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ABHYANKAR-MOH PROPERTY AND UNIQUE AFFINE EMBEDDINGS

by Jerzy Gurycz

Abstract. In this paper we analyze some examples of affine varieties with non-unique embeddings and compare the Abhyankar-Moh Property and unique embedding property for affine varieties.

1. Introduction. The aim of this paper is to study some examples of affine varieties with nonunique embeddings in $\mathbb{C}^n$. Abhyankar and Sathaye have shown in [**2**] a family of affine plane curves which have the unique embedding in affine plane and also gave a family of curves with (at least) two different embeddings. Shpilrain and Yu in their paper [**8**] presented a simple procedure which produces all possible varieties isomorphic to a given one and in the case of hypersurfaces they proposed the number of zeros of the gradient as an invariant of equivalent hypersurfaces. The procedure and the invariant can be combined to get some examples of isomorphic but non-equivalent varieties. We give an explanation of why it is possible to construct such examples and compare the unique embedding property defined by the authors (in [**8**] and [**9**]) and Abhyankar-Moh property of affine varieties.

2. Terminology. Throughout this paper we fix the field of complex numbers $\mathbb{C}$ as a ground field. P_n denotes $\mathbb{C}$–algebra $\mathbb{C}[x_1, \dots, x_n]$.

If $p, p_1, \dots, p_r \in P_n$ and $a \in \mathbb{C}^n$ then by $J(p_1, \dots, p_r)$ and $J_a(p_1, \dots, p_r)$ we denote jacobian matrices $\left(\frac{\partial p_i}{\partial x_j}\right)$ and $\left(\frac{\partial p_i}{\partial x_j}(a)\right)$, respectively, and by $\operatorname{grad}(p) = (\frac{\partial p}{\partial x_1}, \dots, \frac{\partial p}{\partial x_n})$, $\operatorname{grad}_a(p) = (\frac{\partial p}{\partial x_1}(a), \dots, \frac{\partial p}{\partial x_n}(a))$ we denote gradients.

If $Y \subset \mathbb{C}^n$, we denote $I(Y) = \{f \in P_n \colon f|_Y = 0\}$. If I is an ideal of A_n we denote $X(I) = \bigcap_{f \in I} f^{-1}(0)$. By $\overline{f}^{\,I}$ we denote the residue class of an element

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$f \in P_n$ ($\overline{f}^I$ is an element of the quotient ring P_n/I) and by $\mathrm{Aut}(P_n)$ the group of automorphisms of P_n.

3. Equivalence and isomorphism of varieties. We follow [**8**] and [**9**] in definitions. By a variety we mean a collection of polynomials $p_1, \dots, p_r$ in the polynomial algebra P_n. It can be seen as the ideal $\langle p_1, \dots, p_r\rangle$ or as a set of equations $\{p_1{=}0, \dots, p_r{=}0\}$. Such variety we denote by $\widetilde{V}(p_1, \dots, p_r)$ or $\langle x_1, \dots, x_n; p_1{=}0, \dots, p_r{=}0\rangle$ when we need to refer explicitly to the list of variables. We write $\widetilde{V}$ instead of V in order to distinguish between varieties defined in this way and affine closed sets defined as subsets (in $\mathbb{C}^n$) of solutions of sets of polynomial equations. The set of all varieties in P_n will be denoted by $\mathcal{V}(P_n)$. In the language of schemes our varieties are closed subschemes of an affine scheme $\operatorname{Spec} P_n$ (see Chapter II, Corollary 5.10 in [**3**]).

Two varieties $\widetilde{V}(p_1, \dots, p_r)$ and $\widetilde{V}(q_1, \dots, q_s)$ are considered the same if they have equal ideals $\langle p_1, \dots, p_r\rangle = \langle q_1, \dots, q_s\rangle$ (after possibly renaming and permuting their lists of variables) but we will sometimes distinguish their presentations in computations.

