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AN ABSTRACT SEMILINEAR FIRST ORDER DIFFERENTIAL EQUATIONS IN THE HYPERBOLIC CASE

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Abstract. Using the extrapolation spaces, the existence and uniqueness of the solution of a semilinear first order equation in the hyperbolic case are studied.

1. Introduction. Let (X,$\|\cdot\|$) be a Banach space and for each $t \in [0,T]$ let $A(t) : X \supset D_t \to X$ be a linear closed operator with domain D_t dependent on t. Let u be an unknown function from $[0,T]$ into X, f be a nonlinear function from $[0,T] \times X$ into X and $x_0 \in X$. We consider the abstract semilinear initial value problem

$$\begin{cases} u'(t) = A(t)u(t) + f(t,u(t)), & t \in (0,T] \\ u(0) = x_0 \in X. \end{cases} \tag{1}$$

Our purpose is to study the existence and uniqueness of solution of (1). First we shall reduce problem (1) to a problem with densely defined operator whose domain can depend on t. Next, using the same method as in [**4**], we shall introduce the extrapolation space and reduce our problem to the problem with an operator whose domain is independent of t.

2. Preliminaries. Let $(X, \|\cdot\|)$ be a Banach space. Let for each $t \in [0,T]$, $\rho(A(t))$ denote the resolvent set and $R(\lambda, A(t)) = (\lambda I - A(t))^{-1}$, $\lambda \in \rho(A(t))$ be the resolvent of $A(t)$. We make the following assumptions:

(Z_1) For each $t \in [0,T]$, $A(t) : X \supset D_t \to X$ is a closed densely defined linear operator with the domain D_t dependent on t.

(Z_2) The resolvent set $\rho(A(t))$ does not depend on t and 0 belongs to $\rho(A(t))$.

(Z_3) The family $\{A(t)\}$, $t \in [0,T]$, is stable in the sense that there exist real numbers $M \geq 1$ and ω such that

$$\| \prod_{j=1}^{k} R(\lambda, A(t_j)) \| \leq M(\lambda - \omega)^{-k}$$

for all $\lambda > \omega, 0 \leq t_1 \leq ... \leq t_k \leq T, k \in \mathbb{N}$.

(Z_4) For each $x \in X$, the function $[0,T] \ni t \to R(\lambda, A(t))x \in X$ is of class C^1.

(Z_5) For each t, $s \in [0,T]$ the operator $A^{-1}(t)A(s)$ is closable and for each fixed $s \in [0,T]$ the mapping $t \to \overline{A^{-1}(t)A(s)}$ is continuous in $t = s$ on [0,T] in the sense that $\lim_{t \to s} \|\overline{A^{-1}(t)A(s)} - I\| = 0$.

From the Hille–Yosida Theorem ([**3**], Th.1.5.3) and from (Z_1), (Z_3) it follows that for each $t \in [0,T]$, $A(t)$ is the generator of a C_0-semigroup on X.

For each fixed $\mu \in \rho(A(t))$

$$|x|_t := \|R(\mu, A(t))x\|, \qquad x \in X, \quad t \in [0,T] \tag{2}$$

defines a new norm on X.

THEOREM 2.1. ([**4**], Th.3.1). *If assumptions* (Z_1)–(Z_5) *hold then for each* $t \in [0,T]$ *the norms* $|\cdot|_0$ *and* $|\cdot|_t$ *are equivalent.*

We remark that from Theorem 2.1 it follows that $X_0 := (X, |\cdot|_0)$ is not a Banach space. Since X_0 is the normed space, we can complete it in the sense of norm $|\cdot|_0$ to the complete space $\hat{X}_0$. The extrapolation space $\hat{X}_0$ is a Banach space and does not depend on t.

Next, for each $t \in [0,T]$, we extend $A(t)$. We denote by $\hat{A}(t)$ the extension of $A(t)$ with domain $D(\hat{A}(t)) = X$ independent of t and X is dense in $\hat{X}_0$. We collect some facts about $\hat{A}(t)$ in the following theorem.

