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THE MIXED PROBLEM FOR AN INFINITE SYSTEM OF FIRST ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS

by Tomasz Człapiński

Abstract. We consider the mixed problem for the infinite system of nonlinear partial functional differential equations

$$D_t z_i(t,x) = f_i\big(t,x,z_{(t,x)},D_x z_i(x,y)\big), \quad i \in \mathbb{N},$$

where $z_{(x,y)} = \big\{(z_i)_{(t,x)}\big\}$ denotes an infinite sequence of functions defined by the formula $z_{(t,x)}(\tau,s) = z(t+\tau,x+s)$, $(\tau,s) \in [-\tau_0,0]\times[0,h]$. Using the method of bicharacteristics and the quasi-iteration method for a certain integral-functional system, we prove, under suitable assumptions, a theorem on the local existence of generalized solutions of the mixed problem.

1. Introduction. We denote by ℓ^∞ the space of all infinite sequences $p = \{p_i\}$, $p_i \in \mathbb{R}$, $i \in \mathbb{N}$, such that $|p|_\infty = \sup\{|p_i|;\, i \in \mathbb{N}\} < +\infty$, where $\mathbb{N}$ denotes the set of natural numbers. Put $B = [-\tau_0,0]\times[0,h] \subset \mathbb{R}^{1+n}$, where $h = (h_1,\dots,h_n) \in \mathbb{R}^n_+$, $\tau \in \mathbb{R}_+$, ($\mathbb{R}_+ = [0,+\infty)$) and

$$[0,h] = \{x = (x_1,\dots,x_n) \in \mathbb{R}^n;\ 0 \le x_j \le h_j,\, 1 \le j \le n\}\,.$$

Let $z = \{z_i\} : [-\tau_0,\bar a]\times[-b,b+h] \to \ell^\infty$ be a given function, where $\bar a > 0$, $b = (b_1,\dots,b_n)$, $b_j > 0$, $j = 1,\dots,n$. Then for a point $(t,x) = (t,x_1,\dots,x_n) \in [0,\bar a]\times[-b,b]$, we consider the function $z_{(t,x)} = \big\{(z_i)_{(t,x)}\big\} : B \to \ell^\infty$ defined by

$$z_{(t,x)}(\tau,s) = z(t+\tau,x+s), \quad (\tau,s) \in B.$$

In this paper the operator $(t,x) \mapsto z_{(t,x)}$ is used to describe the functional dependence in a differential system.

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For any $a \in (0, \bar{a}]$ we define the sets

$$E_0^* = [-\tau_0, 0] \times [-b, b+h], \qquad E_a = [0,a] \times [-b,b],$$
$$\partial_0 E_a = [0,a] \times [-b, b+h] \setminus [0,a] \times [-b,b), \qquad E_a^* = E_0^* \cup \partial_0 E_a \cup E_a,$$

and for any $X \subset E_a^*$ we denote by $C(X, \ell^\infty)$ the space of all continuous functions from X to ℓ^∞.

Given the sequences of functions $f = \{f_i\}$ and $\phi = \{\phi_i\}$ such that

$$f_i : E_{\bar{a}} \times C(B, \ell^\infty) \times \mathbb{R}^n \to \mathbb{R} \quad \text{and} \quad \phi_i : E_0^* \cup \partial_0 E_{\bar{a}} \to \mathbb{R},$$

where $\bar{a} > 0$, we consider the mixed problem for the following weak coupled infinite system

$$D_t z_i(t,x) = f_i\big(t, x, z_{(t,x)}, D_x z_i(t,x)\big), \qquad (t,x) \in E_{\bar{a}}^*,\ i \in \mathbb{N}, \tag{1}$$
$$z(t,x) = \phi(t,x), \qquad (t,x) \in E_0^* \cup \partial_0 E_{\bar{a}}, \tag{2}$$

where $D_x z_i = (D_{x_1} z_i, \ldots, D_{x_n} z_i)$. Note that since we use the operator $(t,x) \mapsto z_{(t,x)}$ in the right-hand side of (1), the functions f_i, $i \in \mathbb{N}$, actually become functional operators with respect to the third variable. In this setting differential systems with a deviated argument and differential-integral systems are particular cases of (1).

We call $z = \big\{z_i\big\} : E_a^* \to \ell^\infty$, where $0 < a \le \bar{a}$, a solution of problem (1),(2) if

(i) $z \in C(E_a^*, \ell^\infty)$ and the derivative $D_x z = \big\{D_x z_i(t,x)\big\}$ exists on E_a,

(ii) $z(\cdot, x) : [0,a] \to \ell^\infty$ is absolutely continuous on $[0,a]$ for each $x \in [-b,b]$,

(iii) for any fixed $i \in \mathbb{N}$ and $x \in [-b,b]$, the i-th equation of system (1) is satisfied for almost all $t \in [0,a]$, and condition (2) holds true for all $(t,x) \in E_0^* \cup \partial_0 E_a$.

In other words we wish to investigate the local (with respect to t) existence of generalized solutions of problem (1),(2).

In this paper we consider the mixed problem for the infinite nonlinear functional differential problem. We use the method of bicharacteristics that was introduced and developed for non-functional equations by Cinquini-Cibrario [**9**],[**10**] and Cinquini [**8**] for quasilinear as well as nonlinear problems. This method was adapted by Cesari [**6**],[**7**] and Bassanini [**1**],[**2**] for quasilinear systems in the second canonical form. Some extensions of Cesari's results to differential-functional systems were given in [**3**],[**11**],[**18**]. The results obtained in the papers mentioned above by means of the method of bicharacteristics pertain generalized (in the "almost everywhere" sense) solutions.

Generalized solutions to quasilinear or nonlinear functional mixed problems were investigated by Leszczyński [**16**], Turo [**19**], and Kamont and Topolski [**15**]. Classical solutions in the functional setting were considered in [**12**].

In this paper we deal with the mixed problem for infinite nonlinear functional differential systems. Among the literature concerning the Cauchy problem for infinite nonlinear functional differential systems we mention papers by Szarski [**17**] and Jaruszewska-Walczak [**14**]. We prove existence of solutions using the quasi-iteration method for a certain integral-functional system which was introduced by Ważewski [**20**]. This method was used by Brandi, Kamont and Salvadori [**5**] in the case of the Cauchy problem (see also [**13**]). An existence result for this equation was also obtained by Brandi and Ceppitelli [**4**] by means of the method of successive approximations.

2. Notation and assumptions. Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space with the norm $|\cdot|$ defined by $|x| = \max_{1\le i\le n} |x_i|$. We use the symbol ℓ_n^∞ to denote the space of infinite sequences $r = \{r_i\}$, $r_i = (r_{i1}, \dots, r_{in}) \in \mathbb{R}^n$ for $i \in \mathbb{N}$, such that $|r|_\infty = \sup\{|r_i|;\, i \in \mathbb{N}\} < +\infty$. Note that we use the same symbols for the norms in ℓ^∞ and ℓ_n^∞ for simplicity.

Let $C^{0,1}(B, \ell^\infty)$ be the set of all continuous functions $\omega : B \to \ell^\infty$ in the variables $(\tau, s) = (\tau, s_1, \dots, s_n)$ such that the derivatives $D_s\omega_i = (D_{s_1}\omega_i, \dots, D_{s_n}\omega_i)$, exist and $D_s\omega = \{D_s\omega_i\} \in C(B, \ell_m^\infty)$. If $\|\cdot\|_0$ denotes the supremum norm in $C(B, \ell_m^\infty)$ then the norm in $C^{0,1}(B, \mathbb{R})$ is defined by $\|\omega\|_1 = \|\omega\|_0 + \|D_s\omega\|_0$.

For any $\omega \in C(B, \ell_m^\infty)$ let

$$\|\omega\|_L = \sup\Big\{\frac{|\omega(\tau, s) - \omega(\tau, \bar{s})|_\infty}{|s - \bar{s}|} :\ (\tau, s), (\tau, \bar{s}) \in B,\ s \neq \bar{s}\Big\}.$$

If we put $\|\omega\|_{0,L} = \|\omega\|_0 + \|\omega\|_L$, $\|\omega\|_{1,L} = \|\omega\|_1 + \|D_s\omega\|_L$, then we define by $C^{0,i+L}(B, \ell^\infty)$, $i = 0, 1$, the space of all functions $\omega \in C^{0,i}(B, \ell^\infty)$ such that $\|\omega\|_{i,L} < +\infty$ with the norm $\|\cdot\|_{i,L}$.

Let $\Omega^{(0)} = E_{\bar{a}} \times C(B, \ell^\infty) \times \mathbb{R}^n$ and $\Omega^{(1)} = E_{\bar{a}} \times C^{0,1}(B, \ell^\infty) \times \mathbb{R}^n$. We denote by Θ the set of all functions $\theta : [0, \bar{a}] \times \mathbb{R}_+ \to \mathbb{R}_+$ such that $\theta(t, \cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing for almost all $t \in [0, \bar{a}]$ and $\theta(\cdot, p) : [0, \bar{a}] \to \mathbb{R}_+$ is Lebesgue integrable for all $p \in \mathbb{R}_+$.

The norms in $C(E_a^*, \ell^\infty)$ and $C(E_a^*, \ell_n^\infty)$ will be denoted by the same symbol $\|\cdot\|_{E_a^*}$.

