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THE OPTIMIZATION OF EIGENVALUE PROBLEMS INVOLVING THE p-LAPLACIAN

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Abstract. Given a bounded domain $\Omega \subset \mathbb{R}^n$ and numbers $p > 1$, $\alpha \geq 0$, $A \in [0, |\Omega|]$, consider the following optimization problem: find a subset $D \subset \Omega$, of measure A, for which the first eigenvalue of the operator $-\Delta_p + \alpha\, \chi_D\, \varphi_p$ with the Dirichlet boundary condition is as small as possible. We prove the existence of optimal solutions and study their qualitative properties. We also obtain the radial symmetry of optimal solutions in the case when Ω is a ball.

1. Introduction. Let Ω be a bounded domain in the space $\mathbb{R}^n$ $(n \geq 1)$ with the closure $\overline{\Omega}$ and boundary $\partial\Omega$. We denote by $|\Omega|$ the Lebesgue measure of Ω. Given numbers $p > 1$, $\alpha \geq 0$ and a measurable subset D of Ω, we shall be concerned with the eigenvalue problem of the form

$$\begin{cases} -\Delta_p(u) + \alpha\, \chi_D\, \varphi_p(u) = \lambda\, \varphi_p(u) & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Δ_p is the p-Laplacian, χ_D is the characteristic function of D, while φ_p is a function defined by

$$\varphi_p(u) := \begin{cases} |u|^{p-2}u, & \text{if } u \neq 0, \\ 0, & \text{if } u = 0. \end{cases}$$

We also remind that the p-Laplacian is a nonlinear differential operator of the form

$$\Delta_p(u) = \operatorname{div}(|\nabla u|^{p-2}\nabla u) = \operatorname{div}(\varphi_p(\nabla u)),$$

which coincides with the Laplacian Δ for $p = 2$.

In this paper we deal with real function spaces only. For brevity, we introduce the following notation:

$$L^p := L^p(\Omega), \qquad W^{1,p} := W^{1,p}(\Omega), \qquad W_0^{1,p} := W_0^{1,p}(\Omega).$$

Here $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$, with $1 < p < \infty$, are standard Sobolev spaces (see, e.g., [**1**] for more details). It is customary to use solutions of (1) in a weak sense. In particular, any nontrivial function $u\colon \Omega \to \mathbb{R}$ is said to be an eigenfunction of problem (1) if and only if $u \in W_0^{1,p}$ and

$$\int_\Omega \varphi_p(\nabla u)\nabla v + \alpha \int_\Omega \chi_D\, \varphi_p(u)\, v = \lambda \int_\Omega \varphi_p(u)\, v, \qquad \forall\, v \in W_0^{1,p}. \tag{2}$$

Let $\lambda(\alpha, D)$ stand for the lowest eigenvalue λ of problem (1). It is well known that $\lambda(\alpha, D)$ is positive and its eigenfunction is unique up to a scalar multiple (see for instance [**5**], [**10**]). Let us fix $A \in [0, |\Omega|]$ and define

$$\Lambda(\alpha, A) := \inf\{\, \lambda(\alpha, D)\colon D \subset \Omega,\ |D| = A\,\}. \tag{3}$$

Any minimizer in (3) is called an *optimal configuration.* If u is an eigenfunction of problem (1) with $\lambda = \Lambda(\alpha, A)$ and with an optimal configuration D, then (u, D) is said to be an *optimal pair* (or *optimal solution*).

This paper has been inspired by that of S. Chanilo et al. [**3**], in which the linear case $p = 2$ was extensively studied. The authors addressed the question of existence and uniqueness of optimal solutions. They obtained various results on qualitative properties of optimal configurations. In this paper we are interested in obtaining similar results for the p-Laplacian with arbitrary $p > 1$. Our terminology follows that of [**3**].

2. Some auxiliary results. Let us begin with well-known results concerning the eigenvalue problem (1). They can be found, e.g., in [**5**] or [**10**]. If u is an eigenfunction corresponding to the first eigenvalue of problem (1), then u does not change sign in Ω. From now on it will be chosen positive in Ω. Unless otherwise stated, we shall always assume that the eigenfunction u is normalized in such a way that

$$\int_\Omega |u|^p = 1,$$

which is possible according to uniqueness up to a scalar multiple. Using the Rayleigh quotient

$$R(u, \alpha, D) := \frac{\int_\Omega |\nabla u|^p + \alpha \int_\Omega \chi_D\, |u|^p}{\int_\Omega |u|^p},$$

the first eigenvalue of problem (1) may be characterized by

$$\lambda(\alpha, D) = \inf\bigl\{R(u, \alpha, D)\colon u \in W_0^{1,p} \setminus \{0\}\bigr\}.$$

Consequently,

$$\Lambda(\alpha, A) = \inf\big\{R(u,\alpha,D)\colon |D| = A,\ u \in W_0^{1,p} \setminus \{0\}\big\}. \tag{4}$$

In what follows we shall use an estimate of the L^∞-norm of eigenfunctions of problem (1) provided by the following lemma.