THEOREM 2.2. ([**4**], Sec.4). *Suppose that assumptions* (Z_1)–(Z_5) *hold. Then*

(i) *if* $\lambda \in \rho(A(t))$, *then* $\lambda \in \rho(\hat{A}(t))$ *and* $R(\lambda, A(t)) = R(\lambda, \hat{A}(t))|_X$, $t \in [0,T]$,

(ii) *the family* $\{\hat{A}(t)\}$, $t \in [0,T]$ *is stable on* $\hat{X}_0$,

(iii) *the mapping* $[0,T] \ni t \to \hat{A}(t)x$, $x \in X$, *is of class* C^1.

Let assumptions (Z_1)–(Z_5) hold. We adapt the following definition.

DEFINITION 2.3. A function $u \in C([0,T], X)$ given by

$$u(t) = \hat{U}(t,0)x_0 + \int_0^t \hat{U}(t,s)f(s)ds, \quad t \in [0,T],$$

where $\{\hat{U}(t,s)\}, 0 \le s \le t \le T$ is the evolution system of the problem

$$\begin{cases} u'(t) = \hat{A}(t)u(t), & t \in (0,T] \\ u(0) = x_0, \end{cases}$$

is called a mild solution of the linear problem

$$\begin{cases} u'(t) = A(t)u(t) + f(t), & t \in (0,T] \\ u(0) = x_0 \in X. \end{cases} \tag{3}$$

THEOREM 2.4. ([**4**], Sec.6). *Let assumptions* (Z_1)–(Z_5) *hold. If* $f \in L^1(0,T;X)$, *then for every* $x_0 \in X$ *there exists exactly one mild solution of linear problem* (3).

The mild solution of initial value problem (1) is defined analogously to the mild solution of (3).

THEOREM 2.5. ([**4**], Sec.7). *Let assumptions* (Z_1)–(Z_5) *hold. If* $f : [0,T] \times X \to X$ *is such that*

(i) *for each* $x \in X$, $f(\cdot, x) \in L^1(0,T;X)$,

(ii) *there exists* $L > 0$ *such that for* $t \in [0,T]$, u, $v \in X$

$$\|f(t,u) - f(t,v)\| \le L\|u - v\|,$$

then for every $x_0 \in X$ *there exists exactly one mild solution of initial value problem* (1).

3. The family of operators $\{A_0(t)\}$**,** $t \in [0,T]$**.** Let the family $\{A(t)\}$, $t \in [0,T]$, satisfy assumptions (Z_2)–(Z_5) from Section 2 and the following assumption:

(Z_1') Y_0 is a closed subspace of X and for each $t \in [0,T]$

$$Y_0 := \overline{D_t}^{\|\cdot\|}, \qquad Y_0 \subset X, \qquad Y_0 \neq X.$$

We remark that assumption (Z_1') holds particularly if $D_t = D$ does not depend on $t \in [0,T]$ and $\overline{D} \neq X$.

Let for each $t \in [0,T]$, $A_0(t)$ be the part of $A(t)$ in Y_0.

We shall prove that the family $\{A_0(t)\}$, $t \in [0,T]$, satisfies assumptions (Z_1)–(Z_5) from Section 2.

Since the family $\{A(t)\}$, $t \in [0,T]$ is stable on X, it follows from ([**2**], Theorem 3.1.10) that

PROPOSITION 3.1. *For each* $t \in [0,T]$ *the operator*

$$A_0(t) : Y_0 \supset D_t^0 \to Y_0$$

generates a C_0*-semigroup* $S_t^0(s)$, $s \ge 0$ *on* Y_0 *and*

$$R(\lambda, A_0(t)) = R(\lambda, A(t))|_{Y_0}, \qquad \lambda \in \rho(A(t)) \subset \rho(A_0(t)).$$

Consequently, for each $t \in [0,T]$, $A_0(t)$ is a linear closed operator whose domain D_t^0 can depend on t and $\overline{D_t^0}^{\|\cdot\|} = Y_0$.