Assumption H_1. *Let $f = \{f_i\}$, $f_i : \Omega^{(0)} \to \mathbb{R}$ be the infinite sequence of functions in the variables (t, x, w, q) and let δ be any of these variables. Suppose that*

1° the derivatives $D_\delta f_i$, $i \in \mathbb{N}$ exist on $\Omega^{(1)}$, the sequence $D_\delta f = \{D_\delta f_i\}$ is measurable with respect to t and there is a nondecreasing function $\theta_1 : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|D_\delta f(t, x, w, q)|_\infty \le \theta_1(\|w\|_1) \quad \text{on } \Omega^{(1)},$$

2° there is $\theta_2 \in \Theta$ such that

$$|D_\delta f(t,x,w,q) - D_\delta f(t,\bar{x},w+h,\bar{q})|_\infty \\ \leq \theta_2(t,\|w\|_{1,L})\big[|x-\bar{x}| + \|h\|_1 + |q-\bar{q}|\big],$$

for all $(t,x,h,q) \in \Omega^{(1)}$, $\bar{x},\bar{q} \in \mathbb{R}^n$, $w \in C^{0,1+L}(B,\ell^\infty)$.

Remark 1. Note that if $\delta = w$ then for each $(t,x,w,q) \in \Omega^{(1)}$ the derivative $D_\delta f(t,x,w,q)$ is a continuous linear operator from $C^{0,1}(B,\ell^\infty)$ to ℓ^∞. This means that in that case the norm of $D_\delta f(t,x,w,q)$ is a norm of a linear operator while if $\delta = x$ or $\delta = q$ it is a norm in the space ℓ^∞_n. These norms should be distinguished but for simplicity we use the same symbol $|\cdot|_\infty$ in both cases.

Assumption H_2. *Suppose that*

1° $\phi = \{\phi_i\} \in C(E_0^* \cup \partial_0 E_{\bar{a}}, \ell^\infty)$ and the derivative $D_x\phi = \{D_x\phi_i\}$ exists on $E_0^* \cup \partial_0 E_{\bar{a}}$;

2° there are constants $\Lambda_0, \tilde{\Lambda}_1, \Lambda_1, \tilde{\Lambda}_2, \Lambda_2 \in \mathbb{R}_+$, such that on $E_0^* \cup \partial_0 E_{\bar{a}}$ we have

$$|\phi(t,x)|_\infty \leq \Lambda_0, \quad |\phi(t,x) - \phi(\bar{t},x)|_\infty \leq \tilde{\Lambda}_1|t-\bar{t}|, \quad |D_x\phi(t,x)|_\infty \leq \Lambda_1$$
$$|D_x\phi(t,x) - D_x\phi(\bar{t},\bar{x})|_\infty \leq \tilde{\Lambda}_2|t-\bar{t}| + \Lambda_2|x-\bar{x}|;$$

3° the derivatives $D_t\phi_i(t,x)$, $i \in \mathbb{N}$, exist on $\partial_0 E_{\bar{a}} \cap E_{\bar{a}}$ and the consistency condition

$$D_t\phi_i(t,x) = f_i\big(t,x,\phi_{(t,x)},D_x\phi_i(t,x)\big), \quad i \in \mathbb{N}, \tag{3}$$

holds true on $\partial_0 E_{\bar{a}} \cap E_{\bar{a}}$.

Now, analogously to [**5**],[**13**], we define two functional spaces such that the solution z of (1) will belong to the first space, while $D_x z$ to the other. Let ϕ be the function fulfilling Assumption H_2 and $0 < a \leq \bar{a}$.

If $Q_j \geq \Lambda_i$ for $j = 0,1,2$, and $\tilde{Q}_j \geq \tilde{\Lambda}_j$ for $j = 0,1$, then we denote by $C^{0,1+L}_{\phi,a}(Q)$ the set of all functions $z = \{z_i\} : E_a^* \to \ell^\infty$ such that the derivative $D_x z = \big\{D_x z_i\big\}$ exists on E_a^* and

(i) $z(t,x) = \phi(t,x)$ on $E_0^* \cup \partial_0 E_a$;

(ii) $|z(t,x)|_\infty \leq Q_0$, $|D_x z(t,x)|_\infty \leq Q_1$ on E_a;

(iii) for $t,\bar{t} \in [0,a]$, $x,\bar{x} \in [-b,b]$, we have

$$|z(t,x) - z(\bar{t},x)|_\infty \leq \tilde{Q}_1|t-\bar{t}|,$$
$$|D_x z(t,x) - D_x z(\bar{t},\bar{x})|_\infty \leq \tilde{Q}_2|t-\bar{t}| + Q_2|x-\bar{x}|.$$

If $P_j \geq \Lambda_{j+1}$ for $j = 0,1$, and $\tilde{P}_1 \geq \tilde{\Lambda}_2$, then we denote by $C^{0,L}_{D_x\phi,a}(P)$ the set of all functions $u = \big\{u_i\big\} : E_a \to \ell^\infty_n$ such that

(i) $u(t,x) = D_x\phi(t,x) \quad \text{on } \partial_0 E_a \cap E_a$;
(ii) $|u(t,x)|_\infty \le P_0 \quad \text{on } E_a$;
(iii) for $t, \bar{t} \in [0,a]$, $x, \bar{x} \in [-b,b]$, we have

$$|u(t,x) - u(\bar{t},\bar{x})|_\infty \le \tilde{P}_1 |t-\bar{t}| + P_1|x-\bar{x}|.$$

3. Bicharacteristics. Let ϕ be a given function satisfying Assumption H_2 and $0 < a \le \bar{a}$. Then, for any $z \in C^{0,1+L}_{\phi,a}(Q)$, $u \in C^{0,L}_{D_x\phi,a}(P)$ and $i \in \mathbb{N}$, we consider the Cauchy problem

$$\begin{aligned} &\frac{d\eta}{d\tau}(\tau) = -D_q f_i\left(\tau, \eta(\tau), z_{(\tau,\eta(\tau))}, u_i(\tau,\eta(\tau))\right), \qquad \tau \in [0,a],\\ &\eta(t) = x, \end{aligned} \tag{4}$$

and we denote by $g_i[z,u](\cdot,t,x) = (g_{i1}[z,u](\cdot,t,x), \ldots, g_{in}[z,u](\cdot,t,x))$ its Caratheodory solution. This solution we call the i-th bicharacteristic of system (1) corresponding to $[z,u]$. From classical theorems it follows that the unique solution to problem (4) exists if Assumption H_1 holds with $\delta = q$. Let $\lambda_i[z,u](t,x)$ be the left end of the maximal interval on which the solution $g_i[z,u](\cdot,t,x)$ is defined. If $D_{q_j} f_i(t,x,w,q) \ge 0$, $j = 1,\ldots,n$, on $\Omega^{(0)}$ then

$$\big(\lambda_i[z,u](t,x), g_i[z,u](\lambda_i[z,u](t,x),t,x)\big) \in (E_0^* \cup \partial_0 E_a) \cap E_a$$

and we may define the following two sets:

$$\begin{aligned} E^{(i)}_{a0}[z,u] &= \{(t,x) \in E_a;\ \lambda_i[z,u](t,x) = 0\},\\ E^{(i)}_{ab}[z,u] &= \{(t,x) \in E_a;\ g_{ij}[z,u](\lambda_i[z,u](t,x),t,x) = b_j,\\ &\qquad\qquad\qquad\qquad \text{for some } 1 \le j \le n\}. \end{aligned}$$

Write

$$R_1 = 1 + Q_1 + Q_2 + P_1, \qquad \Upsilon(\tau,t) = \exp\Big\{R_1 \Big|\int_t^\tau \theta_2^*(\xi)\,d\xi\Big|\Big\},$$

where $\theta_2^*(\xi) = \theta_2(\xi, Q_0 + Q_1 + Q_2)$. In the sequel we will also write θ_1^* instead of $\theta_1(Q_0 + Q_1)$ for simplicity.

LEMMA 1. *Suppose that ϕ fulfills Assumption H_2 and that Assumption H_1 is satisfied for $\delta = q$. If z, $\bar{z} \in C^{0,1+L}_{\phi,a}(Q)$, u, $\bar{u} \in C^{0,L}_{D_x\phi,a}(P)$, are given functions and $i \in \mathbb{N}$ then for $(t,x), (\bar{t},\bar{x}) \in E_a$ such that the intervals*

$$K_1 = \big[\max\{\lambda_i[z,u](t,x), \lambda_i[z,u](\bar{t},\bar{x})\}, \min\{t,\bar{t}\}\big],$$
$$K_2 = \big[\max\{\lambda_i[z,u](t,x), \lambda_i[\bar{z},\bar{u}](t,x)\}, t\big]$$

are nonempty we have the estimates

$$|g_i[z,u](\tau,t,x) - g_i[z,u](\tau,\bar{t},\bar{x})| \le \Upsilon(\tau,t)\big\{\theta_1^*|t-\bar{t}| + |x-\bar{x}|\big\} \tag{5}$$

for $\tau \in K_1$, and

$$(6)\quad |g_i[z,u](\tau,t,x)-g_i[\bar z,\bar u](\tau,t,x)| \le \Upsilon(\tau,t)\Big|\int_t^\tau \theta_2^*(\xi)\big\{\|z-\bar z\|_{E_\xi} + \|D_x z - D_x\bar z\|_{E_\xi} + \|u-\bar u\|_{E_\xi}\big\}d\xi\Big| \qquad \textit{for } \tau\in K_2.$$

Proof. Let (t,x), $(\bar t,\bar x)\in E_a$ be such that the intervals K_1, K_2 are nonempty. If we transform (4) into an integral equation then by virtue of Assumption H_1 we have

$$\begin{aligned}
&|g_i[z,u](\tau,t,x)-g_i[z,u](\tau,\bar t,\bar x)|\\
&\quad\le |x-\bar x| + \Big|\int_t^{\bar t}|D_q f_i(P_i[z,u](\xi,\bar t,\bar x))|d\xi\Big|\\
&\qquad + \Big|\int_t^\tau |D_q f_i(P_i[z,u](\xi,t,x)) - D_q f_i(P_i[z,u](\xi,\bar t,\bar x))|d\tau\Big|\\
&\quad\le |x-\bar x| + \theta_1^*|t-\bar t| + \Big|\int_t^\tau \theta_2^*(\xi)\big\{|g_i[z,u](\xi,t,x)-g_i[z,u](\xi,\bar t,\bar x)|\\
&\qquad + \|z_{(\xi,g_i[z,u](\xi,t,x))} - z_{(\tau,g_i[z,u](\xi,\bar t,\bar x))}\|_1\\
&\qquad + |u_i(\tau,g_i[z,u](\tau,x,y)) - u_i(\xi,g_i[z,u](\xi,\bar t,\bar x))|\big\}d\xi\Big|\\
&\quad\le |x-\bar x| + \theta_1^*|t-\bar t|\\
&\qquad + \Big|\int_t^\tau \theta_2^*(\xi)R_1|g_i[z,u](\xi,t,x)-g_i[z,u](\xi,\bar t,\bar x)|d\xi\Big|
\end{aligned}$$

for $\tau\in K_1$, where

$$(7)\quad P_i[z,u](\xi,t,x) = \big(\xi, g_i[z,u](\xi,t,x), z_{(\xi,g_i[z,u](\xi,t,x))}, u_i(\xi,g_i[z,u](\xi,t,x))\big).$$

Thus (5) follows from the Gronwall lemma.