Lemma 1. *There exists a constant $C > 0$ which depends on $\Omega, p, \alpha, \lambda$ only and stays bounded for α, λ bounded, such that*

$$\|u\|_{L^\infty} \le C\,\|u\|_{L^p} \tag{5}$$

for any eigenfunction of problem (1).

Proof. It is known that the eigenfunctions corresponding to problem (1) are essentially bounded. More precisely, there exist constants $C_0 > 0$ and $q > p$ (which depend on $\Omega, p, \alpha, \lambda$ only and stay bounded for α, λ bounded) such that

$$\|u\|_{L^\infty} \le C_0\,\|u\|_{L^q} \tag{6}$$

for such eigenfunctions. In this connection refer to [**5**] and [**11**]. We now apply standard arguments involving the interpolation inequality (see [**7**], Section 7.1). Taking $p < q < 2q$ and $\mu = \frac{q-p}{2pq^2}$, by (6) we get the estimate

$$\|u\|_{L^\infty} \le C_0\big(\varepsilon\,\|u\|_{L^{2q}} + \varepsilon^{-\mu}\|u\|_{L^p}\big), \tag{7}$$

which holds for any $\varepsilon > 0$. Further, since the embedding $L^\infty \hookrightarrow L^{2q}$ is continuous, there is a constant $C_1 > 0$ such that

$$\|u\|_{L^{2q}} \le C_1\,\|u\|_{L^\infty}, \qquad \forall\, u \in L^\infty. \tag{8}$$

It follows from (7) and (8) that

$$\|u\|_{L^\infty} \le C_0\,C_1\,\varepsilon\,\|u\|_{L^\infty} + C_0\,\varepsilon^{-\mu}\|u\|_{L^p}.$$

Choosing ε so that $C_0\,C_1\,\varepsilon = \frac{1}{2}$, we find that

$$\|u\|_{L^\infty} \le 2\,C_0\,\varepsilon^{-\mu}\|u\|_{L^p},$$

which is the desired result (with $C = (2\,C_0)^{\mu+1}C_1^{\mu}$). □

3. Main properties of optimal pairs. The following theorems are analogous to those of [**3**]. We recall that Ω is any bounded domain in $\mathbb{R}^n$ and $p \in (1,\infty)$.

Theorem 1. *For any $\alpha \ge 0$ and $A \in [0, |\Omega|]$ there exists an optimal pair. Every optimal pair (u, D) has the following properties:*

(*a*) *$u \in W_0^{1,p} \cap L^\infty$ and ∇u is locally Hölder continuous, i.e., for every compact $K \subset \Omega$ there exists $\beta \in (0,1)$ such that $\nabla u \in C^{0,\beta}(K)$,* (*b*) *there is a number* $t \geq 0$ *such that*

$$\{ u < t \} \subset D \subset \{ u \leq t \}.$$

Here we write $\{ u < t \}$ instead of $\{ x \in \Omega \colon u(x) < t \}$ and similarly we put $\{ u \leq t \} := \{ x \in \Omega \colon u(x) \leq t \}$.

Remark. In the classical case $p = 2$, assertion (*b*) gets a stronger form

$$D = \{ u \leq t \}$$

(see [**3**], Theorem 1). Although we conjecture that the same is true for arbitrary $p > 1$, we have not been able to prove such a result.

Proof of Theorem 1. The regularity properties of eigenfunctions, stated in assertion (*a*), are rather well known. By Lemma 1, any eigenfunction of problem (1) belongs to $W_0^{1,p} \cap L^\infty$. The local Hölder continuity of ∇u was proved, e.g., by DiBenedetto [**4**].