Applying Proposition 3.1 we obtain the following theorem.

THEOREM 3.2. *Suppose that assumptions* (Z_1')–(Z_5) *hold. Then*

(i) *the family* $\{A_0(t)\}$, $t \in [0,T]$ *is stable on* Y_0,

(ii) *the mapping* $[0,T] \ni t \to R(\lambda, A_0(t))y \in (Y_0, \|\cdot\|)$ *is of class* C^1,

(iii) *for each* t, $s \in [0,T]$, *the operator* $A_0^{-1}(t)A_0(s)$ *is closable and for each fixed* $s \in [0,T]$ *the mapping* $[0,T] \ni t \to \overline{A_0^{-1}(t)A_0(s)}$ *is continuous in* $t = s$.

4. The family of operators $\{\hat{A}_0(t)\}, t \in [0,T]$**.** Let assumptions (Z_1')–(Z_5) be satisfied.

Since the family $\{A_0(t)\}$, $t \in [0,T]$ satisfies assumptions (Z_1)–(Z_5) from Section 2, we can construct the extrapolation space of Y_0.

Analogously to norm (2), for each fixed $\mu \in \rho(A(t)) \subset \rho(A_0(t))$ define a new norm on Y_0 as

$$|y|_t := \|R(\mu, A_0(t))y\|, \qquad y \in Y_0, t \in [0,T].$$

Analogously as in Section 2, there exists a space $\hat{X}_0$ which is the closure of Y_0 in the norm $|\cdot|_0$.

From ([**2**], Theorem 3.1.10), the next theorem follows.

THEOREM 4.1. $\hat{X}_0$ *is isomorphic to the space which is the closure of* X *in the norm:*

$$|x|_0 := \|R(\mu, A(0))x\|, \qquad x \in X.$$

In the sequel, for each $t \in [0,T]$, we extend the operator $A_0(t)$ to the operator

$$\hat{A}_0(t) : \hat{X}_0 \supset (Y_0, |\cdot|_0) \to \hat{X}_0.$$

The domains $D(\hat{A}_0(t)) = Y_0$ do not depend on t and Y_0 is dense in $\hat{X}_0$.

Applying Theorem 2.2, we obtain the following theorem.

THEOREM 4.2. *Suppose that assumptions* (Z_1')–(Z_5) *hold. Then*

(i) *if* $\lambda \in \rho(A_0(t))$, *then* $\lambda \in \rho(\hat{A}_0(t))$ *and* $R(\lambda, A_0(t)) = R(\lambda, \hat{A}_0(t))|_{Y_0}$, $t \in [0,T]$,

(ii) *the family* $\{\hat{A}_0(t)\}$, $t \in [0,T]$ *is stable on* $\hat{X}_0$,

(iii) *the mapping* $[0,T] \ni t \to \hat{A}_0(t)y$, $y \in Y_0$ *is of class* C^1.

From this theorem it follows that the norm on $\hat{X}_0$ is given by

$$\|\hat{x}\|_{\hat{X}_0} = |\hat{x}|_0 = \|R(\mu, \hat{A}_0(0))\hat{x}\|, \qquad \hat{x} \in \hat{X}_0, \quad \mu \in \rho(A(0)).$$

5. The linear case. In this section we consider the following linear problem

$$
\begin{cases} u'(t) = A(t)u(t) + f(t), & t \in (0,T] \\ u(0) = x_0, \end{cases} \tag{4}
$$

where $\{A(t)\}$, $t \in [0,T]$, satisfies assumptions (Z_1')–(Z_5) from Section 3.

We remark that from ([**3**], Theorem 5.4.8) it follows that under assumptions (Z_1')–(Z_5) there exists the unique evolution system $\{\hat{U}(t,s)\}$, $0 \le s \le t \le T$ of the problem

$$
\begin{cases} u'(t) = \hat{A}_0(t)u(t), & t \in (0,T] \\ u(0) = x_0 \in \hat{X}_0. \end{cases} \tag{5}
$$

Now we recall the following definition.