In the same way, by Assumption H_1, we get the estimate

$$\begin{aligned}
&|g_i[z,u](\tau,t,x)-g_i[\bar z,\bar u](\tau,t,x)|\\
&\quad\le \Big|\int_t^\tau \theta_2^*(\xi)\big\{\|z-\bar z\|_{E_\xi} + \|D_x z - D_x\bar z\|_{E_\xi} + \|u-\bar u\|_{E_\xi}\big\}d\xi\Big|\\
&\qquad + \Big|\int_t^\tau \theta_2^*(\xi)R_1|g_i[z,u](\xi,t,x)-g_i[\bar z,\bar u](\xi,t,x)|d\xi\Big|
\end{aligned}$$

for $\tau\in K_2$. Now, again using the Gronwall lemma, we get (6), which completes the proof of Lemma 1. □

Lemma 2. *Suppose that ϕ fulfills Assumption H_2 and that Assumption H_1 is satisfied for $\delta = q$. Furthermore, suppose that for every $p\in\mathbb{R}_+$ there is $\delta(p)>0$ such that we have $D_{q_j} f_i(t,x,w,q)\ge\delta(p)$, $i\in\mathbb{N}$, $j=1,\dots,n$, for all $(t,x,w,q)\in\Omega^{(1)}$, such that $\|w\|_1\le p$. If $i\in\mathbb{N}$ and z, $\bar z\in C^{0,1+L}_{\phi,a}(Q)$,*

$u,\ \bar{u} \in C^{0,L}_{D_x\phi,a}(P)$, *are given functions then for all* $(t,x), (\bar{t},\bar{x}) \in E_a$ *we have*

$$|\lambda_i[z,u](t,x) - \lambda_i[z,u](\bar{t},\bar{z})| \le \frac{1}{\delta^*}\Upsilon(0,t)\big\{\theta_1^*|t-\bar{t}| + |x-\bar{x}|\big\}, \tag{8}$$

$$\begin{aligned}|\lambda_i[z,u](t,x) - \lambda_i[\bar{z},\bar{u}](t,x)| \le \frac{1}{\delta^*}\Upsilon(0,t)\int_0^t \theta_2^*(\xi)\big\{&\|z-\bar{z}\|_{E_\xi} \\ &+ \|D_x z - D_x\bar{z}\|_{E_\xi} + \|u-\bar{u}\|_{E_\xi}\big\}d\xi,\end{aligned} \tag{9}$$

where $\delta^* = \delta(Q_0 + Q_1)$.

Proof. Let $i \in \mathbb{N}$ be fixed and $g_i = g_i[z,u]$, $\lambda_i = \lambda_i[z,u]$, $\bar{g}_i = g_i[\bar{z},\bar{u}]$, $\bar{\lambda}_i = \lambda_i[\bar{z},\bar{u}]$. Since (8) is obviously satisfied if (t,x), $(\bar{t},\bar{x}) \in E^{(i)}_{a0}[z,u]$, without loss of generality we may assume that $\lambda_i(\bar{t},\bar{x}) \le \lambda_i(t,x)$ and $(t,x) \in E^{(i)}_{ab}[z,u]$. There exists $1 \le j \le n$ such that $g_{ij}(\lambda_i(t,x),t,x) = b_j$. Then we have

$$\begin{aligned}
&g_{ij}(\lambda_i(t,x),t,x) - g_{ij}(\lambda_i(t,x),\bar{t},\bar{x}) \\
&\quad\ge g_{ij}(\lambda_i(\bar{t},\bar{x}),\bar{t},\bar{x}) - g_{ij}(\lambda_i(t,x),\bar{t},\bar{x}) \\
&\quad= \int_{\lambda_i(\bar{t},\bar{x})}^{\lambda_i(t,x)} D_{q_j} f_i\left(\xi, g_i(\xi,\bar{t},\bar{x}), z_{(\xi,g_i(\xi,\bar{t},\bar{x}))}, u_i(\xi, g_i(\xi,\bar{t},\bar{x}))\right) d\tau \\
&\quad\ge \delta^*[\lambda_i(t,x) - \lambda_i(\bar{t},\bar{x})].
\end{aligned}$$

The above estimate together with (5) gives (8).

Analogously, since (9) is obviously satisfied if $(t,x) \in E^{(i)}_{a0}[z,u] \cap E^{(i)}_{a0}[\bar{z},\bar{u}]$, we may assume that $\bar{\lambda}_i(t,x) \le \lambda_i(t,x)$ and $(t,x) \in E^{(i)}_{ab}[z,u]$. Then for $1 \le j \le n$ such that $g_{ij}(\lambda_i(t,x),t,x) = b_j$ we have

$$\begin{aligned}
&g_{ij}(\lambda_i(t,x),t,x) - \bar{g}_{ij}(\lambda_i(t,x),t,x) \\
&\quad\ge \bar{g}_{ij}(\bar{\lambda}_i(t,x),t,x) - \bar{g}_{ij}(\lambda_i(t,x),t,x) \\
&\quad= \int_{\bar{\lambda}_i(t,x)}^{\lambda_i(t,x)} D_{q_j} f_i\left(\xi, \bar{g}_i(\xi,t,x), \bar{z}_{(\xi,\bar{g}_i(\xi,t,x))}, \bar{u}_i(\xi, \bar{g}_i(\xi,t,x))\right) d\xi \\
&\quad\ge \delta^*[\lambda_i(t,x) - \bar{\lambda}_i(t,x)],
\end{aligned}$$

which together with (6) gives (9). □

4. A certain system of integral-functional equations.

Assumption H_3. *Suppose that*

1° $f = \{f_i\}$, $f_i : \Omega^{(0)} \to \mathbb{R}$ *is an infinite sequence of functions in the variables* (t,x,w,q), *measurable in* t, *and there is a nondecreasing function* $\theta_0 : \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$|f(t,x,w,q)|_\infty \le \theta_0(\|w\|_0) \qquad \text{on } \Omega^{(0)},$$

Furthermore, let Assumption H_1 be satisfied with $\delta = x, w, q$.

2° For every $p \in \mathbb{R}_+$ there is a constant $\delta(p) > 0$ such that we have $D_{q_j} f_i(t,x,w,q) \geq \delta(p)$, $i \in \mathbb{N}$, $j = 1,\dots,n$, for all $(t,x,w,q) \in \Omega^{(1)}$, such that $\|w\|_1 \leq p$.

Let ϕ and f satisfy assumptions H_2,H_3, respectively, and for $a \in (0,\bar{a}]$ let $z = \{z_i\} \in C^{0,1+L}_{\phi,a}(Q)$, $u = \{u_i\} \in C^{0,L}_{D_x\phi,a}(P)$, where $u_i = (u_{i1},\dots,u_{in})$, $i \in \mathbb{N}$ be given functions. Note that u consists of n infinite sequences which we denote by $\hat{u}_j = \{u_{ij}\}$, $1 \leq j \leq n$. We define the operators $T_i[z,u]$, $V_{ij}[z,u]$, $i \in \mathbb{N}$, $j = 1,\dots,n$, by

$$\begin{aligned}
T_i[z,u](t,x) &= \phi_i(\lambda_i[z,u](t,x), g_i[z,u](\lambda_i[z,u](t,x),t,x)) \\
&\quad + \int_{\lambda_i[z,u](t,x)}^{t} \Big[f_i(P_i[z,u](\xi,t,x)) \\
&\quad - \sum_{k=1}^{n} D_{q_k} f_i(P_i[z,u](\xi,t,x)) u_{ik}(\tau, g_i[z,u](\tau,t,x)) \Big] d\xi, \\
V_{ij}[z,u](t,x) &= D_{x_j}\phi_i(\lambda_i[z,u], g_i[z,u](\lambda_i[z,u](t,x),t,x)) \\
&\quad + \int_{\lambda_i[z,u](t,x)}^{t} \Big[D_{x_i} f_i(P_i[z,u](\xi,t,x)) \\
&\quad + D_w f_i(P_i[z,u](\xi,t,x)) \circ (\hat{u}_j)_{(\tau, g_i[z,u](\xi,t,x))} \Big] d\xi
\end{aligned}$$