We now prove the existence of optimal pairs. Let us fix numbers $\alpha \geq 0$ and $A \in [0, |\Omega|]$. For shortness, we set

$$\Lambda := \Lambda(\alpha, A) \qquad \text{and} \qquad \lambda(D) := \lambda(\alpha, D).$$

Following [**3**], we let $\{D_j\}$ be a minimizing sequence in the sense that

$$\lambda(D_j) \to \Lambda \quad \text{as} \quad j \to \infty. \tag{9}$$

Let $u_j \in W_0^{1,p}$ be the L^p-normalized positive eigenfunction corresponding to $\lambda(D_j)$ so that, according to (2),

$$\int_\Omega \varphi_p(\nabla u_j)\nabla v + \alpha \int_\Omega \chi_{D_j}\, \varphi_p(u_j)\, v = \lambda(D_j) \int_\Omega \varphi_p(u_j)\, v, \qquad \forall\, v \in W_0^{1,p}. \tag{10}$$

Substituting $v = u_j$ into (10) we infer that

$$\int_\Omega |\nabla u_j|^p + \alpha \int_\Omega \chi_{D_j}\, |u_j|^p = \lambda(D_j), \qquad \forall\, j \in \mathbb{N}. \tag{11}$$

By (9), the sequence $\{\lambda(D_j)\}$ is bounded and therefore the sequence $\{u_j\}$ is bounded in $W_0^{1,p}$, by (11). Next, $\{\chi_{D_j}\}$ is a sequence bounded in $L^\infty = (L^1)^*$. Thus, by the Banach-Alaoglu theorem (see for instance [**12**], Theorem 3.17), we may choose subsequences, again denoted by $\{u_j\}$, $\{\chi_{D_j}\}$, and functions $u \in W_0^{1,p}$, $\eta \in L^\infty$ such that $u_j \rightharpoonup u$ in $W_0^{1,p}$ (weak convergence) and $\chi_{D_j} \overset{*}{\rightharpoonup} \eta$ in L^∞ (weak* convergence). The compactness of the embedding $W_0^{1,p} \hookrightarrow L^p$ implies that $u_j \to u$ (strongly) in L^p. Since $|D_j| = A$ and $\chi_{D_j} \overset{*}{\rightharpoonup} \eta$ in L^∞, we see that

$$\int_\Omega \eta = \lim_{j\to\infty} \int_\Omega \chi_{D_j} = A.$$

We may also assume that

$$\int_\Omega \chi_{D_j}\,|u_j|^p \;\to\; \int_\Omega \eta\,|u|^p \quad\text{as}\quad j\to\infty. \tag{12}$$

To see this, note that $|u_j|^p \to |u|^p$ in L^1, because $u_j \to u$ in L^p. Consequently,

$$\int_\Omega \chi_{D_j}\,|u_k|^p \;\to\; \int_\Omega \chi_{D_j}\,|u|^p \quad\text{as } k\to\infty \qquad (j=1,2,\dots).$$

Moreover, the weak* convergence $\chi_{D_j} \overset{*}{\rightharpoonup} \eta$ in L^∞ implies

$$\int_\Omega \chi_{D_j}\,|u|^p \;\to\; \int_\Omega \eta\,|u|^p \quad\text{as}\quad j\to\infty.$$

Now we can use a standard diagonal argument to deduce that (12) holds for an appropriate subsequence.

Next, note that, by convexity, the functional $W_0^{1,p} \ni v \mapsto \int_\Omega |\nabla v|^p \in \mathbb{R}$ is weakly lower semicontinuous and hence

$$\int_\Omega |\nabla u|^p \le \liminf_{j\to\infty} \int_\Omega |\nabla u_j|^p. \tag{13}$$

On the other hand, we see by (9), (11) and (12) that

$$\lim_{j\to\infty} \int_\Omega |\nabla u_j|^p = \Lambda - \alpha \int_\Omega \eta\,|u|^p. \tag{14}$$

It follows from (13) and (14) that

$$\int_\Omega |\nabla u|^p + \alpha \int_\Omega \eta\,|u|^p \le \Lambda. \tag{15}$$

Since $0 \le \chi_{D_j} \le 1$ for $j\in\mathbb{N}$ and $\chi_{D_j} \overset{*}{\rightharpoonup} \eta$ in L^∞, we conclude that

$$0 \le \eta \le 1 \qquad \text{a.e. in } \Omega.$$

Hence the function η in inequality (15) can be replaced by a characteristic function. Indeed, it follows from the “bathtub principle”, Theorem 1.18 of [**9**], that the minimization problem

$$\inf\left\{ \int_\Omega \eta\,|u|^p \colon 0\le\eta\le 1,\ \int_\Omega \eta = A \right\}$$

has a solution χ_D, where D is any set such that $|D| = A$ and

$$\{\, u < t \,\} \subset D \subset \{\, u \le t \,\},$$

where

$$t := \sup\{\, s\colon |\{u<s\}| \le A \,\}. \tag{16}$$

Therefore, in view of (15), we get the inequality

$$\int_\Omega |\nabla u|^p + \alpha \int_\Omega \chi_D\, |u|^p \le \Lambda.$$

Taking into account that $\int_\Omega |u|^p = 1$, we see by (4) that this must actually be an equality, and so the pair (u, D) is an optimal solution.