Definition 5.1. A function $u : [0,T] \to \hat{X}_0$ is a classical solution of the problem

$$
\begin{cases} u'(t) = \hat{A}_0(t)u(t) + f(t), & t \in (0,T] \\ u(0) = x_0 \in \hat{X}_0, \end{cases} \tag{6}
$$

if u is continuous on [0,T], continuously differentiable on (0,T], $u(t) \in Y_0$ for $t \in [0,T]$ and (6) is satisfied.

Applying Theorem 4.1 and ([**3**], Theorem 5.5.3) we obtain the following theorem.

Theorem 5.2. *Suppose that assumptions* (Z_1')–(Z_5) *hold.*
If $f \in C^1([0,T],X)$, *then for each* $x_0 \in Y_0$ *problem* (6) *has exactly one classical solution* u *given by*

$$
u(t) = \hat{U}(t,0)x_0 + \int_0^t \hat{U}(t,s)f(s)ds, \tag{7}
$$

where $\{\hat{U}(t,s)\}, 0 \le s \le t \le T$ *is the evolution system of* (5).

Furthermore, from the proof of Theorem 5.5.3 in [**3**] it follows that the function u given by (7) is of class $C^1([0,T],\hat{X}_0)$.

Theorem 5.3. *Let assumptions* $(Z_1')-(Z_5)$ *be satisfied. If* $f \in C^1([0,T],X)$ *and* $x_0 \in Y_0$, *then the function* u *given by* (7) *is continuous in* $(X, \|\cdot\|)$.

Proof. From Theorem 5.2 and Definition 5.1 it follows that $u(t) \in X$ for $t \in [0,T]$.

The norm

$$
\|y\|_{D(\hat{A}_0(0))} := |y|_0 + |\hat{A}_0(0)y|_0, \qquad y \in Y_0
$$

is equivalent to the norm $\|\cdot\|$ (see [**4**], Prop. 5.3). Thus for each fixed $t_0 \in [0,T]$ and for each $t \in [0,T]$

$$\|u(t)-u(t_0)\| \leq$$
$$\leq M[|u(t)-u(t_0)|_0 + |[\hat{A}_0(0)(\hat{A}_0(t))^{-1}][\hat{A}_0(t)u(t) - \hat{A}_0(t_0)u(t_0)]|_0$$
$$+ |[\hat{A}_0(0)(\hat{A}_0(t))^{-1}][\hat{A}_0(t_0)u(t_0) - \hat{A}_0(t)u(t_0)]|_0],$$

where $M := max\{|\mu|, 1\}$. By Definition 5.1

$$|u(t)-u(t_0)|_0 \to 0, \qquad t \to t_0.$$

Therefore from Theorem 4.2

$$|[\hat{A}_0(0)(\hat{A}_0(t))^{-1}][\hat{A}_0(t_0)u(t_0) - \hat{A}_0(t)u(t_0)]|_0 \to 0, \qquad t \to t_0.$$

From Definition 5.1 it follows that

$$\hat{A}_0(t)u(t) = u^{'}(t) - f(t).$$

Since $u^{'} \in C([0,T], \hat{X}_0)$ and $f \in C^1([0,T], \hat{X}_0)$, there is

$$|[\hat{A}_0(0)(\hat{A}_0(t))^{-1}][\hat{A}_0(t)u(t) - \hat{A}_0(t_0)u(t_0)]|_0 \to 0, \qquad t \to t_0.$$

Hence u given by (7) is continuous in $(X, \|\cdot\|)$. □

A mild solution of initial value problem (4) is defined analogously to a mild solution of (3).

From Theorem 5.2 and Theorem 5.3, it follows the following theorem.