for $(t,x) \in E^*_a$, and

$$T_i[z,u](t,x) = \phi_i(t,x), \quad V_{ij}[z,u](t,x) = D_{x_j}\phi_i(t,x)$$

for $(t,x) \in E^*_0 \cup \partial_0 E_a$, where $g_i[z,u]$ is a solution of (4), $\lambda_i[z,u]$ is the left end of the maximal interval on which this solution is defined and $P_i[z,u]$ is given by (7). In the definition of $V_{ij}[z,u]$ the derivative $D_w f_i(P_i[z,u](\xi,t,x))$ is a continuous linear operator and by $\circ$ we denote the value of this operator taken on the function $(\hat{u}_j)_{(\tau, g_i[z,u](\xi,t,x))} \in C^{0,1}(B,\ell^\infty)$. Put $T[z,u] = \{T_i[z,u]\}$ and $V[z,u] = \{V_i[z,u]\}$, where $V_i[z,u] = (V_{i1}[z,u],\dots,V_{in}[z,u])$. We will consider the system of integral-functional equations

$$z = T[z,u], \qquad u = V[z,u]. \tag{10}$$

Remark 2. The integral-functional system (10) arises in the following way. We introduce an additional unknown function $u = D_x z$ in the i-th equation of system (1). Then we consider the linearization of this equation with respect to $u_i = (u_{i1},\dots,u_{in})$, which yields

$$D_t z_i(t,x) = f_i(\tilde{P}_i) + \sum_{k=1}^{n} D_{q_k} f_i(\tilde{P}_i)\big(D_{x_k} z_i(t,x) - u_{ik}(t,x)\big), \tag{11}$$

where $\tilde{P}_i = (t, x, z_{(t,x)}, u_i(t,x))$. Differentiating (1) with respect to x_j and substituting $u = D_x z$, we get

$$D_t u_{ij}(t,x) = D_{x_j} f_i(\tilde{P}_i) + D_w f_i(\tilde{P}_i) \circ (\hat{u}_j)_{(t,x)} + \sum_{k=1}^{n} D_{q_k} f_i(\tilde{P}_i) D_{x_j} u_{ik}(x,y), \quad j = 1, \ldots, n. \tag{12}$$

Making use of (4) we have

$$\frac{d}{d\xi} z_i(\xi, g_i[z,u](\xi,t,x)) = D_t z_i(\xi, g_i[z,u](\xi,t,x)) - \sum_{k=1}^{n} D_{q_k} f_i(P_i[z,u](\xi,t,x)) D_{x_k} z_i(\xi, g_i[z,u](\xi,t,x)).$$

Substituting (11) in the above relation and integrating the resulting equation with respect to ξ on $[\lambda[z,u](t,x), t]$, we get the first of the equations in (10) on E_a. Repeating these considerations for (12) and taking into account that $z = \phi$, $u = D_x \phi$, on $E_0^* \cup \partial_0 E_a$, we get the second equation in (10).

Suppose that ϕ and f satisfy Assumptions H_2, H_3, respectively. Under these assumptions we prove that the solution of (12) exists, using the quasi-iteration method, which general idea was given by Ważewski [**20**]. We define a sequence $\{z^{(m)}, u^{(m)}\}$ in the following way.

1° Let $\tilde{\phi}$ be any extension of ϕ onto the set E_a^* such that $\tilde{\phi}$ satisfies conditions 1°,2° of Assumption H_2 on E_a^*. We put

$$z^{(0)}(t,x) = \tilde{\phi}(t,x), \qquad u^{(0)}(t,x) = D_x \tilde{\phi}(t,x), \tag{13}$$

and then $z^{(0)} \in C^{0,1+L}_{\phi,a}(Q)$, $u^{(0)} \in C^{0,L}_{D_x\phi,a}(P)$.

2° If $z^{(m)} \in C^{0,1+L}_{\phi,a}(Q)$, $u^{(m)} \in C^{0,L}_{D_x\phi,a}(P)$ are already defined functions then $u^{(m+1)}$ is a solution of the equation

$$u = V^{(m)}[u], \tag{14}$$

and $z^{(m+1)}$ is defined by

$$z^{(m+1)} = T[z^{(m)}, u^{(m+1)}]. \tag{15}$$

The operator $V^{(m)} = \{V_i^{(m)}\}$, where $V_i^{(m)}[u] = (V_{i1}^{(m)}[u], \dots, V_{in}^{(m)}[u])$, $i \in \mathbb{N}$, is given by

$$\begin{aligned}
(16)\quad V_{ij}^{(m)}&[u](t,x) \\
&= D_{x_j}\phi_i(\lambda_i[z^{(m)},u](t,x), g_i[z^{(m)},u](\lambda_i[z^{(m)},u](t,x),t,x)) \\
&\quad + \int_{\lambda_i[z^{(m)},u](t,x)}^{t} \Big[D_{x_j} f_i(P_i[z^{(m)},u](\xi,t,x)) \\
&\qquad + D_w f_i(P_i[z^{(m)},u](\xi,t,x)) \circ (\hat{u}_j^{(m)})_{(\xi, g_i[z^{(m)},u](\xi,t,x))}\Big] d\xi
\end{aligned}$$

for $(t,x) \in E_a$, and

$$V_{ij}^{(m)}[u](x,y) = D_{x_j}\phi_i(x,y) \qquad \text{for } (t,x) \in E_0^* \cup \partial_0 E_a.$$

Remark 3. Note that the operators $V^{(m)}$ and $V[z^{(m)},\cdot]$ are not identical since in the last term of (16) we have $\hat{u}_j^{(m)}$ instead of $\hat{u}_j$. In Theorem 3 we prove that $D_x z^{(m)} = u^{(m)}$ and therefore, analogously as in Remark 2, we may say that $\hat{u}_j^{(m)}$ appears in (16) as a substitution for $D_x \hat{z}_j^{(m)}$.

In the next section we prove the existence of the sequence $\{z^{(m)}, u^{(m)}\}$ under the assumption that the constants defining classes $C_{\phi,a}^{0,1+L}(Q)$ and $C_{D_x\phi,a}^{0,L}(P)$ are sufficiently large. More precisely, they fulfill the following assumption.

Assumption H_4. *Suppose that* $Q_0 > \Lambda_0$, $Q_1 > \Lambda_1$,

$$\begin{aligned}
&Q_2 > \Lambda_2\big[\frac{1}{\delta^*}(1+\theta_1^*) + 1\big] + S_1\theta_1^*\frac{1}{\delta^*} \\
&\tilde{Q}_1 > \max\Big\{\tilde{\Lambda}_1, \Lambda_1\big[\frac{1}{\delta^*}(1+\theta_1^*) + 1\big]\theta_1^* + \big[1 + \frac{1}{\delta^*}\theta_1^*\big](\theta_0(Q_0) + \theta_1^* P_0)\Big\}, \\
&\tilde{Q}_2 > \max\Big\{\tilde{\Lambda}_2, \Lambda_2\big[\frac{1}{\delta^*}(1+\theta_1^*) + 1\big]\theta_1^* + \big[S_1 + S_1\theta_1^*\frac{1}{\delta^*}\big]\theta_1^*\Big\},
\end{aligned}$$

and let $P_0 = Q_1$, $P_1 = Q_2$, $\tilde{P}_1 = \tilde{Q}_2$.

Write

$$\begin{aligned}
\Gamma_0(t) &= \Lambda_1 + \theta_1^* S_1 t, \\
\tilde{\Gamma}_0(t) &= \Lambda_1 \Upsilon(0,t)\big[\frac{1}{\delta^*}(1+\theta_1^*) + 1\big]\theta_1^* \\
&\quad + \big[1 + \frac{1}{\delta^*}\Upsilon(0,t)\theta_1^*\big](\theta_0(Q_0) + \theta_1^* P_0) \\
&\quad + \int_0^t \big\{\theta_1^* + \theta_2^*(\tau)P_0\big\} R_1 \Upsilon(0,\tau)\, d\tau,
\end{aligned}$$

$$\Gamma_1(t) = \Lambda_2 \Upsilon(0,t)\big[\frac{1}{\delta^*}(1+\theta_1^*)+1\big] + S_1\theta_1^*\frac{1}{\delta^*} + \int_0^t \big\{\theta_2^*(\tau)R_1S_1 + \theta_1^*P_1\big\}\Upsilon(0,\tau)d\tau,$$

where $S_1 = 1 + P_0$.

REMARK 4. Note that, since $\lim_{t\to 0}\Upsilon(0,t) = 1$, by Assumption H_4, we may choose $a \in (0,\bar{a}]$ sufficiently small for the following estimates to hold

$$\Lambda_0 + \big[\theta_0^* + \theta_1^* P_0\big]a \le Q_0, \qquad \tilde{\Gamma}_0(a) \le \tilde{Q}_1, \qquad \Gamma_0(a) \le Q_1 = P_0,$$
$$\theta_1^*[S_1 + \Gamma_1(a)] \le \tilde{Q}_2 = \tilde{P}_1, \qquad \Gamma_1(a) \le Q_2 = P_1.$$

5. The existence of the sequence of successive approximations. In this section we prove that the sequence $\{z^{(m)}, u^{(m)}\}$ exists, provided that $a \in (0,\bar{a}]$ is sufficiently small.

THEOREM 3. *If Assumptions H_2–H_4 are satisfied and $a \in (0,\bar{a}]$ is sufficiently small then for any $m \in \mathbb{N}$ we have*

(I_m) $z^{(m)}$, $u^{(m)}$ are defined on E_a^* and we have $z^{(m)} \in C^{0,1+L}_{\phi,a}(Q)$, $u^{(m)} \in C^{0,L}_{D_x\phi,a}(P)$;

(II_m) $D_x z^{(m)}(t,x) = u^{(m)}(t,x)$ on E_a.

PROOF. We will prove (I_m) and (II_m) by induction. It follows from (15) that (I_0),(II_0) are satisfied. Suppose that conditions (I_m) and (II_m) hold true for a given $m \in \mathbb{N}$. We first prove that $u^{(m+1)} : E_a^* \to \mathbb{R}^n$ exists and $u^{(m+1)} \in C^{0,L}_{D_x\phi,a}(P)$.