It remains to prove statement (b). Let (u, D) be any optimal pair (with $u > 0$ in Ω and $\int_\Omega |u|^p = 1$). Let us define the number t by formula (16). We now claim that

$$\{\, u < t \,\} \subset D \tag{17}$$

and

$$D \subset \{\, u \le t \,\} \tag{18}$$

(up to a set of measure zero). If the inclusion (17) did not hold, then it would be possible to reduce the integral $\int_\Omega \chi_D\, |u|^p$ by shifting a part of D from $\{\, u \ge t \,\}$ to $\{\, u < t \,\}$. Similarly, if the inclusion (18) was not true, then we could reduce $\int_\Omega \chi_D\, |u|^p$ by shifting a part of D from $\{\, u > t \,\}$ to $\{\, u \le t \,\}$. Both cases lead to a contradiction with variational property (4). This completes the proof. □

Next, let us denote by μ_0 the first eigenvalue of the problem

$$\begin{cases} -\Delta_p(u) = \mu\, \varphi_p(u) & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{cases}$$

We proceed to study the dependence of the optimal eigenvalue $\Lambda(\alpha, A)$ on parameters α and A.

Theorem 2. *Let $\Lambda(\alpha, A)$ be the optimal eigenvalue defined by* (3)*. Then the following assertions hold:*

- (a) *The function $(\alpha, A) \mapsto \Lambda(\alpha, A)$ is Lipschitz continuous, uniformly on bounded sets. More precisely, for any $\alpha, \alpha' \ge 0$ and $A, A' \in [0, |\Omega|]$ the inequality*

$$\begin{aligned} &|\Lambda(\alpha, A) - \Lambda(\alpha', A')| \\ &\qquad \le |\alpha - \alpha'|\, \frac{\max\{A, A'\}}{|\Omega|} + |A - A'|\, \min\{\alpha, \alpha'\}\, C_{\max\{\alpha,\alpha'\}} \end{aligned} \tag{19}$$

  holds, with C_m bounded for m bounded.
- (b) *$\Lambda(\alpha, A)$ is strictly increasing in A for fixed $\alpha > 0$ and strictly increasing in α for fixed $A > 0$.*
- (c) *$\Lambda(\alpha, A) - \alpha$ is strictly decreasing in α for fixed $A < |\Omega|$.*

(*d*) *If* $A < |\Omega|$, *then there is the unique value* $\alpha = \overline{\alpha}(A)$ *such that*

$$\Lambda(\overline{\alpha}(A), A) = \overline{\alpha}(A). \tag{20}$$

The function $A \mapsto \overline{\alpha}(A)$ *is continuous and strictly increasing,* $\overline{\alpha}(0) = \mu_0$ *and*

$$\overline{\alpha}(A) \to \infty \quad \text{as} \quad A \to |\Omega|. \tag{21}$$

Proof. To prove statements (*a*) and (*b*), we can argue in almost the same way as in the proof of Proposition 10 in [**3**]. Namely, letting (u, D) and (u', D') be minimizers for $\Lambda(\alpha, A)$ and $\Lambda(\alpha', A')$ respectively, we may assume that $u, u' > 0$ in Ω and $\int_\Omega u^p = \int_\Omega (u')^p = 1$, so that according to (4) there is

$$\Lambda(\alpha, A) = \int_\Omega |\nabla u|^p + \alpha \int_D u^p, \qquad |D| = A,$$

and similarly

$$\Lambda(\alpha', A') = \int_\Omega |\nabla u'|^p + \alpha' \int_{D'} (u')^p, \qquad |D'| = A'.$$

By symmetry of (19) we may assume that $A' \geq A$. If $A' = 0$, inequality (19) is obvious. Now let us fix $A' > 0$. We choose a set $D_1 \subset D'$ in such a way that $|D_1| = A$. Next, we choose a set $D_1' \supset D$ so that $|D_1'| = A'$ and

$$\{\, u < s \,\} \subset D_1' \subset \{\, u \leq s \,\}$$

for a suitable number $s > 0$. Observe that

$$\frac{\int_{D_1'} u^p}{|D_1'|} \leq \frac{\int_\Omega u^p}{|\Omega|}$$

(like in the case of $p = 2$ and $D_1' = \{\, u \leq s \,\}$, which was treated in [**3**]). By Lemma 1, we can adapt the proof of Proposition 10 in [**3**] to our situation, replacing the exponent 2 by p. We omit the lengthy details here.

