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FORMULAE FOR THE SINGULARITIES AT INFINITY OF PLANE ALGEBRAIC CURVES

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Abstract. This paper collects together formulae concerning singularities at infinity of plane algebraic curves. For every polynomial $f : \mathbb{C}^2 \to \mathbb{C}$ with isolated critical points we consider the well known topological invariants $\mu(f)$ (the global Milnor number) and $\lambda(f)$ (the sum of all the "jumps" in the Milnor number at infinity). We prove new estimations for $\mu(f) + \lambda(f)$ and show that the number of critical values at infinity of f is less than or equal to $((d-1)^2 - \mu(f) - \lambda(f))/d$ where d is the degree of f. We give also some estimations for the Łojasiewicz exponent at infinity.

Introduction. The aim of this paper is to present some old and new results concerning the invariants at infinity of plane algebraic curves. Our intention is to complete the collection of formulae given by Pham in the appendix to his article [**34**]. We base our considerations on the properties of intersection numbers (see [**6**, Appendix D.3]) and on a variant of the Riemann-Hurwitz formula.

If $f,\ g \in \mathbb{C}[X,Y]$ and $p = (a,b) \in \mathbb{C}^2$ then $(f,g)_p = \dim_{\mathbb{C}} \mathcal{O}_p/(f,g)$ where $\mathcal{O}_p$ is the local ring of $\mathbb{C}^2$ at p and (f,g) is the ideal generated by f and g in $\mathcal{O}_p$. Let $(f,g)_{\mathbb{C}^2} = \sum_{p\in\mathbb{C}^2}(f,g)_p$. Then $(f,g)_{\mathbb{C}^2} < +\infty$ if and only if the system of equations $f = g = 0$ has finite number of solutions in $\mathbb{C}^2$. We put $\mu_p(f) = \left(\frac{\partial f}{\partial X}, \frac{\partial f}{\partial Y}\right)_p$, $r_p(f) =$ the number of irreducible factors of $f(a+X, b+Y)$ in the ring of formal power series $\mathbb{C}[[X,Y]]$, and $\operatorname{ord}_p f = \inf\{k \in \mathbb{N} : d^k f(p) \neq 0\}$.

Let us recall Teissier's lemma: $(f, \frac{\partial f}{\partial Y})_p = \mu_p(f) + (f, X-a)_p - 1$ *for every point* $p = (a,b)$ *of the reduced curve* $f(X,Y) = 0$ *such that the line* $X = a$

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intersects the curve in a finite number of points. This is the two dimensional case of a formula due to Teissier (see [**38**, Chap. II, Theorem 5] or [**39**, Chap. II, Proposition 1.2]). Note here that the expression $(f, \frac{\partial f}{\partial Y})_p - (f, X - a)_p + 1$ appears in many classical texts (see for example [**27**, Chap. X, p. 181] where a notion equivalent to the Milnor number of a plane branch is introduced).

In Section 1, a projective version of Teissier's lemma enables us to give very simple proofs of Plücker's and Noether's formulae and of the formula due to Krasiński (see [**30**, Chap. I, Theorem 6.4]) for the degree of the discriminant.

In Section 2 we consider polynomial functions on affine algebraic curves and introduce some notions which are useful later.

Section 3 is central to this paper; we give a global version of Teissier's lemma (Theorem 3.3) and present some applications. It turns out that the well-known invariants $\mu(f)$ (the total Milnor number) and $\lambda(f)$ (the jump of Milnor numbers at infinity) (see [**7**], [**11**], [**17**]) of a polynomial $f : \mathbb{C}^2 \to \mathbb{C}$ with a finite number of critical points can be described in terms of the restriction of f to the generic polar $\frac{\partial f}{\partial Y} = 0$. This permits us to give new estimations for $\mu(f) + \lambda(f)$ and for the number of critical values at infinity of a polynomial f (Theorems 3.4, 3.5).

In Section 4 we present new proofs of well-known formulae due to different authors. In particular we give a simple proof of the description of the Euler characteristic of the fiber $f^{-1}(t)$ due to Suzuki [**36**], [**37**] and Gavrilov [**17**], [**18**].

We end the paper with some estimations of the Łojasiewicz exponent (Theorem 5.2) and open questions (Section 6). Earlier versions of this paper included results of polynomials without critical points; these are to appear in [**22**]

1. Projective Teissier's lemma, Applications. If $C \subset \mathbb{P}^2(\mathbb{C})$ is a (reduced) projective plane curve then for every $p \in C$ we put $\mu_p(C)$ = the Milnor number of C at p, $r_p(C)$ = the number of branches of C centered ar p, $\mathrm{ord}_p(C)$ = the order of C at p.

If we choose a local reduced equation $f(X, Y) = 0$ of C, where $f \in \mathbb{C}[X, Y]$ is a polynomial in affine coordinates X, Y, then $\mu_p(C) = \mu_p(f)$, $r_p(C) = r_p(f)$ and $\mathrm{ord}_p(C) = \mathrm{ord}_p f$.

If C, D are two projective curves (possibly with multiple components) then we denote by $(C, D)_p$ the intersection multiplicity of C, D at p. If $f(X, Y) = 0$ and $g(X, Y) = 0$ are local equations of C and D respectively in affine coordinates X, Y then $(C, D)_p = (f, g)_p$.

Let $C \subset \mathbb{P}^2(\mathbb{C})$ be a reduced projective curve and let $q = (q_0{:}q_1{:}q_2) \in \mathbb{P}^2(\mathbb{C})$. If $F(X, Y, Z) = 0$ is the minimal equation of C and $q_0\frac{\partial F}{\partial X} + q_1\frac{\partial F}{\partial Y} + q_2\frac{\partial F}{\partial Z} \neq 0$ in $\mathbb{C}[X, Y, Z]$ then the polar $\nabla_q C$ is defined to be the curve (possibly with multiple components) given by the equation $q_0\frac{\partial F}{\partial X} + q_1\frac{\partial F}{\partial Y} + q_2\frac{\partial F}{\partial Z} = 0$. Recall that the notion of the polar is projectively invariant. The following projective version of Teissier's lemma is basic for us.

LEMMA 1.1. *For every $p \in C$, $p \neq q$*

$$(C, \nabla_q C)_p = \mu_p(C) + (C, \overline{qp})_p - 1 \tag{1.1}$$

where $\overline{qp}$ is the line passing through q and p.

PROOF. We choose the coordinates so that $p = (0{:}0{:}1)$ and $q = (0{:}1{:}0)$. Then (1.1) reduces to Teissier's local formula: $\left(f, \frac{\partial f}{\partial Y}\right)_0 = \left(\frac{\partial f}{\partial X}, \frac{\partial f}{\partial Y}\right)_0 + (f, X)_0 - 1$. $\square$

Let C be a reduced projective curve of degree d and L be a line such that $L \not\subset C$. We put $\epsilon(C, L) = \sum_{p \in C \cap L}\big((C, L)_p - \operatorname{ord}_p C\big)$ and call $\epsilon(C, L)$ the multiplicity of the tangent L to C. Note that $\epsilon(C, L) > 0$ iff there is a point $p \in C$ such that L is a tangent to C at p. Let $q \notin C$. If L is a tangent to C at p passing through q then $\epsilon(C, L) = \sum\big((C, \nabla_q C)_p - (\mu_p(C) + \operatorname{ord}_p(C) - 1)\big)$ the sum being over all $p \in C \cap L$ by the projective version of Teissier's lemma. Applying Bezout's theorem to the curves C and $\nabla_q C$, we get

PLÜCKER'S FORMULA. *The number of tangents to C passing through q is equal to*

$$d(d-1) - \sum_{p \in C}(\mu_p(C) + \operatorname{ord}_p(C) - 1).$$

The above formulation of Plücker's formula and the generalization to hypersurfaces are due to Teissier [**40**, Appendix 2]. Another application of 1.1 is the following

MAX NOETHER'S FORMULA. *The Euler characteristic $\chi(C)$ of the curve $C \subset \mathbb{P}^2(\mathbb{C})$ of degree $d > 0$ is equal to*

$$\chi(C) = -d(d-3) + \sum_{p \in C} \mu_p(C). \tag{1.2}$$

PROOF. Let $q \notin C$ and let us identify the pencil of lines through q with the projective line $\mathbb{P}(\mathbb{C})$. Let $\pi_q : C \to \mathbb{P}(\mathbb{C})$ be given by the formula $\pi_q(p) = \overline{qp}$. It is easy to check that $\operatorname{mult}_p \pi_q = (C, \overline{qp})_p$ and $\deg \pi_q = d$. Then by the Riemann–Hurwitz formula (cf. Appendix A.3) we get $\chi(C) = d\chi(\mathbb{P}^2(\mathbb{C})) - \sum_{p \in C}((C, \overline{qp})_p - 1) = 2d - \sum_{p \in C}((C, \nabla_q C)_p - \mu_p(C)) = -d(d-3) + \sum_{p \in C} \mu_p(C)$ by (1.1) and Bezout's theorem applied to C and $\nabla_q C$. $\square$

COROLLARY 1.2. *Let $\nu : \tilde{C} \to C$ be the normalization of the curve C. Then $\chi(\tilde{C}) = -d(d-3) + \sum_{p \in C}(\mu_p(C) + r_p(C) - 1)$. The genus g of an irreducible curve C satisfies $2g = (d-1)(d-2) - \sum_{p \in C}(\mu_p(C) + r_p(C) - 1)$.*

Proof. We have $\chi(\tilde{C}) = \chi(C) + \sum_{p\in C}(r_p(C) - 1)$ (Appendix A.2) hence the formula for $\chi(\tilde{C})$ follows from (1.2). If C is irreducible, we get the formula for g by using the relation $\chi((\tilde{C}) = 2 - 2g$. □

The classical version of Max Noether's formula gives the genus of $\tilde{C}$ in terms of infinitely near points ([**32**], [**5**]). Formula (1.2) appears in [**16**] and in [**25**] (Lemma 8).

