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THE DERIVED LIE PSEUDOALGEBRA OF AN ANCHORED MODULE

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Abstract. The aim of this paper is to construct two functors, called *the derived functors* of anchored modules which allow linear connections. Some applications are given in the cases of vector bundles with differentials and Lie algebras. It is also shown that a linear connection and a skew-symmetric form on a preinfinitesimal module lift on the derived Lie pseudoalgebra to a null curvature connection and to a closed skew-symmetric form, respectively.

The basic algebraic properties of the category of vector bundles are systematically presented in [**4**]. Using this point of view, the algebraic properties of Lie algebroids are studied in [**5**]. The abstract versions of Lie algebroids are Lie pseudoalgebras and Lie–Rinehart algebras, which are studied in [**8**] and [**6**] respectively. Using the same point of view, the first author studied in [**11**] the categories of vector bundles with differentials and their abstract versions, the modules with differentials, as extensions of Lie algebroids and Lie pseudoalgebras, respectively. As explained in [**8**] or [**11**], there are two categories of vector bundles and two categories of modules. Similarly, there are also two kinds of functors from the categories of vector bundles with differentials to the corresponding categories of modules with differentials. The aim of this paper is to construct two functors, called *the derived functors* of anchored modules which allow linear connections. At the level of objects, these two functors associate the same Lie pseudoalgebra with an anchored module which allows a linear connection. The isomorphism class of the Lie pseudoalgebra does not depend on the linear connection or on the bracket. Concerning the morphisms, the two functors are compatible with the covariant and the contravariat class of modules. Some applications concerning vector bundles with differentials and Lie algebras are given. It is also proved that a linear connection and a skew-symmetric form on a preinfinitesimal module lift on the derived Lie pseudoalgebra to a null curvature connection and to a closed skew-symmetric form, respectively.

1. The derived Lie pseudoalgebra of an anchored module.

A *module* is a couple (A, L), where A is a commutative and associative $\mathbf{k}$-algebra, and L is an A-module, where $\mathbf{k}$ is a commutative ring. An *anchored module* (or a *module with arrows* in [**15, 11**]) is a module (A, L) which has an *anchor* (or an *arrow*), i.e. an A-linear map $D : L \longrightarrow \mathcal{D}er(A)$; we denote $D(X)(a) = [X, a]_L$. A *preinfinitesimal module* is an anchored module (A, L) with the anchor D and a *bracket* $[\cdot, \cdot]$, i.e. a $\mathbf{k}$-bilinear map $[\cdot, \cdot]_L : L \times L \longrightarrow L$ which is skew symmetric and enjoys the property $[X, a \cdot Y]_L = [X, a]_L \cdot Y + a \cdot [X, Y]_L$, $(\forall) X, Y \in L$, $a \in A$. A preinfinitesimal module (A, L) is an *infinitesimal module* if the condition $[D(X), D(Y)] = D\left([X, Y]_L\right)$holds. An infinitesimal module (A, L) is a *Lie pseudoalgebra* (or a *Lie–Rinehart algebra* in [**6**]) if $\mathcal{J} = 0$, where $\mathcal{J}(X, Y, Z) = \left[[X, Y]_L, Z\right]_L + \left[[Y, Z]_L, X\right]_L + [[Z, X]_L, Y]_L$ is the Jacobiator of the bracket. Notice that if (A, L) is a preinfinitesimal module, then $\mathcal{D} : L \wedge L \longrightarrow \mathcal{D}er(A)$, $\mathcal{D}(X \wedge Y) = [D(X), D(Y)] - D\left([X, Y]_L\right)$is an anchor map for $L \wedge L$, i.e. $(A, L \wedge L)$ is an anchored module (the wedge product is the skew-symmetrization of L as an A-module). Consider the morphism $f : L \longrightarrow L'$ of A-modules. Then f is a *morphism* of anchored module if $D = D' \circ f$ and a morphism of preinfinitesimal module, respectively if $[f(X), f(Y)]_{L'} = f([X, Y]_L)$. A morphism of infinitesimal modules or of Lie pseudoalgebras over A is a morphism of preinfinitesimal modules over A. A *linear connection* ∇ on a module (A, M), related to an anchored module (A, L), or a *linear L-connection*, is a map $\nabla : L \times M \longrightarrow M$, $\nabla(X, u) \overset{not.}{=} \nabla_X u$, such that the Koszul conditions hold. If (A, L) is a preinfinitesimal module, the *curvature* of ∇ is $\nabla_{X \wedge Y} = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]_L}$. In particular, a linear L-connection on a preinfinitesimal module (A, L) can be defined. Its *torsion* is $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]_L$. The formula $[X, Y]_L = \nabla_X Y - \nabla_Y X$ defines a bracket, thus (A, L) becomes a preinfinitesimal module, through which ∇ becomes torsion free. Notice that *an anchored module which allows a linear connection is the same thing as a preinfinitesimal module which allows a torsion free linear connection.* Using the Koszul arguments (according to [**6**, Proposition 2.13]), we can prove that for a given anchored module (A, L) and a projective module (A, M), there is a linear L-connection on M. As a Corollary, we obtain that an anchored module (A, L), which is projective as a module, allows linear connections and brackets.

Notice that the curvature of a linear connection ∇ on L is a linear connection on the anchored module on $L \wedge L$ defined above. The formula $[X \wedge Y, U \wedge V]_{L \wedge L} = \nabla_{X \wedge Y}(U \wedge V) - \nabla_{U \wedge V}(X \wedge Y)$ defines a bracket on

$L \wedge L$, thus $(A, L \wedge L)$ becomes a preinfinitesimal module. The formulas $[X,Y]_{L^{(1)}} = L(X,Y) + X \wedge Y$, $[X \wedge Y, Z]_{L^{(1)}} = \nabla_{X \wedge Y} Z - \nabla_Z (X \wedge Y)$ and $[X \wedge Y, Z \wedge T]_{L^{(1)}} = \nabla_{X \wedge Y} (Z \wedge T) - \nabla_{Z \wedge T} (X \wedge Y)$ define a bracket on $L^{(1)} = L \oplus (L \wedge L)$, where the anchor is given by $D^{(1)}(X + (Y \wedge Z)) = D(X) + \mathcal{D}(Y \wedge Z)$. We call the preinfinitesimal module $(A, L^{(1)})$ the (*first*) *derived* preinfinitesimal module of (A, L), given by ∇. The following result can be proved by a straightforward computation.