Our next aim is to prove statements (*c*) and (*d*). Let

$$0 \leq \alpha < \alpha' \tag{22}$$

and let (u, D) be a minimizer for $\Lambda(\alpha, A)$ (with $u > 0$ in Ω and $\int_\Omega u^p = 1$). Note that

$$\begin{aligned} \Lambda(\alpha, A) - \alpha &= \int_\Omega |\nabla u|^p + \alpha \int_\Omega \chi_D\, u^p - \alpha \\ &= \int_\Omega |\nabla u|^p - \alpha \int_\Omega (1 - \chi_D)\, u^p. \end{aligned} \tag{23}$$

Since $|D| = A < |\Omega|$ by assumption, we see that $1 - \chi_D \not\equiv 0$ and hence

$$\int_\Omega (1 - \chi_D)\, u^p > 0. \tag{24}$$

It follows from (22), (23) and (24) that

$$\Lambda(\alpha, A) - \alpha > \int_\Omega |\nabla u|^p - \alpha' \int_\Omega (1-\chi_D)\, u^p \ge \Lambda(\alpha', A') - \alpha', \tag{25}$$

and thus statement (c) is proved. Next, by (24) and (25) we find that

$$\lim_{\alpha\to\infty} [\Lambda(\alpha, A) - \alpha] = -\infty.$$

On the other hand, $\Lambda(0, A) = \mu_0 > 0$. By continuity and monotonicity, there is the unique value $\alpha = \overline{\alpha}(A)$ such that (20) holds. Moreover, it is clear by (a) and (b) that the function $A \mapsto \overline{\alpha}(A)$ is continuous and strictly increasing. Now, let us suppose that property (21) is not satisfied, and so there exists a constant M such that

$$\overline{\alpha}(A) \le M, \qquad \forall\, A < |\Omega|. \tag{26}$$

Let (u, D) be a minimizer for $\Lambda(\overline{\alpha}(A), A) = \overline{\alpha}(A)$, with a positive eigenfunction u normalized this time in such a way that

$$\int_\Omega |\nabla u|^p = 1. \tag{27}$$

Since, in view of (4),

$$\overline{\alpha}(A) = \frac{\int_\Omega |\nabla u|^p + \overline{\alpha}(A) \int_\Omega \chi_D\, u^p}{\int_\Omega u^p},$$

we see that

$$\int_\Omega |\nabla u|^p + \overline{\alpha}(A) \int_\Omega \chi_D\, u^p = \overline{\alpha}(A) \int_\Omega u^p,$$

or equivalently

$$\overline{\alpha}(A) \int_\Omega (1-\chi_D)\, u^p = \int_\Omega |\nabla u|^p = 1. \tag{28}$$

We now claim that

$$\int_\Omega (1-\chi_D)\, u^p = \int_{\Omega\setminus D} u^p \to 0 \quad \text{as} \quad A \to |\Omega|. \tag{29}$$

To see this, note that by Lemma 1

$$\int_{\Omega\setminus D} u^p \le |\Omega \setminus D| \sup_\Omega u^p = (|\Omega| - A) \sup_\Omega u^p \le (|\Omega| - A)\, C\, \|u\|^p_{L^p}, \tag{30}$$

where C is a constant independent of A, by (26). Moreover, since the embedding $W_0^{1,p} \hookrightarrow L^p$ is continuous,

$$\|u\|^p_{L^p} \le C'\, \|u\|^p_{W_0^{1,p}}, \qquad \forall\, u \in W_0^{1,p}. \tag{31}$$

By virtue of the Poincaré inequality (see for instance [**1**], Section 6.26), we obtain the estimate

$$\|u\|^p_{W^{1,p}_0} \le C'' \int_\Omega |\nabla u|^p, \qquad \forall\, u \in W^{1,p}_0. \tag{32}$$

Combining (30)–(32) with (27), we conclude that

$$\int_{\Omega\setminus D} u^p \le (|\Omega| - A)\,\widetilde{C},$$

where $\widetilde{C} = C\,C'C''$ is a constant independent of A. This gives (29). Now (26) and (29) contradict (28), and hence assertion (d) follows. $\square$

The following result is an easy consequence of Theorems 1 and 2.

Theorem 3. *Fix $\alpha > 0$, $A > 0$, and let D be an optimal configuration. Then*

(a) *D contains a tubular neighbourhood of the boundary $\partial\Omega$,*

(b) *if $\alpha < \overline{\alpha}(A)$, then every connected component D_0 of the interior of D hits the boundary, i.e.,*

$$\overline{D_0} \cap \partial\Omega \ne \emptyset.$$

In particular, if Ω is simply connected, then D is connected.