Note that the unique presentation of a variety can be achieved by the Gröbner basis algorithm after choosing some order in $\mathbb{N}^n$. It is natural to associate, with each variety $\widetilde{V}{=}\langle x_1, \dots, x_n; p_1{=}0, \dots, p_r{=}0\rangle$, $\mathbb{C}$–algebra $\mathbb{C}[\widetilde{V}] = P_n/\langle p_1, \dots, p_r\rangle$ (it's coordinate ring) and the zero-set $X(\widetilde{V}) = \cap_{i=1}^{r} p_i^{-1}(0)$ (a closed affine algebraic subset of $\mathbb{C}^n$). It should be noted that there is no assumption that the variety is reduced (it can have multiples of components; equivalently ideal is not assumed to be radical) nor is it irreducible (ideal is not necessarily prime).

Definition 3.1. By an isomorphism of two varieties $\widetilde{V}_1$ and $\widetilde{V}_2$ we mean a $\mathbb{C}$–isomorphism of their coordinate rings. If $\widetilde{V}_1$ and $\widetilde{V}_2$ are isomorphic, we write $\widetilde{V}_1 \cong \widetilde{V}_2$.

Let $\widetilde{V}_1$, $\widetilde{V}_2$ be varieties in P_n. They are called equivalent if there exists a polynomial automorphism which takes one of them onto the other; we write $\widetilde{V}_1 \equiv \widetilde{V}_2$ if it is so. Hence the following

Definition 3.2. Two varieties $\widetilde{V}_1 = \widetilde{V}(p_1, \dots, p_r)$, $\widetilde{V}_2 = \widetilde{V}(q_1, \dots, q_s)$ in P_n are equivalent if there exists an isomorphism $\phi\colon P_n \to P_n$ such that $\phi(\langle p_1, \dots, p_r\rangle) = \langle q_1, \dots, q_s\rangle$ or, equivalently, $\langle \phi(p_1), \dots, \phi(p_r)\rangle = \langle q_1, \dots, q_s\rangle$.

This definition is motivated by the equivalence of curves and hypersurfaces. Plane curves p, q are called equivalent if $q = \phi(p)$ for some automorphism ϕ of $\mathbb{C}[x, y]$. In the case of hypersurfaces $\widetilde{V}(p) \equiv \widetilde{V}(q)$ is the same as $\phi(p) = q$ for some automorphism ϕ of P_n because $\langle p\rangle = \langle q\rangle$ iff $p = cq$ for some nonzero constant $c \in \mathbb{C}^*$.

For a variety $\widetilde{V}=\widetilde{V}(p_1,\ldots,p_r)$ we denote by $\mathrm{Isom}(\widetilde{V})$ the set of all varieties in P_n isomorphic with $\widetilde{V}$ and by $\mathrm{Equiv}(\widetilde{V})$ the set of all varieties equivalent to $\widetilde{V}$. It should be noted that $\mathrm{Equiv}(\widetilde{V}) \subset \mathrm{Isom}(\widetilde{V})$. In the case of hypersurfaces (or curves for $n=2$) we simply write $\mathrm{Isom}(p)$ and $\mathrm{Equiv}(p)$.

Definition 3.3. A variety $\widetilde{V}$ is said to have the unique embedding in P_n if any variety isomorphic to $\widetilde{V}$ is equivalent to $\widetilde{V}$.

This means $\mathrm{Isom}(\widetilde{V}) \subset \mathrm{Equiv}(\widetilde{V})$ and hence $\mathrm{Isom}(\widetilde{V}) = \mathrm{Equiv}(\widetilde{V})$. Equivalently this means that $\mathrm{Isom}(\widetilde{V})$ contains only one orbit of the action of $\mathrm{Aut}(P_n)$ on $\mathcal{V}(P_n)$.