THEOREM 5.4. *Assume* (Z_1')–(Z_5). *If* $f \in C^1([0,T], X)$ *and* $x_0 \in Y_0$, *then problem* (4) *has the unique mild solution.*

6. The semilinear case. In this section we consider nonlinear problem (1), mentioned in the introduction, where $\{A(t)\}$, $t \in [0,T]$ satisfies (Z_1')–(Z_5).

We remark that if the function $f : [0,T] \times Y_0 \to Y_0$ satisfies assumption from Theorem 2.5 we obtain the theorem on the existence and uniqueness for the problem (1).

But if the function $f : [0,T] \times X \to X$, we need the following assumption.

(Z_6) The function $f : [0,T] \times X \to X$ is of class C^1 and

$$\|\frac{\partial f}{\partial x}(t,x)\|_{X\to X} \leq L \quad \text{and} \quad \|\frac{\partial f}{\partial x}(t,x)\|_{\hat{X}_0\to\hat{X}_0} \leq L_0,$$

where $L > 0$ and $L_0 > 0$ independent of t and x.

From this assumption it follows that

$$\|f(t,x_1) - f(t,x_2)\| \leq L\|x_1 - x_2\|, \qquad x_1, x_2 \in X, \quad t \in [0,T], \tag{8}$$

and

$$|f(t,x_1) - f(t,x_2)|_0 \leq L_0|x_1 - x_2|_0, \qquad x_1, x_2 \in X, \quad t \in [0,T]. \tag{9}$$

The classical solution of the problem

$$(10)\qquad \begin{cases} u'(t) = \hat{A}_0(t)u(t) + f(t,u(t)), & t \in (0,T] \\ u(0) = x_0 \in \hat{X}_0 \end{cases}$$

is defined analogously to the classical solution of (6) (Def. 5.1).

The following theorem holds true.

THEOREM 6.1. *Let assumptions* (Z_1')–(Z_6) *hold. If* u *is a classical solution of* (10)*, then* u *satisfies the integral equation*

$$(11)\qquad u(t) = \hat{U}(t,0)x_0 + \int_0^t \hat{U}(t,s)f(s,u(s))ds,$$

where $\{\hat{U}(t,s)\}$, $0 \le s \le t \le T$ *is the evolution system of* (5).

In the sequel, we shall need the following lemma.

LEMAT 6.2. *Let assumptions* (Z_1')–(Z_6) *hold. Suppose that* $u \in C([0,T],X) \cap C^1([0,T],\hat{X}_0)$. *Then the function* $g : [0,T] \ni t \to f(t,u(t))$ *is of class* $C^1([0,T],(\hat{X}_0,|\cdot|_0))$.

PROOF. Let $t,\ t+h \in [0,T]$.

$$\begin{aligned} \frac{1}{h}[g(t+h)-g(t)] &= \frac{1}{h}[f(t+h,u(t+h)) - f(t,u(t))] \\ &= \frac{1}{h}[f(t+h,u(t+h)) - f(t,u(t+h))] \\ &\quad + f_x'(t,u(t))\frac{1}{h}[u(t+h)-u(t)] + \eta(t,u(t),h). \end{aligned}$$

This, together with Theorem 4.1 and assumption (Z_6), shows that g' exists in $\hat{X}_0$ and for each $t \in [0,T]$

$$g'(t) = f_t'(t,u(t)) + f_x'(t,u(t))u'(t).$$

Now for $t_0 \in [0,T]$ and $t \in [0,T]$ there is

$$\begin{aligned} |g'(t) - g'(t_0)|_0 &\le |f_t'(t,u(t)) - f_t'(t,u(t))\ |_{t=t_0}\ |_0 \\ &\quad + |f_x'(t,u(t))u'(t) - f_x'(t,u(t))\ |_{t=t_0}\ u'(t_0)|_0 \\ &\le C\frac{M}{\mu-\omega}\|f_t'(t,u(t)) - f_t'(t,u(t))\ |_{t=t_0}\ \| \\ &\quad + |f_x'(t,u(t))[u'(t)-u'(t_0)]|_0 + |[f_x'(t,u(t)) - f_x'(t,u(t))\ |_{t=t_0}]u'(t_0)|_0. \end{aligned}$$

Thus the function $[0,T] \ni t \to g'(t) \in \hat{X}_0$ is continuous. Hence the function $g : [0,T] \ni t \to f(t,u(t))$ is of class $C^1([0,T],(\hat{X}_0,|\cdot|_0))$. This concludes the proof of Lemma 6.2. □

Now we shall prove the following theorem.

