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SYMMETRIES IN 4-DIMENSIONAL LORENTZ MANIFOLDS

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Abstract. This paper presents a brief discussion of the description of symmetries in 4-dimensional Lorentz manifolds (with a view to the space-time of general relativity). The orbit structure in terms of foliations is particularly stressed. The main symmetry discussed is local isometry, but other symmetries are briefly mentioned.

1. Introduction. The aim of this paper is to present a brief, reasonably modern approach to the study of symmetry in general relativity theory, that is, on a 4-dimensional manifold admitting a Lorentz metric. Throughout, M will be a smooth, connected, Hausdorff manifold admitting a smooth, Lorentz metric g of signature $(-,+,+,+)$ (and hence M is paracompact [**3**]). If $m \in M, T_mM$ will denote the tangent space to M at m. A Lie derivative is denoted by $\mathcal{L}$. When component notation is used, a partial derivative and a covariant derivative with respect to the Levi-Civita connection Γ associated with g are denoted, respectively, by a comma and a semi-colon.

In Einstein's general relativity theory, M plays the role of the space-time and the geometrical objects g, Γ and the curvature tensor on M derived from Γ collectively describe the gravitational field. Einstein's equations provide the physical restrictions on these objects. However, they will not be required in this paper.

Of course, there are many different types of symmetry studied in general relativity, for example, (local) isometries, homotheties, conformal isometries, affine and projective collineations and symmetries of the curvature and related tensors (for reviews see [**1, 4**]). The purpose of this paper, however, is more general, and will concentrate on techniques rather than the specific symmetry involved. Nevertheless, local isometries will finally be studied as an application.

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So far as the present author is aware, the mathematical study of symmetry in general relativity theory has not taken into account the progress made in the recent studies of the integrability of vector fields and foliations. The main purpose of this paper is to attempt a small step in this direction and to set on a more rigorous basis the general theory of symmetries and their associated orbits.

2. Space-Time Geometry and Decomposition. Let $m \in M$ and $0 \neq v \in T_mM$. Then v is called *spacelike* (respectively, *timelike* or *null*) if $g(v,v) > 0$ (respectively, $g(v,v) < 0$ or $g(v,v) = 0$). A 1-dimensional subspace of T_mM is called a *direction* (at m), and is referred to as a *spacelike* (respectively, *timelike* or *null*) *direction* if it is spanned by a spacelike (respectively, timelike or null) vector at m. If U is a 2-dimensional subspace of T_mM, then U is called *spacelike* (respectively, *timelike* or *null*) if all non-zero members of U are spacelike (respectively, if U contains exactly two distinct null directions or if U contains exactly one null direction). If U is a 3-dimensional subspace of T_mM, the same definitions as in the 2-dimensional case apply except that in the timelike case, one insists that at least two (or, equivalently, infinitely many) distinct null directions are contained in U. These definitions are exclusive and exhaustive of all non-zero members of T_mM and all 1-, 2- and 3-dimensional subspaces of T_mM. A (smooth) submanifold N of M of dimension $1, 2$ or 3 is called *spacelike at* $m \in M$ if its tangent space is a spacelike direction or subspace of T_mM and *spacelike* if it is spacelike *at each* $m \in M$ (and similarly for timelike and null). If N is a spacelike (respectively, timelike) submanifold of M, then g induces a positive definite (respectively, Lorentz) metric on N.

It should be pointed out here that the term (smooth) *submanifold* of M means what is sometimes referred to as a (smooth) *immersed* submanifold of M. Thus, if M' is a submanifold of M, then M' is a subset of M which has a manifold structure, and is such that the inclusion map $i : M' \to M$ is a (smooth) immersion. If, in addition, the manifold topology (from the manifold structure) on M' equals its subspace topology as a subspace of M when the latter has its manifold topology, then M is called a *regular* or *embedded* submanifold. One of the advantages of regular submanifolds is that if M_1 and M_2 are smooth manifolds and $f : M_1 \to M_2$ is a smooth map whose range $f(M_1)$ lies inside a smooth *regular* submanifold N_2 of M_2, then the map $f : M_1 \to N_2$ is also smooth. If N_2 is not regular, this latter map may not even be continuous (but if it is continuous then $f : M_1 \to N_2$ is smooth). There is a type of submanifold introduced, as far as the author is aware, by Stefan [**13, 14**], and which is intermediate between submanifolds and regular submanifolds. A *leaf* of M is a connected (immersed) submanifold N of M with the additional property that, if T is any locally connected topological space, and $f : T \to M$