When we fix the variety $\widetilde{V}$ we can consider $\mathrm{Isom}(\widetilde{V})$ and $\mathrm{Equiv}(\widetilde{V})$. Sometimes we want to restrict considerations to some subclass $\mathcal{K}$ of varieties in P_n (for example hypersurfaces, nonsingular curves, varieties from birational equivalence class of some fixed variety etc.). We say that we found $\cong$–models for class $\mathcal{K}$ if we can split $\mathcal{K}$ into a disjoint union $\mathcal{K} = \cup\mathcal{K}_\alpha$ (each $\mathcal{K}_\alpha$ is a set of varieties) and for each $\mathcal{K}_\alpha$ we have a representant $\widetilde{V}_\alpha \in \mathcal{K}_\alpha$ (hopefully with some nice description) such that all other members $\widetilde{V} \in \mathcal{K}_\alpha$ are isomorphic to $\widetilde{V}_\alpha$. In a similar manner we define $\equiv$–models. Now we can restate our definition: a variety $\widetilde{V}$ has the unique embedding if for this variety $\cong$–class of $\widetilde{V}$ coincides with $\equiv$–class of $\widetilde{V}$.

Several natural (and rather hard) classification questions arise when we try to find $\cong$–models and $\equiv$–models. Remarkable results were obtained in the case of affine curves: Abhyankar-Moh theorem states that a line in $\mathbb{C}^2$ has unique embedding [**1**], theorem of Lin and Zaidenberg [**7**] gives a list of $\equiv$–models for irreducible simply connected curves (they are just $x^n - y^m$ with $(n,m)=1$).

4. Gradients and jacobians. For a variety $\widetilde{V}=\widetilde{V}(p_1,\ldots,p_r)$ in P_n it may happen that the deletion of some of the generators $p_1,\ldots,p_r$ from the generating set does not change the ideal. We shall say that $p_1,\ldots,p_r$ is a *reduced set of generators* of $\widetilde{V}$ if any proper subset of $\{p_1,\ldots,p_r\}$ generates a proper subideal of $\langle p_1,\ldots,p_r\rangle$. For a variety with a reduced set of generators we define

$$Z(\widetilde{V}) = \{a \in \mathbb{C}^n \colon \mathrm{rank} J_a(p_1,\ldots,p_r) < n - \dim V\}$$

and $z(\widetilde{V})$ the cardinality of the set $Z(\widetilde{V})$. If $\widetilde{V}$ is a hypersurface $\widetilde{V}(p)$ then $Z(\widetilde{V}) = \{a \in \mathbb{C}^n \colon \mathrm{grad}_a(p) = 0\}$. In general, Z and z depend on the choice of ideal generators but not in the case of hypersurfaces. Note that if $\phi\colon \mathbb{C}[\widetilde{V}_1] \to \mathbb{C}[\widetilde{V}_2]$ is an isomorphism of $\widetilde{V}_1$ and $\widetilde{V}_2$ then $\phi_*(Z(\widetilde{V}_2)) = Z(\widetilde{V}_1)$.

Since hypersurfaces $\widetilde{V}(p)$, $\widetilde{V}(q)$ are equivalent if and only if $p=\phi(q)$ for some polynomial automorphism ϕ of P_n, chain rule for derivatives implies that the number of zeroes of the gradient is an invariant of equivalence. We can distinguish non-equivalent varieties using this invariant. Examples can be found in [**8**] and in the section *Non-equivalence* of this paper.

5. Elementary transformations. Let $\widetilde{V}{=}\langle p_1,\dots,p_r\rangle$. Then for each polynomial $p\in P_n$ the graph map $y{=}p(x_1,\dots,x_n)$ induces an isomorphism $\phi_p\colon\mathbb{C}[\widetilde{V}]\to\mathbb{C}[\widetilde{W}]$ where $\widetilde{W}=\langle x_1,\dots,x_n,y;p_1,\dots,p_r,y-p\rangle$. We call such an isomorphism an elementary transformation of type (P1) or *introducing a new variable*. Inverses of transformations of type (P1) are called (P2) or *cancelling a variable*. Transformations which permute and rename variables are called (P3). Each isomorphism of varieties is a composition of transformations of type (P1), (P2) and (P3) (see theorem 1.1 in [**8**]).

6. Non-equivalence. The aim of this section is to give an insight into constructions of non-equivalent isomorphic hyperurfaces which have different gradient invariant.

Examples of isomorphic but non-equivalent varieties in [**8**] are obtained in the following way: Start with a hypersurface $\widetilde{V}_1$, apply a sequence of elementary transformations and get another hypersurface $\widetilde{V}_2$ such that $z(\widetilde{V}_1)$ and $z(\widetilde{V}_2)$ differ.