THEOREM 6.3. *Assume* (Z_1')–(Z_6) *and let* $x_0 \in Y_0$. *Then there exists exactly one solution of* (11) *continuous in* $(X, \|\cdot\|)$.

PROOF. Let

$$u_0(t) := x_0, \qquad t \in [0,T],$$
$$g_0(t) := f(t, u_0(t)), \qquad t \in [0,T].$$

Applying Theorem 5.2, we see that the problem

$$\begin{cases} u'(t) = \hat{A}_0(t)u(t) + g_0(t), & t \in (0,T] \\ u(0) = x_0 \in Y_0 \end{cases}$$

has exactly one classical solution u_1 given by

$$u_1(t) = \hat{U}(t,0)x_0 + \int_0^t \hat{U}(t,s)g_0(s)ds,$$

where $\{\hat{U}(t,s)\}$, $0 \le s \le t \le T$ is the evolution system of (5). Therefore, from Theorem 5.3 and ([**3**], Theorem 5.5.3), we have

$$u_1 \in C([0,T],X) \cap C^1([0,T],\hat{X}_0).$$

Let

$$g_1(t) := f(t, u_1(t)), \qquad t \in [0,T].$$

By Lemma 6.2, the function $[0,T] \ni t \to g_1(t) \in \hat{X}_0$ is of class C^1. Once again using Theorem 5.5.3 in [**3**] and Theorem 5.3, we see that the problem

$$\begin{cases} u'(t) = \hat{A}_0(t)u(t) + g_1(t), & t \in (0,T] \\ u(0) = x_0 \in Y_0 \end{cases}$$

has exactly one classical solution u_2 given by

$$u_2(t) = \hat{U}(t,0)x_0 + \int_0^t \hat{U}(t,s)g_1(s)ds$$

and

$$u_2 \in C([0,T],X) \cap C^1([0,T],\hat{X}_0).$$

After n steps, we conclude that there exists exactly one function $u_{n+1} \in C([0,T],X)$ given by

$$u_{n+1}(t) = \hat{U}(t,0)x_0 + \int_0^t \hat{U}(t,s)f(s,u_n(s))ds, \quad n = 0,1,2,...,$$

where

$$u_n \in C([0,T],X) \cap C^1([0,T],\hat{X}_0).$$

Let $k := \sup\{\|\hat{U}(t,s)\| : 0 \le s \le t \le T\}$. In the space $C([0,T],X)$, consider two equivalent norms:

$$\|u\| := \sup\{\|u(t)\| : 0 \le t \le T\},$$
$$\|u\|' := \sup\{e^{-kLt}\|u(t)\| : 0 \le t \le T\},$$

where $L > 0$ is the Lipschitz constant (see (8)). Therefore

$$\begin{aligned}
\|u_{n+1} - u_n\|' &= \sup_{t\in[0,T]} \{e^{-kLt}\|u_{n+1}(t) - u_n(t)\|\} \\
&\le \sup_{t\in[0,T]} \{e^{-kLt} \int_0^t \|\hat{U}(t,s)[f(s,u_n(s)) - f(s,u_{n-1}(s))]\| ds\} \\
&\le \sup_{t\in[0,T]} \{e^{-kLt} k \int_0^t L\|u_n(s) - u_{n-1}(s)\| ds\} \\
&\le kL \sup_{t\in[0,T]} \{e^{-kLt}\|u_n - u_{n-1}\|' \int_0^t e^{kLs} ds\} \le (1 - e^{-CLT})\|u_n - u_{n-1}\|'.
\end{aligned}$$