We end this section by proving a formula for the degree of the discriminant due to Krasiński [**30**, Theorem 6.4]. Let $f(X,Y) \in \mathbb{C}[X,Y]$ be a reduced (without multiple factors) polynomial such that $\deg_Y f = \deg f = d > 0$ and let C be the projective closure of the affine curve $f(X,Y) = 0$. We put $C_\infty =$ the set of points at infinity of C and $c = \# C_\infty$

Krasiński's formula. *Let $\Delta(X) = \operatorname{disc}_Y f(X,Y)$ be the Y–discriminant of the polynomial f. Then*

$$\deg \Delta(X) = d(d-2) + c - \sum_{p\in C_\infty} \mu_p(C).$$

Proof. Let $F(X,Y,Z)$ be the homogeneous polynomial corresponding to $f(X,Y)$. Then $\frac{\partial F}{\partial Y}$ corresponds to $\frac{\partial f}{\partial Y}$, because we have assumed $\deg_Y f = \deg f$. Let $q = (0{:}1{:}0)$. Then C is given by $F(X,Y,Z) = 0$ and $\nabla_q C$ by $\frac{\partial F}{\partial Y} = 0$. By a classical formula (see [**45**], p. 111) we have $\deg \Delta(X) = \deg \operatorname{Res}_Y(f, \frac{\partial f}{\partial Y}) = \sum_{p\in\mathbb{C}^2}(f, \frac{\partial f}{\partial Y})_p$. Consequently

$$\begin{aligned}\deg \Delta(X) = &\sum_{p\in C\setminus L_\infty} (C, \nabla_q C)_p = \sum_{p\in C}(C,\nabla_q C)_p - \sum_{p\in L_\infty}(C,\nabla_q C)_p = \\ &= d(d-1) - \sum_{p\in C_\infty}(\mu_p(C) + (C, \overline{qp})_p - 1) = d(d-2) + c - \sum_{p\in C_\infty}\mu_p(C).\end{aligned}$$

□

We have applied the projective version of Teissier's lemma and Bezout's theorem to C and L_∞.

2. Polynomials on affine plane curves. Let $f = f(X,Y)$, $g = g(X,Y)$ be nonconstant polynomials and suppose that $g|f^{-1}(0)$ has finite fibers. Then g is nonconstant on irreducible components of $f^{-1}(0)$.

Let $\text{g.deg}(g|f) = \sup\{\, (f, g-t)_{\mathbb{C}^2} : t \in \mathbb{C}\,\}$ ("geometric degree of g with respect to the curve $f = 0$") and for every $t\in\mathbb{C}$: $\delta^t(g|f) = \text{g.deg}(g|f) - (f, g - t)_{\mathbb{C}^2}$ ("the defect of the fiber of $g|\{f = 0\}$ over $t \in \mathbb{C}$"). It is easy to see that $0 \le \delta^t(g|f) \le \text{g.deg}(g|f) \le (\deg f)(\deg g)$.

Let $[x]$ be the integral part of $x \in \mathbb{R}$.

Proposition 2.1. *The set* $D = \{\, t \in \mathbb{C} : \delta^t(g|f) > 0 \,\}$ *is finite, moreover,*

$$\#D \leq \left[\deg f - \frac{\mathrm{g.deg}(g|f)}{\deg g}\right].$$

The mapping $f^{-1}(0) \setminus g^{-1}(D) \to \mathbb{C} \setminus D$ *induced by* g *is proper. The set* D *is the smallest set with this property.*

Proposition 2.1 is an easy consequence of two lemmas presented below.

Lemma 2.2. *Suppose that* f *is a* Y*–monic polynomial and let* $R(X,T)$ *be the* Y*–resultant of* $f(X,Y)$, $g(X,Y) - T$. *Let us write* $R(X,T) = R_0(T)X^N + \cdots + R_N(T)$, $R_0(T) \neq 0$ *in* $\mathbb{C}[T]$. *Then*

(i) $\mathrm{g.deg}(g|f) = \deg_X R(X,T) = N$,
(ii) $\delta^t(g|f) = N - \deg_X R(X,t)$,
(iii) $\deg R_0(T) \leq \left[\deg f - \frac{N}{\deg g}\right]$.

Proof. Properties (i) and (ii)follow from the classical formula $\deg_X R(X,t) = (f, g-t)_{\mathbb{C}^2}$ (cf. [**45**], p. 111) and from the definitions given above. In order to check the estimation for $\deg R_0(T)$, let us put weight $X = 1$, weight $T = \deg g$. Then by the well-known property of the resultant, weight $R(X,T) \leq (\deg f)(\deg g)$ and consequently weight $R_0(T)X^N \leq (\deg f)(\deg g)$, that is weight $R_0(T) \leq (\deg f)(\deg g) - N$. But weight $R_0(T) = (\deg R_0(T))(\text{weight}\, T) = (\deg R_0(T))(\deg g)$ and we get $\deg R_0(T) \leq \left[\deg f - \frac{N}{\deg g}\right]$. $\square$

Let $R(X,T) = R_0(T)X^N + \cdots + R_N(T) \in \mathbb{C}[X,T]$ be a polynomial such that $N > 0$ and $R_0(T) \neq 0$ in $\mathbb{C}[T]$. Set $V = \{\, (x,t) \in \mathbb{C}^2 : R(x,t) = 0 \,\}$, $D = \{\, t \in \mathbb{C} : R_0(t) = 0 \,\}$ and let $\pi : V \to \mathbb{C}$ be given by $\pi(x,t) = t$.

Lemma 2.3. *The mapping* $V \setminus \pi^{-1}(D) \to \mathbb{C} \setminus D$ *induced by* π *is proper. The set* D *is the smallest set with this property.*

Proof. Let K be a compact subset of $\mathbb{C}$. Then $\pi^{-1}(K)$ is compact if and only if $K \cap D = \emptyset$, by continuity of roots, (see [**41**], Lemma 1) and the lemma follows. $\square$

In order to give another description of the introduced notions we need a formula for intersection multiplicity of two affine curves $f(X,Y) = 0$, $g(X,Y) = 0$. Let us consider the field $\mathbb{C}((X^{-1}))$ of Laurent series centered at infinity. If $\phi(X) \in \mathbb{C}((X^{-1}))$ then $\phi(X) = c_n X^n + \cdots + c_1 X + c_0 + c_{-1}X^{-1} + \cdots + c_{-m}X^{-m} + \ldots$. We put $\deg \phi = \sup\{\, j : c_j \neq 0 \,\}$ with the convention $\sup \emptyset = -\infty$. We have (see [**1**])

PUISEUX'S THEOREM 2.4. *If* $f(X,Y) \in \mathbb{C}((X^{-1}))[Y]$ *is a monic irreducible in* $\mathbb{C}((X^{-1}))[Y]$ *polynomial of degree* d *then there is a Laurent series* $y(\tau) \in \mathbb{C}((\tau^{-1}))$, τ *a new variable such that*

$$f(\tau^d, Y) = \prod_{\epsilon^d=1} (Y - y(\epsilon\tau)) \qquad \text{in } \mathbb{C}((\tau^{-1}))[Y].$$

Now we can formulate

ZEUTHEN'S RULE 2.5. (*For the number of intersections of affine curves*) *Let* f, $g \in \mathbb{C}[X,Y]$ *be coprime,* $f = f(X,Y)$ *a monic polynomial with respect to* Y. *Suppose that* $f(X,Y) = \prod_{i=1}^{s} f_i(X,Y)$ *where* $f_i(X,Y) \in \mathbb{C}((X^{-1}))[Y]$ *is irreducible of degree* d_i. *Let* $y_i(\tau) \in \mathbb{C}((\tau^{-1}))$ *be such that* $f_i(\tau^{d_i}, y_i(\tau)) = 0$ *for* $i = 1, \ldots, s$. *Then*

$$(f,g)_{\mathbb{C}^2} = \sum_{i=1}^{s} \deg g(\tau^{d_i}, y_i(\tau)).$$

PROOF. We have $(f,g)_{\mathbb{C}^2} = \sum_{p\in\mathbb{C}^2}(f,g)_p = \deg \mathrm{Res}_Y(f,g)$, then we use the expression of the resultant in terms of roots and Puiseux's theorem. $\square$

With the assumptions of 2.5 we call the pairs $(\tau^{d_i}, y_i(\tau))$ cycles at infinity of the curve $f(X,Y) = 0$. The lemma below follows immediately from Zeuthen's Rule.

LEMMA 2.6. *Let* $(\tau^{d_i}, y_i(\tau))$, $i = 1, \ldots, s$ *be the cycles at infinity of the curve* $f(X,Y) = 0$. *Let* I_+ *(resp.* I_0, I_-*) be the set of all* $i \in \{1, \ldots, s\}$ *such that* $\deg g(\tau^{d_i}, y_i(\tau)) > 0$ *(resp.* $= 0$, < 0*). For every* $i \in I_0$ *we put* $t_i =$ *the unique non–zero complex number such that* $\deg(g(\tau^{d_i}, y_i(\tau)) - t_i) < 0$. *Then*

(1) $\mathrm{g.deg}(g|f) = \sum_{i\in I_+} \deg g(\tau^{d_i}, y_i(\tau))$,
(2) $\delta^t(g|f) = 0$ *if* $t \notin \{t_i\}_{i\in I_0} \bigcup \{0\}$,
(3) $\delta^0(g|f) = -\sum_{i\in I_-} \deg g(\tau^{d_i}, y_i(\tau))$,
(4) *if* $t \in \{t_i\}_{i\in I_0}$ *then* $\delta^t(g|f) = -\sum \deg(g(\tau^{d_j}, y_j(\tau)) - t_j)$ *the sum being over all* $j \in I_0$ *such that* $t_j = t$.

In the sequel we denote $\delta(g|f) = \sum_{t\in\mathbb{C}} \delta^t(g|f)$.

Let $\mathbb{C}((X^{-1}))^* = \bigcup_{n\geq 1} \mathbb{C}((X^{-1/n}))$ be the field of Puiseux series "centered at infinity". From Puiseux's theorem it follows that $\mathbb{C}((X^{-1}))^*$ is an algebraically closed field. We extend the notion of degree to the field of Puiseux series in a natural way. Then $-\deg$ is a valuation of $\mathbb{C}((X^{-1}))^*$. The following version of Lemma 2.6 enables us to calculate $\mathrm{g.deg}(g|f)$ and $\delta(g|f)$ without using the resultant.