We claim that given $z^{(m)} \in C^{0,1+L}_{\phi,a}(Q)$, the operator $V^{(m)}$ maps $C^{0,L}_{D_x\phi,a}(P)$ into itself for sufficiently small $a \in (0,\bar{a}]$. It follows from Assumptions H_2,H_3 and inequality (5) that given $u \in C^{0,L}_{D_x\phi,a}(P)$, for all $(t,x),(\bar{t},\bar{x}) \in E_a$ we have the estimates

$$|V_i^{(m)}[u](t,x)| \le \Lambda_1 + \int_{\lambda_i[z^{(m)},u](t,x)}^t \theta_1^* S_1 d\xi \le \Gamma_0(a),$$

$$
\begin{aligned}
&|V_i^{(m)}[u](t,x) - V^{(m)}[u](\bar{t},\bar{x})| \\
&\quad \le \Lambda_2 \Upsilon(0,t)\big\{[1+\theta_1^*]\frac{1}{\delta^*}+1\big\}\big\{\theta_1^*|t-\bar{t}|+|x-\bar{x}|\big\} \\
&\qquad + \Big|\int_t^{\bar{t}} \theta_1^* S_1 d\xi\Big| + \Big|\int_{\lambda_i[z^{(m)},u](t,x)}^{\lambda_i[z^{(m)},u](\bar{t},\bar{x})} \theta_1^* S_1 d\xi\Big| \\
&\qquad + \big\{\theta_1^*|t-\bar{t}|+|x-\bar{x}|\big\}\cdot \int_{\lambda(t,x)}^{t} \big\{\theta_2^*(\xi) R_1 S_1 + \theta_1^* P_1\big\}\Upsilon(\xi,t)d\xi \\
&\quad \le \theta_1^*[S_1+\Gamma_1(a)]|t-\bar{t}| + \Gamma_1(a)|x-\bar{x}|.
\end{aligned}
$$

Hence by Assumption H_4 we may take $a \in (0,\bar{a}]$ so small that $\Gamma_0(a) \le P_0$, $\Gamma_1(a) \le P_1$, $\theta_1^*[S_1+\Gamma_1(a)] \le \tilde{P}_1$, and then

$$
\begin{gathered}
|V^{(m)}[u](t,x)|_\infty \le P_0, \\
|V^{(m)}[u](x,y) - V^{(m)}[u](\bar{t},\bar{x})|_\infty \le P_1\big[|t-\bar{t}|+|x-\bar{x}|\big]
\end{gathered}
\tag{17}
$$

for $(t,x),(\bar{t},\bar{x}) \in E_a$. Since $V^{(m)}[u] = D_x\phi$ on $E_0^* \cup \partial E_a$, it follows from (17) that $V^{(m)}$ maps $C^{0,L}_{D_x\phi,a}(P)$ into itself.

If $u \in C^{0,L}_{D_x\phi,a}(P)$, $\bar{u} \in C^{0,L}_{D_x\bar{\phi},a}(P)$ then analogously, by Assumptions H_2, H_3, formulas (6), (9) and the relation $V^{(m)}[u] = V^{(m)}[\bar{u}] = D_x\phi$ on E_0, we get

$$
\|V^{(m)}[u] - V^{(m)}[\bar{u}]\|_{E_a^*} \le \int_0^a G(\tau)\|u-\bar{u}\|_{E_\tau^*} d\tau,
$$

where

$$
\begin{aligned}
G(t) &= \Lambda_2\Upsilon(0,t)\theta_2^*(t)\big[\frac{1}{\delta^*}(1+\theta_1^*)+1\big] + \theta_1^* S_1 \frac{1}{\delta^*}\Upsilon(0,t)\theta_2^*(t) \\
&\quad + \big[\theta_2^*(t)R_1S_1 + \theta_1^*P_1\big]\Upsilon(0,t)\int_0^t \theta_2^*(\tau)d\tau + \theta_2^*(t)S_1.
\end{aligned}
$$

We may take $a \in (0,\bar{a}]$ so small that $\int_0^a G(\tau)d\tau < 1$ and consequently $V^{(m)}$ is a contraction with the norm $\|\cdot\|_{E_a^*}$. By the Banach fixed point theorem, there exists the unique solution $u \in C^{0,L}_{D_x\phi,a}$ of (14) which is $u^{(m+1)}$.

Our next goal is to prove that $z^{(m+1)}$ given by (15) satisfies (II_{m+1}). For $t \in [0,a]$, $x,\bar{x} \in [-b,b]$ put $\Delta(t,x,\bar{x}) = \big\{\Delta_i(t,x,\bar{x})\big\}$, where

$$
\Delta_i(t,x,\bar{x}) = z_i^{(m+1)}(t,x) - z_i^{(m+1)}(t,\bar{x}) - u_i^{(m+1)}(t,x)\cdot(x-\bar{x}),
\tag{18}
$$

and "$\cdot$" denotes the scalar product. We will prove that there is a constant $\tilde{C} \in \mathbb{R}_+$ such that

$$
|\Delta(t,x,\bar{x})|_\infty \le \tilde{C}|x-\bar{x}|^2
\tag{19}
$$

For simplicity we put $\lambda_i(t,x) = \lambda_i[z^{(m)}, u^{(m+1)}](t,x)$ and

$$P_i(\xi) = P_i[z^{(m)}, u^{(m+1)}](\xi, t, x), \quad g_i(\xi) = g_i[z^{(m)}, u^{(m+1)}](\xi, t, x),$$

and replacing x with $\bar{x}$ we get analogous formulas for $\bar{P}_i(\xi)$, $\bar{g}_i(\xi)$. In view of (15) we may write (18) in the form

$$\begin{aligned}
\Delta_i(t,x,\bar{x}) &= \phi_i(\lambda_i(t,x), g_i(\lambda_i(t,x))) - \phi_i(\lambda_i(t,\bar{x}), \bar{g}_i(\lambda_i(t,\bar{x}))) \\
&\quad - D_x\phi_i(\lambda_i(t,x), g_i(\lambda(t,x))) \cdot (x - \bar{x}) \\
&\quad + \int_{\lambda(t,x)}^{t} \big(f_i(P_i(\xi)) - f_i(\bar{P}_i(\xi)) \big)\, d\xi \\
&\quad - \int_{\lambda(t,x)}^{t} \big\{ D_q f_i(P_i(\xi)) \cdot u_i^{(m+1)}(\xi, g_i(\xi)) \\
&\qquad\quad - D_q f_i(\bar{P}_i(\xi)) \cdot u_i^{(m+1)}(\xi, \bar{g}_i(\xi)) \big\} d\xi \\
&\quad + \int_{\lambda_i(t,x)}^{\lambda_i(t,\bar{x})} \{ f_i(\bar{P}_i(\xi)) - D_q f_i(\bar{P}_i(\xi)) \cdot u_i^{(m+1)}(\xi, \bar{g}(\xi) \} d\xi \\
&\quad - \int_{\lambda_i(t,x)}^{t} \big\{ D_x f_i(P_i(\xi)) + D_w f_i(P_i(\xi)) \circ (u^{(m)})_{(\xi, g_i(\xi))} \big\} d\xi \cdot (x - \bar{x}),
\end{aligned}$$

where

$$\begin{aligned}
&D_w f_i(P_i(\xi)) \circ (u^{(m)})_{(\xi, g_i(\xi))} \\
&\qquad = \big(D_w f_i(P_i(\xi)) \circ (\hat{u}_1^{(m)})_{(\xi, g_i(\xi))}, \ldots, D_w f_i(P_i(\xi)) \circ (\hat{u}_n^{(m)})_{(\xi, g_i(\xi))} \big)
\end{aligned}$$

In the above formula we apply the Hadamard mean value theorem to the difference $f_i(P_i(\xi)) - f_i(\bar{P}_i(\xi))$, whence

$$\begin{aligned}
&\int_{\lambda_i(t,x)}^{t} \big(f_i(P_i(\xi)) - f_i(\bar{P}_i(\xi)) \big)\, d\xi \\
&\qquad = \int_{\lambda_i(t,x)}^{t} \int_0^1 D_x f_i(Q_i(s,\xi)) [g_i(\xi) - \bar{g}_i(\xi)] ds\, d\xi \\
&\qquad + \int_{\lambda_i(t,x)}^{t} \int_0^1 D_w f_i(Q_i(s,\xi)) \circ \Big[z^{(m)}_{(\xi, g_i(\xi))} - z^{(m)}_{(\xi, \bar{g}_i(\xi))} \Big] ds\, d\xi \\
&\qquad + \int_{\lambda_i(t,x)}^{t} \int_0^1 D_q f_i(Q_i(s,\xi)) \cdot \Big[u_i^{(m+1)}(\xi, g_i(\xi)) - u_i^{(m+1)}(\xi, \bar{g}_i(\xi)) \Big] ds\, d\xi,
\end{aligned}$$

where

$$Q_i(s,\xi) = s P_i[z^{(m)}, u^{(m+1)}](\xi, t, x) + (1-s) P_i[z^{(m)}, u^{(m+1)}](\xi, t, \bar{x}).$$