is a continuous map whose range lies inside N, then the map $f : T \to N$ is continuous. It follows [**13**] that if M_1 and M_2 are smooth manifolds and N_2 is a leaf of M_2, and $f : M_1 \to M_2$ is a smooth map whose range lies in N_2, then the map $f : M_1 \to N_2$ is continuous, and hence smooth. If N is a subset of M admitting two structures N_1 and N_2 as smooth *regular* submanifolds of M, then, from earlier remarks in this paragraph, the identity maps $N_1 \to N_2$ and $N_2 \to N_1$ are each smooth and so $N_1 = N_2$ and the regular submanifold structure is unique (see, e.g. [**2**]). The same uniqueness conclusion also holds if *regular submanifold* is replaced by *leaf* [**13**]. Clearly, every connected regular submanifold is a leaf, but the three types of (connected) submanifold structures (immersed, embedded and leaf) are distinct since the *irrational wrap* on the torus is a leaf which is not regular [**13**], whilst the well known *figure of eight* in $\mathbf{R}^2$ (see, e.g. [**2**]) is a connected submanifold which is easily shown not to be a leaf.

Now let A be a vector space of global, smooth vector fields on M and define the *distribution* Δ *on* M *associated with* A by [**5**]

$$m \to \Delta(m) = \{X(m) : X \in A\} \subseteq T_m M. \tag{1}$$

Then, for $i = 0, 1, 2, 3, 4$ and $p = 1, 2, 3$, define subsets V_i, S_p, T_p and N_p by

$$V_i = \{m \in M : \dim \Delta(m) = i\} \tag{2}$$

$$\begin{aligned} S_p &= \{m \in M : \dim \Delta(m) = p \textit{ and } \Delta(m) \textit{ is spacelike}\} \\ T_p &= \{m \in M : \dim \Delta(m) = p \textit{ and } \Delta(m) \textit{ is timelike}\} \\ N_p &= \{m \in M : \dim \Delta(m) = p \textit{ and } \Delta(m) \textit{ is null}\}. \end{aligned} \tag{3}$$

Thus, $M = \cup_{i=0}^4 V_i$ and $V_p = S_p \cup T_p \cup N_p$ $(p = 1, 2, 3)$. This decomposition of M can be refined topologically by appealing to the rank theorem to see that $M = \cup_{i=k}^4 V_i$ is open in M for $k = 0, \ldots, 4$. This can then be used to reveal the following disjoint decompositions of M [**5**]

$$M = V_4 \cup \bigcup_{i=0}^{3} \operatorname{int} V_i \cup Z_1 \tag{4}$$

$$M = V_4 \cup \bigcup_{p=1}^{3} \operatorname{int} S_p \cup \bigcup_{p=1}^{3} \operatorname{int} T_p \cup \bigcup_{p=1}^{3} \operatorname{int} N_p \cup \operatorname{int} V_0 \cup Z \tag{5}$$

where int denotes the topological interior (and $\operatorname{int} V_4 = V_4$) and where Z and Z_1 are closed subsets of M *each with empty interior*.

3. Local Space-Time Symmetries. With A as in the last section, let $A_1, \ldots, A_k \in A$ and let $\phi^1_{t_1}, \ldots, \phi^k_{t_k}$ be the smooth, local diffeomorphisms associated with them, for appropriate values of t. Then consider the set of all such local diffeomorphisms (where defined) of the form

$$m \to \phi^1_{t_1}(\phi^2_{t_2}(\cdots \phi^k_{t_k}(m) \cdots)) \quad (m \in M) \tag{6}$$

for each choice of $k, X_1, \cdots, X_k$ and admissible $(t_1, \cdots, t_k) \in \mathbf{R}^k$. There is an equivalence relation on M given by $m_1 \sim m_2$ if some local diffeomorphism of the form (6) maps m_1 into m_2. The associated equivalence classes in M are called the *orbits* of A and it is known that *these orbits can each be given the structure of a connected, smooth submanifold of* M [**15, 13, 14**]. In fact, Stefan has shown that these submanifolds constitute a *foliation with singularities, so that each has the extra property of being a leaf.* He also showed that if O is any such leaf and $m \in O$, then the tangent space to O at m is the subspace $\{f_* v : v \in \Delta(m')\}$ of $T_m M$ for each f of the form (6) and each $m' \in M$ such that $f(m') = m$. This subspace need not equal $\Delta(m)$. The condition that it does so for each $m \in M$ is equivalent to the condition that the orbits are *integral manifolds* of the set A and then A is *integrable* [**15, 13, 14**].