Proof. Statement (a) is an immediate consequence of Theorem 1. Next, let us fix an optimal pair (u, D) (with $u > 0$ in Ω) and suppose that assertion (b) is false. Then we can find such a connected component D_0 of $\operatorname{Int} D$ that

$$\overline{D_0} \cap \partial\Omega = \emptyset. \tag{33}$$

By Theorem 1 (b),

$$D_0 \subset D \subset \{\, u \le t \,\}, \tag{34}$$

and, consequently, $\partial D_0 \subset \{\, u = t \,\} \cup \partial\Omega$, because $\{\, u < t \,\} \subset \operatorname{Int} D$ by continuity of u. Now (33) shows that, actually, $\partial D_0 \subset \{\, u = t \,\}$. By assumption, $\alpha < \overline{\alpha}(A)$, which implies that $\Lambda(\alpha, A) - \alpha > 0$, by assertions (c) and (d) of Theorem 2. Thus it follows that

$$-\,\Delta_p(u) = (\Lambda(\alpha, A) - \alpha\,\chi_D)\,\varphi_p(u) > 0 \quad \text{in } D_0. \tag{35}$$

If we let $v := u - t$, then

$$\begin{cases} -\Delta_p(v) = -\Delta_p(u) > 0 & \text{in } D_0, \\ v = 0 & \text{on } \partial D_0. \end{cases}$$

We thus conclude from the maximum principle (see for instance [**6**], Theorem 2) that $v \ge 0$ in D_0. This means that $u \ge t$ in D_0, which together with (34) shows that $u \equiv t$ in D_0 and hence $\Delta_p(u) \equiv 0$ in D_0, contrary to (35). This proves (b). $\square$

4. Symmetry. In this section we address the questions of symmetry and uniqueness of optimal configurations. In the case $p = 2$ more general results were obtained in [**3**], including a reflection symmetry in the presence of the domain convexity. The Steiner symmetrization played a key role in this case. Using general ideas from Theorem 4 of [**3**] and the Schwarz symmetrization, we obtain the following theorem on radial symmetry in the case of $p > 1$.

THEOREM 4. *Let Ω be a ball, $\Omega = \{ x \in \mathbb{R}^n \colon |x| < R \}$. Given any $A \in [0, |\Omega|]$ and $\alpha \geq 0$ such that $\alpha \neq \overline{\alpha}(A)$, there exists the unique optimal configuration D, and D is an annular region of the form*

$$D = \{ x \in \mathbb{R}^n \colon r(A) < |x| < R \}.$$

Moreover, if (u, D) is an optimal pair with $u > 0$ in Ω, then the eigenfunction u is radially symmetric and strictly decreasing in $|x|$.

PROOF. To begin with, we recall some properties of the Schwarz symmetrization (radially symmetric decreasing rearrangement) $u \mapsto u^*$. They can be found, e.g., in [**8**] and [**9**]. We remind that Ω is now a ball centered at 0. First, let us recall that for $u \in L^p$ (with $1 \leq p \leq \infty$)

$$(|u|^p)^* = (u^*)^p, \tag{36}$$

and hence

$$\|u\|_{L^p} = \|u^*\|_{L^p}. \tag{37}$$

Secondly, if $u \in W_0^{1,p}$ then $u^* \in W_0^{1,p}$ and

$$\int_\Omega |\nabla u^*|^p \leq \int_\Omega |\nabla u|^p. \tag{38}$$

Moreover, for $f, g \geq 0$ in Ω, there is

$$\int_\Omega f\, g \leq \int_\Omega f^* g^*. \tag{39}$$

Now fix any $\alpha \geq 0$ and $A \in [0, |\Omega|]$. Let (u, D) be an optimal pair (with $u > 0$ in Ω). We define D_* by

$$\chi_{D_*} := 1 - (\chi_{D^c})^*.$$

It is easy to see that D_* is (up to a set of measure zero) an annulus of the form

$$D_* = \{ x \in \mathbb{R}^n \colon r(A) < |x| < R \}.$$

We claim that

$$\int_\Omega \chi_{D_*} (u^*)^p \le \int_\Omega \chi_D \, u^p. \tag{40}$$

To see this, observe that

$$\begin{aligned}\int_\Omega \chi_D \, u^p &= \int_\Omega (1-\chi_{D^c})\, u^p = \int_\Omega u^p - \int_\Omega \chi_{D^c} u^p \\ &\ge \int_\Omega u^p - \int_\Omega (\chi_{D^c})^* (u^p)^* = \int_\Omega (u^p)^* - \int_\Omega (\chi_{D^c})^* (u^p)^* = \\ &= \int_\Omega (1-(\chi_{D^c})^*)\,(u^p)^* = \int_\Omega \chi_{D_*} (u^*)^p,\end{aligned}$$

by (39), (37) and (36) respectively. It now follows from (37), (38) and (40) that

$$\begin{aligned} R(u^*,\alpha,D_*) &= \frac{\int_\Omega |\nabla u^*|^p + \alpha \int_\Omega \chi_{D_*} \, |u^*|^p}{\int_\Omega |u^*|^p} \\ &\le \frac{\int_\Omega |\nabla u|^p + \alpha \int_\Omega \chi_D \, |u|^p}{\int_\Omega |u|^p} = \Lambda(\alpha, A). \end{aligned} \tag{41}$$