We answer the following question: In which of elementary transformations is this equivalence invariant broken?

It is worth considering a simple example of two isomorphic but non-equivalent curves in $\mathbb{C}[x,y]$. Take $p=x^2y-1$ and $q=uv-1$ (q is just x^2+y^2-1 after linear change of coordinates $u=x+iy, v=x-iy$). Explicit isomorphisms $\phi\colon\mathbb{C}[x,y]/\langle x^2y-1\rangle\to\mathbb{C}[u,v]/\langle uv-1\rangle$ and $\psi\colon\mathbb{C}[u,v]/\langle uv-1\rangle\to\mathbb{C}[x,y]/\langle x^2y-1\rangle$ are given by $\phi(\bar{x})=\bar{u}$, $\phi(\bar{y})=\bar{v}^2$ and $\psi(\bar{u})=\bar{x}$, $\psi(\bar{v})=\overline{xy}$. ϕ and ψ are well defined $\mathbb{C}$–algebra homomorphisms. To see that ϕ is well defined we need to check if from $f\in\langle xy^2-1\rangle$ there follows $f(u,v^2)\in\langle uv-1\rangle$. Let $f=p(x,y)(x^2y-1)$. Then $f(u,v^2)=p(u,v^2)(u^2v^2-1)=p(u,v^2)(uv-1)(uv+1)\in\langle uv-1\rangle$. In the same manner, to check if ψ is well defined we need to check if from $g(u,v)\in\langle uv-1\rangle$ there follows that $g(x,xy)\in\langle x^2y-1\rangle$. It is so, because if $g(u,v)=q(u,v)(uv-1)$ then $g(x,xy)=q(x,xy)(x^2y-1)$. One sees that ϕ and ψ are inverses of each other so they are isomorphisms.

We can decompose ϕ into a sequence of elementary transformations:

$$\begin{gathered}
\widetilde{V}_1=\langle x,y;x^2y-1\rangle\cong_{T_1}\\
\widetilde{V}_2\langle x,y,u,v;u-x,v-xy,x^2y-1\rangle\cong_{T_2}\\
\widetilde{V}_3\langle x,y,u,v;u-x,v-xy,xv-1\rangle\cong_{T_3}\\
\widetilde{V}_4\langle x,y,u,v;u-x,y-v^2,xv-1\rangle\cong_{T_4}\\
\widetilde{V}_5\langle x,y,u,v;u-x,y-v^2,uv-1\rangle\cong_{T_5}\\
\widetilde{V}_6\langle u,v;uv-1\rangle.
\end{gathered}$$

Transformation T_1 is (P1) simultaneously for u and v, T_2, T_3 and T_4 are identities (we change ideal generators), T_5 is (P3) for x and y simultaneously.

Now we look at Z in each step of decomposition. In the computations below $J(\widetilde{V})$ denotes the jacobian matrix of the generators of the ideal of $\widetilde{V}$.

$$J(\widetilde{V}_1) = J(\langle x, y \colon x^2 - y = 1 \rangle) = [2x, -1]$$

and $Z(\widetilde{V}_1) = 0 \times \mathbb{C}$.

$$J(\widetilde{V}_2) = J(\langle x, y, u, v \colon x = u, xy = v, x^2 y = 1 \rangle) = \begin{bmatrix} 1 & 0 & -1 & 0 \\ y & x & 0 & -1 \\ 2xy & x^2 & 0 & 0 \end{bmatrix}$$

and $Z(\widetilde{V}_2) = 0 \times \mathbb{C}^3$.

$$J(\widetilde{V}_3) = J(\langle x, y, u, v \colon x = u, xy = v, xv = 1 \rangle) = \begin{bmatrix} 1 & 0 & -1 & 0 \\ y & x & 0 & -1 \\ v & 0 & 0 & x \end{bmatrix}$$

and $Z(\widetilde{V}_3) = 0 \times \mathbb{C} \times \mathbb{C} \times 0$.