LEMMA 2.6'. *Let* $Y_j(X)$, $j = 1, \ldots, d$ *be the roots of* $f(X,Y)$ *(*Y*–monic polynomial) counted with multiplicities. Let* J_+ *(resp.* J_0, J_-*) be the set of all*

$j \in \{1, \ldots, d\}$ *such that* $\deg g(X, Y_j(X)) > 0$ *(resp.* $= 0$, < 0*). For every* $j \in J_0$ *we put* $t_j =$ *the unique nonzero complex number such that* $\deg(g(X, Y_j(X)) - t_j) < 0$. *Then*

(1) $\text{g.deg}(g|f) = \sum_{j \in J_+} \deg g(X, Y_j(X))$,
(2) $\delta^t(g|f) = 0$ *if* $t \notin \{t_j\}_{j \in J_0} \cup \{0\}$,
(3) $\delta^0(g|f) = -\sum_{j \in J_-} \deg g(X, Y_j(X))$,
(4) *if* $t \in \{t_i\}_{i \in J_0}$ *then* $\delta^t(g|f) = -\sum \deg(g(X, Y_j(X)) - t_j)$ *the sum being over all* $j \in J_0$ *such that* $t_j = t$.

For every $z = (x, y) \in \mathbb{C}^2$ we put $|z| = \max(|x|, |y|)$. The Łojasiewicz exponent at infinity $\mathcal{L}_\infty(g|f)$ of the polynomial function $g|f^{-1}(0)$ is the supremum of the set $\{\, \theta \in \mathbb{R} : \exists C, R > 0 \forall z \in f^{-1}(0) \quad |g(z)| \geq C|z|^\theta \text{ if } |z| \geq R \,\}$. Since $g|f^{-1}(0)$ has finite fibers the aforementioned set is not empty. We omit a simple proof of the following

Lemma 2.7. *Suppose that* $\deg_Y f = \deg f = d > 0$ *and let* $(\tau^{d_i}, y_i(\tau))$ $(i = 1, \ldots, s)$ *be the cycles at infinity of the curve* $f(X, Y) = 0$. *Then*

$$\mathcal{L}_\infty(g|f) = \min_{i=1}^{s} \left\{ \frac{1}{d_i} \deg g(\tau^{d_i}, y_i(\tau)) \right\}.$$

Note that the above formula can also be written in the following form

Lemma 2.7'. *With the notations introduced above*

$$\mathcal{L}_\infty(g|f) = \min_{j=1}^{d} \{ \deg g(X, Y_j(X)) \}.$$

The Newton diagram at infinity $\Delta_\infty(F)$ of the polynomial $F(X, T) = \sum a_{\alpha,\beta} X^\alpha T^\beta$ is the convex hull of the set $\{\, (\alpha, \beta) \in \mathbb{N}^2 : a_{\alpha,\beta} \neq 0 \,\} \cup \{(0, 0)\}$. The sides of $\Delta_\infty(F)$ are the straight lines passing through successive vertices of $\Delta_\infty(F)$. The last side passes through the vertex of $\Delta_\infty(F)$ lying on the horizontal axis. Recall that the slope of the line $\beta = \lambda\alpha + \mu$ is λ. The lines parallel to the vertical axis have the slope ∞. We put $1/\infty = 0$. The following proposition is a version of the result due to Chądzyński and Krasiński [**8**, Theorem 3.1].

Proposition 2.8. *Let* $\deg_Y f = \deg f = d > 0$. *Consider* $R(X, T) = \text{Res}_Y(f(X, Y), g(X, Y) - T)$. *If* θ *is the slope of the last side of* $\Delta_\infty(R)$ *then*

$$\mathcal{L}_\infty(g|f) = -\frac{1}{\theta}.$$

Proof. Expressing the resultant in terms of its roots we get $R(X, T) = (-1)^d \prod_{j=1}^{d} (T - g(X, Y_j(X)))$ in $\mathbb{C}((X^{-1}))^*[T]$. By Property B.1 (Appendix B) applied to the field $\mathbb{C}((X^{-1}))^*$ with valuation $-\deg$, we check that

$\min_{j=1}^{d} \deg g(X, Y_j(X)) = -1/\theta$. Now Proposition 2.8 follows from Lemma 2.7'. $\square$

REMARK TO PROPOSITION 2.8. Write $R(X,T) = R_0(T)X^N + \cdots + R_N(T)$, $R_0(T) \neq 0$. Then:

$\theta = -\max_{j=1}^{N} \frac{\deg R_j}{j}$ if $R_0(T) = \text{const}$,
$\theta = \infty$ if $R_0(0) \neq 0$ and $R_0(T) \neq \text{const}$,
$\theta = \min_{j=0}^{r} \frac{\text{ord}_0(R_j)}{r+1-j}$ if $R_0(0) = \cdots = R_r(0) = 0$ and $R_{r+1}(0) \neq 0$.

Now we are able to give some estimations of the Łojasiewicz exponent.

THEOREM 2.9. *With the notation introduced above:*

(i) $\mathcal{L}_\infty(g|f) > 0$ *if and only if* $g|f^{-1}(0)$ *is proper. Moreover* $\mathcal{L}_\infty(g|f) \geq 1/\deg f$. *If the curve* $f(X,Y) = 0$ *has exactly one cycle at infinity then* $\mathcal{L}_\infty(g|f) = (f,g)_{\mathbb{C}^2}/\deg f$.
(ii) $\mathcal{L}_\infty(g|f) < 0$ *if and only if* $\delta^0(g|f) > 0$. *If* $\mathcal{L}_\infty(g|f) < 0$ *then* $\deg f - \frac{\text{g.deg}(g|f)}{\deg g} \geq 1$ *and* $-\delta^0(g|f) \leq \mathcal{L}_\infty(g|f) \leq -\delta^0(g|f) \left[\deg f - \frac{\text{g.deg}(g|f)}{\deg g}\right]^{-1}$.
(iii) $\mathcal{L}_\infty(g|f)$ *is a rational number. If* $\mathcal{L}_\infty(g|f) = p/q$ *with coprime integers then* $q \leq \deg f$. *If* $\mathcal{L}_\infty(g|f) < 0$ *then* $\deg f > 1$ *and* $q \leq \deg f - 1$.

PROOF. Part (i) follows immediately from 2.7, 2.6 and 2.1. The inequality $\mathcal{L}_\infty(g|f) < 0$ is equivalent to $\delta^0(g|f) > 0$ by 2.7 and 2.6. Moreover $\mathcal{L}_\infty(g|f) \geq -\delta^0(g|f)$. The estimation $\deg f - \frac{\text{g.deg}(g|f)}{\deg g} \geq 1$ follows from Proposition 2.1. To get the upper bound for $\mathcal{L}_\infty(g|f)$, consider the resultant $R(X,T) = \text{Res}_Y(f(X,Y), g(X,Y) - T)$ and write $R(X,T) = R_0(T)X^N + \cdots + R_N(T)$, $R_0(T) \neq 0$. By Lemma 2.2 we have g.deg$(g|f) = \deg_X R(X,T) = N$,
(b) $\delta^0(g|f) = N - \deg_X R(X,0)$,
(c) $\deg R_0(T) \leq \left[\deg f - \frac{N}{\deg g}\right]$.

Let $\Delta_\infty(R)$ be the Newton polygon at infinity of $R(X,T)$. The points $(\deg_X R(X,0), 0)$ and $(N, \deg R_0(T))$ are vertices of $\Delta_\infty(R)$. Since the last side of $\Delta_\infty(R)$ passes through $(\deg_X R(X,0), 0)$, its slope θ is less than or equal to $\deg R_0(T)/(N - \deg_X R(X,0))$. Hence by (a), (b) and (c) $\theta \leq \deg R_0(T)/(N - \deg_X R(X,0)) \leq \left[\deg f - \frac{\text{g.deg}(g|f)}{\deg g}\right] / \delta^0(g|f)$. By Proposition 2.8

$$\mathcal{L}_\infty(g|f) = \frac{-1}{\theta} \leq -\delta^0(g|f) \left[\deg f - \frac{\text{g.deg}(g|f)}{\deg g}\right]^{-1}.$$

Part (iii) follows easily from 2.7. If $\mathcal{L}_\infty(g|f) < 0$ then the curve $f(X,Y) = 0$ has at least two cycles at infinity by (i). Therefore $\deg f > 1$ and $d_i < d$ for all $i = 1, \ldots, s$ in the formula of Lemma 2.7. Hence $q \leq \deg f - 1$. $\square$

Let us put $\mathcal{L}_{\infty,\min}(g|f) = \inf\{\mathcal{L}_\infty(g-t|f) : t \in \mathbb{C}\}$. From this definition and from Theorem 2.9, we get

THEOREM 2.10.

(i) *If* $g|f^{-1}(0)$ *is proper then* $\mathcal{L}_{\infty,min}(g|f) = \mathcal{L}_\infty(g|f)$,

(ii) $\mathcal{L}_{\infty,min}(g|f) \neq 0$,

(iii) *If* $\mathcal{L}_{\infty,min}(g|f) < 0$ *then* $\mathcal{L}_{\infty,min}(g|f) = \min\{\mathcal{L}_\infty(g-t|f) : t \in D\}$ *and* $-\delta_{max}(g|f) \leq \mathcal{L}_{\infty,min}(g|f) \leq -\delta_{max}(g|f)\left[\deg f - \frac{\mathrm{gdeg}(g|f)}{\deg g}\right]^{-1}$ *where* $\delta_{max}(g|f) = \max\{\delta^t(g|f) : t \in D\}$,

(iv) $\mathcal{L}_{\infty,min}(g|f)$ *is a rational number. If* $\mathcal{L}_{\infty,min}(g|f) = p/q$ *with coprime integers then* $q \leq \deg f$. *If* $\mathcal{L}_{\infty,min}(g|f) < 0$ *then* $q \leq \deg f - 1$.

3. Global version of Teissier's lemma, Estimations. Let $f : \mathbb{C}^2 \to \mathbb{C}$ be a polynomial mapping, $d = \deg f > 1$. *We assume that f has a finite number of critical points.* This means that the partial derivatives $\frac{\partial f}{\partial X}$, $\frac{\partial f}{\partial Y}$ do not have a common factor, that is, for all $t \in \mathbb{C}$ the polynomials $f - t$ are reduced. We put $\mu^t(f) = \sum_{p \in f^{-1}(t)} \mu_p(f)$ (the fiber Milnor number) and $\mu(f) = \sum_{t \in \mathbb{C}} \mu^t(f)$ (the total Milnor number).