Thus, from the equality $|D_*| = |D|$ and variational property (4), we conclude that (u^*, D_*) is another optimal pair and therefore

$$-\Delta_p(u^*) = (\Lambda(\alpha,A) - \alpha\,\chi_{D_*})\,\varphi_p(u^*) \quad \text{in } \Omega. \tag{42}$$

Since $u > 0$ in Ω, there is $u^* \ge 0$ in Ω, and consequently $u^* > 0$ in Ω, by the Harnack type inequality of Trudinger ([**14**], Theorem 1.1).

We can write u^* in the form

$$u^*(x) = \tilde{u}(r), \qquad \text{where } r = |x|. \tag{43}$$

It is easy to check that the function $\tilde{u}$ is strictly decreasing in the interval $[0, R)$. Indeed, $\tilde{u}$ is decreasing by the definition of $\tilde{u}$. As a result, if

$$\tilde{u}(r_1) = \tilde{u}(r_2) \qquad \text{for some } 0 \le r_1 < r_2 < R, \tag{44}$$

then $\tilde{u} \equiv \text{const}$ in (r_1, r_2), and so $u^* \equiv \text{const}$ in the annulus

$$P := \{\, x \in \mathbb{R}^n \colon r_1 < |x| < r_2 \,\}.$$

Thus

$$\Delta_p(u^*) \equiv 0 \quad \text{in } P.$$

However, the right-hand side of (42) is different from zero at any point, because $\Lambda(\alpha, A) > 0$, $\Lambda(\alpha, A) - \alpha \ne 0$ by assumption, and $u^* > 0$ in Ω. This contradiction shows that (44) is impossible.

We now proceed to show that $u = u^*$. First, note that in view of (41) there is

$$\int_\Omega |\nabla u^*|^p = \int_\Omega |\nabla u|^p.$$

By Theorem 1.1 of Brothers and Ziemer [**2**], the above equality implies that $u = u^*$, provided the set

$$\{ x \in \Omega \colon \nabla u^*(x) = 0 \}$$

has measure zero. By (43) it suffices to prove that

$$|\{ r \in (0, R) \colon \tilde{u}'(r) = 0 \}| = 0. \tag{45}$$

To this end, we recall that

$$r^{n-1} \Delta_p(u^*) = \big(r^{n-1}\, \varphi_p\big(\tilde{u}'(r)\big)\big)'$$

(see for instance [**13**]). It now follows from (42) that

$$-\,\big(r^{n-1}\, \varphi_p\big(\tilde{u}'(r)\big)\big)' = r^{n-1}\, (\Lambda(\alpha, A) - \alpha\, \chi_E)\, \varphi_p(\tilde{u}(r)), \tag{46}$$

where $E := (r(A), R)$. Let us put

$$g(r) := r^{n-1}\, \varphi_p\big(\tilde{u}'(r)\big), \qquad \forall\, r \in (0, R).$$

Obviously,

$$g(r) = 0 \quad \Longleftrightarrow \quad \tilde{u}'(r) = 0, \qquad \forall\, r \in (0, R).$$

From (46) there follows that the function $g \colon (0, R) \to \mathbb{R}$ satisfies the equation

$$-\, g'(r) = r^{n-1}\, (\Lambda(\alpha, A) - \alpha\, \chi_E)\, \varphi_p(\tilde{u}(r)). \tag{47}$$

Suppose that (45) is false. Then the set $\{ g = 0 \}$ has positive measure and we deduce by Lemma 7.7 of [**7**] that $g'(r) = 0$ for almost all $r \in \{ g = 0 \}$. This contradicts equation (47), because the right-hand side is again different from zero, and the proof is complete. □

5. A related eigenvalue problem. We now state another eigenvalue problem, which is closely related to the original problem (1). In the case $p = 2$, it corresponds to the problem of building a body of prescribed shape out of given materials (of varying densities) in such a way that the body has a prescribed mass and so that the basic frequency of the resulting membrane with fixed boundary is as small as possible, as indicated in [**3**].

