. \\ \left. + 4(\lambda+\mu)h\frac{\partial H u_3}{\partial \bar z} + 2\mu h\left(H\frac{\partial u_3}{\partial \bar z} + Q\frac{\partial u_3}{\partial z}\right) + \mu(Hv_+ + Q\overline{v_+})\right] \\ +\varepsilon^2 R^2 \mu\left[(2K - H^2 - Q\overline{Q})u_+ - 2HQ\overline{u_+}\right] + h^2 X_+ = 0, \\ \\ \mu(h^2\nabla^2 u_3 + h\rho) + \varepsilon R\left\{2\mu h Re\left[\frac{1}{\Lambda}\frac{\partial(Hu_+ + Q\overline{u_+})}{\partial z}\right.\right. \\ \left.\left. + 4\overline{Q}\frac{\partial}{\partial \bar z}\left(\frac{1}{\Lambda}u_+\right)\right] + 2(\lambda+\mu)h^2 H\theta + 2\lambda h v_3\right\} \\ -4R^2\varepsilon^2\left[(\lambda+\mu)H^2 + 2\mu Q\overline{Q}\right]u_3 + h^2 X_3 = 0, \end{array}\right. \tag{19}$$

$$\left[\begin{array}{l} 4\mu h^2 \frac{\partial}{\partial \bar z}\left(\frac{1}{\Lambda}\frac{\partial v_+}{\partial z}\right) + 2(\lambda+\mu)h^2\frac{\partial\rho}{\partial \bar z} - 3\mu\left(2h\frac{\partial v_3}{\partial \bar z} + v_+\right) \\ -\varepsilon R\left[\frac{4\mu h}{\Lambda}\frac{\partial \Lambda Q v_3}{\partial z} + 4(\lambda+\mu)h\frac{\partial H v_3}{\partial \bar z} + 2\mu h\left(H\frac{\partial v_3}{\partial z} + Q\frac{\partial v_3}{\partial \bar z}\right)\right. \\ \left. +3\mu(Hu_+ + Q\overline{u_+})\right] + \varepsilon^2 R^2\mu\left[(2K - H^2 - Q\overline{Q})v_+ - 2HQ\overline{v_+}\right] \\ +h^2 Y_+ = 0, \\ \\ \mu h^2 \nabla^2 v_3 - 3(\lambda h\theta + (\lambda+2\mu)v_3) \\ +\varepsilon R\left\{2\mu h Re\left[\frac{1}{\Lambda}\frac{\partial(Hv_+ + Q\overline{v_+})}{\partial z} + 4\overline{Q}\frac{\partial}{\partial \bar z}\left(\frac{1}{\Lambda}v_+\right)\right]\right. \\ \left. + 2(\lambda+\mu)h^2 H\rho + 6\lambda H v_3\right\} - 4R^2\varepsilon^2\left[(\lambda+\mu)H^2 + 2\mu Q\overline{Q}\right]v_3 \\ +h^2 Y_3 = 0. \end{array}\right. \tag{20}$$

To determine the components of the displacement vector and the stress tensor we shall use expansion with respect to the small parameter ε:

$$(u_i, v_i) = (\overset{(0)}{u}_i, \overset{(0)}{v}_i) + \varepsilon(\overset{(1)}{u}_i, \overset{(1)}{v}_i) + \varepsilon^2(\overset{(2)}{u}_i, \overset{(2)}{v}_i) + ...,$$

$$(T_{ij}, S_{ij}) = (\overset{(0)}{T}_{ij}, \overset{(0)}{S}_{ij}) + \varepsilon(\overset{(1)}{T}_{ij}, \overset{(1)}{S}_{ij}) + \varepsilon^2(\overset{(2)}{T}_{ij}, \overset{(2)}{S}_{ij}) + ...,$$

$$(X_i, Y_i) = (\overset{(0)}{X}_i, \overset{(0)}{Y}_i) + \varepsilon(\overset{(1)}{X}_i, \overset{(1)}{Y}_i) + \varepsilon^2(\overset{(2)}{X}_i, \overset{(2)}{Y}_i) + ...$$

