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REMARKS ON BOUNDED SOLUTIONS FOR SOME NONAUTONOMOUS ODE

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Abstract. A Borsuk–Ulam type argument is used in order to prove existence of nontrivial bounded solutions to some nonautonomous linear differential equations.

1. Introduction. The main purpose of the paper is to present a topological method of detecting bounded solutions of some nonautonomous differential equations. We confine ourselves to the simplest linear case in order to be clear in presentation. The idea is to some extent connected with way of thinking of the Cracow school (comp. [**8**], [**13**]). We consider a process defined by the equation in the extended phase space. Since the invariant sets are noncompact, we propose to define another topologically equivalent dynamical system which can be extended to a compact space. To this end we use Poincaré's old idea, which has been used to analyse planar systems.

The aproach seems geometrically simpler than the use of skew-symmetric flows as in [**12**], [**15**]. Perhaps one can here also try to apply techniques from the Conley index theory [**2**]. We use an argument of a Borsuk–Ulam-type instead. Actually, we use the topological fact that there are no homotopically nontrivial maps $f : S^n \to S^k$, when $n < k$.

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2. Poincaré compactification of polynomial vector fields. In this section we shortly recall the procedure described in [**7**] and [**4**], following the

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ideas of Poincaré [**10**], how a polynomial vector field on $\mathbf{R}^n$ induces a vector field on S^n.

Let $X = (P_1, P_2, ..., P_n)$ be a polynomial vector field in $\mathbf{R}^n$. We can identify $\mathbf{R}^n$ with the hyperplane $\Pi = \{y \in \mathbf{R}^{n+1} | \ y_{n+1} = 1\}$ tangent to the unit sphere $S^n = \{y \in \mathbf{R}^{n+1} | \ \sum_{i=1}^{n+1} y_i^2 = 1\}$ at the north pole. Denote by S_+^n and S_-^n the open northern and southern hemisphere, respectively. We consider the following two diffeomorphisms $\Phi^+ : \mathbf{R}^n \to S_+^n$ and $\Phi^- : \mathbf{R}^n \to S_-^n$ given by $\Phi^{\pm}(x) = \pm\frac{1}{\Delta(x)}(x_1, x_2, ..., x_n, 1)$, where $\Delta(x) = (1 + \sum_{i=1}^n x_i^2)^{\frac{1}{2}}$. In this form X induces a vector field Y in $S_+^n \cup S_-^n$ defined by $Y(y) = D\Phi_x^{\pm} X(x)$ if $y = \Phi^{\pm}(x)$.

Now assume that k is the maximum of the degrees of P_i. The following theorem has been proved in [**7**], see also [**1**], [**4**]. One can find detailed description of the vector field in local charts there.

THEOREM 1. ([**7**]) *The vector field Y can be extended analytically to the whole sphere S^n after multiplication by the factor y_{n+1}^{k-1} and in such a way that the equator $S^{n-1} = \{y \in S^n | \ y_{n+1} = 0\}$ is invariant.*

The above theorem has been used to investigate the behaviour of the vector field at infinity. We need this for another purpose. First, observe that in the case of a linear vector field in $\mathbf{R}^n$ you do not need to multiply the induced vector field in order to extend it to the whole sphere. Thus we can formulate an immediate consequence of Theorem 1.

COROLLARY 2. *The flow in $\mathbf{R}^n$ given by a linear vector field X is conjugate to the flow in the upper hemisphere S_+^n given by the induced vector field Y and the latter has such an extension to the closed hemisphere that the equator is an invariant set.*

PROPOSITION 3. *Given a flow in $\mathbf{R}^n$ defined by a linear vector field X, there exists a flow defined on the closed unit disc $D^n = \{y \in \mathbf{R}^n \ ||y|| \leq 1\}$ such that the open disc and the boundary are invariant sets and the flow on the open disc is topologically equivalent to the original flow in $\mathbf{R}^n$.*