Let C^t be the projective closure of the fiber $f^{-1}(t)$. If $F(X,Y,Z)$ is the homogeneous form corresponding to f, then C^t is given by the equation $F(X,Y,Z) - tZ^d = 0$. Let $L_\infty \subset \mathbb{P}^2(\mathbb{C})$ be the line at infinity given by $Z = 0$ and let $(C^t)_\infty = C^t \cap L_\infty$. Obviously $(C^t)_\infty = (C^0)_\infty$. In the sequel we write $C = C^0$ and $C_\infty = (C^0)_\infty$. Let $\mu_p^t = \mu_p(C^t)$ for every $p \in C_\infty$. The following proposition is due to Broughton [**7**] (see also [**30**] for a simple direct proof).

PROPOSITION 3.1. *There exists a family* $(\mu_p^{gen})_{p \in C_\infty}$ *of nonnegative integers such that*

(1) $\forall t \in \mathbb{C} \quad \mu_p^t \geq \mu_p^{gen}$,

(2) *for all but a finite number of* $t \in \mathbb{C}$: $\mu_p^t = \mu_p^{gen}$.

For every $t \in \mathbb{C}$ according to Broughton, we put $\lambda^t(f) = \sum_{p \in C_\infty}(\mu_p^t - \mu_p^{\mathrm{gen}})$ and $\lambda(f) = \sum_{t \in \mathbb{C}} \lambda^t(f)$. Moreover we set $\Lambda(f) = \{\, t \in \mathbb{C} : \lambda^t(f) \neq 0 \,\}$. We call every element of $\Lambda(f)$ a critical value at infinity of f. Equivalent definitions of this notion and numerous examples are discussed in [**15**]. In order to give descriptions of $\lambda^t(f)$ and $\Lambda(f)$ let us assume that $\deg_Y f = \deg f = d$. Then we may write

$$f(X,Y) = Y^d + a_1(X)Y^{d-1} + \cdots + a_d(X).$$

Let $\Delta(X,T) = \mathrm{disc}_Y(f(X,Y) - T) = \mathrm{Res}_Y(f(X,Y) - T, \frac{\partial f}{\partial Y})$ and let us write $\Delta(X,T) = \Delta_0(T)X^N + \cdots + \Delta_N(T)$, $\Delta_0(T) \neq 0$ in $\mathbb{C}[T]$.

PROPOSITION 3.2.

(1) $\lambda^t(f) = N - \deg_X \Delta(X,t)$,

(2) $\Lambda(f) = \{\, t \in \mathbb{C} : \Delta_0(t) = 0 \,\}$.

PROOF. By Krasiński's formula we have $\deg_X \Delta(X,t) = d(d-2) + c - \sum_{p\in C_\infty} \mu_p^t$, $\deg_X \Delta(X,T) = d(d-2) + c - \sum_{p\in C_\infty} \mu_p^{\mathrm{gen}}$. Hence follows (1). Property (2) is an immediate consequence of (1). □

The theorem below can be considered as a global version of Teissier's lemma. Some versions were proved in [**2**] (Lemma 6.1) and [**3**] (Proposition 2.1) but our result gives more information. In [**21**] we proved a version of the theorem below by using Viro's method of integration [**42**]. This kind of results in n dimensions has been recently obtained by topological methods in [**12**].

THEOREM 3.3. *Let $f = f(X,Y) \in \mathbb{C}[X,Y]$ be Y–monic polynomial, $\deg_Y f > 1$, with finite number of critical points. Then the mapping $f|\{\frac{\partial f}{\partial Y} = 0\}$ has finite fibers and we have*

(i) $\mathrm{g.deg}(f|\frac{\partial f}{\partial Y}) = \mu(f) + \delta(f|\frac{\partial f}{\partial Y}) + (f,X)_{\mathbb{C}^2} - 1$,

(ii) *if $\deg_Y f = \deg f$ then $\lambda^t(f) = \delta^t(f|\frac{\partial f}{\partial Y})$ for all $t \in \mathbb{C}$, in particular $\lambda(f) = \delta(f|\frac{\partial f}{\partial Y})$ and $\Lambda(f)$ is the smallest set such that $f|\{\frac{\partial f}{\partial Y} = 0\}$ induces a proper mapping over $\mathbb{C} \setminus \Lambda(f)$,*

(iii) *if $\deg_Y f = \deg f$ then $\mathcal{L}_\infty(\frac{\partial f}{\partial X}|\frac{\partial f}{\partial Y}) + 1 = \mathcal{L}_{\infty,min}(f|\frac{\partial f}{\partial Y})$.*

Note that if the mapping $f|\{\frac{\partial f}{\partial Y} = 0\}$ is proper then we have

$$\big(f, \frac{\partial f}{\partial Y}\big)_{\mathbb{C}^2} = \mu(f) + (f,X)_{\mathbb{C}^2} - 1$$

which is a counterpart of Teissier's local result: $\big(f, \frac{\partial f}{\partial Y}\big)_0 = \mu_0(f) + (f,X)_0 - 1$.

PROOF OF THEOREM 3.3. Let us write $f(X,Y) = Y^d + a_1(X)Y^{d-1} + \cdots + a_d(X)$, $\frac{\partial f}{\partial Y}(X,Y) = d\prod_{i=1}^s g_i(X,Y)$, $g_i = g_i(X,Y) \in \mathbb{C}((X^{-1}))[Y]$ irreducible, monic of degrees $d_i = \deg_Y g_i$.

Let $y_i(\tau) \in \mathbb{C}((\tau^{-1}))$ be such that $g_i(\tau^{d_i}, y_i(\tau)) = 0$. Then by Zeuthen's Rule we get

$$(f, \frac{\partial f}{\partial Y})_{\mathbb{C}^2} = \sum_{i=1}^s \deg f(\tau^{d_i}, y_i(\tau)) \tag{1}$$

It is easy to check that the mapping $f|\{\frac{\partial f}{\partial Y} = 0\}$ has finite fibers, consequently the both sides of (1) are finite.

Let $I_0 = \{\, i : \deg f(\tau^{d_i}, y_i(\tau)) = 0 \,\}$, $f(\tau^{d_i}, y_i(\tau)) = t_i +$ terms of negative degrees, $t_i \in \mathbb{C} \setminus \{0\}$ for $i \in I_0$.

Obviously

$$(2)\qquad \begin{cases} \deg f(\tau^{d_i}, y_i(\tau)) = \deg \frac{\partial f}{\partial X}(\tau^{d_i}, y_i(\tau)) + d_i & \text{if } i \notin I_0 \\ \deg\big(f(\tau^{d_i}, y_i(\tau)) - t_i\big) = \deg \frac{\partial f}{\partial X}(\tau^{d_i}, y_i(\tau)) + d_i & \text{if } i \in I_0. \end{cases}$$

From (1) and (2) we obtain $\big(f, \frac{\partial f}{\partial Y}\big)_{\mathbb{C}^2} = \sum_{i=1}^{s}(\deg \frac{\partial f}{\partial X}(\tau^{d_i}, y_i(\tau)) + d_i) - \sum_{i\in I_0} \deg\big(f(\tau^{d_i}, y_i(\tau)) - t_i\big) = \big(\frac{\partial f}{\partial X}, \frac{\partial f}{\partial Y}\big)_{\mathbb{C}^2} + \sum_{i=1}^{s} d_i + \sum_{t\neq 0} \delta^t(f|\frac{\partial f}{\partial Y}) = \mu(f) + d - 1 + \delta(f|\frac{\partial f}{\partial Y}) - \delta^0(f|\frac{\partial f}{\partial Y})$ by Zeuthen's Rule and Lemma 2.6. Moreover

$$\text{g.deg}(f|\frac{\partial f}{\partial Y}) = (f, \frac{\partial f}{\partial Y})_{\mathbb{C}^2} + \delta^0(f|\frac{\partial f}{\partial Y}) = \mu(f) + \delta(f|\frac{\partial f}{\partial Y}) + (f, X)_{\mathbb{C}^2} - 1$$

and we get (i).

By Lemma 2.2, $\delta^t(f|\frac{\partial f}{\partial Y}) = N - \deg_X \Delta(X, t)$ where $N = \deg_X \Delta(X, T)$. If $\deg_Y f = \deg f$ then, by Proposition 3.2, $\lambda^t(f) = N - \deg_X \Delta(X, t)$ and the second part of Theorem 3.3 follows.

Using Lemma 2.7 and formulae (2) we get

$$\begin{aligned} &\mathcal{L}_\infty(\frac{\partial f}{\partial X}|\frac{\partial f}{\partial Y}) + 1 = \inf_{i=1}^{s} \left\{ \frac{1}{d_i} \deg \frac{\partial f}{\partial X}(\tau^{d_i}, y_i(\tau)) + 1 \right\} \\ &= \inf \left\{ \inf_{i\notin I_0} \{ \frac{1}{d_i} \deg f(\tau^{d_i}, y_i(\tau)) \}, \inf_{i\in I_0} \{ \frac{1}{d_i} \deg(f(\tau^{d_i}, y_i(\tau)) - t_i) \} \right\} \\ &= \mathcal{L}_{\infty,\min}(f|\frac{\partial f}{\partial Y}) \end{aligned}$$

□

Corollary to Theorem 3.3. *Let $f = f(X, Y) \in \mathbb{C}[X, Y]$ be a polynomial such that* $\deg_Y f = \deg f = d > 0$. *Let* $\Delta(X) = \operatorname{disc}_Y f(X, Y)$. *Then*

$$\deg \Delta(X) = \mu(f) + \lambda^*(f) + d - 1$$

where $\lambda^*(f) = \sum_{t\neq 0} \lambda^t(f)$.