$$J(\widetilde{V}_4) = J(\langle x, y, u, v \colon x = u, y = v^2, xv = 1 \rangle) = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -2v \\ v & 0 & 0 & x \end{bmatrix}$$

and $Z(\widetilde{V}_4) = 0 \times \mathbb{C} \times \mathbb{C} \times 0$.

$$J(\widetilde{V}_5) = J(\langle x, y, u, v \colon u = x, y = v^2, uv = 1 \rangle) = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -2v \\ 0 & 0 & v & u \end{bmatrix}$$

and $Z(\widetilde{V}_5) = \mathbb{C} \times \mathbb{C} \times 0 \times 0$. Finally

$$J(\widetilde{V}_6) = J(\langle u, v \colon uv = 1 \rangle) = [v, u]$$

and $Z(\widetilde{V}_6) = 0 \times 0$.

$Z(\widetilde{V}_1)$ is just $\pi_{uv}(Z(\widetilde{V}_2))$ and $Z(\widetilde{V}_6) = \pi_{xy}(Z(\widetilde{V}_5))$. Note the difference between $Z(\widetilde{V}_2)$ and $Z(\widetilde{V}_3)$. $\widetilde{V}_2$ and $\widetilde{V}_3$ is the same variety. From the geometric point of view it is an intersection of $v = xy$ with two different cylinders, for V_2 for intersection we take the cylinder $x^2 y = 1$ (along v, u) and for V_3 the cylinder $xv - 1$ (along y, u). This suggests that breaking Z is possible if there exists a suitable change of ideal generators.

7. Extension property (Abhyankar-Moh Property) and unique embeddings. The aim of this section is to compare the Abhyankar-Moh Property and the unique embedding property.

Let $\widetilde{V}=\widetilde{V}(p_1,\dots,p_r)$ be a variety in P_n. Let $Y=X(\widetilde{V})=\cap\, p_i^{-1}(0)$ (it is a closed affine algebraic subset of $\mathbb{C}^n$). Recall that a regular mapping $\phi\colon Y\to\mathbb{C}^n$ is called an embedding if the induced map of coordinate rings $\phi^*\colon P_n\to\mathbb{C}[Y]$ is an epimorphism. If $F\colon X\to Y$ is a regular mapping of closed affine sets $X\subset\mathbb{C}^m$, $Y\subset\mathbb{C}^n$ then the induced map F^* is defined as $F^*\colon P_n/_{I(Y)}\ni\overline{u}\to\overline{u\circ F}\in P_n/_{I(X)}$.

Definition 7.1. We say that a closed affine algebraic subset Y has the Abhyankar-Moh Property (embedding extension property or A.M.P. for short) if for any polynomial embedding $F:Y\to\mathbb{C}^n$ there exists a polynomial automorphism $\overline{F}$ of $\mathbb{C}^n$ such that F is a restriction of $\overline{F}$.

Several results ([**5**], [**6**]) show that if a variety Y is of a small dimension compared to n then it has A.M.P. For example in the smooth case each embedding of Y in the ambient space of dimension $n=2\dim Y+2$ (and higher of course) extends to an automorphism of $\mathbb{C}^n$. The exact connection between A.M.P. of Y and the unique embedding property (see definition 3.3) is clarified by the following

Proposition 7.2. *Let I be a radical ideal in P_n and $\widetilde{V}=\widetilde{V}(I)$, $Y=\bigcap_{f\in I}f^{-1}(0)$. If Y has A.M.P. then $\widetilde{V}$ has unique embedding.*

Proof. We will need Hilbert's Nullstellensatz (see [**3**] p. 4). Let $\phi\colon P_n/J\to P_n/I$ be an isomorphism, i.e. $\widetilde{V}(I)\cong\widetilde{V}(J)$, I be radical. Note that then J is radical. Assume $Y=\cap_{f\in I}f^{-1}(0)$ has A.M.P., i.e. for any polynomial embedding $F\colon Y\to\mathbb{C}^n$ there exists an isomorphism $G\colon\mathbb{C}^n\to\mathbb{C}^n$ such that $G|_Y=F$. We need to show that there exists an isomorphism $\psi\colon P_n\to P_n$ such that $\psi(J)=I$.