Proposition 1.1. *The following properties hold true:*
1. $[D^{(1)}(X), D^{(1)}(Y)] = D^{(1)}([X,Y]_{L^{(1)}})$, $(\forall) X, Y \in L$.
2. If the linear connection ∇ *has no torsion, then* $\mathcal{J}^{(1)}(X,Y,Z) = 0$, $(\forall) X, Y, Z \in L$, *where* $\mathcal{J}^{(1)}$ *denotes the Jacobiator of* $[\cdot,\cdot]_{L^{(1)}}$.

The preinfinitesimal module $(A, L^{(1)})$, with the anchor $D^{(1)}$, the bracket $[\cdot,\cdot]_{L^{(1)}}$ and the linear connection ∇ on L, defines a torsion free linear connection $\nabla^{(1)}$ on $L^{(1)}$, according to the formulas $\nabla^{(1)}_X Y = \nabla_X Y + \frac{1}{2} X \wedge Y$, $\nabla^{(1)}_X (Y \wedge Z) = \nabla_X (Y \wedge Z)$, $\nabla^{(1)}_{X \wedge Y} Z = \nabla_{X \wedge Y} Z$, $\nabla^{(1)}_{X \wedge Y} (Z \wedge T) = \nabla_{X \wedge Y} (Z \wedge T)$. This linear connection can be used to define the *second derived preinfinitesimal module* of (A, L) as the derived ALS $(A, L^{(2)})$ of the (first) derived preinfinitesimal module $(A, L^{(1)})$. The order $p \in \mathbb{N}$ *derived preinfinitesimal module* (or the *p-derived preinfinitesimal module*), denoted as $(A, L^{(p)})$, is obtained inductively for $p \geq 2$. These p-derived preinfinitesimal modules define inductively the module $(A, L_0^{(\infty)})$. We denote as $L^{(\infty)}$ the A-submodule of $L_0^{(\infty)}$, which consists of $X_{(\infty)} = X_0 + X_1 + \cdots$ which has the property that there is an $n \in \mathbb{N}$ such that $D^{(p)}(X_p) = 0$, $(\forall) p > n$; the minimum of such n is called the *degree* of $X_{(\infty)}$ and it is denoted by $\deg X_{(\infty)}$. The anchor of $X_{(\infty)}$ is $D^{(\infty)}(X_{(\infty)}) = \sum_{i=0}^{n} D^{(i)}(X_i)$, where $n = \deg X_{(\infty)}$. Considering also $Y_{(\infty)} = Y_0 + Y_1 + \cdots \in L^{(\infty)}$, it follows that $\left[X_{(\infty)}, Y_{(\infty)}\right]_{L^{(\infty)}} = \sum_{p,q \in \mathbb{N}} [X_p, Y_q]_{L^{(\infty)}}$.

Theorem 1.1. *The couple* $(A, L^{(\infty)})$ *is a Lie pseudoalgebra.*

Notice that a Lie pseudoalgebra (A, L) is not isomorphic to $(A, L^{(\infty)})$. In order to construct a natural functor from the category of preinfinitesimal modules to the category of Lie pseudoalgebra, the above construction must be improved.

Consider now a preinfinitesimal module (A, L'), which allows a linear connection ∇, a Lie pseudoalgebra (A, L) and an anchored module morphism $f : L' \longrightarrow L$ (i.e. $a' = a \circ f$, where a' and a are the anchors on L' and L respectively). Consider the derived modules $L'^{(1)} = L' \oplus (L' \wedge L'), \ldots,$

$L'^{(k+1)} = L'^{(k)} \oplus \left(L'^{(k)} \wedge L'^{(k)}\right), \ldots$ Denote $f = f_0$ and define $f_1 : L'^{(1)} \longrightarrow L$, $f_{1|L'} = f_0$ and $f_1(X \wedge Y) = [f_0(X), f_0(Y)]_L - f_0([X,Y]_{L'})$, $(\forall)X, Y \in L'$. Thus $a \circ f_1(X \wedge Y) = a([f_0(X), f_0(Y)]_L) - a \circ f_0([X,Y]_{L'}) = [a \circ f_0(X), a \circ f_0(Y)]_L - a \circ f_0([X,Y]_{L'}) = [a'(X), a'(Y)]_L - a'([X,Y]_{L'}) = D^{(1)}(X \wedge Y)$. It follows that f_1 is an anchored module morphism. Inductively, assume that f_k is constructed, then let $f_{k+1} : L'^{(k+1)} \longrightarrow L$, $f_{k+1|L'^{(k)}} = f_k$ and $f_{k+1}(X \wedge Y) = [f_k(X), f_k(Y)]_L - f_k([X,Y]_{L'^{(k)}})$, $(\forall)X, Y \in L'^{(k)}$, which is an anchored module morphism. The maps $(f_k)_{k \in I\!N}$ define a map $f_{(\infty)} : L'^{(\infty)} \longrightarrow L$.

Proposition 1.2. *a) The map $f_{(\infty)} : L'^{(\infty)} \longrightarrow L$ is a Lie pseudoalgebra morphism.*

b) Let $f : L' \longrightarrow L$ be an anchored module morphism of two preinfinitesimal modules over the same algebra, which allow linear connections and lead to the derived modules $L'^{(\infty)}$ and $L^{(\infty)}$, respectively. Then f induces a Lie pseudoalgebra morphism $f_{(\infty)} : L'^{(\infty)} \longrightarrow L^{(\infty)}$.