Setting $Q := 1 - e^{-CLT}$, by induction we obtain

$$\|u_{n+1} - u_n\|' \le Q^n \|u_1 - u_0\|', \qquad n = 0, 1, 2, \dots$$

Consequently, for $n < m$

$$\|u_n - u_m\|' \le \frac{Q^n}{1-Q}\|u_1 - u_0\|'.$$

Since $\lim_{n\to\infty} Q^n = 0$, $\{u_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Thus, by letting $n \to \infty$, we see that $u \in C([0,T],X)$. □

A mild solution of initial value problem (1) is defined analogously to a mild solution of (3).

From Theorem 6.3 the next theorem follows.

Theorem 6.4. *Assume* (Z_1')–(Z_6) *and let* $x_0 \in Y_0$. *Then there exists exactly one mild solution of initial value problem* (1).

7. Example. We shall give an example of the family $\{\mathcal{A}(t)\}$, $t \in [0,T]$ with a domain which is not dense and can depend on t. For each $t \in [0,T]$, the operator $\mathcal{A}(t)$ will hold assumptions (Z_2)–(Z_5).

Let $\Omega := \Omega_1 \setminus \Omega_2$, where

$$\Omega_1 := \{(x,y) \in \mathbb{R}^2 : x > 0 \quad y > 0\},$$
$$\Omega_2 := \{(x,y) \in \mathbb{R}^2 : \quad 0 < x < 1, \quad 0 < y < 1, \quad (x-1)^2 + (y-1)^2 \ge 1\}.$$

We shall consider the differential operator of second order:

$$A(t;x,y;D) := B(x,y;D) + b(t;x,y)I, \qquad (x,y) \in \Omega, \quad t \in [0,T], \tag{12}$$

where

$$B(x,y;D) := b_1(x,y)\frac{\partial^2}{\partial x^2} + 2b_2(x,y)\frac{\partial^2}{\partial x \partial y} + b_3(x,y)\frac{\partial^2}{\partial y^2}. \tag{13}$$

We make the following assumptions:

(P_1) For each $t \in [0,T]$, the operator $A(t;x,y;D)$ is uniformly strongly elliptic on Ω in the sense that there is a constant $C > 0$ such that for all $(x,y) \in \Omega$ and $(\xi_1, \xi_2) \in \mathbb{R}^2$

$$b_1(x,y)\xi_1^2 + 2b_2(x,y)\xi_1\xi_2 + b_3(x,y)\xi_2^2 \geq C(\xi_1^2 + \xi_2^2).$$

(P_2) The coefficients b_1, b_2, b_3 are uniformly continuous on Ω; are continuous and uniformly bounded on $\overline{\Omega}$. We remark that from (P_1) it follows that the inverse operator B^{-1} exists and there is $K > 0$ that $\|B^{-1}\| \leq K$.

Moreover, we assume that:

(P_3) The coefficient b on $[0,T] \times \overline{\Omega}$ is of class C^1 and $|b(t;x,y)| < \frac{1}{K}$.

With the family $\{A(t;x,y;D)\}$, $t \in [0,T]$, we associate the family of linear operators $\{A(t)\}$, $t \in [0,T]$, on the space

$$C_0(\overline{\Omega}) := \{u \in C(\overline{\Omega}) : \lim_{Q \to \infty} u(Q) = 0, \quad Q \in \overline{\Omega}\}.$$

The norm in $C_0(\overline{\Omega})$ is defined by

$$\|u\| := \max\{|u(Q)| : \quad Q \in \overline{\Omega}\}.$$

Let

$$D(A) := \{u \in C_0(\overline{\Omega}) : \quad u \in W^{2,q}_{loc}, \quad A(t;x,y;D)u \in C_0(\overline{\Omega}), \quad u\,|_{\partial\Omega} = 0\}$$

be the domain of the operator $A(t)$ for each $t \in [0,T]$ and let

$$A(t)u = A(t;x,y;D)u, \qquad u \in D(A).$$

$W^{2,q}_{loc}$ denotes the set of all functions which are in $W^{2,q}(\Omega \cap \Gamma)$ for all closed bounded sets Γ.