In general relativity, the situations of interest occur when A is a Lie algebra (under the Lie bracket operator) of global, smooth vector fields on M and then attention is directed to the nature of the *orbits* of the symmetries represented by A and whether they are integral manifolds of A. If $\dim \Delta(m)$ is constant on M, the Fröbenius theorem (see e.g. [**2**]) guarantees that the orbits are submanifolds and, in fact, integral manifolds of A. The work of Stefan then ensures that the orbits are leaves of a foliation on M. If $\dim \Delta(m)$ is not constant, then integrability need not follow. If, however, A satisfies the *locally finitely generated* condition (i.e. that each $m \in M$ has an open neighbourhood U and a finite subset A' of A such that each $X \in A$, when restricted to U, is a combination of members of A' (restricted to U) with coefficients which are smooth maps $U \to \mathbf{R}$), then Hermann [**10**] has shown that A is integrable (in fact, he showed more than this). *Thus, if A is a finite-dimensional Lie algebra, it is integrable and, again* [**13, 14**]*, the orbits are leaves of a foliation with singularities.*

The symmetries usually studied in general relativity are described by a Lie algebra of global, smooth vector fields on the space-time M, with each particular symmetry being characterised by insisting upon the appropriate property being possessed by the resulting local diffeomorphisms of the type (6) (see, e.g. [**1, 4**]). Thus, *projective symmetry* is defined by insisting that each map (6) takes geodesics to geodesics and the resulting Lie algebra A, now labelled $P(M)$, is the set of all global, smooth vector fields on M with this property. The vector fields in $P(M)$ are called *projective* and are characterised by the condition that, in any chart of M

$$(7) \qquad \begin{aligned} X_{a;b} &= \frac{1}{2} h_{ab} + F_{ab} \qquad (h_{ab} = h_{ba},\ F_{ab} = -F_{ba}) \\ h_{ab;c} &= 2g_{ab}\psi_c + g_{ac}\psi_b + g_{bc}\psi_a \end{aligned}$$

for some closed 1-form field ψ and 2-form field F on M. Special cases are the *affine* vector fields (for which $\psi \equiv 0$ on M and whose associated maps (6) preserve also the geodesic affine parameter), the *homothetic* vector fields (which are affine and satisfy $h_{ab} = cg_{ab}, c \in \mathbf{R}$) and the *Killing* vector fields which are homothetic with $c = 0$ and so $\mathcal{L}_X g = 0$ (and for which each map (6) is a local isometry). The sets of all affine, homothetic and Killing vector fields on M are labelled $A(M), H(M)$ and $K(M)$ respectively, and $K(M) \subseteq H(M) \subseteq A(M) \subseteq P(M)$, with each being a subalgebra of $P(M)$. *Conformal symmetry* is defined by insisting that each map f in (6) is a local conformal diffeomorphism, that is, $f^*g = \alpha g$ for some appropriate local, smooth real valued function α. The resulting set of all global, smooth vector fields on M with this property is labelled $C(M)$ and its members are called *conformal.* Then $X \in C(M)$ is characterised in any chart of M by

$$X_{a;b} = \phi g_{ab} + F_{ab} \qquad (F_{ab} = -F_{ba}) \tag{8}$$

where $\phi : M \to \mathbf{R}$ and F is a 2-form field on M. The set $C(M)$ is a Lie algebra and $H(M)$ and $K(M)$ above are subalgebras of it. Now it is well-known that $P(M)$ and $C(M)$ are finite-dimensional with $\dim P(M) \leq 24$ and $\dim C(M) \leq 15$ and so it follows from the discussion above that *the orbits of $P(M)$ and $C(M)$ are each foliations with singularities and are integral manifolds of $P(M)$ and $C(M)$, respectively, and similarly for their subalgebras mentioned above.* [It is remarked that the local action on M provided by the local diffeomorphisms described in the above Lie algebras need not lead to a *global* Lie group action on M. This occurs if and only if each vector field in the Lie algebra is *complete* [**12**].]