Proof. By Theorem 3.3 we have $\text{g.deg}(f|\frac{\partial f}{\partial Y}) = \mu(f) + \lambda^*(f) + \delta^0(f|\frac{\partial f}{\partial Y}) + d - 1$. Hence we get the corollary because $\deg \Delta(X) = \big(f, \frac{\partial f}{\partial Y}\big)_{\mathbb{C}^2} = \text{g.deg}(f|\frac{\partial f}{\partial Y}) - \delta^0(f|\frac{\partial f}{\partial Y})$. □

Theorem 3.4. *Let $f : \mathbb{C}^2 \to \mathbb{C}$ be a polynomial of degree $d > 1$ with a finite number of critical points. Then $\mu(f) + \lambda(f) \leq (d - 1)^2$ and*

$$\#\Lambda(f) \leq \left[\frac{(d-1)^2 - \mu(f) - \lambda(f)}{d}\right].$$

PROOF. We may assume that $\deg f = \deg_Y f$. By 3.3 and Bezout's theorem we get $\mu(f) + \lambda(f) = \text{g.deg}(f|\frac{\partial f}{\partial Y}) - d + 1 \leq d(d-1) - d + 1 = (d-1)^2$. From 3.3(ii) and 2.1 we get

$$\#\Lambda(f) \leq \left[\deg \frac{\partial f}{\partial Y} - \frac{\text{g.deg}(f|\frac{\partial f}{\partial Y})}{\deg f}\right] = \left[d - 1 - \frac{\mu(f) + \lambda(f) + d - 1}{d}\right] =$$
$$= \left[\frac{(d-1)^2 - \mu(f) - \lambda(f)}{d}\right].$$

□

COROLLARY TO THEOREM 3.4.

(1) $\mu(f) + \lambda(f) \leq (d-1)^2 - d(\#\Lambda(f))$,
(2) *if f has at least one critical value at infinity, then* $\mu(f) + \lambda(f) \leq (d-1)^2 - d$,
(3) $\#\Lambda(f) \leq \max(1, d-3)$.

PROOF. Part (1) is an equivalent form of Theorem 3.4. Part (2) follows from (1). To check (3) we may assume $d > 3$ (if $d = 3$ then $\lambda(f) \leq 1$ and $\#\Lambda(f) = 1$, if $d = 2$ then $\Lambda(f) = \emptyset$). If $\lambda(f) \leq d-3$ then $\#\Lambda(f) \leq \lambda(f) \leq d-3$. Suppose that $\lambda(f) \geq d-2$. We have $\frac{(d-1)^2 - \mu(f) - \lambda(f)}{d} \leq \frac{(d-1)^2 - (d-2)}{d} = d - 3 + \frac{3}{d}$ and (3) follows. □

EXAMPLE. Let $f(X,Y) = Y((XY-1)^2 + X^2Y)$. Then f is of degree 5 with two critical values at infinity. It is easy to check, by computing the Milnor numbers at infinity, that $\lambda^0(f) = 2$, $\lambda^1(f) = 1$. Moreover $\mu(f) = 0$.

Question: Is the bound for $\#\Lambda(f)$, obtained above, optimal for $d > 5$?

We complete above estimations by

THEOREM 3.5. *Let $f : \mathbb{C}^2 \to \mathbb{C}$ be a polynomial with a finite number of critical points of degree $d > 1$ with c points at infinity. Then*

(i) *(i)* $\mu(f) + \lambda(f) \geq d(c-2) + 1$,
(ii) *(ii)* $\#\Lambda(f) \leq d - c$.

Note that *the projective degeneracy at infinity of f* introduced in [**31**] is equal to $d - c$ and estimation (ii) is Theorem 2.4 of [**31**]. We restrict our attention to polynomials with a finite number of critical points, however an easy modification of our arguments gives (ii) for all polynomials, as in [**31**].

PROOF OF THEOREM 3.5. It is easy to see that Part (ii) follows from Theorem 3.4 and from Part (i). To prove Part (i) we may assume that $\deg_Y f = \deg f$. Thus we can write $f(X,Y) = Y^d + a_1(X)Y^{d-1} + \cdots + a_d(X) = \prod_{j=1}^{d}(Y - Y_j(X))$ in $\mathbb{C}((X^{-1}))^*[Y]$ and $\frac{\partial f}{\partial Y} = d\prod_{k=1}^{d-1}(Y - Z_k(X))$.

Let $(1{:}m_i{:}0)$, $i = 1, \dots, c$, be the points at infinity of the curve $f(X,Y) = 0$. We may assume that $Y_j(X) = m_j X +$ terms of degree less than 1 for $j = 1, \dots, c$. Therefore

* $\deg(Y_i(X) - Y_j(X)) = 1$ for $1 \le i < j \le c$,

** for every $k \in \{1, \dots, d\}$ there is a unique $i \in \{1, \dots, c\}$ such that $\deg(Y_k(X) - Y_i(X)) < 1$.

By Property B.4 (Appendix B) applied to the field $\mathbb{C}((X^{-1}))^*$ with valuation $-\deg$, we get

(1) $\#\{\, k \in \{1, \dots, d-1\} : \deg(Z_k(X) - Y_i(X)) \ge 1 \text{ for all } i = 1, \dots, d\,\} = c - 1$

On the other hand, by Lemma 2.6'

$$\text{g.deg}(f|f_Y) = \sum_{j \in J_+} \deg f(X, Z_j(X)). \tag{2}$$

Using (1) we check that $\deg f(X, Z_j(X)) = \sum_{i=1}^{d} \deg(Z_j(X) - Y_i(X)) \ge d$ for exactly $c-1$ values of $j \in \{1, \dots, d-1\}$. Therefore $\text{g.deg}(f|f_Y) \ge (c-1)d$ by (2). Hence and from Theorem 3.3 we obtain $\mu(f) + \lambda(f) = \text{g.deg}(f|f_Y) - d + 1 \ge d(c-2) + 1$. □

Remark to Theorem 3.5. If f has at least one critical value at infinity and $\mu(f) + \lambda(f) = (d-1)^2 - d$ then the curve $f = 0$ has $d-1$ points at infinity.

Proof. With the notation introduced above we have $\sum_{j \in J_+} \deg f(X, Z_j(X)) = \text{g.deg}(f|f_Y) = \mu(f) + \lambda(f) + d - 1 = (d-2)d$ and $\#J_+ \le d - 2$. Thus $\#J_+ = d - 2$ and $\deg f(X, Z_j(X)) = d$ for all $j \in J_+$. Therefore $c - 1 = \#J_+ = d - 2$. □

Examples.

1. Let $f(X,Y) = Y^d + X^{c-1}Y^{d-c+1} + Y$, $1 < c \le d$. Then f is of degree d with c points at infinity and $\mu(f) + \lambda(f) = d(c-2)+1$. Thus the inequality 3.5(i) is optimal.

Indeed, let $\frac{\partial f}{\partial Y} = dY^{d-1} + (d-c+1)X^{c-1}Y^{d-c} + 1 = d\prod_{j=1}^{d-1}(Y - z_j(X))$ in $\mathbb{C}((X^{-1}))^*[Y]$. Using the Newton polygon (appendix B) we find that $c-1$ roots of $\frac{\partial f}{\partial Y}$ are of degree 1 and $d-c$ roots are of degree $-(c-1)/(d-c)$, say $\deg z_1(X) = \cdots = \deg z_{c-1}(X) = 1$ and $\deg z_c(X) = \cdots = \deg z_{d-1}(X) = -(c-1)/(d-c)$. It is easy to check that $\deg f(X, z_j(X)) = d$ for $j = 1, \dots, c-1$ and $\deg f(X, z_j(X)) < 0$ for $j = c, \dots, d-1$. Therefore $\text{g.deg}(f|\frac{\partial f}{\partial Y}) = \sum_{j=1}^{c-1} \deg f(X, z_j(X)) = d(c-1)$ and $\mu(f) + \lambda(f) = d(c-2)+1$ by Theorem 3.3. Moreover, $\Lambda(f) = \{0\}$.

2. Let $f(X,Y) = Y^d + X^{d-2}Y^2 + Y$, $d > 2$. Here $\mu(f) + \lambda(f) = (d-1)^2 - d$ and $\Lambda(f) = \{0\}$. This example shows that the inequality $\mu(f) + \lambda(f) \le$

$(d-1)^2 - d$ is optimal in the class of polynomials having at least one critical value at infinity.

Now, we give a global counterpart of the local formula $(f, \frac{\partial(f,g)}{\partial(X,Y)})_p = \mu_p(f) + (f,g)_p - 1$ (see for example [**35**, Proposition 4.1]). In the case of meromorphic curves a formula for the number of intersections of the curves $f=0$ and $\frac{\partial(f,g)}{\partial(X,Y)} = 0$ was given by Assi [**4**]. The following version of Assi's result was found independently by Cassou-Nogués and Maugendre [**14**], and by the authors.

PROPOSITION 3.6. *Let $f : \mathbb{C}^2 \to \mathbb{C}$ be a polynomial with a finite number of critical points and let $g : \mathbb{C}^2 \to \mathbb{C}$ be a polynomial such that the mapping $g|f^{-1}(0)$ has finite fibers. Put $J = \frac{\partial f}{\partial X}\frac{\partial g}{\partial Y} - \frac{\partial f}{\partial Y}\frac{\partial g}{\partial X}$. Then*

$$(f, J)_{\mathbb{C}^2} + \delta(g|f) = \mu(f) + \lambda^*(f) + \mathrm{g.deg}(g|f) - 1.$$

PROOF. Let $(\tau^{d_i}, y_i(\tau))$, $i = 1, \dots, s$ be the cycles at infinity of the curve $f(X,Y) = 0$ (we assume $\deg_Y f = \deg f = d$). Then we have

$$\begin{aligned}
&\frac{\partial f}{\partial X}(\tau^{d_i}, y_i(\tau)) d_i \tau^{d_i - 1} + \frac{\partial f}{\partial Y}(\tau^{d_i}, y_i(\tau)) y_i'(\tau) = 0\\
&\frac{\partial g}{\partial X}(\tau^{d_i}, y_i(\tau)) d_i \tau^{d_i - 1} + \frac{\partial g}{\partial Y}(\tau^{d_i}, y_i(\tau)) y_i'(\tau) = \frac{d}{d\tau} g(\tau^{d_i}, y_i(\tau)).
\end{aligned}$$