$D(A)$ is clearly independent of t and from [**5**] it follows that this is not dense in $C_0(\overline{\Omega})$.

We remark that from (P_3) it follows that for each $u \in D(A)$,
$[0,T] \ni t \to A(t)u \in C_0(\overline{\Omega})$ is of class C^1.

We collect some facts about $A(t)$ in the following theorem.

THEOREM 7.1. *Let assumptions (P_1)–(P_3) hold. Then*

(i) $0 \in \rho(A(t))$,

(ii) *for each t, $s \in [0,T]$ the operator $A^{-1}(t)A(s)$ is closable and for each fixed $s \in [0,T]$ the mapping $t \to \overline{A^{-1}(t)A(s)}$ is continuous in $t = s$.*

From (P_1) it follows that the operator $B : C_0(\overline{\Omega}) \supset D(A) \to C_0(\overline{\Omega})$ is uniformly strongly elliptic on Ω. Consequently ([**3**], Sec.7.3) there is the operator $B^{\frac{1}{2}} : C_0(\overline{\Omega}) \supset D(B^{\frac{1}{2}}) \to C_0(\overline{\Omega})$ given by

$$B^{\frac{1}{2}}u = \frac{1}{\pi}\int_0^\infty z^{-\frac{1}{2}} BR(z,B)u dz, \quad u \in D(A)$$

and such that $[B^{\frac{1}{2}}]^2 = B$.

Thus from [**6**] (Prop.2.7 and Prop.2.6), we have the following theorem.

Theorem 7.2. *The operator*

$$\mathcal{B} := \begin{bmatrix} 0 & I \\ B & 0 \end{bmatrix},$$

with domain $D(A) \times D(B^{\frac{1}{2}})$ *is a Hille–Yosida operator on* $[D(B^{\frac{1}{2}})] \times C_0(\overline{\Omega})$, *where* $[D(B^{\frac{1}{2}})]$ *denotes the linear space* $D(B^{\frac{1}{2}})$ *with the norm*

$$|u| := \|u\| + \|B^{\frac{1}{2}}u\|, \qquad u \in D(B^{\frac{1}{2}}).$$

Therefore, for each $t \in [0,T]$ we may define the operator

$$\mathcal{A}(t) : [D(B^{\frac{1}{2}})] \times C_0(\overline{\Omega}) \supset D(\mathcal{A}) \to [D(B^{\frac{1}{2}})] \times C_0(\overline{\Omega})$$

by

$$\mathcal{A}(\mathbf{t}) := \begin{bmatrix} 0 & I \\ A(t) & 0 \end{bmatrix}.$$

The domain of $\{\mathcal{A}(t)\}$, $t \in [0,T]$ is $D(\mathcal{A}) = D(A) \times D(B^{\frac{1}{2}})$. For each $t \in [0,T]$, the operator $\mathcal{A}(t)$ is not densely defined.

Theorem 7.3. *Suppose assumptions* (P_1)–(P_3) *hold. Then*

(i) $0 \in \rho(\mathcal{A}(t))$,

(ii) *the family* $\{\mathcal{A}(t)\}$, $t \in [0,T]$ *is stable,*

(iii) *the mapping* $[0,T] \ni t \to \mathcal{A}(t)x \in [D(B^{\frac{1}{2}})] \times C_0(\overline{\Omega})$, $x \in D(\mathcal{A})$ *is of class* C^1.

(iv) *for each* t, $s \in [0,T]$ *operator* $\mathcal{A}^{-1}(t)\mathcal{A}(s)$ *is closable and for an arbitrary* $s \in [0,T]$ *the mapping* $[0,T] \ni t \to \overline{\mathcal{A}^{-1}(t)\mathcal{A}(s)}$ *is continuous in* $t = s$.

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