It can be proved that if $i : L \longrightarrow L$ is the identity morphism of a preinfinitesimal module which allows a linear connection, then $i_{(\infty)} : L^{(\infty)} \longrightarrow L^{(\infty)}$ is a projector, i.e. $i^2_{(\infty)} = i_{(\infty)}$. Thus we have not got a functor yet. It follows that $L^{(\infty)}$ splits as $L^{(\infty)} = im(i_{(\infty)}) \oplus \ker(i_{(\infty)})$. We denote $im(i_{(\infty)})$ as L^∞ and we call it the *derived Lie pseudoalgebra* of L which corresponds to the linear connection ∇. The following result follows using the above constructions.

Theorem 1.2. *If (A, L) is an anchored module which allows linear connections, then L^∞ belongs to an isomorphism class of Lie pseudoalgebras over A, which depend neither on the linear connections nor on the brackets. The correspondences $L \longrightarrow L^\infty$ and $f \longrightarrow f_\infty$ define a covariant functor from the category of anchored modules over a commutative algebra A which allow linear connections to the category of Lie pseudoalgebras over A. It induces also a functor from the category of preinfinitesimal modules which allow linear connections to the category of Lie pseudoalgebras.*

Notice that the Lie pseudoalgebra (A, L) is a Lie pseudoalgebra quotient of (A, L^∞). Notice also that the inclusion $j : L \longrightarrow L^\infty$ is not a Lie pseudoalgebra morphism. The Lie pseudoalgebra L^∞ has the following universal property, which can be proved using its definition and Proposition 1.2.

Proposition 1.3. *If (A, L') is a Lie pseudoalgebra and $f : L \to L'$ is an anchored module morphism, then there is a unique infinitesimal module morphism $f_{(\infty)} : L^{(\infty)} \to L'$ induced as in Proposition 1.2 such that $f = f_{(\infty)} \circ i$, where $i : L \to L^{(\infty)}$ is the inclusion.*

Now we extend the functor to both the categories of anchored modules (covariant and contravariant ones). Let us denote by $\mathcal{A}$ a subcategory of the category of commutative $\mathbf{k}$-algebras or $\mathbf{k}$-rings. The *contravariant* category of modules, which we denote by $\mathcal{M}od^{\mathcal{A}}$, is considered in [**8**] and [**11**]. It has as morphisms the *con-morphisms* $(A', L') \stackrel{(\varphi,\psi)}{\longrightarrow} (A, L)$, where $\varphi : A \longrightarrow A'$ is a morphism of commutative rings (or algebras) and $\psi : L' \longrightarrow A' \otimes_A L$ is an A'-module morphism. If $(A'', L'') \stackrel{(\varphi',\psi')}{\longrightarrow} (A', L') \stackrel{(\varphi,\psi)}{\longrightarrow} (A, L)$ are morphisms of modules, then $(\varphi, \psi) \circ (\varphi', \psi') = (\varphi \circ \varphi', \bar{\psi} \circ \psi')$, where $\bar{\psi}$ is the natural A''-module morphism $\bar{\psi} : A'' \otimes_{A'} L' \longrightarrow A'' \otimes_A L$ induced by the A'-module morphism $\psi : L' \longrightarrow A' \otimes_A L$; explicitly, if $\psi(X') = \sum_i a'_i \otimes_A X_i$, then $\bar{\psi}(a'' \otimes_{A'} X') = \sum_i a'' \varphi'(a'_i) \otimes_A X_i$. We have used the name *contravariant* for the category of modules used above in order to make a distinction from the usual category of modules $\mathcal{M}od_{\mathcal{A}}$, which is called covariant. A *covariant* category of modules is defined to have as morphisms the *cov-morphisms* denoted also by $(A', L') \stackrel{(\varphi,\psi)}{\longrightarrow} (A, L)$, where $\varphi : A' \longrightarrow A$ is a morphism of commutative rings (or algebras) and $\psi : L' \longrightarrow L$ is an A'-module morphism. Notice that a co(ntra)variant category of modules need not to contain all the modules. It can be defined using functors as in [**11**].

Let $\mathcal{C}$ be a category and $\mathcal{C}M$ be a contravariant or a covariant category of modules. A *covariant pseudofunctor* on $\mathcal{C}$ to $\mathcal{C}M$ is a couple $F' = (F'_0, F'_1)$, where $F'_0 : \mathcal{O}b(\mathcal{C}) \longrightarrow \mathcal{O}b(\mathcal{C}M)$ and $F'_1 : \mathcal{H}om_{\mathcal{C}} \longrightarrow \mathcal{H}om_{\mathcal{C}M}$ are maps such that if $f \in \mathcal{H}om_{\mathcal{C}}(U, V)$ and $g \in \mathcal{H}om_{\mathcal{C}}(V, W)$, then $F'_1(f) : F'_0(U) \longrightarrow F'_0(V)$, $F'_1(g) : F'_0(V) \longrightarrow F'_0(W)$ and $F'_1(g \circ f) = F'_1(g) \circ F'_1(f)$. For every $U \in \mathcal{O}b(\mathcal{C})$, denote by $i_U \in \mathcal{H}om_{\mathcal{C}}(U, U)$ the identity morphism. A pseudofunctor misses to be a functor because the condition $F'_1(i_U) = id_{F'_0(U)}$ is not fulfilled. This obstacle can be very easy removed. From $F'_1(i_U) \circ F'_1(i_U) = F'_1(i_U)$ it follows that $F'_1(i_U) : F'_0(U) \longrightarrow F'_0(U)$ is a projector; denote by $F_0(U)$ the module which is its image and by $I_U : F_0(U) \longrightarrow F'_0(U)$ the inclusion morphism. For every morphism $f \in \mathcal{H}om_{\mathcal{C}}(U, V)$, put $F_1(f) = F'_1(i_V) \circ F'_1(f) \circ I_U$. It follows that the couple $F = (F_0, F_1)$ is a covariant functor on the category $\mathcal{C}$ to the category of modules $\mathcal{C}M$. We call F the *canonical functor* associated with the pseudofunctor F'. Notice that the above construction can be adapted by duality to the contravariant case.