Let us define

$$\begin{aligned}
\Delta_i^{(1)}(t,x,\bar{x}) &= \phi_i(\lambda_i(t,x), g_i(\lambda_i(t,x))) - \phi_i(\lambda_i(t,\bar{x}), \bar{g}_i(\lambda_i(t,\bar{x}))) \\
&\quad - D_x\phi_i(\lambda_i(t,x), g_i(\lambda(t,x)))[\lambda_i(t,x) - \lambda_i(t,\bar{x})] \\
&\quad - D_x\phi_i(\lambda_i(t,x), g_i(\lambda(t,x))) \cdot [g_i(\lambda_i(t,x)) - g_i(\lambda_i(t,\bar{x}))] \\
&\quad + \int_{\lambda_i(t,x)}^{t} \int_0^1 \big[D_x f_i(Q_i(s,\xi)) - D_x f_i(P_i(\xi))\big] \cdot [g_i(\xi) - \bar{g}_i(\xi)] ds\, d\xi \\
&\quad + \int_{\lambda_i(t,x)}^{t} \int_0^1 \big[D_w f_i(Q_i(s,\xi)) - D_w f(P_i(\xi))\big] \\
&\qquad\quad \circ \Big[z^{(m)}_{(\xi, g_i(\xi))} - z^{(m)}_{(\xi,\bar{g}_i(\xi))}\Big] ds\, d\xi \\
&\quad + \int_{\lambda_i(t,x)}^{t} \int_0^1 \big[D_q f_i(Q_i(s,\xi)) - D_q f_i(\bar{P}_i(\xi))\big] \\
&\qquad\quad \cdot \Big[u_i^{(m+1)}(\xi, g_i(\xi)) - u_i^{(m+1)}(\xi, \bar{g}_i(\xi))\Big] ds\, d\xi \\
&\quad + \int_{\lambda_i(t,x)}^{t} D_w f_i(P_i(\xi)) \circ \Big[z^{(m)}_{(\xi, g_i(\xi))} - z^{(m)}_{(\xi,\bar{g}_i(\xi))} \\
&\quad - (u^{(m)})_{(\xi, g_i(\xi))} \cdot [g_i(\xi) - \bar{g}_i(\xi)]\Big] d\xi, \\
\Delta_i^{(2)}(t,x,\bar{x}) &= [\lambda_i(t,x) - \lambda_i(t,\bar{x})] \cdot D_t\phi(\lambda_i(t,x), g_i(\lambda(t,x))) \\
&\quad - \int_{\lambda_i(t,\bar{x})}^{\lambda_i(t,x)} f_i(\bar{P}_i(\xi)) d\xi \\
&\quad + [\bar{g}_i(\lambda_i(t,x)) - \bar{g}_i(\lambda_i(t,\bar{x}))] \cdot D_x\phi_i(\lambda_i(t,x), g_i(\lambda(t,x))) \\
&\quad + \int_{\lambda_i(t,\bar{x})}^{\lambda_i(t,x)} D_q f(\bar{P}_i(\xi)) \cdot u_i^{(m+1)}(\xi, \bar{g}_i(\xi)) d\xi,
\end{aligned}$$

and

$$\begin{aligned}
&\tilde{\Delta}_i^{(1)}(t,x,\bar{x}) \\
&\qquad = D_x\phi_i(\lambda_i(t,x), g_i(\lambda_i(t,x))) \cdot [g_i(\lambda_i(t,x)) - \bar{g}_i(\lambda_i(t,x)) - (x - \bar{x})] \\
&\tilde{\Delta}_i^{(2)} = \int_{\lambda_i(t,x)}^{t} D_x f_i(P_i(\xi)) \cdot [g_i(\xi) - \bar{g}_i(\xi) - (x - \bar{x})] d\xi \\
&\qquad + \int_{\lambda_i(t,x)}^{t} D_w f_i(P_i(\xi)) \circ (u^{(m)})_{(\xi, g_i(\xi))} \cdot [g_i(\xi) - \bar{g}_i(\xi) - (x - \bar{x})] d\xi \\
&\tilde{\Delta}_i^{(3)} = -\int_{\lambda_i(t,x)}^{t} \big[D_q f_i(P_i(\xi)) - D_q f_i(\bar{P}_i(\xi))\big] \cdot u_i^{(m+1)}(\xi, g_i(\xi)) d\xi.
\end{aligned}$$

With the above definitions, we have

$$\Delta_i(t,x,\bar{x}) = \sum_{k=1}^{2} \Delta_i^{(k)}(t,x,\bar{x}) + \sum_{k=1}^{3} \tilde{\Delta}_i^{(k)}(t,x,\bar{x}). \tag{20}$$

Since g_i is a solution of (4), we see that

$$g_i(\xi) - \bar{g}_i(\xi) - (x-\bar{x}) = \int_\xi^t \big[D_q f_i(P_i(\zeta)) - D_q f_i(\bar{P}_i(\zeta))\big] d\zeta.$$

Substituting the above relation in $\tilde{\Delta}_i^{(2)}$ and in $\tilde{\Delta}_i^{(1)}$ with $\xi = 0$ and changing the order of integrals where necessary, we get

$$\begin{aligned}
\sum_{k=1}^{3} \tilde{\Delta}_i^{(k)}(t,x,\bar{x}) &= \int_{\lambda_i(t,x)}^{t} \big[D_q f_i(P_i(\xi)) - D_q f_i(\bar{P}_i(\xi))\big] \\
&\cdot \Big[D_x\phi_i(0,g_i(0)) + \int_{\lambda_i(t,x)}^{\xi} D_x f_i(P_i(\zeta)) d\zeta \\
&+ \int_{\lambda_i(t,x)}^{\xi} D_w f_i(P_i(\zeta)) \circ (u^{(m)})_{(\zeta, g_i(\zeta))} d\zeta - u_i^{(m+1)}(\xi, g_i(\xi))\Big] d\xi \\
&= \int_{\lambda_i(t,x)}^{t} \big[D_q f_i(P_i(\xi)) - D_q f_i(\bar{P}_i(\xi))\big] \\
&\cdot \Big[V^{(m)}[u^{(m+1)}](\xi, g_i(\xi)) - u_i^{(m+1)}(\xi, g_i(\xi))\Big] d\xi = 0,
\end{aligned}$$

from which and from (20) we get $\Delta_i(t,x,\bar{x}) = \sum_{k=1}^{2} \Delta_i^{(k)}(t,x,\bar{x})$. In the above transformations we have also used the group property for the bicharacteristic g_i. Assumptions H_2, H_3, inequality (5) and the existence of derivatives $D_x\phi$, $D_x z^{(m)} = u^{(m)}$ yield the existence of a constant $C_1 \in \mathbb{R}_+$ such that

$$|\Delta_i^{(1)}(t,x,\bar{x})| \le C_1 |x-\bar{x}|^2, \qquad t \in [0,a],\ x,\bar{x} \in [-b,b].$$

Writing $\Delta_i^{(2)}$ in the form

$$\begin{aligned}
\Delta_i^{(2)}(t,x,\bar{x}) &= \int_{\lambda_i(t,\bar{x})}^{\lambda_i(t,x)} \big[D_t\phi_i(\lambda_i(t,x), g_i(\lambda_i(t,x))) - f_i(\bar{P}_i(\xi))\big] d\xi \\
&+ \int_{\lambda_i(t,\bar{x})}^{\lambda_i(t,x)} D_q f_i(\bar{P}_i(\xi)) \\
&\qquad \cdot \big[u_i^{(m+1)}(\xi, \bar{g}_i(\xi)) - D_x\phi_i(\lambda_i(t,x), g_i(\lambda_i(t,x), t, x))\big] d\xi
\end{aligned}$$

making additional use of the consistency condition (3), and taking into account the relation $u^{(m+1)} = D_x\phi$ on $\partial_0 E \cap E_a$, we get the estimate of the same type for $\Delta_i^{(2)}$ with a constant $C_2 \in \mathbb{R}_+$. This means that (19) holds true with $\tilde{C} = C_1 + C_2$, which completes the proof of (II_{m+1}).

Finally, we prove that $z^{(m+1)}$ defined by (15) belongs to the class $C^{0,1+L}_{\phi,a}(Q)$. Since $D_x z^{(m+1)} = u^{(m+1)}$, it follows from (17) that

$$|D_x z^{(m+1)}(t,x)|_\infty \le Q_1,$$
$$|D_x z^{(m+1)}(t,x) - D_x z^{(m+1)}(\bar{t},\bar{x})|_\infty \le \tilde{Q}_2|t-\bar{t}| + |x-\bar{x}|]$$

for $(t,x),(\bar{t},\bar{x}) \in E_a$. By Assumptions H_2–H_4 we easily get

$$|z^{(m+1)}(t,x)|_\infty \le \Lambda_0 + [\theta_0(Q_0) + \theta_1^* P_0]a,$$
$$|z^{(m+1)}(t,x) - z^{(m+1)}(\bar{t},x)|_\infty \le \tilde{\Gamma}_0(a)|t-\bar{t}|,$$

for $(t,x),(\bar{t},x) \in E_a$. By Assumption H_4 we may choose $a \in (0,\bar{a}]$ so small that additionally $\Lambda_0 + [\theta_0(Q_0) + \theta_1^* P_0]a \le Q_0$, $\tilde{\Gamma}_0(a) \le \tilde{Q}_1$. This together with the relation $z^{(m+1)} = \phi$ on $E_0^* \cup \partial_0 E_a$ gives $z^{(m+1)} \in C^{0,1+L}_{\phi,a}(Q)$, which completes the proof of (I_{m+1}). Thus Theorem 3 follows by induction. □

6. The main result. Write

$$H^*(t) = H(t) + H(t)\exp\Big\{\int_0^t G(\xi)d\xi\Big\}\int_0^t G(\xi)d\xi,$$

where

$$H(t) = \Lambda_1 \Upsilon(0,t)\theta_2^*(t)\Big[\frac{1}{\delta^*}(1+\theta_1^*) + 1\Big] + \theta_1^* S_1 \frac{1}{\delta^*}\Upsilon(0,t)\theta_2^*(t)$$
$$+ \big[\theta_2^*(t)R_1P_0 + \theta_1^*R_1\big]\Upsilon(0,t)\int_0^t \theta_2^*(\tau)d\tau + \theta_1^* + \theta_2^*(t)P_0.$$