and then we equate to zero the factors of ε^n. These equations may be written as:

a) equilibrium equations:

$$
\begin{cases}
\dfrac{h}{\Lambda}\dfrac{\partial}{\partial z}(\overset{(n)}{T}_{11}-\overset{(n)}{T}_{22}+2i\overset{(n)}{T}_{12})+h\dfrac{\partial}{\partial \bar z}\overset{(n)}{T}{}^{\alpha}_{\alpha} \\
=-h\overset{(n)}{X}_{+}+R(H\overset{(n-1)}{T}_{+}+Q\overline{\overset{(n-1)}{T}_{+}}), \\
\\
\dfrac{h}{\Lambda}\left(\dfrac{\partial\overset{(n)}{T}_{+}}{\partial z}+\dfrac{\partial\overline{\overset{(n)}{T}_{+}}}{\partial \bar z}\right)=-h\overset{(n)}{X}_{3} \\
-R\left[H\overset{(n-1)}{T}{}^{\alpha}_{\alpha}+Re(\overline{Q}(\overset{(n-1)}{T}{}^{1}_{1}-\overset{(n-1)}{T}{}^{2}_{2}+2i\overset{(n-1)}{T}{}^{1}_{2}))\right], \\
\\
\dfrac{h}{\Lambda}\dfrac{\partial}{\partial \bar z}(\overset{(n)}{S}_{11}-\overset{(n)}{S}_{22}+2i\overset{(n)}{S}_{12})+h\dfrac{\partial}{\partial \bar z}\overset{(n)}{S}{}^{\alpha}_{\alpha}-3\overset{(n)}{T}_{+} \\
=-h\overset{(n)}{Y}_{+}+R(H\overset{(n-1)}{S}_{+}+Q\overline{\overset{(n-1)}{S}_{+}}), \\
\\
\dfrac{h}{\Lambda}\left(\dfrac{\partial\overset{(n)}{S}_{+}}{\partial z}+\dfrac{\partial\overline{\overset{(n)}{S}_{+}}}{\partial \bar z}\right)-3\overset{(n)}{T}_{33} \\
=-h\overset{(n)}{Y}_{3}-R\left[H\overset{(n-1)}{S}{}^{\alpha}_{\alpha}+Re(\overline{Q}(\overset{(n-1)}{S}{}^{1}_{1}-\overset{(n-1)}{S}{}^{2}_{2}+2i\overset{(n-1)}{S}{}^{1}_{2}))\right];
\end{cases}
\tag{21}
$$

b) Hooke's law

$$
\begin{cases}
\left[\begin{array}{l}
h(\overset{(n)}{T}_{11}-\overset{(n)}{T}_{22}+2i\overset{(n)}{T}_{12})=4\mu\Lambda\left[h\dfrac{\partial}{\partial\bar z}\left(\dfrac{1}{\Lambda}\overset{(n)}{u}_{+}\right)-RQ\overset{(n-1)}{u}_{3}\right], \\
hT^{\alpha}_{\alpha}=2(\lambda+\mu)(h\overset{(n)}{\theta}-2HR\overset{(n-1)}{u}_{3})+2\lambda\overset{(n)}{v}_{3}, \\
h\overset{(n)}{T}_{+}=\mu\left[2h\dfrac{\partial\overset{(n)}{u}_{3}}{\partial\bar z}+R(H\overset{(n-1)}{u}_{+}+Q\overline{\overset{(n-1)}{u}_{+}})+\overset{(n)}{v}_{+}\right], \\
h\overset{(n)}{T}_{33}=\lambda(h\overset{(n)}{\theta}-2HR\overset{(n-1)}{u}_{3})+(\lambda+2\mu)\overset{(n)}{v}_{3},
\end{array}\right. \\
\left[\begin{array}{l}
h(\overset{(n)}{S}_{11}-\overset{(n)}{S}_{22}+2i\overset{(n)}{u}_{12})=4\mu\Lambda\left[h\dfrac{\partial}{\partial\bar z}\left(\dfrac{1}{\Lambda}\overset{(n)}{v}_{+}\right)-RQ\overset{(n-1)}{u}_{3}\right], \\
h\overset{(n)}{S}{}^{\alpha}_{\alpha}=2(\lambda+\mu)(h\overset{(n)}{\rho}-2HR\overset{(n-1)}{v}_{3}), \\
h\overset{(n)}{S}_{+}=\mu\left[2h\dfrac{\partial\overset{(n)}{v}_{3}}{\partial\bar z}+R(H\overset{(n-1)}{v}_{+}+Q\overline{\overset{(n-1)}{v}_{+}})\right], \\
h\overset{(n)}{S}_{33}=\lambda(h\overset{(n)}{\rho}-2HR\overset{(n-1)}{v}_{3});
\end{array}\right.
\end{cases}
\tag{22}
$$