Given $0 \leq h < H$ and $M \in [h\, |\Omega|, H\, |\Omega|]$, consider measurable "density functions" $\varrho \colon \Omega \to \mathbb{R}$ satisfying conditions

$$h \leq \varrho \leq H \qquad \text{and} \qquad \int_\Omega \varrho = M.$$

Then the objective is to find ϱ and u which realize the minimum of the form

$$\Theta(h,H,M) := \inf\left\{ \frac{\int_\Omega |\nabla u|^p}{\int_\Omega \varrho\, |u|^p} : u \in W_0^{1,p} \setminus \{0\},\ h \le \varrho \le H,\ \int_\Omega \varrho = M \right\}. \tag{48}$$

The corresponding eigenvalue problem is

$$\begin{cases} -\Delta_p(u) = \Theta(h,H,M)\,\varrho\,\varphi_p(u) & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{49}$$

Eigenvalue problem (49) seems to be worth considering, even though it is not exactly the vibrating membrane equation. Eigenvalue problems (1), (49) and their optimization are mutually related in the following way:

Theorem 5. *Let us fix $p \in (1,\infty)$.*

(a) *If (u,ϱ) is a minimizer for problem* (49)*, then the function ϱ has the form*

$$\varrho_D = h\,\chi_D + H\,\chi_{D^c},$$

where D is a set satisfying $\{\, u < t\,\} \subset D \subset \{\, u \le t\,\}$ with some $t \ge 0$.

(b) *The pair (u,ϱ_D) is a minimizer for problem* (49)*, with parameter values (h,H,M), if and only if (u,D) is an optimal pair for problem* (1)*, with parameter values (α,A) given by*

$$\alpha = (H-h)\,\Theta(h,H,M), \qquad A = \frac{H\,|\Omega| - M}{H-h}.$$

The optimal eigenvalues are related by

$$\Lambda(\alpha,A) = H\,\Theta(h,H,M).$$

(c) *The values of (α,A) that occur when (h,H,M) vary are precisely those satisfying*

$$A \in [0,|\Omega|), \qquad\qquad 0 \le \alpha \le \overline{\alpha}(A)$$

or

$$A = |\Omega|, \qquad\qquad 0 < \alpha < \infty,$$

where $\overline{\alpha}(A)$ is defined by (20)*. In particular, $\alpha = \overline{\alpha}(A)$ corresponds to $h = 0$.*

The proof of Theorem 5 is essentially the same as of Theorem 13 in [**3**], with obvious modifications. Therefore we omit the proof.

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References

1. Adams R.A., *Sobolev Spaces*, Academic Press, New York, 1975.
2. Brothers J.E., Ziemer W.P., *Minimal rearrangements of Sobolev functions,* J. Reine Angew. Math., **384** (1988), 153–179.
3. Chanilo S., Grieser D., Imai M., Kurata K., Ohnishi I., *Symmetry breaking and other phenomena in the optimization of eigenvalues for composite membranes,* Comm. Math. Phys., **214** (2000), 315–337.
4. DiBenedetto E., $C^{1+\alpha}$ *local regularity of weak solutions of degenerate elliptic equations,* Nonlinear Analysis, **7** (1983), 827–850.
5. Drábek P., *Strongly nonlinear degenerated and singular elliptic problems,* in: Nonlinear Partial Differential Equations (Fès, 1994), 112–146, Pitman Research Notes in Mathematics Series **343**, Longman, Harlow, 1996.
6. Garcia-Melian J., de Lis S., *Maximum and comparison principles for operators involving the p-Laplacian,* J. Math. Anal. Appl., **218** (1998), 49–65.
7. Gilbarg D., Trudinger N.S., *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin, 1983.
8. Kawohl B., *Rearrangements and Convexity of Level Sets in Partial Differential Equations*, Lecture Notes in Mathematics, **1150**, Springer-Verlag, Berlin, 1985.
9. Lieb E.H., Loss M., *Analysis*, AMS Graduate Studies in Mathematics, **14**, Providence, 2001.
10. Pielichowski W., *On the first eigenvalue of a quasilinear elliptic operator,* Selected Problems of Mathematics, Anniversary Issue, **6**, Cracow University of Technology, Cracow, 1995, 235–241.
11. ———, *A nonlinear eigenvalue problem related to Gabriella Bognár's conjecture,* Studia Sci. Math. Hungar., **33** (1997), 441–454.
12. Rudin W., *Functional Analysis*, McGraw-Hill, New York, 1973.
13. de Thélin F., *Quelques résultats d'existence et de non-existence pour une E.D.P. elliptique nonlinéaire,* C. R. Acad. Sci. Paris Sér. I Math., **299** (1984), 911–914.
14. Trudinger N.S., *On Harnack type inequalities and their application to quasilinear elliptic equations,* Comm. Pure Appl. Math., **20** (1967), 721–747.

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