Now we return to anchored modules (or modules with arrows of [**11**]). Let (A', L') and (A, L) be two anchored modules. A *con-morphism of anchored module* is a couple (φ, ψ), denoted $(A', L') \stackrel{(\varphi,\psi)}{\longrightarrow} (A, L)$, where $\varphi : A \longrightarrow A'$ is an algebra morphism and $\psi : L' \longrightarrow A' \otimes_A L$ is an A'-module morphism, such that for every $X' \in L'$ which has the ψ-decomposition $\psi(X') = \sum_i a'_i \otimes X_i$, $[X', \varphi(a)]_{L'} = \sum_i a'_i \varphi([X_i, a]_L)$. (We have used the notation $[X, a]_L = D(X)(a)$, where D is the anchor of the anchored module (A, L) and $X \in L$, $a \in A$.) A *cov-morphism of anchored modules* is a couple (φ, ψ), denoted also $(A', L') \underset{(\varphi,\psi)}{\rightarrow} (A, L)$, where $\varphi : A' \longrightarrow A$ is an algebra morphism and $\psi : L' \longrightarrow L$ is an A'-module morphism, such that for every $X' \in L'$ and $a' \in A'$, $[\psi(X'), \varphi(a')]_L = \varphi([X', a']_{L'})$.

Now consider two preinfinitesimal modules (A', L') and (A, L). A *con-morphism of preinfinitesimal module* (defined in [**11**]) is a con-morphism of anchored modules $(A', L') \overset{(\varphi,\psi)}{\longrightarrow} (A, L)$ such that for every X', $Y' \in L'$ with the ψ-decompositions $\psi(X') = \sum_i a'_i \otimes X_i$ and $\psi(Y') = \sum_\alpha b'_\alpha \otimes Y_\alpha$ respectively, $\psi([X', Y']_{L'}) = \sum_i a'_i b'_\alpha \otimes [X_i, Y_\alpha] + \sum_\alpha [X', b'_\alpha]_{L'} \otimes Y_\alpha - \sum_i [Y', a'_i]_{L'} \otimes X_i$. A *cov-morphism of preinfinitesimal module* [**11**] is a cov-morphism of anchored module $(A', L') \underset{(\varphi,\psi)}{\rightarrow} (A, L)$ such that for every X', $Y' \in L'$ the condition $\psi([X', Y']_{L'}) = [\psi(X'), \psi(Y')]_L$ is fulfilled. Morphisms of infinitesimal modules and Lie pseudoalgebras are the very morphisms of preinfinitesimal module structures, forgetting the restrictive conditions for the brackets. In the sequel we call a cov-morphism or a con-morphism of modules with differentials (i.e. AM–anchored modules, PM–preinfinitesimal modules or IM–infinitesimal modules) simply a morphism of modules with differentials. We can prove the following result, which extends Theorem 1.2:

THEOREM 1.3. *If (A, L) is an anchored module which allows linear connections, then L^∞ belongs to an isomorphism class of Lie pseudoalgebras, which do not depend on the linear connections or the brackets. The correspondence $(A, L) \longrightarrow (A, L^\infty)$, $(\varphi, \psi) \longrightarrow (\varphi, \psi_\infty)$ and $(A, L) \longrightarrow (A, L^\infty)$, $(\varphi, \psi) \longrightarrow (\varphi, \psi^\infty)$ defines two covariant functors from the covariant and contravariant categories of anchored modules over commutative algebras which allow linear connections to the corresponding categories of Lie pseudoalgebras (the covariant and the contravariant one, respectively). It induces also functors from the covariant and contravariant categories of preinfinitesimal modules which allow linear connections to the categories of Lie pseudoalgebras.*

A particular case is when the anchor on L is null. In this case one has an infinitesimal module structure given by a skew-symmetric map $b : L \wedge L \longrightarrow L$. Moreover, the Lie pseudoalgebra L^∞ is a Lie algebra and it does not depend on b. As in the case of modules over the same base, the module $(A, L^{(\infty)})$ has the following universal property:

PROPOSITION 1.4. *If (A', L') is a Lie pseudoalgebra and $(A, L) \overset{(\varphi,\psi)}{\longrightarrow} (A', L')$ (or $(A, L) \underset{(\varphi,\psi)}{\rightarrow} (A', L')$) is an anchored module con-morphism (cov-morphism),* *then there is a unique Lie pseudoalgebra con-morphism* $(A, L) \overset{(\varphi,\psi^{\infty})}{\longrightarrow} (A', L')$ *(or cov-morphism* $(A, L) \underset{(\varphi,\psi_{\infty})}{\rightarrow} (A', L')$*), such that* $(\varphi, \psi) = (\varphi, \psi^{\infty}) \circ (id_A, i)$ *(or* $(\varphi, \psi) = (\varphi, \psi_{\infty}) \circ (id_A, i)$*), where* $(id_A, i) : (A, L) \rightarrow (A, L^{(\infty)})$ *is the inclusion.*

If (A, L') is an anchored submodule of (A, L), then it is natural to ask if there is a similar relation between the derived (Lie) algebroids.

THEOREM 1.4. *If* (A, L') *is an anchored submodule of* (A, L) *and* L *allows a linear connection* ∇ *such that* $\nabla_{X'} Y' \in L'$, $(\forall) X', Y' \in L'$, *then* $(A, L'^{(\infty)})$ *is a Lie sub-pseudoalgebra of* $(A, L^{(\infty)})$ *and* (A, L'^{∞}) *is a Lie sub-pseudoalgebra of* (A, L^{∞}). *In particular, if* (A, L') *is an anchored submodule of* (A, L) *such that the inclusion* $i : L' \rightarrow L$ *splits and* L *allows a linear connection* ∇, *then the same conclusion holds.*