c) equilibrium equations in terms of the displacement vector components:

$$\begin{cases} 4\mu h^2 \dfrac{\partial}{\partial \bar{z}}\left(\dfrac{1}{\Lambda}\dfrac{\partial \overset{(n)}{u}_+}{\partial z}\right) + 2(\lambda+\mu)h^2 \dfrac{\partial \overset{(n)}{\theta}}{\partial \bar{z}} + 2\lambda h \dfrac{\partial \overset{(n)}{v}_3}{\partial \bar{z}} = \overset{(n)}{L}_+, \\ \mu h^2 \nabla^2 \overset{(n)}{v}_3 - 3(\lambda h \overset{(n)}{\theta} + (\lambda+2\mu)\overset{(n)}{v}_3) = \overset{(n)}{M}_3, \end{cases} \tag{23}$$

$$\begin{cases} 4\mu h^2 \dfrac{\partial}{\partial \bar{z}}\left(\dfrac{1}{\Lambda}\dfrac{\partial \overset{(n)}{v}_+}{\partial z}\right) + 2(\lambda+\mu)h^2 \dfrac{\partial \overset{(n)}{\rho}}{\partial \bar{z}} - 3\mu\left(2h \dfrac{\partial \overset{(n)}{u}_3}{\partial \bar{z}} + \overset{(n)}{v}_+\right) = \overset{(n)}{M}_+, \\ \mu h (h \nabla^2 \overset{(n)}{u}_3 + \overset{(n)}{\rho}) = \overset{(n)}{L}_3, \end{cases} \tag{24}$$

where

$$\begin{aligned} \overset{(n)}{L}_+ = & -h^2 \overset{(n)}{X}_+ + R\left[\frac{4\mu h}{\Lambda}\frac{\partial \Lambda Q \overset{(n)}{u}_3}{\partial z} + 4(\lambda+\mu)h \frac{\partial H \overset{(n-1)}{u}_3}{\partial \bar{z}} \right. \\ & \left. + 2\mu h\left(H \frac{\partial \overset{(n-1)}{u}_3}{\partial \bar{z}} + Q \frac{\partial \overset{(n-1)}{u}_3}{\partial z}\right) + \mu\left(H \overset{(n-1)}{v}_+ + Q \overline{\overset{(n-1)}{v}_+}\right)\right] \\ & -\mu R^2 \left[(2K - H^2 - Q\overline{Q})\overset{(n-2)}{u}_+ - 2HQ \overline{\overset{(n-2)}{u}_+}\right], \end{aligned}$$

$$\begin{aligned} \overset{(n)}{M}_+ = & -h^2 \overset{(n)}{Y}_+ + R\left[\frac{4\mu h}{\Lambda}\frac{\partial \Lambda Q \overset{(n-1)}{v}_3}{\partial z} + 4(\lambda+\mu)h \frac{\partial H \overset{(n-1)}{v}_3}{\partial \bar{z}} \right. \\ & \left. + 2\mu h\left(H \frac{\partial \overset{(n-1)}{v}_3}{\partial \bar{z}} + Q \frac{\partial \overset{(n-1)}{v}_3}{\partial z}\right) + 3\mu\left(H \overset{(n-1)}{u}_+ + Q \overline{\overset{(n-1)}{u}_+}\right)\right] \\ & -\mu R^2 \left[(2K - H^2 - Q\overline{Q})\overset{(n-2)}{v}_+ - 2HQ \overline{\overset{(n-2)}{v}_+}\right], \end{aligned}$$