4. The Killing Algebra K(M). Consider the finite-dimensional Lie algebra of Killing vector fields $K(M)$ on M. The material of section 3 shows that *the orbits associated with $K(M)$ are leaves of a foliation with singularities* and are integral manifolds of $K(M)$. It also shows that, if O is any orbit of $K(M)$, and f any associated local isometry of $K(M)$ whose domain and range are the open subsets U and U' of M, then f gives rise to a smooth map $U \cap O \to U'$ whose range lies in the leaf O. Hence, it gives rise to a smooth map $U \cap O \to U' \cap O$, since $U' \cap O$ is an open and hence, regular submanifold of O. Then if $m \in U \cap O$, $f_*(T_m O) = T_{f(m)} O$. The definitions at the beginning of section 2 then show that, since f is a local isometry, O *is either spacelike, timelike or null.* If O is spacelike (respectively, timelike), then g induces a metric $h = i^*g$ on O which is positive-definite (respectively, Lorentz). If $X \in K(M)$ then X is tangent to O and so there is a unique smooth, global vector field $\tilde{X}$ on O such that $i_*\tilde{X} = X$. If O is non-null with induced metric h, then the condition that $X \in K(M)$, that is $\mathcal{L}_X g = 0$, is easily shown to imply that $\mathcal{L}_{\tilde{X}} h = 0$ and so $\tilde{X}$ is a Killing vector field on O with metric h, that is, $\tilde{X} \in K(O)$. In fact, the map $k : X \to \tilde{X}$ is a Lie algebra homomorphism $K(M) \to K(O)$.

In general, the map k is neither injective nor surjective. That the map k is not surjective can be seen from the space-time metric given in a global chart on $\{(x, y, z, t) \in \mathbf{R}^4 : t > 0\} \equiv M$ by

$$ds^2 = -dt^2 + t dx^2 + e^{2t} dy^2 + e^{3t} dz^2. \tag{9}$$

Here $K(M)$ is 3-dimensional, being spanned by the vector fields $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$. However, each subset O of constant t is an orbit of $K(M)$ and is, with its induced metric, flat Euclidean 3-space and so $\dim K(O) = 6$.

To investigate whether k is injective or not, let $0 \neq X \in K(M)$ and let $m \in M$ with $X(m) = 0$. Then the local isometries ϕ_t associated with X satisfy $\phi_t(m) = m$ and m is called a *zero* of X (or a *fixed point* of each ϕ_t). If U is a coordinate neighbourhood of m with coordinates y^a, then the linear isomorphism $\phi_{t*} : T_m M \to T_m M$ is represented in the basis $(\frac{\partial}{\partial y^a})_m$ by the matrix

$$e^{tB} = \exp t \Big(\frac{\partial X^a}{\partial y^b}\Big)_m \tag{10}$$

where $B^a_b \equiv \left(\frac{\partial X^a}{\partial y^b}\right)_m$ is the *linearisation* of X at m. Thus, since $X \in K(M)$, it follows from (7) that $B^a_b = (F^a{}_b)_m$. Also, since X is affine, if χ is the usual exponential diffeomorphism from some open neighbourhood of $0 \in T_m M$ onto some open neighbourhood V of m, then [**11**]