2. Vector bundles with differentials.

An *anchored vector bundle (AVB),* (or a relative tangent space of [**9, 10, 11**]) is a couple (θ, D), where $\theta = (R, q, M)$ is a vector bundle and $D : \theta \longrightarrow \tau M$ is a vector bundle morphism, called an *anchor*. A triple $(\theta, D, [\cdot, \cdot]_\theta)$ is an *almost Lie structure* (ALS) if (θ, D) is an AVB and $[\cdot, \cdot]_\theta : \Gamma(\theta) \times \Gamma(\theta) \longrightarrow \Gamma(\theta)$ is an *almost Lie map* (or a *bracket*), i.e. an $I\!R$-linear and skew symmetric map which enjoys the property $[X, f \cdot Y]_\theta = (DX)(f) \cdot Y + f \cdot [X, Y]_L$, $(\forall) X, Y \in \Gamma(\theta)$ and $f \in \mathcal{F}(M)$ (cf. [**9**]). We denote as $\mathcal{J}(X, Y, Z) = [[X, Y]_\theta, Z]_\theta + [[Y, Z]_\theta, X]_\theta + [[Z, X]_\theta, Y]_\theta$, which we call the *Jacobiator* of the bracket. If $(\theta, D, [\cdot, \cdot]_\theta)$ is an ALS, then $\mathcal{D} : \Gamma(\theta \wedge \theta) \longrightarrow \mathcal{X}(M)$, $\mathcal{D}(X \wedge Y) = [D(X), D(Y)] - D([X, Y]_\theta)$ is an anchor map for $\theta \wedge \theta$, i.e. $(\theta \wedge \theta, \mathcal{D})$ is an AVB. An *algebroid* is an ALS for which $\mathcal{D} = 0$ and a *Lie algebroid* is an algebroid for which $\mathcal{J} = 0$.

We give now two examples which motivate the reason to enlarge the study of (Lie) algebroids. The following example was originally considered by Sussmann in [**17**]. Consider the subalgebra $\mathcal{A}_0 \subset \mathcal{F}(I\!R)$ of real smooth function which are null for the negative reals and increase for the positive reals. Denote by $\mathcal{M}_0 \subset \mathcal{X}(I\!R^2)$ the submodule of vector fields $X = f(x, y)\frac{\partial}{\partial x} + \varphi(x) g(x, y)\frac{\partial}{\partial y}$, where $\varphi \in \mathcal{A}_0$. Then $\mathcal{M}_0$ is neither integrable, nor finitely generated. The following example shows that there are integrable (thus involutive) distributions which are not finitely generated. For a smooth real function ψ which is null together with all its derivatives, only in 0, consider the submodule $\mathcal{M}_\psi \subset \mathcal{X}(I\!R^2)$ constructed as in the previous example, where φ is ψ or a derivative $\psi^{(n)}$, $n \geq 0$. Neither $\mathcal{M}_0$, nor $\mathcal{M}_\psi$ can be isomorphic with a module $D(\Gamma(\theta))$, where $(D, \theta, [\cdot, \cdot]_\theta)$ is a (Lie) algebroid.

For an AVB (θ, D), the couple $(\mathcal{F}(M), \Gamma(\theta))$ is an anchored module. If $(\theta, D, [\cdot,\cdot]_\theta)$ is an ALS, then the couple $(\mathcal{F}(M), \Gamma(\theta))$ is a preinfinitesimal module. If $(\theta, D, [\cdot,\cdot]_\theta)$ is an algebroid or a Lie algebroid, then $(\mathcal{F}(M), \Gamma(\theta))$ is an infinitesimal module or a Lie pseudoalgebra, respectively. These correspondences are functorial (see [**11**]). All the previous constructions can be adapted for vector bundles with differentials (AVB, ALS, algebroids and Lie algebroids). For example, given an AVB (θ, D) and a vector bundle ξ over the same base as θ, a *linear R-connection* on ξ, related to the AVB (θ, D), is a linear connection on $(\mathcal{F}(M), \Gamma(\xi))$, related to the anchored module $(\mathcal{F}(M), \Gamma(\theta))$.

In the case when $(\theta, D, [\cdot,\cdot]_\theta)$ is a Lie algebroid, then $(\mathcal{F}(M), \Gamma(\theta))$ is a Lie pseudoalgebra, the cohomology of this Lie pseudoalgebra is the *cohomology of the Lie algebroid*. In the case when $\theta = \tau M$ is the tangent bundle of the differentiable manifold M, the cohomology of the Lie pseudoalgebra $(\mathcal{F}(M), \tau M)$ is the de Rham cohomology of M. The cohomology of the Lie pseudoalgebra $(\mathcal{F}(M), \mathcal{X}(M)^\infty)$ can be viewed as the *generalized de Rham cohomology* of the manifold M.

There are two categories which have as objects the vector bundles, but different morphisms. One of these categories of vector bundles is the usual one, when the morphisms are the usual morphisms of vector bundles $\xi' \xrightarrow{(g,f)} \xi$: if $\xi' = (E', \pi', M')$ and $\xi = (E, \pi, M)$, $g : M' \longrightarrow M$ are the vector bundles and $f : E' \longrightarrow E$ are such that $g \circ \pi' = \pi \circ f$ and f restricted to fibres, $f_{|\pi^{-1}(x')} : \pi'^{-1}(x') \longrightarrow \pi^{-1}(g(x'))$, is linear. The other category of vector bundles has as morphisms the comorphisms of vector bundles $\xi' \underset{(g,f)}{\rightarrow} \xi$, $g : M' \longrightarrow M$ and $f : E \longrightarrow E'$ are such that $g \circ \pi' \circ f = \pi$ and f restricted to fibres, $f_{|\pi^{-1}(g(x'))} : \pi^{-1}(g(x')) \longrightarrow \pi'^{-1}(x')$, is linear. Notice that for the vector bundles over the same base, where g is the identity of the base, the two categories of vector bundles have the same morphisms. One can also use the module morphisms of sections. (See [**4**] and [**7**] for more details.) If $\xi = (E, \pi, M)$ is a vector bundle, then $(\mathcal{F}(M), \Gamma(\xi))$ is a module. A morphism of vector bundles $\xi' \xrightarrow{(g,f)} \xi$ defines and it is defined by a con-morphism of modules $(\mathcal{F}(M'), \Gamma(\xi')) \xrightarrow{(g^*,f_*)} (\mathcal{F}(M), \Gamma(\xi))$, i.e. $g^* : \mathcal{F}(M) \longrightarrow \mathcal{F}(M')$, $g^*(u) = u \circ g$ and $f_* : \Gamma(\xi') \longrightarrow \mathcal{F}(M') \otimes_{\mathcal{F}(M)} \Gamma(\xi)$, which is a morphism of $\mathcal{F}(M')$-modules. Notice that there is an isomorphism of modules $\mathcal{F}(M') \otimes_{\mathcal{F}(M)} \Gamma(\xi) \cong \Gamma(f^*\xi)$. A comorphism of vector bundles $\xi' \underset{(g,f)}{\rightarrow} \xi$ defines and it is defined by a cov-morphism of modules $(\mathcal{F}(M'), \Gamma(\xi')) \underset{(g^*,f_*)}{\rightarrow} (\mathcal{F}(M), \Gamma(\xi))$, i.e. $g^* : \mathcal{F}(M) \longrightarrow \mathcal{F}(M')$, $g^*(u) = u \circ f_0$ and $f^* : \Gamma(\xi) \longrightarrow \Gamma(\xi')$, which is a morphism of $\mathcal{F}(M)$-modules.