PROOF. By means of the projection $\pi(y_1, y_2, ..., y_{n+1}) = (y_1, y_2, ..., y_n)$ we obtain a homeomorphism $h : S_+^n \to intD^n$, which gives the equivalence of the flows on the northern hemisphere and on the disc. On the other hand, the flow on the hemisphere is topologically equivalent to the one in $\mathbf{R}^n$ because they are conjugate. $\square$

Let us observe that the described procedure works for vector fields whose rate of growth at infinity is not bigger that the polynomial growth of degree n (in Corr. 2 and Prop. 3 with at most linear growth).

3. G–spaces and G–index. We are going to use cohomology of the Čech type. The Čech cohomology theory has the continuity property, which says that if a cohomology class vanishes on a closed set, then it vanishes on a neighbourhood of this set. Throughout the paper, the group Z_2 of integers mod 2 will be used as a coefficient group in cohomology.

Let G be the group Z_2. Assume that G acts freely on a paracompact space X. We call X a *G–space*. Any such G–space admits an equivariant map $h : X \to EG$ into a classifying space EG; any two such maps are equivariantly homotopic (see [**5**] Thm 8.12 and Thm 6.14). The map h induces a map $\hat{h} : X/G \to BG := EG/G$ on the orbit spaces. Consequently, one has a uniquely determined homomorphism

$$\hat{h}^* : H^*(BG, Z_2) \to H^*(X/G, Z_2).$$

In our special case $G = Z_2$, the space EG can be identified with the sphere of infinite dimension S^∞ with a free antipodal action of G. The orbit space is the infinite dimensional projective space P^∞.

Let us recall the definition of the G–index $ind_G X$, for a G–space X (see e.g. [**14**]).

Definition 1. We say that the *G–index of X* is not less than k if the homomorphism $\hat{h}^k : H^k(BG, Z_2) \to H^k(X/G, Z_2)$ is a monomorphism.

Most of the properties of the $G-$index are immediate consequences of this definition. In particular, *monotonicity* says:

If G acts freely on X and Y, and $f : X \to Y$ is an equivariant map, then $ind_G Y \geq ind_G X$.

The *dimension* property:

If $dim X < m$ then $ind_G X < m$, where dim denotes the covering dimension.

An important special case of the above says:

If $ind_g X = 0$ then $X \neq \emptyset$.

The consequence of the continuity of Čech cohomology is the following *continuity* property:

Let G act freely on X and $A \subset X$ be a compact G–space. Then there is an open neighbourhood U of A in X which is a G–space such that $ind_G U = ind_G A$.

We shall use the important property that $ind_G S^n = n$. The concept of the G–index was first defined by Yang [**16**] for $G = Z_2$ and extended to other more general settings by several authors, notably to actions of compact Lie groups by Fadell and Husseini [**6**].

4. Nonautonomous systems. We consider the following linear nonautonomous system of differential equations in $\mathbf{R}^n$:

$$x'(t) = A(t)x(t) \tag{1}$$

where $A : \mathbf{R} \to M_{n\times n}$ is a continuous map from the real numbers to the space of square matrices.

Let us make the following assumptions

(A1) $A(t) = A_+$ for $t \geq t_1$,

(A2) $A(t) = A_-$ for $t \leq t_2$,

(A3) the matrices A_+, A_- are hyperbolic i.e. have no eigenvalues with real part 0.

Let us denote by k the number of eigenvalues λ of A_- with $Re\lambda < 0$, and by l the number of eigenvalues λ of A_+ with $Re\lambda < 0$.

Now we are ready to formulate our main result.

THEOREM 4. *If $k \neq l$, the equation* (1) *has a nontrivial bounded solution.*

PROOF. It is well known that the equation (1) determines a process in the extended phase space $\mathbf{R}^{n+1}$ given by the vector field $X(x,t) = (A(t)x, 1)$. In the meaning of [**11**] this is a skew-product flow on $\mathbf{R}^{n+1}$.