$$\begin{aligned} \overset{(n)}{L}_3 = & -h^2 \overset{(n)}{X}_3 - R\left\{2\mu h Re\left[\frac{1}{\Lambda}\frac{\partial\left(H \overset{(n-1)}{u}_+ + Q \overline{\overset{(n-1)}{u}_+}\right)}{\partial z} + 4\overline{Q}\frac{\partial}{\partial \bar{z}}\left(\frac{1}{\Lambda}\overset{(n-1)}{u}_+\right)\right] \right. \\ & \left. + 2(\lambda+\mu)hH \overset{(n-1)}{\theta} + 2\lambda \overset{(n-1)}{v}_3\right\} + 4R^2 \varepsilon^2 \left[(\lambda+\mu)H^2 + 2\mu Q\overline{Q}\right] \overset{(n-2)}{u}_3, \end{aligned}$$

$$\begin{aligned} \overset{(n)}{M}_3 = & -h^2 \overset{(n)}{Y}_3 - R\left\{2\mu h Re\left[\frac{1}{\Lambda}\frac{\partial\left(H \overset{(n-1)}{v}_+ + Q \overline{\overset{(n-1)}{v}_+}\right)}{\partial z} + 4\overline{Q}\frac{\partial}{\partial \bar{z}}\left(\frac{1}{\Lambda}\overset{(n-1)}{v}_+\right)\right] \right. \\ & \left. + 2(\lambda+\mu)hH \overset{(n-1)}{\rho} + 6\lambda H \overset{(n-1)}{v}_3\right\} + 4R^2 \left[(\lambda+\mu)H^2 + 2\mu Q\overline{Q}\right] \overset{(n-2)}{v}_3; \end{aligned}$$

d) for the boundary conditions (7), we have:

$$\begin{cases} T_{(ll)}+iT_{(ls)}=\dfrac{1}{2}\left[T^{\alpha}_{\alpha}-(T_{11}-T_{22}+2iT_{12})\left(\dfrac{d\bar z}{ds}\right)^2\right]=\overset{(0)}{f}_1+i\overset{(0)}{f}_2,\\ T_{(ln)}=-Im\left(T_+\dfrac{d\bar z}{ds}\right)=\overset{(0)}{f}_3,\\ S_{(ll)}+iS_{(ls)}=\dfrac{1}{2}\left[S^{\alpha}_{\alpha}-(S_{11}-S_{22}+2iS_{12})\left(\dfrac{d\bar z}{ds}\right)^2\right]=\overset{(1)}{f}_1+i\overset{(1)}{f}_2,\\ S_{(ln)}=-Im\left(S_+\dfrac{d\bar z}{ds}\right)=\overset{(1)}{f}_3, \end{cases} \Rightarrow$$