By Cramer's identities we get

$$d_i \tau^{d_i-1} J(\tau^{d_i}, y_i(\tau)) = \big(-\frac{d}{d\tau} g(\tau^{d_i}, y_i(\tau))\big) \frac{\partial f}{\partial Y}(\tau^{d_i}, y_i(\tau))$$

hence

$$\deg J(\tau^{d_i}, y_i(\tau)) + d_i - 1 = \deg \frac{d}{d\tau} g(\tau^{d_i}, y_i(\tau)) + \deg \frac{\partial f}{\partial Y}(\tau^{d_i}, y_i(\tau)). \tag{1}$$

From (1) we get, by Zeuthen's Rule,

$$(f, J)_{\mathbb{C}^2} + d = \sum_{i=1}^{s} [\deg \frac{d}{d\tau} g(\tau^{d_i}, y_i(\tau)) + 1] + \big(f, \frac{\partial f}{\partial Y}\big)_{\mathbb{C}^2}. \tag{2}$$

Let $I_0 = \{\, i : \deg g(\tau^{d_i}, y_i(\tau)) = 0 \,\}$ and let us write $g(\tau^{d_i}, y_i(\tau)) = t_i +$ terms of negative degrees, $t_i \in \mathbb{C} \setminus \{0\}$ for $i \in I_0$. We have

$$\begin{cases} \deg \frac{d}{d\tau} g(\tau^{d_i}, y_i(\tau)) = \deg g(\tau^{d_i}, y_i(\tau)) - 1 & \text{if } i \notin I_0 \\ \deg \frac{d}{d\tau} g(\tau^{d_i}, y_i(\tau)) = \deg(g(\tau^{d_i}, y_i(\tau)) - t_i) - 1 & \text{if } i \in I_0. \end{cases} \tag{3}$$

Using (3) we obtain

$$
(4)\quad
\begin{aligned}
&\sum_{i=1}^{s}[\deg \frac{d}{d\tau} g(\tau^{d_i}, y_i(\tau)) + 1] \\
&\quad = \sum_{i \notin I_0} \deg g(\tau^{d_i}, y_i(\tau)) + \sum_{i \in I_0} \deg(g(\tau^{d_i}, y_i(\tau)) - t_i) = \\
&\quad = \sum_{i=1}^{s} \deg g(\tau^{d_i}, y_i(\tau)) + \sum_{i \in I_0} \deg(g(\tau^{d_i}, y_i(\tau)) - t_i) \\
&\quad = (f, g)_{\mathbb{C}^2} - \sum_{t \neq 0} \delta^t(g|f)
\end{aligned}
$$

From (2) and (4) we get

$$
(5)\qquad (f, J)_{\mathbb{C}^2} + d = (f, g)_{\mathbb{C}^2} - \sum_{t \neq 0} \delta^t(g|f) + \big(f, \frac{\partial f}{\partial Y}\big)_{\mathbb{C}^2}.
$$

By Corollary to Theorem 3.3 $\big(f, \frac{\partial f}{\partial Y}\big)_{\mathbb{C}^2} = \mu(f) + \lambda^*(f) + d - 1$, consequently $(f, J)_{\mathbb{C}^2} + \sum_{t \neq 0} \delta^t(g|f) = (f, g)_{\mathbb{C}^2} + \mu(f) + \lambda^*(f) - 1$ but $(f, g)_{\mathbb{C}^2} = \text{g.deg}(g|f) - \delta^0(g|f)$ and Proposition 3.6 follows. □

Remark. If $\delta^t(g|f) = 0$ for $t \neq 0$ then (5) reduces to Assi's formula

$$
(f, J)_{\mathbb{C}^2} + d = (f, g)_{\mathbb{C}^2} + \big(f, \frac{\partial f}{\partial Y}\big)_{\mathbb{C}^2}.
$$

4. The Euler characteristic of the fibers. Recall that $f : \mathbb{C}^2 \to \mathbb{C}$ is a polynomial with a finite number of critical points. Let us begin with an elementary proof of a formula due to Cassou–Noguès [**11**, Proposition 12].

Cassou–Noguès' formula. *Let $f = f(X, Y)$ be a polynomial of degree $d > 0$. With the notation introduced above,*

$$
d(d-3) + c + 1 = \sum_{p \in C_\infty} \mu_p^{gen} + \mu(f) + \lambda(f).
$$

Proof. We may assume that $\deg_Y f = \deg f$. Let $\Delta = \operatorname{disc}_Y f$. By Krasiński's formula for the degree of the discriminant, we have

$$
\deg \Delta = d(d-2) + c - \sum_{p \in C_\infty} \mu_p(C).
$$

On the other hand, by Corollary to Theorem 3.3

$$
\deg \Delta = \mu(f) + \sum_{t \neq 0} \lambda^t(f) + d - 1
$$

and Cassou–Noguès' formula follows. □

Now we are able to prove the formula for the Euler characteristic of the fibers $f^{-1}(t)$. Recall that C^t is the projective closure of $f^{-1}(t)$. The following lemma is well-known (see for example [**26**], Lemma 2.3).

Lemma 4.1. *If F is a finite subset of a curve $C \subset \mathbb{P}^2(\mathbb{C})$ then $\chi(C \setminus F) = \chi(C) - \#F$.*

In particular, the Euler characteristic of an affine curve is equal to the Euler characteristic of its projective closure minus the number of points at infinity. In [**37**] (Proposition 2, p. 533) Suzuki gave a formula for the Euler characteristic of the fibers of a primitive polynomial. In the case of polynomials with a finite number of critical points, Suzuki's result was improved by Gavrilov ([**17**], Theorem 3.3 and [**18**], Theorem 2.2).

Suzuki–Gavrilov's formula. *If $f : \mathbb{C}^2 \to \mathbb{C}$ is a polynomial with isolated critical points then the Euler characteristic of the fiber $f^{-1}(t)$ is given by*

$$\chi(f^{-1}(t)) = 1 - \mu(f) - \lambda(f) + \mu^t(f) + \lambda^t(f).$$

Proof. By Max Noether's formula

$$\chi(C^t) = -d(d-3) + \mu^t(f) + \sum_{p \in C_\infty} \mu_p^t = -d(d-3) + \mu^t(f) + \lambda^t(f) + \sum_{p \in C_\infty} \mu_p^{\mathrm{gen}}.$$

By Cassou-Noguès' formula

$$\sum_{p \in C_\infty} \mu_p^{\mathrm{gen}} = d(d-3) + c + 1 - \mu(f) - \lambda(f).$$

Then we get $\chi(C^t) = c + 1 - \mu(f) - \lambda(f) + \mu^t(f) + \lambda^t(f)$ and, by the lemma, $\chi(f^{-1}(t)) = 1 - \mu(f) - \lambda(f) + \mu^t(f) + \lambda^t(f)$. $\square$

Let $f : \mathbb{C}^2 \to \mathbb{C}$ be a polynomial with a finite number of critical points. The number $t \in \mathbb{C}$ is a *typical* value of f if $\mu^t(f) = \lambda^t(f) = 0$. It is an *atypical* value if $\mu^t(f) > 0$ or $\lambda^t(f) > 0$.

Corollary 4.2. *The value $t \in \mathbb{C}$ is typical iff $\chi(f^{-1}(t)) = 1 - \mu(f) - \lambda(f)$. If the value $t \in \mathbb{C}$ is atypical then $\chi(f^{-1}(t)) > 1 - \mu(f) - \lambda(f)$.*

According to [**25**] the set of atypical values is the smallest set A such that the mapping $\mathbb{C}^2 \setminus f^{-1}(A) \to \mathbb{C} \setminus A$ induced by f is a smooth locally trivial fibration.

Assume that the polynomial f is irreducible and let $\Gamma = f^{-1}(0)$. Let us put $\mu^*(f) = \sum_{t \neq 0} \mu^t(f)$. The following result was proved in [**2**].

Abhyankar–Sathaye's formula. *Let γ be the genus of the Riemann surface corresponding to Γ. Then*

$$2\gamma + \sum_{p\in\Gamma}(r_p(f)-1) + (r_\infty(f)-1) = \mu^*(f) + \lambda^*(f)$$

where $r_\infty(f)$ is the number of branches at infinity of Γ.

Proof. Let C be the projective closure of Γ. Then $\chi(C) = c + \chi(\Gamma) = c+1-\mu^*(f)-\lambda^*(f)$ by Gavrilov's formula and $\chi(\tilde{C}) = \chi(C)+\sum_{p\in C}(r_p-1) = c+1-\mu^*(f)-\lambda^*(f)+\sum_{p\in C}(r_p-1) = c+1-\mu^*(f)-\lambda^*(f)+\sum_{p\in\Gamma}(r_p-1)+\sum_{p\in C_\infty} r_p - c = 1-\mu^*(f)-\lambda^*(f)+\sum_{p\in\Gamma}(r_p-1)+r_\infty(f)$.

On the other hand, $\chi(\tilde{C}) = 2-2\gamma$ and Abhyankar-Sathaye's formula follows. $\square$

Remark 4.3. Let $\Gamma\subset\mathbb{C}^2$ be an affine irreducible curve. Then it is easy to check that $\chi(\Gamma) = 2-2\gamma-\sum_{p\in\Gamma}(r_p(f)-1)-r_\infty(f)$ (see **[26]**, Proposition 2.4).

5. The Łojasiewicz exponent and critical values at infinity. Let $f:\mathbb{C}^2\to\mathbb{C}$ be a polynomial with a finite number of critical points. Put $\operatorname{grad} f = (\frac{\partial f}{\partial X}, \frac{\partial f}{\partial Y})$. By the Łojasiewicz exponent at infinity we mean $\mathcal{L}_\infty(f) = \sup\{\,\theta\in\mathbb{R} : \exists C, R>0 \forall z\in\mathbb{C}^2 \quad |\operatorname{grad} f(z)| \geq C|z|^\theta \text{ for } |z|\geq R\,\}$. The following lemma is implicit in **[24]** and **[13]**.