$$\phi_t \circ \chi = \chi \circ \phi_{t*}. \tag{11}$$

It is easily checked from this that, in the resulting normal coordinate system x^a with domain V about m, the components X^a of X are *linear functions* of the coordinates x^a. Since $B^a_b = (F^a{}_b)_m$ is skew self- adjoint with respect to $g(m)$, it follows that the rank of B is *even.* If $B = 0$ then $X \equiv 0$ on M and so B has rank 2 or 4. The zeros of X in V have coordinates satisfying $B^a{}_b x^b = 0$ and so, if $\operatorname{rank} B = 4$, the zero m is isolated, whereas if $\operatorname{rank} B = 2$, the zeros of X in V can be given the structure of a 2-dimensional, *regular* submanifold N of the open submanifold V [**6, 7**]. Now return to the map k and suppose it is *not* injective. Let O be the orbit of $K(M)$ through m. Then there exists $X \in K(M), X \not\equiv 0$, such that X vanishes on O, that is, $\tilde{X} = 0$. Since m is thus not isolated, $\operatorname{rank} B = 2$, and so the zeros of X *in* V are exactly the points on the 2-dimensional regular submanifold N of V. Let $O' = O \cap V$. Then O' is an open subset (and hence an open submanifold) of O. It follows that O' is a submanifold of M contained in the open (hence regular) submanifold V of M and hence O' is a submanifold of V [**2**]. But then $O' \subseteq N \subseteq V$, with O' and N submanifolds of V with N regular. It follows that O' is a submanifold of

N and so $\dim O' \leq \dim N$ and hence $\dim O(= \dim O') \leq 2$. *Hence, if* $\dim O$ *is* 3 *or* 4*, k is injective.* If, however, $\dim O \leq 2$, k can fail to be injective, as the following example shows. Let M_1 and M_2 be 2-dimensional, connected, smooth manifolds with $M_2 = \mathbf{R}^2$. Let g_1 be a positive definite metric on M_1 with $K(M_1)$ 1-dimensional and spanned by a Killing vector field with a single zero at $m \in M_1$. Let g_2 be the usual Minkowski metric on M_2, so that $\dim K(M_2) = 3$. Then the space-time $M_1 \times M_2$ with metric $g_1 \otimes g_2$ is such that $\dim K(M) = 4$ and $O = \{m_1\} \times M_2$ is a 2-dimensional, timelike orbit of $K(M)$ with $\dim K(O) = 3$. Thus, the map $K(M) \to K(O)$ is *not* injective.

If O is an orbit of $K(M)$, it was pointed out above that O is either spacelike, timelike or null. Thus, if $\dim O = p$ $(1 \leq p \leq 3)$ and $O \cap S_p \neq \emptyset$, then $O \subseteq S_p$ (and similarly for T_p and N_p). It is convenient at this point to distinguish between orbits which are, in some sense, *stable* with respect to their type and dimension and those which are not. Thus, an orbit is called *stable* if it is contained in one of the subsets $\operatorname{int} S_p, \operatorname{int} T_p$ or $\operatorname{int} N_p (1 \leq p \leq 3)$. Actually, since the inner product of a Killing vector field and the tangent vector to an affinely parameterised geodesic is constant along the geodesic, an argument based on the normal geodesics to orbits contained in S_3 and T_3 and an appeal to the rank theorem similar to that made at the end of section 2 shows that *S_3 and T_3 are open. Thus, all orbits in S_3 and T_3 are stable.* Regarding the stability of orbits, it is easy to show that, if O is any orbit of $K(M)$ such that $O \cap \operatorname{int} S_p \neq \emptyset$ $(1 \leq p \leq 3)$, then $O \subseteq \operatorname{int} S_p$ (and similarly for T_p and N_p). It is now possible to prove a number of results about how the existence of a certain type of *stable* orbit restricts the dimension of $K(M)$. These results are often used in the relativistic literature without justification. Some similar (but, as yet, incomplete) results are available in a similar context for *unstable* orbits [**8**].

In summary then (see [**8, 9**] for further discussion), the Lie algebra $K(M)$ of global, smooth Killing vector fields on a space-time M with smooth, Lorentz metric g is finite-dimensional and the orbits resulting from the maps (6) constitute a foliation with singularities. The maps (6) are smooth (local) maps $M \to M$ (and also $O \to O$, for any orbit O) and give rise to a Lie group (global) action on M if and only if each member of $K(M)$ is complete. A convenient decomposition of M with respect to the Lorentz metric g on M is provided by (2)-(5). The tangency of the members of $K(M)$ to an orbit leads to a natural Lie algebra homomorphism $K(M) \to K(O)$ which is easily seen to be not necessarily surjective and which is, perhaps less obviously, not necessarily injective, but is injective if $\dim O \geq 3$. This latter remark stems from a study of the zeros of the members of $K(M)$. The orbits of $K(M)$ were then divided into stable and unstable ones and the known (and used) results in orbit theory in general relativity can then be shown to apply to the stable orbits.

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