Let us take polynomials $\phi_k\in\phi(\overline{x_k}^{\,J})$ and let F be the polynomial map $F(x)=(\phi_1(x),\dots,\phi_n(x))$ from Y to $\mathbb{C}^n$. Then for $F^*\colon P_n\ni p\to\overline{p\circ F}^{\,I(Y)}\in P_n/I(Y)$ elements $F^*(x_k)=\overline{\phi_k}^{\,I(Y)}$ generate $P_n/I(Y)$, because $\overline{\phi_k}^{\,I}$ generate P_n/I and $I\subset I(Y)$. Hence F is an embedding and by the assumption there exists a polynomial automorphism $G\colon\mathbb{C}^n\to\mathbb{C}^n$ such that $G|_Y=F|_Y$. Let $\psi:=G^*$ be the induced automorphism of P_n.

First we check that $\psi(q)\in I$ for $q\in J$. For any $g\in P_n$, $\psi(g)=g\circ G$ and $\psi(g)|_Y=g\circ F|_Y$ and hence $\psi(g)|_Y=g\circ(\phi_1,\dots,\phi_n)|_Y$. But $\overline{q}^{\,J}=\overline{0}^{\,J}$ gives $\overline{0}^{\,I}=\phi(\overline{q}^{\,J})=\phi(q(\overline{x_1}^{\,J},\dots,\overline{x_n}^{\,J}))=q(\phi(\overline{x_1}^{\,J}),\dots\phi(\overline{x_n}^{\,J}))=q(\overline{\phi_1}^{\,I},\dots,\overline{\phi_n}^{\,I})=\overline{q\circ(\phi_1,\dots,\phi_n)}^{\,I}$. Hence we have $q\circ(\phi_1,\dots,\phi_n)\in I$ and $\psi(q)-q\circ(\phi_1,\dots,\phi_n)|_Y=0$ so $\psi(q)-q\circ(\phi_1,\dots,\phi_n)\in I(Y)$. By

Hilbert's Nullstellensatz this means that $\psi(q) - q \circ (\phi_1, \ldots, \phi_n) \in \sqrt{I}$ and hence $\psi(q) \in \sqrt{I} = I$.

Now we need to show that for $f \in I$ there exists a polynomial $g \in J$ such that $\psi(g) = f$. Let us take $g := f \circ G^{-1}$ then $\psi(g) = g \circ G = f$. Observe that $g|_{F(Y)} = 0$ so $g \in I(F(Y))$. Observe also that $X(J) = F(Y)$. But J is a radical ideal and hence $I(F(Y)) = I(X(J)) = J$ so we have $g \in J$. $\square$

It is not clear to the author if the converse implication of proposition 7.2 is true (i.e. if the zero-set of a reduced affine variety which has the unique embedding has A.M.P.).

The problem is in the fact that the definition of unique embedding (i.e. definition 3.3) is stated in a slightly different manner than the Abhyankar-Moh Property. In A.M.P. we require that for each embedding $\phi\colon X \to \mathbb{C}^n$ there exists an extension of this embedding. On the other hand, in the definition of unique embedding of the variety $\widetilde{V}(I)$ we require only that if $\widetilde{V}(J) \cong \widetilde{V}(I)$ (i.e. there exists an isomorphism $\phi\colon P_n/I \to P_n/J$) then I and J are equivalent by some automorphism Φ of P_n (i.e $\Phi(I) = J$) and do not require any relation between Φ and ϕ. This suggests another definition of the unique embedding.

Before we state the new definition we introduce two simple notations.

Let I and J be ideals of P_n. Let us denote by π_I the canonical epimorphism

$$P_n \ni f \xrightarrow{\pi_I} \overline{f}^I \in P_n/I$$

Let us denote an endomorphism F of P_n such that $F(I) \subset J$ by

$$(P_n, I) \xrightarrow{F} (P_n, J)$$

and an automorphism G of P_n such that $G(I) = J$ and $H = G^{-1}$ by

$$(P_n, I) \underset{H}{\overset{G}{\longleftrightarrow}} (P_n, J).$$

The letter above the arrow denotes the map from left to right and the one below the arrow denotes the inverse of this map.