$$\begin{cases} \dfrac{\lambda+\mu}{2\mu}h\overset{(n)}{\theta}+\dfrac{\lambda}{2\mu}\overset{(n)}{v}_3-h\Lambda\dfrac{\partial}{\partial\bar z}\left(\dfrac{1}{\Lambda}\overset{(n)}{u}_+\right)\left(\dfrac{d\bar z}{ds}\right)^2\\ =\dfrac{h}{2\mu}\overset{(0,n)}{f}_++R\left[\dfrac{\lambda+\mu}{\mu}H-\Lambda Q\left(\dfrac{d\bar z}{ds}\right)^2\right]\overset{(n-1)}{u},\\ 2hIm\left(\dfrac{\partial\overset{(n)}{v}_3}{\partial\bar z}\dfrac{d\bar z}{ds}\right)=-\dfrac{h}{\mu}\overset{(1,n)}{f}_3-RIm\left[\left(H\overset{(n-1)}{v}_++Q\overline{\overset{(n-1)}{v}_+}\right)\left(\dfrac{d\bar z}{ds}\right)\right],\\ h\left[\dfrac{\lambda+\mu}{2\mu}\overset{(n)}{\rho}-\Lambda\dfrac{\partial}{\partial\bar z}\left(\dfrac{1}{\Lambda}\overset{(n)}{v}_+\right)\left(\dfrac{d\bar z}{ds}\right)^2\right]\\ =\dfrac{h}{2\mu}\overset{(1,n)}{f}_++R\left[\dfrac{\lambda+\mu}{\mu}H-\Lambda Q\left(\dfrac{d\bar z}{ds}\right)^2\right]\overset{(n-1)}{v}_3,\\ Im\left[\left(2h\dfrac{\partial\overset{(n)}{u}_3}{\partial\bar z}+\overset{(n)}{v}_+\right)\dfrac{d\bar z}{ds}\right]\\ =-\dfrac{h}{\mu}\overset{(0,n)}{f}_3-RIm\left[\left(H\overset{(n-1)}{u}_++Q\overline{\overset{(n-1)}{u}_+}\right)\dfrac{d\bar z}{ds}\right]. \end{cases}$$

The general solution of the homogeneous systems (23) and (24) can be written in an explicit form [1],[2]:

$$\begin{aligned} u_+ &= -\frac{\lambda h}{6(\lambda+2\mu)}\frac{\partial\omega}{\partial\bar z}-\frac{5\lambda+6\mu}{3\lambda+2\mu}\frac{1}{\pi}\int\limits_s\int\frac{\varphi'(\zeta)dS}{\overline{\zeta}-\bar z}\\ &+\left(\frac{1}{\pi}\int\limits_s\int\frac{dS}{\overline{\zeta}-\bar z}\right)\overline{\varphi'(z)}-\overline{\psi(z)},\\ v_3 &= \omega-\frac{2\lambda h}{3\lambda+2\mu}[\varphi'(z)+\overline{\varphi'(z)}], \end{aligned}$$

$$v_+ = i\frac{\partial\chi}{\partial\bar z} - 2h\overline{\Psi'(z)} - \frac{1}{\pi}\int_s\int\frac{\Phi'(\zeta)dS}{\overline{\zeta}-\bar z} - \left(\frac{1}{\pi}\int_s\int\frac{dS}{\overline{\zeta}-\bar z}\right)\overline{\Phi'(z)}$$

$$+ \frac{4(\lambda+2\mu)h^2}{3\mu}\overline{\Phi''(z)},$$

$$u_3 = \Psi(z) + \overline{\Psi(z)} - \frac{1}{h}\frac{1}{\pi}\int_s\int\left(\Phi'(\zeta)+\overline{\Phi'(\zeta)}\right)\ln|\zeta - z|ds$$

$$(dS = \Lambda(\zeta,\overline{\zeta})d\xi\eta, \quad \zeta = \xi + i\eta)$$

where $\varphi(z)$, $\psi(z)$, $\Phi(z)$ and $\Psi(z)$ are analytic functions of $z = x^1 + ix^2$, and ω, χ are general solutions of the following homogeneous equations

$$\nabla^2\omega - \frac{3(\lambda+\mu)}{(\lambda+2\mu)h^2}\omega = 0,$$

$$\nabla^2\chi - \frac{3}{h^2}\chi = 0,$$

$$(\nabla^2 = \frac{4}{\Lambda}\frac{\partial^2}{\partial z\partial\bar z}).$$

References

[1] Vekua, I.N., Some General Methods for Constructing different Versions of Shell Theory. Boston-London-Melburne, 1986.

[2] Vekua, I.N., On Construction of Approximate Solutions of Equations of the Shallow Spherical Shell. Int. J. Solids Structures, 5 (1969), 991-1003.

[3] Meunargia, T.V., The Method of a Small Parameter for Non-Linear Spherical Shells. Reports of Enlarged Session of the Seminar of I. Vekua Institute of Applied Mathematics, vol. 14, N2, 1999.

Received December, 12, 2003; revised May, 21, 2004; accepted June, 6, 2004.