THEOREM 4. *If Assumptions H_2–H_4 are satisfied then the sequences $\{z^{(m)}\}$, $\{u^{(m)}\}$ are uniformly convergent on E_a for sufficiently small $a \in (0,\bar{a}]$.*

PROOF. Suppose that $a \in (0,\bar{a}]$ is such that the conclusion of Theorem 3 holds true. For any $\tau \in [0,a]$ and $m \in \mathbb{N}$ we put

$$Z^{(m)}(\tau) = \sup\big\{|z^{(m)}(t,x) - z^{(m-1)}(t,x)|_\infty;\ (t,x) \in E_\tau^*\big\},$$
$$U^{(m)}(\tau) = \sup\big\{|u^{(m)}(t,x) - u^{(m-1)}(t,x)|_\infty;\ (t,x) \in E_\tau^*\big\}.$$

Using the same technique as in the proof of Theorem 3, by Assumptions H_2, H_3 and inequality (6), for any $t \in [0,a]$ and $m \in \mathbb{N}$, we get the estimate

$$U^{(m+1)}(t) \le \int_0^t G(\xi)U^{(m+1)}(\xi)d\xi$$
$$+ \int_0^t G(\xi)\big[Z^{(m)}(\xi) + U^{(m)}(\xi)\big]d\xi.$$

Making use of the Gronwall lemma, we have

$$U^{(m+1)}(t) \le \exp\Big\{\int_0^t G(\xi)d\xi\Big\}\int_0^t G(\xi)\big[Z^{(m)}(\xi)+U^{(m)}(\xi)\big]d\xi. \tag{21}$$

By Assumptions H_2, H_3 and relations (10), (21), we get the estimate

$$Z^{(m+1)}(t) \le \int_0^t H^*(\xi)\big[Z^{(m)}(\xi)+U^{(m)}(\xi)\big]d\xi, \qquad t\in[0,a]. \tag{22}$$

Thus if we take

$$M(t) = \exp\Big\{\int_0^a G(\xi)d\xi\Big\}G(t) + H^*(t),$$

then using (21), (22) for any $t\in[0,a]$, we have

$$Z^{(m+1)}(t)+U^{(m+1)}(t) \le \int_0^t M(\xi)\big[Z^{(m)}(\xi)+U^{(m)}(\xi)\big]d\xi.$$

Now, by induction, it is easy to get

$$Z^{(m)}(t)+U^{(m)}(t) \le \frac{\big(\int_0^t M(\xi)d\xi\big)^{m-1}}{(m-1)!}\big[Z^{(1)}(a)+U^{(1)}(a)\big], \qquad t\in[0,a],$$

and consequently

$$\sum_{i=k}^{m}\big[Z^{(i)}(a)+U^{(i)}(a)\big] \le \big[Z^{(1)}(a)+U^{(1)}(a)\big]\sum_{i=k-1}^{m-1}\frac{\big(\int_0^a M(\xi)d\xi\big)^i}{i!}. \tag{23}$$

Since the series $\sum_{i=1}^{\infty}\frac{(\int_0^a M(\xi)d\xi)^i}{i!}$ is convergent, it follows from (23) that the sequences $\{z^{(m)}\}$, $\{u^{(m)}\}$ satisfy the uniform Cauchy condition on E_a^*, which means that they are uniformly convergent on E_a^*. This completes the proof of Theorem 4. □

THEOREM 5. *If Assumptions H_2–H_4 are satisfied then there is a solution of the problem* (1), (2).

PROOF. It follows from Theorem 4 that there exist functions $\bar z, \bar u$ such that $\{z^{(m)}\}$, $\{u^{(m)}\}$ are uniformly convergent on E_a^* to $\bar z, \bar u$, respectively, if $a\in(0,\bar a]$ is sufficiently small. Furthermore, $D_x\bar z$ exists on E_a^* and $D_x\bar z=\bar u$. We prove that $\bar z$ is a solution of (1).

From (12) it follows that for any $i\in\mathbb{N}$ and $(t,x)\in E_{a0}^{(i)}[\bar z, D_x\bar z]$ we have

$$\begin{aligned}\bar z_i(t,x) = \phi_i(0,\bar g_i(0,t,x)) + \int_0^t \Big[& f_i(P_i[\bar z, D_x\bar z](\xi,t,x)) \\ &- \sum_{k=1}^{n} D_{q_k} f_i(P_i[\bar z, D_x\bar z](\xi,t,x)) D_{x_k}\bar z_i(\xi,t,x)\Big]d\xi,\end{aligned} \tag{24}$$

where $\bar g_i = g_i[\bar z, D_x\bar z]$.

For a fixed t we define the transformation $x \mapsto \bar{g}_i(0,t,x) = \zeta$. Then by the group property $\bar{g}_i(\tau,t,x) = \bar{g}_i(\tau,0,\zeta)$ and by (24), we get

$$\begin{aligned}
\bar{z}_i(t,\bar{g}_i(i,0,\zeta)) = \phi_i(0,\zeta)& \\
+ \int_0^t \Big[& f_i\big(\xi,\bar{g}_i(\xi,0,\zeta),\bar{z}_{(\xi,\bar{g}_i(\xi,0,\zeta))},D_x\bar{z}(\xi,\bar{g}_i(\xi,0,\zeta))\big) \\
& - \sum_{k=1}^{n} D_{q_j} f_i\big(\xi,\bar{g}_i(\xi,0,\zeta),\bar{z}_{(\xi,\bar{g}_i(\xi,0,\zeta))},D_x\bar{z}(\xi,\bar{g}_i(\xi,0,\zeta))\big) \\
& \qquad D_{x_k}\bar{z}_i(\xi,\bar{g}_i(\xi,0,\zeta))\Big] d\xi.
\end{aligned}$$

Differentiating the above relation with respect to t and making use of the reverse transformation $\zeta \mapsto \bar{g}_i(t,0,\zeta) = x$, we see that the i-th equation of system (1) is satisfied for almost all t with fixed x on $E^{(i)}_{a0}[\bar{z},D_x\bar{z}]$.

Analogously, for any $(t,x) \in E^{(i)}_{ab}[\bar{z},D_x\bar{z}]$ we have

$$\begin{aligned}
(25) \quad \bar{z}_i(t,x) = \phi_i(0,\bar{g}_i(0,t,x)) + \int_{\bar{\lambda}_i(t,x)}^t \Big[& f_i(P_i[\bar{z},D_x\bar{z}](\xi,t,x)) \\
& - \sum_{k=1}^{n} D_{q_j} f_i(P_i[\bar{z},D_x\bar{z}](\xi,t,x)) D_{x_k}\bar{z}_i(\xi,t,x)\Big] d\xi,
\end{aligned}$$

where $\bar{\lambda}_i = \lambda_i[\bar{z},D_x\bar{z}]$.
Without loss of generality we may suppose that $\bar{g}_{ij}(\bar{\lambda}_i(t,x),t,x) = b_j$ for $j = n$ and for simplicity we write $\zeta' = (\zeta_1,\dots,\zeta_{n-1})$, $\bar{g}'_i = (\bar{g}_{i1},\dots,\bar{g}_{i,n-1})$. For a fixed t we define the transformation $x \mapsto \big(\bar{g}'_i(\bar{\lambda}_i(t,x),t,x),\bar{\lambda}(t,x)\big) = (\zeta',\eta)$. Then by (25) and the group property we get

$$\begin{aligned}
& \bar{z}_i(t,\bar{g}_i(t,\eta,\zeta',b_n)) = \phi_i(\eta,\zeta',b_n) \\
& \quad + \int_\eta^t \Big[f\big(\xi,\bar{g}_i(\xi,\eta,\zeta',b_n),\bar{z}_{(\xi,\bar{g}_i(\xi,\eta,\zeta',b_n))},D_x\bar{z}(\xi,\bar{g}_i(\xi,\eta,\zeta',b_n))\big) \\
& \quad - \sum_{k=1}^{n} D_{q_k} f_i\big(\xi,\bar{g}_i(\xi,\eta,\zeta',b_n),\bar{z}_{(\xi,\bar{g}_i(\xi,\eta,\zeta',b_n))},D_x\bar{z}(\xi,\bar{g}_i(\xi,\eta,\zeta',b_n))\big) \\
& \quad \cdot D_{x_k}\bar{z}_i(\xi,\bar{g}_i(\xi,\eta,\zeta',b_n))\Big] d\xi.
\end{aligned}$$

As previously, differentiating the above relation with respect to t and making use of the reverse transformation $(\zeta',\eta) \mapsto \bar{g}_i(t,\eta,\zeta',b_n) = x$, we see that the i-th equation of system (1) is satisfied for almost all t with fixed x also on $E^{(i)}_{ab}[\bar{z},D_x\bar{z}]$. Since $\bar{z}$ obviously fulfills condition (2), the proof of Theorem 5 is complete. □

Remark 5. If in Theorem 5 we assume that f is continuous then we get existence of classical solutions of problem (1), (2).

7. Solutions with a generalized Lipschitz condition. We may consider solutions of problem (1), (2) that satisfy a generalized Lipschitz condition with respect to the first variable. In this case we modify the assumption on the initial function ϕ.