Definition 7.3. A variety $\widetilde{V}(I)$ in P_n is said to have the extendible embeddings property if for each isomorphism of affine varieties $\phi\colon P_n/I \to P_n/J$ there exists an automorphism $\Phi \in \mathrm{Aut}(P_n)$ such that the following diagram commutes

$$\begin{tikzcd}
(P_n, I) \arrow[r, "\Phi"] \arrow[d, "\pi_I"'] & (P_n, J) \arrow[d, "\pi_J"] \\
P_n/I \arrow[r, "\phi"] & P_n/J
\end{tikzcd}$$

i.e. $\pi_J \circ \Phi = \phi \circ \pi_I$.

Note that if a variety $\widetilde{V}(I)$ has the extendible embeddings property then it automatically has the unique embedding.

PROPOSITION 7.4. *Let I be an ideal in P_n, $\widetilde{V}=\widetilde{V}(I)$ and $Y = \cap_{f\in I} f^{-1}(0)$. If $\widetilde{V}$ has extendible embedding property then Y has Abhyankar-Moh Property.*

PROOF. Let $F=(F_1, \dots, F_n)\colon Y \to \mathbb{C}^n$ be a regular embedding of $Y=X(I)$, i.e. regular map such that the induced map $F^*\colon P_n \ni p \to \overline{p \circ F}^I \in P_n/I$ is an epimorphism. Let $J = \operatorname{Ker} F^*$ (observe that $J = I(F(Y))$). Then we have a canonical map $\overline{F^*}\colon P_n/J \to P_n/I$, which is an isomorphism of $\widetilde{V}(I)$ and $\widetilde{V}(J)$. Since $\widetilde{V}$ has the extendible embeddings property, there exists $\Phi \in \operatorname{Aut}(P_n)$ such that $I = \Phi(J)$ and the following diagram commutes

$$\begin{tikzcd}
(P_n, J) \arrow[r, shift left, "\Phi"] \arrow[d, "\pi_J"'] & (P_n, I) \arrow[l, shift left, "\Psi"] \arrow[d, "\pi_I"] \\
P_n/J \arrow[r, "\overline{F^*}"] & P_n/I
\end{tikzcd}$$

Let $G := (\phi_1, \dots, \phi_n)$ be an automorphism of $\mathbb{C}^n$ induced by Φ (i.e. $\phi_k = \Phi(x_k)$, $k = 1, \dots, n$). We need to check that $G|_Y = F$. It is enough to check that $\phi_k|_Y = F_k$ for $k = 1, \dots, n$. By the definitions

$$\overline{\phi_k}^I = \overline{\Phi(x_k)}^I = \overline{F^*}(\overline{x_k}^J) = \overline{x_k \circ F}^I = \overline{F_k}^I$$

This clearly implies $\phi_k|_Y = F_k$ and hence $G|_Y = F$. □

Definitions 3.3 and 7.3 raise new questions: (1) Is it true that each variety which has the unique embedding has the extendible embedding property? (2) What are the varieties for which the notions of the unique embedding and extendible embedding property coincide?

From propositions 7.4 and 7.2 we know when and how we can safely switch between the geometric language of A.M.P. and the algebraic language of unique embeddings.

It is worth noting that although the number of zeroes of the gradient is a simple and powerful invariant of equivalent hypersurfaces, there are also other algebraic and geometric methods to check that certain algebraic sets admit different affine embeddings. For example Jelonek in [**4**] showed that for any $n > 1$ the hypersurface $\Gamma_n = \{x \in \mathbb{C}^n : x_1 \cdot \ldots \cdot x_n = 1\}$ has infinitely many regular embeddings. Kaliman in [**6**] gave an example of two isomorphic curves in $\mathbb{C}^3$ which cannot be mapped to each other by an automorphism of $\mathbb{C}^3$.

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Jagiellonian University
Institute of Mathematics
Reymonta 4
30-059 Kraków
Poland
e-mail: `gurycz@im.uj.edu.pl`