Lemma 5.1. *Let $\deg_X f = \deg_Y f = \deg f > 1$. Then*

$$\mathcal{L}_\infty(f) = \min\{\,\mathcal{L}_{\infty,min}(f|\frac{\partial f}{\partial X}), \mathcal{L}_{\infty,min}(f|\frac{\partial f}{\partial Y})\,\} - 1.$$

Proof. By a theorem due to Chądzyński and Krasiński (see **[9]**, Theorem 1), we get

$$\mathcal{L}_\infty(f) = \min\{\,\mathcal{L}_\infty(\frac{\partial f}{\partial Y}|\frac{\partial f}{\partial X}), \mathcal{L}_\infty(\frac{\partial f}{\partial X}|\frac{\partial f}{\partial Y})\,\}.$$

Then we use Theorem 3.3 (iii). $\square$

Our main result concerning the Łojasiewicz exponent is

Theorem 5.2. *Let $f:\mathbb{C}^2\to\mathbb{C}$ be a polynomial of degree $d>1$ with a finite number of critical points. Then*

(i) *if f has no critical values at infinity, then $\mathcal{L}_\infty(f)\geq -1+1/(d-1)$,*
(ii) *if f has at least one critical value at infinity, then $-\lambda_{max}(f)\leq\mathcal{L}_\infty(f)+1\leq -\lambda_{max}(f)\left[d-2-\frac{\mu(f)+\lambda(f)-1}{d}\right]^{-1}$ where $\lambda_{max}(f) = \max\{\lambda^t(f) : t\in\mathbb{C}\}$.*

(iii) *The Łojasiewicz exponent is a rational number; if* $\mathcal{L}_\infty(f) = p/q$ *with coprime integers* p, q *then* $q \leq d-1$. *If* $\mathcal{L}_\infty(f) < -1$ *then* $d > 2$ *and* $q \leq d-2$.

PROOF. We may assume that $\deg_X f = \deg_Y f = d$. Suppose that f has no critical values at infinity, that is $\Lambda(f) = \emptyset$. Then the mappings $f|\{\frac{\partial f}{\partial X} = 0\}$ and $f|\{\frac{\partial f}{\partial Y} = 0\}$ are proper by (3.3)(ii) and $\mathcal{L}_{\infty,\min}(f|\frac{\partial f}{\partial X})$, $\mathcal{L}_{\infty,\min}(f|\frac{\partial f}{\partial Y}) \geq \frac{1}{d-1}$ by Theorem 2.9(i). Part (i) of the theorem follows from Lemma 6.1.

Suppose that $\Lambda(f) \neq \emptyset$. By Lemma 6.1 we may assume

$$\mathcal{L}_\infty(f) = \mathcal{L}_{\infty,\min}(f|\frac{\partial f}{\partial Y}) - 1. \tag{1}$$

The mapping $f|\{\frac{\partial f}{\partial Y} = 0\}$ is not proper by (3.3)(ii). Moreover, $\delta_{\max}(f|\frac{\partial f}{\partial Y}) = \lambda_{\max}(f)$. By Theorem 2.10 we get

$$-\lambda_{\max}(f) \leq \mathcal{L}_{\infty,\min}(f|\frac{\partial f}{\partial Y}) \leq -\lambda_{\max}(f)\left[d - 1 - \frac{\mathrm{g.deg}(f|\frac{\partial f}{\partial Y})}{d}\right]^{-1} \tag{2}$$

and Part (ii) follows from (1), (2) and Theorem 3.3. To get Part (iii) it suffices to use Lemma 5.1 and Theorem 2.10(iv). $\square$

COROLLARY TO THEOREM 5.2. *(see* **[24]**, **[8]***) Let* $f : \mathbb{C}^2 \to \mathbb{C}$ *be a polynomial with a finite number of critical points. Then* f *has no critical value at infinity if and only if* $\mathcal{L}_\infty(f) > -1$.

6. More estimations, questions. To illustrate the obtained results let us check the following

THEOREM 6.1. *Let* f *be a polynomial with a finite number of critical points of degree* $d > 1$ *with* $d-1$ *points at infinity. Let* p *be the unique point at which the curve* $f = 0$ *intersects the line at infinity with multiplicity* > 1. *Suppose that* $\Lambda(f) \neq \emptyset$. *Then* f *has exactly one critical value at infinity* t_0 *and* $\mu(f) = d^2 - 2d - \mu_p^{t_0}$, $\lambda(f) = \mu_p^{t_0} - d + 1$, $\mathcal{L}_\infty(f) = d - 2 - \mu_p^{t_0}$ *where* $\mu_p^{t_0}$ *is the Milnor number of the curve* $f = t_0$ *at* p.

PROOF. By theorems 3.4 and 3.5 f has exactly one critical value t_0 at infinity and $\mu(f) + \lambda(f) = d^2 - 3d + 1$. From Cassou–Noguès formula we get $\mu_p^{\mathrm{gen}} = d - 1$. Thus $\lambda(f) = \lambda^{t_0}(f) = \mu_p^{t_0} - \mu_p^{\mathrm{gen}} = \mu_p^{t_0} - d + 1$ and $\mu(f) = d^2 - 3d + 1 - \lambda(f) = d^2 - 2d - \mu_p^{t_0}$. By Theorem 5.2 we get $\mathcal{L}_\infty(f) = -\lambda(f) - 1 = d - 2 - \mu_p^{t_0}$. $\square$

EXAMPLE. Yoshihara **[44]** gave an example of a rational plane curve C of degree 6 with the unique singular point p of order 2 with two branches. Let us take the generic line passing through p as the line at infinity. Let $f = 0$ be the affine equation of C. One checks that $\Lambda(f) = \{0\}$ and $\mu_p^0 = 19$. Then by the above theorem $\lambda(f) = 14$ and $\mathcal{L}_\infty(f) = -15$.

REMARK 6.2. Let us keep the assumptions of 6.1 and suppose that $\Lambda(f) = \emptyset$. Then from the Cassou-Noguès formula we get $\mu(f) = d^2 - 2d - \mu_p^{\text{gen}}$. One can also check that $\mathcal{L}_\infty(f) = d - 2 - \mu_p^{\text{gen}}$.

In the sequel we will need

PROPOSITION 6.3. *Let C be a reduced projective curve of degree d. Suppose that m components of C pass through a point $p \in C$. Then $\mu_p(C) \leq (d-1)(d-2) + m - 1$.*

PROOF. Let $C_1,\dots,C_m$ be the irreducible components of C passing through p. From the formula for the genus of an irreducible curve (Corollary 1.2), we get $\mu_p(C_i) \leq (d_i - 1)(d_i - 2)$ where $d_i = \deg C_i$ for $i = 1,\dots, m$. On the other hand, by a well-known property of Milnor numbers (see **[35]**) we can write

$$\begin{aligned}
&\mu_p(C) + m - 1 = \mu_p\Big(\bigcup_{i=1}^{m} C_i\Big) + m - 1 \\
&= \sum_{i=1}^{m} \mu_p(C_i) + 2 \sum_{1\leq i<j\leq j} (C_i, C_j)_p \leq \sum_{i=1}^{m} (d_i - 1)(d_i - 2) + 2 \sum_{1\leq i<j\leq j} d_i d_j \\
&= \Big(\sum_{i=1}^{m} d_i\Big)^2 - 3\Big(\sum_{i=1}^{m} d_i\Big) + 2m \leq d^2 - 3d + 2m.
\end{aligned}$$

Thus we have proved $\mu_p(C) + m - 1 \leq d^2 - 3d + 2m$ and the proposition follows. □

Now we can prove

PROPOSITION 6.4. *Let f be a polynomial of degree $d > 3$ with $d - 1$ points at infinity. Then $\mu(f) \geq d - 3$.*

PROOF. We may assume $\mu(f) < +\infty$. By theorem 6.1 and remark 6.2 we can write $\mu(f) = d^2 - 2d - \mu_p(C^{t_0})$ for a $t_0 \in \mathbb{C}$. Since $(C^{t_0}, \mathbb{P}(\mathbb{C})_\infty)_p = 2$, at least two components of C^{t_0} pass through the point p. Therefore $\mu_p(C^{t_0}) \leq (d-1)(d-2)+1$ by proposition 6.3 and $\mu(f) = d^2-2d-(d-1)(d-2)-1 = d-3$. □

An application of 6.4 is the following

PROPOSITION 6.5. *Let $f : \mathbb{C}^2 \to \mathbb{C}$ be a polynomial of degree $d > 3$ without critical points in $\mathbb{C}^2$. Then the curve $f = 0$ has at most $d-2$ points at infinity.*

PROOF. Let $f : \mathbb{C}^2 \to \mathbb{C}$ be a polynomial of degree d with c points at infinity. If $c = d$ then $\mu(f) = (d-1)^2 > 0$. If $c = d-1$ then $\mu(f) \geq d-3 > 0$. Thus if $\mu(f) = 0$ then $c \leq d-2$. □

Question: How many points at infinity can a polynomial without critical points of degree $d > 3$ have?

PROPOSITION 6.6. *If $f : \mathbb{C}^2 \to \mathbb{C}$ is a polynomial of degree $d > 3$ with a finite number of critical points then $\lambda(f) \leq d^2 - 3d$.*

PROOF. By Corollary to Theorem 3.4 we have $\mu(f) + \lambda(f) \leq d^2 - 3d + 1$ if $\lambda(f) \neq 0$. Therefore it suffices to show that $\lambda(f) < d^2 - 3d + 1$ for $d > 3$.

If we had $\lambda(f) = d^2 - 3d + 1$ then we would get $\mu(f) = 0$ and f would have $d - 1$ points at infinity by Remark to Theorem 3.5. This contradicts Proposition 6.5. $\square$

Using Theorem 5.2(ii) we get

PROPOSITION 6.7. *With the assumptions of 6.6: $\mathcal{L}_\infty(f) \geq -d^2 + 3d - 1$.*

Question: If $d = 4$ then the above estimations are optimal (take $f(X, Y) = X^4 - X^2Y^2 + 2XY - 1$, then $\lambda(f) = 4$ and $\mathcal{L}_\infty(f) = -5$). What are the optimal bounds on $\lambda(f)$ and $\mathcal{L}_\infty(f)$ in the class of polynomials of degree $d > 4$ with a finite number of critical points?

Appendix A. Traditionally, the Riemann-Hurwitz formula is stated for compact Riemann surfaces. However one may observe that the classical proof gives the following, purely topological result

A.1. *Let X, Y be compact topological spaces, Y triangulable and let $\pi : X \to Y$ be a continuous open surjective mapping with finite fibers. Suppose there is a finite subset $A \subset Y$ such that $\pi : X \setminus \pi^{-1}(A) \to Y \setminus A$ is a covering with d sheets. Then X is triangulable and*

$$\chi(X) = d\chi(Y) + \sum_{y \in A} (\#\pi^{-1}(y) - d)$$

where $\chi(X)$ (resp. $\chi(Y)$) is the Euler characteristic of X (resp. Y).