Assumption H_5. *Suppose that*

1° $\phi = \{\phi_i\} \in C(E_0^* \cup \partial_0 E_{\bar{a}}, \ell^\infty)$ *and the derivative* $D_x\phi = \{D_x\phi_i\}$, $i \in \mathbb{N}$, *exists on* $E_0^* \cup \partial_0 E_{\bar{a}}$;

2° there are constants $\Lambda_0, \Lambda_1, \Lambda_2 \in \mathbb{R}_+$, *and Lebesgue integrable functions* $\omega_1, \omega_2 : \mathbb{R}_+ \to \mathbb{R}_+$ *such that on* $E_0^* \cup \partial_0 E_{\bar{a}}$ *we have* $|\phi(t,x)|_\infty \le \Lambda_0$ *and*

$$|\phi(t,x) - \phi(\bar{t},x)|_\infty \le \left|\int_t^{\bar{t}} \omega_1(\tau)d\tau\right|, \quad |D_x\phi(t,x)|_\infty \le \Lambda_1$$

$$|D_x\phi(t,x) - D_x\phi(\bar{t},\bar{x})|_\infty \le \left|\int_t^{\bar{t}} \omega_2(\tau)d\tau\right| + \Lambda_2|x-\bar{x}|;$$

3° the derivatives $D_t\phi_i(t,x)$, $i \in \mathbb{N}$, *exist on* $\partial_0 E_{\bar{a}} \cap E_{\bar{a}}$ *and the consistency condition* (3) *holds true on* $\partial_0 E_{\bar{a}} \cap E_{\bar{a}}$.

Let ϕ fulfill Assumption H_5 and let $a \in (0, \bar{a}]$. Instead of the functional spaces $C^{0,1+L}_{\phi,a}(Q)$, $C^{0,L}_{D_y\phi,a}(P)$ considered in section 2, we may define two other spaces. By $C^{0,1+L}_{\phi,a}(\mu, Q)$, we define the set of all functions $z : E_a^* \to \ell^\infty$ such that the derivative $D_x z$, exists on E_a^* and

(i) $z(t,x) = \phi(t,x)$ on $E_0^* \cup \partial_0 E_a$;
(ii) $|z(t,x)|_\infty \le Q_0$, $|D_x z(t,x)|_\infty \le Q_1$ on E_a;
(iii) for $t, \bar{t} \in [0,a]$, $x, \bar{x} \in [-b,b]$, we have

$$|z(t,x) - z(\bar{t},x)|_\infty \le \left|\int_t^{\bar{t}} \mu_1(\tau)d\tau\right|,$$

$$|D_x z(t,x) - D_x z(\bar{t},\bar{x})|_\infty \le \left|\int_t^{\bar{t}} \mu_2(\tau)d\tau\right| + Q_2|x-\bar{x}|.$$

We also define by $C^{0,L}_{D_x\phi,a}(\nu, P)$ the set of all functions $u : E_a \to \ell^\infty_n$ such that

(i) $u(t,x) = D_x\phi(t,x)$ on $\partial_0 E_a \cap E_a$;
(ii) $|u(t,x)|_\infty \le P_0$ on E_a;
(iii) for $t, \bar{t} \in [0,a]$, $x, \bar{x} \in [-b,b]$, we have

$$|u(t,x) - u(\bar{t},\bar{x})|_\infty \le \left|\int_t^{\bar{t}} \nu(\tau)d\tau\right| + P_1|x-\bar{x}|.$$

In the above definitions the constants $\tilde{Q}_1, \tilde{Q}_2, \tilde{P}_1$ from section 2 have been replaced by Lebesgue integrable functions μ_1, μ_2, ν, respectively.

Assumption H_6. *Suppose that*

1° $f = \{f_i\}$, $f_i : \Omega^{(0)} \to \mathbb{R}$ *is an infinite sequence of functions in the variables* (t, x, w, q), *measurable in* t, *and there is* $\theta_1 \in \Theta$ *such that*

$$|f(t, x, w, q)|_\infty \leq \theta_1(t, \|w\|_0) \qquad \text{on } \Omega^{(0)}.$$

2° *For* $\delta = x, w, q$ *the derivatives* $D_\delta f_i$, $i \in \mathbb{N}$ *exist on* $\Omega^{(1)}$, *the sequence* $D_\delta f = \{D_\delta f_i\}$ *is measurable with respect to* t, *we have*

$$|D_\delta f(t, x, w, q)|_\infty \leq \theta_1(t, \|w\|_1) \quad \text{on } \Omega^{(1)},$$

and there is $\theta_2 \in \Theta$ *such that*

$$\begin{aligned} &|D_\delta f(t, x, w, q) - D_\delta f(t, \bar{x}, w + h, \bar{q})|_\infty \\ &\qquad\qquad \leq \theta_2(t, \|w\|_{1,L}) \big[|x - \bar{x}| + \|h\|_1 + |q - \bar{q}|\big], \end{aligned}$$

for all $(t, x, h, q) \in \Omega^{(1)}$, $\bar{x}, \bar{q} \in \mathbb{R}^n$, $w \in C^{0,1+L}(B, \ell^\infty)$.

3° *For every* $p \in \mathbb{R}_+$ *there is a constant* $\delta(p) > 0$ *such that we have* $D_{q_j} f_i(t, x, w, q) \geq \delta(p)\theta_1(t, p)$, $i \in \mathbb{N}$, $j = 1, \ldots, n$, *for all* $(t, x, w, q) \in \Omega^{(1)}$, *such that* $\|w\|_1 \leq p$.

Let ϕ be a given function satisfying Assumption H_5 and $0 < a \leq \bar{a}$. As in section 3, for any $z \in C^{0,1+L}_{\phi,a}(\mu, Q)$, $u \in C^{0,L}_{D_x\phi,a}(\nu, P)$ and $i \in \mathbb{N}$, we may define the i-th bicharacteristic $g_i[z, u](\cdot, t, x)$ of system (1) corresponding to $[z, u]$ as a solution of problem (4). We may also prove a lemma about properties of $g_i[z, u](\cdot, t, x)$ analogous to Lemma 1, where instead of the term $\theta_1^*|t - \bar{t}|$ at the right-hand side of (5), we now have $\big|\int_t^{\bar{t}} \theta_1^*(\xi) d\xi\big|$, where $\theta_1^*(\tau) = \theta_1(\tau, Q_1 + Q_2)$. For $\lambda_i[z, u](t, x)$, the left end of the maximal interval on which $g_i[z, u](\cdot, t, x)$ is defined we may prove the following estimates

$$\begin{aligned} \Big|\int_{\lambda_i[z,u](t,x)}^{\lambda_i[z,u](\bar{t},\bar{z})} \theta_1^*(\xi) d\xi\Big| &\leq \frac{1}{\delta^*} \Upsilon(0, t) \Big\{\Big|\int_t^{\bar{t}} \theta_1^*(\xi) d\xi\Big| + |x - \bar{x}|\Big\}, \\ \Big|\int_{\lambda_i[z,u](t,x)}^{\lambda_i[\bar{z},\bar{u}](t,x)} \theta_1^*(\xi) d\xi\Big| &\leq \frac{1}{\delta^*} \Upsilon(0, t) \int_0^t \theta_2^*(\xi) \big\{\|z - \bar{z}\|_{E_\xi} \\ &\qquad + \|D_x z - D_x \bar{z}\|_{E_\xi} + \|u - \bar{u}\|_{E_\xi}\big\} d\xi, \end{aligned}$$

instead of (8) and (9) proved in Lemma 2.

Suppose that Assumptions H_5 and H_6 are satisfied and that there are constants M_1, $M_2 \in \mathbb{R}_+$ such that we have

$$\omega_1(\tau) \leq M_1 \theta_1^*(\tau), \quad \omega_2(\tau) \leq M_2 \theta_1^*(\tau), \qquad \text{for } \tau \in [0, \bar{a}].$$

Then we may choose parameters defining the classes $C^{0,1+L}_{\phi,a}(\mu, Q)$, $u \in C^{0,L}_{D_x\phi,a}(\nu, P)$ such that for sufficiently small $a \in (0, \bar{a}]$ there is a solution $\bar{z}$ of problem (1), (2) belonging to the class $C^{0,1+L}_{\phi,a}(\mu, Q)$ and such that $D_x\bar{z} \in C^{0,L}_{D_x\phi,a}(\nu, P)$.

Finally, we show some examples of infinite functional differential systems which are particular cases of (1).

Example 1. Given $\tilde{f}_i : E_{\bar{a}} \times \ell^\infty \times \mathbb{R}^n \to \mathbb{R}$, $i \in \mathbb{N}$, let us consider the differential system with deviated argument

$$D_t z_i(t, x) = \tilde{f}_i(t, x, z(\alpha(t), \beta(t, x)), D_x z_i(t, x)), \tag{26}$$

where $\alpha : [0, \bar{a}] \to \mathbb{R}$, $\beta : E_{\bar{a}} \to [-b, b]$, and $\alpha(t) \le t$ for $t \in [0, \bar{a}]$. We define a function $f = \{f_i\}$ by

$$f(t, x, w, q) = \tilde{f}(t, x, w(\alpha(t) - t, \beta(t, x) - x), q)$$

for $(t, x, w, q) \in E_{\bar{a}} \times C(B, \ell^\infty) \times \mathbb{R}^n$. If $(\alpha(t) - t, \beta(t, x) - x) \in B$ for $(t, x) \in E_{\bar{a}}$ then (26) is a particular case of (1) under natural assumptions on $\alpha, \beta, \tilde{f}$.

Example 2. With $\tilde{f}_i$ as in the previous example, consider the differential-integral system

$$D_t z_i(t, x) = \tilde{f}_i(t, x, \int_B z(t + \tau, x + s) d\tau ds, D_x z_i(t, x)). \tag{27}$$

If we define a function $f = \{f_i\}$ by

$$f(t, x, w, q) = \hat{f}(t, x, \int_B w(\tau, s) d\tau ds, q)$$

for $(t, x, w, q) \in E_{\bar{a}} \times C(B, \ell^\infty) \times \mathbb{R}^n$, then it is easy to formulate assumptions on $\tilde{f}$ in order to get the existence theorem for (27) as a particular case of (1).

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