Let (θ', D') and (θ, D) be two anchored vector bundles over the bases M' and M, respectively. A *comorphism of anchored vector bundles* $(\theta', D') \underset{(g,f)}{\rightarrow} (\theta, D)$ is a comorphism of vector bundles $\theta' \underset{(g,f)}{\rightarrow} \theta$ (or the cov-morphism of modules $(\mathcal{F}(M'), \Gamma(\theta')) \underset{(g^*,f^*)}{\rightarrow} (\mathcal{F}(M), \Gamma(\theta)))$ such that if $X \in \Gamma(\theta)$ and $u \in \mathcal{F}(M)$, then $D(X)(u) = D'(f^*(X))(g^*(u))$, or $g^*([X,u]_\theta) = [f^*(X), g^*(u)]_{\theta'}$. A *morphism of anchored vector bundles* $(\theta', D') \xrightarrow{(g,f)} (\theta, D)$ is a morphism of vector bundles $\theta' \xrightarrow{(g,f)} \theta$ (or a con-morphism of modules $(\mathcal{F}(M'), \Gamma(\theta')) \xrightarrow{(g,f_*)} (\mathcal{F}(M), \Gamma(\theta)))$ such that if $u \in \mathcal{F}(M)$ and $X' \in \Gamma(\theta')$ allows the decomposition

$$f_*(X') = \sum_i a'_i \otimes_{\mathcal{F}(M)} X_i \in \mathcal{F}(M') \otimes_{\mathcal{F}(M)} \Gamma(\theta), \tag{1}$$

then $[X', g^*(u)]_{\theta'} = \sum_i a'_i \cdot g^*([X_i, u]_\theta)$.

If $(\theta', D', [\cdot,\cdot]_{\theta'})$ and $(\theta, D, [\cdot,\cdot]_\theta)$ are ALS's, then a *comorphism of almost Lie structure* is a comorphism of the anchored vector bundles, such that $[f^*(X), f^*(Y)]_{\theta'} = f^*([X,Y]_\theta)$, $(\forall) X, Y \in \Gamma(\theta)$. A *morphism of almost Lie structure* is a morphism of anchored vector bundles, such that $(\forall) X', Y' \in \Gamma(\theta')$ which allow the decompositions (1) and $f_*(Y') = \sum_\alpha a'_\alpha \otimes_{\mathcal{F}(M)} Y_\alpha \in \mathcal{F}(M') \otimes_{\mathcal{F}(M)} \Gamma(\theta)$, respectively, then $f_*([X', Y']_{\theta'}) = \sum_\alpha [X', b'_\alpha]_{\theta'} \otimes_{\mathcal{F}(M)} Y_\alpha - \sum_i [Y', a'_i]_{\theta'} \otimes_{\mathcal{F}(M)} X_i, + \sum_{i,\alpha} a'_i b'_\alpha \otimes_{\mathcal{F}(M)} [X_i, Y_\alpha]$. Morphisms of algebroids and (Lie) algebroids are the very morphisms of almost Lie structure, forgetting the restrictive conditions for brackets. In the sequel, we consider comorphisms and morphisms of vector bundles with differentials (i.e. anchored vector bundle, almost Lie structure, algebroid or Lie algebroid), simply called (co)morphisms. Since every AVB allows linear connections (see [**9**]), it follows that we can compose the AVB-functors with the corresponding functors $^\infty$ and $_\infty$, respectively. Thus the following result follows.

Theorem 2.1. *There are two covariant functors from the two categories of anchored vector bundles to the two categories of Lie pseudoalgebras.*

Let $(M, \mathcal{D})$ be a regular distribution (particularly, in the integrable case, it can be the tangent bundle $\tau\mathcal{F}$ of a foliation $\mathcal{F}$) on a manifold M, and denote by $\nu\mathcal{D} = \tau M/\mathcal{D}$ its normal bundle. Consider an AVB structure $(\nu\mathcal{D}, D)$ given in the following way: take a Riemannian metric on τM, denote as $\mathcal{D}^\perp$ the orthogonal complenent of $\mathcal{D}$ with respect this metric, as $\phi : \nu\mathcal{D} \longrightarrow \mathcal{D}^\perp$ the canonical vector bundle isomorphism and as $\iota : \mathcal{D}^\perp \longrightarrow \tau M$ the inclusion vector bundle morphism, then take $D = \iota \circ \phi$, $D : \nu\mathcal{D} \longrightarrow \tau M$. Not all the

AVB's obtained in this way are isomorphic, so consider the isomorphic class $\mathcal{D}^{\perp}$ of such AVB's. Applying the derived functor to $\mathcal{D}^{\perp}$, we get a Lie pseudoalgebra, which depends only on the isomorphic class $\mathcal{D}^{\perp}$. It can be called a *transversal Lie pseudoalgebra* of the distribution. A *tangent Lie pseudoalgebra* can be obtained using the AVB defined by the distribution $\mathcal{D}$ itself.