Now we can apply the procedure from section 2. But we do this first with the vector field $x \mapsto A(t)x$ with fixed t.Then for each fixed x, we can apply the same procedure to the vector field $t \mapsto 1$ in $\mathbf{R}$. More explicitly, we can multiply this constant vector field by a smooth, even and positive-valued function $k(t)$ such that $K(0) = 1$ and $\lim_{t\to\infty} k(t) = 0$, e.g. $k(t) = \exp(-|t|)$. In this way we obtain a skew-product flow on a solid cylinder $D^n \times [-1, 1]$. This flow φ considered in the interior of the cylinder is topologically equivalent to the original one.

Let us assume for simplicity that $t_1 = \frac{1}{2}, t_2 = -\frac{1}{2}$ (otherwise we rescale the procedure).

The obtained flow is very simple to observe. Invariant sets are e.g.

$$D^n \times \{-1\},\ D^n \times \{1\},\ \partial D^n \times [-1, 1],\ \{0\} \times [-1, 1]$$

The orbits connecting points $(0, -1)$ and $(0, 1)$ correspond to bounded solutions of the equation (1). Thus it is enough to prove that there exists an orbit different from the trivial one $\{0\} \times (-1, 1)$ which starts from $(0, -1)$ and ends at $(0, 1)$.

On the other hand, observe that, by our assumptions, for each point (x, t) with $t > -1$, the ω- limit set is contained in $D^n \times \{1\}$. Moreover, since A_+ is hyperbolic, it is either $(0, 1)$ or a subset of $\partial D^n \times \{1\}$.

Furthermore, the horizontal sections of the flow below $-\frac{1}{2}$-level are copies of the level -1, and similarly above the level $\frac{1}{2}$.

Let us suppose that there are no nontrivial orbits connecting $(0,-1)$ and $(0,1)$.

Considering the flow on $D^n \times \{-1\}$, the dimension of the unstable manifold of the hyperbolic stationary point 0 is $n-k$. We can take a small sphere $S_\varepsilon = S^{n-k-1}$ in it.

Consider a sphere at a bit higher level $S^{n-k-1} \times \{-1+\delta\}$. An orbit of every point from this set has to approach a neighbourhood of a sphere $S^{n-l-1} \times \{1\} \subset \partial D^n \times \{1\}$ (a sphere in the unstable subspace).

We have a natural antipodal action of $G = Z_2$ on the cylinder $(x,t) \mapsto (-x,t)$. This is an obvious observation that the construction in Section 1 preserves the property that the flow is equivariant, since the central projection is an odd map. We choose a G_2-invariant neighbourhood V of $S^{n-l-1} \times \{1\}$. Since the set $S^{n-k-1} \times \{-1+\delta\}$ is compact, there is a finite time τ_0 such that, for each point $(x,t) \in S^{n-k-1} \times \{-1+\delta\}, \quad \varphi_{\tau_0}(x,t) \in V$.

Therefore, we have defined an equivariant map $\beta : S^{n-k-1} \times \{-1+\delta\} \to V$.

By the continuity of the G–index, $ind_G V = n-l-1$. Thus we have just proved the inequality $n-k-1 \leq n-l-1$.

In the same way, using the reverse time, we prove that $n-l-1 \leq n-k-1$. therefore we obtain $l = k$ contrary to our assumption. $\square$

Remarks. First let us observe that assumptions (A1), (A2) may be weakened by $\lim_{t\to\pm\infty} = A_\pm$.

Note that a similar proof could also works in the nonlinear case with the uniqueness and global existence assumptions satisfied. We should then assume that the right-hand side of the equation is odd, of polynomial growth, with the only hyperbolic stationary point at 0. Our result seems to be complementary to a theorem of Sacker and Sell (see [**3**], [**11**]).

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