PROOF. We assume for simplicity that X and Y are of dimension 2. Let (V, E, F) be a triangulation of Y with the set V of vertices, the set E of edges and the set F of faces. Suppose that $A \subset V$. Then the standard considerations (see, for example [**28**, Appendix C]) show that there exist a triangulation (V', E', F') of X such that $\#F' = d(\#F)$, $\#E' = d(\#E)$ and $V' = \pi^{-1}(V)$. Therefore $\#V' = \sum_{y \in V} \#\pi^{-1}(y) = d(\#V) + \sum_{y \in A}(\#\pi^{-1}(y) - d)$ and we get $\chi(X) = \#V' - \#E' + \#F' = d(\#V - \#E + \#F) + \sum_{y \in A}(\#\pi^{-1}(y) - d) = d\chi(Y) + \sum_{y \in A}(\#\pi^{-1}(y) - d)$. $\square$

Let us note two useful corollaries of A.1

A.2. *Let $\nu : \tilde{X} \to X$ be the normalization of a curve X. Then*

$$\chi(\tilde{X}) = \chi(X) + \sum_{x\in X}(r_x(X) - 1).$$

Proof. Here $d = 1$ and $\#\nu^{-1}(x) = r_x(X)$ (the number of branches through x). Then we use A.1. □

Let X, Y be algebraic projective curves, Y smooth and let $\pi : X \to Y$ be a regular surjective mapping with finite fibers. Then the assumptions of A.1 are satisfied with $d = \deg\pi$ (the geometric degree of π) and for every $x \in X$ the multiplicity $\operatorname{mult}_x \pi$ is defined in such a way that $\sum_{x\in\pi^{-1}(y)} \operatorname{mult}_x \pi = \deg\pi$ and for generic $y \in Y$ $\operatorname{mult}_x \pi = 1$ if $x \in \pi^{-1}(y)$ (see [**32**, Chapter 3]).

A.3. *With the notation introduced above,*

$$\chi(X) = (\deg\pi)\chi(Y) - \sum_{x\in X}(\operatorname{mult}_x \pi - 1).$$

Proof. By A.1 we get

$$\begin{aligned}
\chi(X) &= (\deg\pi)\chi(Y) + \sum_{y\in Y}(\#\pi^{-1}(y) - \deg\pi)\\
&= (\deg\pi)\chi(Y) + \sum_{y\in Y}\Big(\sum_{x\in\pi^{-1}(y)} 1 - \sum_{x\in\pi^{-1}(y)} \operatorname{mult}_x \pi\Big)\\
&= (\deg\pi)\chi(Y) - \sum_{y\in Y}\sum_{x\in\pi^{-1}(y)}(\operatorname{mult}_x \pi - 1)\\
&= (\deg\pi)\chi(Y) - \sum_{x\in X}(\operatorname{mult}_x \pi - 1).
\end{aligned}$$

□

Note that in A.3 we do not assume X to be smooth!

Appendix B. We recall here the Newton method of determining orders of roots and the number of roots of the given order of a polynomial with coefficients in a valued field. Then we use it to locate the roots of the derivative.

Let $\nu : K\setminus\{0\} \to \mathbb{R}$ be a nonzero valuation of a field K. We put $\nu(0) = +\infty$ where the symbol $+\infty$ has usual properties.

B.1. *Let $P(Y) = a_0Y^d + a_1Y^{d-1} + \cdots + a_d$, a_0, $a_d \neq 0$ be a polynomial of degree $d > 0$ with coefficients in K. Assume $P(Y) = a_0(Y - y_1)\cdots(Y - y_d)$ in $K[Y]$ and suppose that $\nu(y_1) \le \cdots \le \nu(y_d)$. Let $0 = k_0 < k_1 < \cdots < k_{s-1} < k_s = d$ and $\Theta_1 < \cdots < \Theta_s$ be two sequences defined by conditions*

$$\nu(y_{k_j+1}) = \cdots = \nu(y_{k_{j+1}}) = \Theta_{j+1}$$

for $j = 0, \ldots, s - 1$.

Set $I_{j+1} = \{\, k \in \mathbb{N} : k_j < k \le d, a_k \ne 0 \,\}$ *for* $j = 0, \dots, s-1$. *Then*

(i) $$\min\left\{ \frac{\nu(a_k) - \nu(a_{k_j})}{k - k_j} : k \in I_{j+1} \right\} = \Theta_{j+1}$$

and

(ii) $$\max\left\{ k \in I_{j+1} : \frac{\nu(a_k) - \nu(a_{k_j})}{k - k_j} = \Theta_{j+1} \right\} = k_{j+1}$$

PROOF. We have $a_k/a_0 = (-1)^k(y_1 \cdots y_k + \cdots)$. It is easy to see that $\nu(a_k/a_0) \ge \nu(y_1 \cdots y_k)$ with the equality for $k = k_j$. Therefore, if $a_k \ne 0$ and $k \ge k_j$ then $\frac{\nu(a_k)-\nu(a_{k_j})}{k-k_j} \ge \frac{\nu(y_1\cdots y_k)-\nu(y_1\cdots y_{k_j})}{k-k_j} = \frac{\nu(y_{k_j+1}\cdots y_k)}{k-k_j} \ge \frac{\Theta_{j+1}(k-k_j)}{k-k_j} = \Theta_{j+1}$ with the equality for $k = k_{j+1}$. To finish the proof it suffices to observe that $\nu(y_{k_j+1} \cdots y_k) > \Theta_{j+1}(k - k_j)$ for $k > k_{j+1}$. □

The polygon with vertices $(\nu(a_{k_j}), d - k_j)$ for $j = 0, \dots, s$ is called the Newton polygon (see [**45**, p.98])

Let A be a set. For any $a_1, \dots, a_p \in A$ we denote by $\langle a_1, \dots, a_p \rangle$ the sequence $a_1, \dots, a_p$ regarded as unordered. From B.1 we get

B.2. *Let* $P(Y) = a_0Y^d + a_1Y^{d-1} + \cdots + a_d = a_0 \prod_{j=1}^d (Y - y_j)$ *and* $\bar{P}(Y) = \bar{a}_0Y^d + \bar{a}_1Y^{d-1} + \cdots + \bar{a}_d = \bar{a}_0 \prod_{j=1}^d (Y - \bar{y}_j)$ *be polynomials of degree* $d > 0$ *with roots in* K. *Suppose that* $\nu(a_i) = \nu(\bar{a}_i)$ *for* $i = 0, \dots, d$.
Then $\langle \nu(y_1), \dots, \nu(y_d) \rangle = \langle \nu(\bar{y}_1), \dots, \nu(\bar{y}_d) \rangle$.

The property below is a version of the Kuo–Lu lemma (see [**29**, Lemma 3.3] and [**20**, Lemma 2.2]). In the sequel we assume that K has zero characteristic.

B.3. *If* $P(Y) = a_0Y^d + a_1Y^{d-1} + \cdots + a_d = a_0 \prod_{i=1}^d (Y - y_i)$, $a_0 \ne 0$, $d > 1$ *and* $P'(Y) = da_0 \prod_{j=1}^{d-1} (Y - z_j)$ *then* $\langle \nu(z_1 - y_i), \dots, \nu(z_{d-1} - y_i) \rangle = \langle \nu(y_1 - y_i), \dots, \nu(y_{i-1} - y_i), \nu(y_{i+1} - y_i), \dots, \nu(y_d - y_i) \rangle$ *for every* $i = 1, \dots, d$.

PROOF. Fix $i = 1, \dots, d$ and consider the special case $y_i = 0$. We see that $P(Y)/Y$ is of degree $d-1$ with roots $y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_d$. Moreover, polynomials $P(Y)/Y$ and $P'(Y)$ satisfy the assumption of B.2. Hence we get $\langle \nu(z_1), \dots, \nu(z_{d-1}) \rangle = \langle \nu(y_1), \dots, \nu(y_{i-1}), \nu(y_{i+1}), \dots, \nu(y_d) \rangle$ which proves B.3 in the special case.

In the general case we consider the polynomial $P_i(Y) = P(Y + y_i)$. The roots of $P_i(Y)$ are $y_1 - y_i, \dots, y_{i-1} - y_i, 0, y_{i+1} - y_i, \dots, y_d - y_i$ while the roots of $P'(Y)$ are $z_1 - y_i, \dots, z_{d-1} - y_i$. Applying the special case to P_i we get the lemma. □

We call $\nu(y_i - y_j)$ the order of contact of y_i, y_j and put by definition $sepP = \inf\{\, \nu(y_i - y_j) : 1 \le i < j \le d \,\}$. Then $sepP = +\infty$ iff P has a unique root.

B.4. *Suppose that $P'(Y) = da_0 \prod_{j=1}^{d-1}(Y - z_j)$ and let c denote the maximal number of roots such that any two have order of contact $sepP$. Then*

$$\#\{\, k \in \{1, \ldots, d-1\} : \nu(z_k - y_i) \leq sepP \text{ for } i = 1, \ldots, d \,\} = c - 1.$$

Proof. We may assume that:

(i) for $1 \leq i < j \leq c \quad \nu(y_i - y_j) = sepP$,

(ii) for every $k \in \{1, \ldots, d\}$ there is an $i \in \{1, \ldots, c\}$ (necessarily unique) such that $\nu(y_k - y_i) > sepP$.

Let $d_i = \#\{k : \nu(y_k - y_i) > sepP\}$. We have $\sum_{i=1}^{c} d_i = d$. Let us consider the sets $K_i = \{k \in \{1, \ldots, d-1\} : \nu(z_k - y_i) > sepP\}$ for $i = 1, \ldots, c$. From B.3 we get $\#K_i = d_i - 1$. Let $J = \{1, \ldots, d-1\} \setminus (K_1 \cup \cdots \cup K_c)$, then $\#J = c - 1$. From definition of J we get:

* if $k \in J$ then $\nu(y_j - z_k) \leq sepP$ for $j = 1 \ldots, d$,

** if $k \notin J$ then $\nu(y_j - z_k) > sepP$ for some $j = 1 \ldots, d$

which ends the proof. $\square$

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