3. Constructions related to the derived Lie pseudoalgebra of an anchored module.

This section contains two applications of the derived Lie pseudoalgebra of an anchored module. The first application shows the existence of an universal Lie algebra L_{∞}^{∞}, such that every at most countable (in particularly finitely) generated Lie algebra over a commutative algebra A is a quotient Lie algebra of L_{∞}^{∞}. The second application is dealing with lifts of linear connections and differential forms to curvature free connections and closed forms, respectively.

3.1. *An application to Lie algebras.* A Lie algebra is a particular Lie pseudoalgebra, which has a null anchor. In this section we consider an application for Lie algebras over a commutative algebra A. Every module is an anchored module, taking a null anchor, and allows a null linear connection. Thus the results obtained for anchored modules which allow linear connections can be adapted for arbitrary modules. In this paper we give only an application using the universal property of the derived Lie pseudoalgebra of an anchored module. All the modules in this subsection are over a commutative algebra A.

Let us suppose that a Lie algebra L'_n has a finite number a generators $\mathcal{G}_n = \{e_i\}_{i=\overline{1,n}}$. Let L_n be the free module generated by $\mathcal{G}_n$, $f : L_n \to L'_n$ be the canonical morphism and L_n^{∞} be the derived Lie algebra of L_n, viewed as an A-module. Using Proposition 1.3, a morphism $f_{\infty} : L_n^{\infty} \to L'_n$ is induced. This morphism is surjective, since $\mathcal{G}_n$ are generators. It follows that L'_n is a quotient of L_n^{∞}. In the same way it follows that every Lie algebra which has n generators is a quotient Lie algebra of L_n^{∞}. If L'_{n+1} is a Lie algebra which has $n+1$ generators, then let us consider as before $\mathcal{G}_{n+1} = \{e_i\}_{i=\overline{1,n+1}}$ a system of generators, let L_{n+1} be the free module generated by $\mathcal{G}_{n+1}$ and L_{n+1}^{∞} be the derived Lie algebra of L_{n+1}. Then L_n is a submodule of L_{n+1}, thus using Theorem 1.4, it follows that L_n^{∞} is a Lie subalgebra of L_{n+1}^{∞}. Since $\mathcal{G}_{n+1}$ is also a system of generators for L_n, it follows that L'_n is also a Lie algebra quotient of L_{n+1}^{∞}. Let us denote as L_{∞}^{∞} the derived Lie algebra of a module generated by a countable set $\mathcal{G}_{\infty} = \{e_i\}_{i \in I\!N}$. In the same way as above it follows that L'_n is also a Lie algebra quotient of L_{∞}^{∞}.

We can summarize this construction in the following result:

Theorem 3.1. *Given a commutative algebra A, there is an ascending sequence of Lie algebras over A : $L_1^{\infty} \subset L_2^{\infty} \subset \cdots \subset L_n^{\infty} \subset \cdots \subset L_{\infty}^{\infty}$, such that* *every Lie algebra L'_n which has n generators is a quotient of every Lie algebra L_k^∞ with $k \geq n$.*

Every Lie algebra over A which has a most countable number of generators is a quotient Lie algebra of the Lie algebra L_∞^∞.

3.2. *Lifts of linear connections and skew symmetric forms.* In this subsection all the modules are defined over a commutative algebra A and L is a preinfinitesimal module over A, which allows a linear connection.

Proposition 3.1. *Let L' be a module, $\nabla : L \times L' \longrightarrow L'$ be a linear L-connection.*

Then there are linear connections $\nabla^{(\infty)} : L^{(\infty)} \times L' \longrightarrow L'$ and $\nabla^\infty : L^\infty \times L' \longrightarrow L'$ which are curvature free and extend ∇.

We say that the linear connections $\nabla^{(\infty)}$ and ∇^∞ are the *lifts* of ∇. It is possible to interpret the result obtained above in terms of representations of anchored modules and Lie pseudoalgebras. In fact, if (A, L) is an anchored module and (A, L') is another module, then a *pre-representation* of L on L' is a linear connection on L' related to L. If (A, L) is a Lie pseudoalgebra, then a *representation* of L on L' is a pre-representation which is defined by a linear connection which is curvature free. Using the terminology of [**6**], (A, L) is a Lie–Rinehart algebra and (A, L') is an (A, L)*-module* (or a *Lie–Rinehart module*). Proposition 3.1 can be reformulated as follows: *Every pre-representation of an anchored module (A, L) on a module (A, L') lifts to representations of the Lie pseudoalgebras $L^{(\infty)}$ and L^∞ on the module L' (i.e. L' is an $(A, L^{(\infty)})$-module, as well as an (A, L^∞)-module).*

Let (A, L) be a preinfinitesimal module. For $k \geq 0$, let $\Omega^k(L)$ be the module of skew symmetric and A-linear maps $\omega : L^k \longrightarrow A$. Define $d_L : \Omega^k(L) \longrightarrow \Omega^{k+1}(L)$: $(d_L\omega)(X_0, \dots, X_k) = \sum_i (-1)^i [X_i, \omega(X_1, \dots, \hat{X}_i, \dots, X_k)]_L + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j]_L, X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_k)$. It is well-defined and satisfies $d_L^2 = 0$ iff L is a Lie pseudoalgebra. If it is the case, then the cohomology of $(\Omega^\bullet(L), d_L)$ is called the *Lie pseudoalgebra cohomology* of L (with trivial coefficients), and it is denoted by $H^\bullet(L)$.

Let (A, L) be a preinfinitesimal module, (A, M) be a module and ∇ be a curvature free linear L-connection on M. Let $\Omega^k(L, M)$ be the module of skew symmetric and A-linear maps $\omega : L^k \longrightarrow M$, called the space of "$k$-forms" on L with values in M. Define $d_{L,M} : \Omega^k(L, M) \longrightarrow \Omega^{k+1}(L, M)$,

$$\begin{aligned}(d_{L,M}\omega)(X_0, \dots, X_k) = &\sum_{i=0}^{k} (-1)^i \nabla_{X_i} \left(\omega\left(X_0, \dots \widehat{X_i}, \dots X_k\right)\right) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_k)\end{aligned}$$

for $k \geq 0$. If (A, L) is a Lie pseudoalgebra, then $d^2_{L,M} = 0$. In this case the cohomology of $(\Omega^*(L, M),\ d_{L,M})$ is called the *cohomology of L with coefficients in M*.

Let (A, L) be an anchored module which allows a linear connection, (A, M) be a module and ∇ be a linear L-connection on M. According to Proposition 3.1, the linear L-connection ∇ lifts to a linear L^∞-connection ∇^∞, which is curvature free. It follows that we can consider the cohomology of L^∞ with coefficients in M, related to ∇^∞.

A special case is when $M = A$, considered as a module over A, $\nabla_X a = [X, a]_L$, $(\forall) X \in L$ and $a \in A$. Then the curvature of ∇ is $\nabla_{X\wedge Y} a = \mathcal{D}(X, Y)(a)$; it is null only if L is an infinitesimal module. The lifted connection ∇^∞ is $\nabla^\infty_{X_\infty} a = D^\infty(X_\infty)(a)$, $(\forall) X_\infty \in L^\infty$ and $a \in A$. It follows that the cohomology of L^∞ with coefficients in A, related to this connection, is just the Lie pseudoalgebra cohomology of L^∞. Another special case is when (A, L) is an arbitrary module and the anchor on L is null. In this case there is the null linear connection on L. It follows that L^∞ is a Lie algebra over A and any pre-representation of L on an other module (A, M) (i.e. an A-bilinear map $\nabla : L \times M \to M$) lifts to a representation of the Lie algebra L^∞ on M.

Now (A, L) is an anchored module which allows a linear connection. Consider a module (A, M) and let ∇ be a linear L-connection on M. Let $\omega \in \Omega^1(L, M)$ be a 1-form on L, which takes values in M. Denote $\omega = \omega_{(0)}$ and define $\omega_{(1)} \in \Lambda^1(L^{(1)})$ by $\omega_{(1)}(X) = \omega_{(0)}(X)$, $\omega_{(1)}(X \wedge_1 Y) = d_{L^{(0)}}\omega_{(0)}(X, Y)$, $(\forall) X, Y \in L = L^{(0)}$. Inductively, assume that $\omega_{(k)}$ is defined for $k \geq 1$, then define $\omega_{(k+1)} \in \Omega^1(L^{(k+1)}, M)$ by $\omega_{(k+1)}(X) = \omega_{(k)}(X)$, $\omega_{(k+1)}(X \wedge_{k+1} Y) = d_{L^{(k)}}\omega_{(k)}(X, Y)$, $(\forall) X, Y \in L^{(k)}$. These forms define $\omega_{(\infty)} \in \Omega^1(L^{(\infty)}, M)$ which in its turn defines $\omega_\infty \in \Omega^1(L^\infty, M)$ by restriction. It can be shown that the forms $\omega_{(\infty)}$ and ω_∞ are closed, i.e. $d_{(\infty)}\omega_{(\infty)} = 0$ and $d_\infty \omega_\infty = 0$. This construction can be extended to any p-form. It can be proved that for $p \geq 1$ any p-form $\omega \in \Omega^p(L, M)$ can be extended canonically to closed p-forms $\omega_{(\infty)} \in \Omega^p(L^{(\infty)}, M)$ and $\omega_\infty \in \Omega^p(L^\infty, M)$. We say that the p-forms $\omega_{(\infty)}$ and ω_∞ are the *lifts* of the p-form ω.

Now consider on A, viewed as an A-module, the linear L-connection $\nabla_X a = [X, a]_L$, $(\forall) X \in L$, $a \in A$. Every p-form $\omega \in \Omega^p(L, A)$ defines the p-forms $\tilde{\omega} \in \Omega^p(L^\infty, A)$ and $\tilde{\omega}' \in \Omega^p(L^{(\infty)}, A)$, $\tilde{\omega} = \pi^*\omega$ and $\tilde{\omega}' = \pi_0^*\omega$, where $\pi : L^\infty \longrightarrow L$ and $\pi_0 : L^{(\infty)} \longrightarrow L$ are the canonical projections viewed as module morphisms. It is easy to see that $\tilde{\omega} = \omega_\infty$ or $\tilde{\omega}' = \omega_{(\infty)}$ iff $d\omega = 0$. Consistent examples can be obtained in the case of an ALS $(\theta, D, [\cdot, \cdot])$. In this case, the lift of a form $\omega \in \Lambda^{top}(\theta^*) = \Omega^{top}(\Gamma(\theta), \mathcal{F}(M))$ is $\tilde{\omega}$ since $d_\theta \omega = 0$.

In the case when (A, L) is a Lie pseudoalgebra, the lifts induce some natural morphisms of additive groups $H^*(L, A) \to H^*(L^{(\infty)}, A)$ and $H^*(L, A) \to H^*(L^\infty, A)$, where $H^*(L, A)$ is the cohomology of the Lie pseudoalgebra (A, L). When $(A, L) = (\mathcal{F}(M), \mathcal{X}(M))$, these morphisms relate the de Rham groups of the manifold M to the cohomologies of the Lie pseudoalgebras $(\mathcal{F}(M), \mathcal{X}(M)^{(\infty)})$ and $(\mathcal{F}(M), \mathcal{X}(M)^\infty)$, respectively.

Notice also that in the case when (A, L) is a Lie algebra defined by a null anchor, then also the lifts induce some natural morphisms of additive groups $H^*(L, L) \to H^*(L^{(\infty)}, L)$ and $H^*(L, L) \to H^*(L^\infty, L)$, where $H^*(L, L)$ is the cohomology of the Lie algebra (A, L). The case $p = 1$ is quite different from the case $p > 1$. It can be proved that an exact 1-form $\omega \in \Omega^1(L^\infty, M)$ has an exact lift ω_∞ (i.e. $\omega_\infty = d_\infty f$ iff $\omega = d_L f$). Notice that an analogous result for a p-form $\omega \in \Omega^p(L^\infty, M)$, $p \geq 2$, seems to fail.

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