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MARKOV INEQUALITY ON A CERTAIN COMPACT SUBSET OF $\mathbb{R}^2$

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Abstract. We construct a compact subset K of $\mathbb{R}^2$ which satisfies the Markov inequality – but K is not polynomially cuspidal at the point (0; 0). The set K is connected and fat (i.e. K is equal to the closure of its interior).

The Markov inequality gives the estimation for the derivative of the polynomial (of the given degree) if the estimation for the norm of the polynomial is known. This inequality is very useful in the theory of polynomial approximation. For multivariate polynomials it is often a very difficult task to prove that the Markov inequality is fulfilled (or not fulfilled) for a given compact set (by the way, for the polynomials of one variable this problem is sometimes also difficult, e.g. for Cantor-type sets). There are several important papers (in the multidimensional case) about the Markov inequality on sets with polynomial cusps (e.g. Pawłucki and Pleśniak [**6**], Baran [**1**], Kroó and Szabados [**5**]). The case of non-polynomial cusps is much more difficult. Some examples of the sets (satisfying the Markov inequality) that are not polynomially cuspidal can be found e.g. in: [**2**], [**4**], [**7**], [**8**]. Recently Erdélyi and Kroó ([**3**]) obtained interesting results: one of the theorems proved in their paper gives the construction of the set (with one non-polynomial cusp) satisfying the Markov-type inequality (i.e. the constant is "worse" than that in the Markov inequality).

We construct the set (with one non-polynomial cusp) satisfying the Markov inequality:

Theorem. *Let* $\gamma = \frac{k}{l}, \gamma \geq 2$ *(k, l are positive integers). Suppose that* f_1, f_2 *are two functions continuous on the interval* $[0; 1]$ *and constant on the interval*

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$[\frac{1}{5}; 1]$. *Suppose also that for* $0 < x \leq \frac{1}{5}$

$$f_1(x) = \frac{1}{2}x^\gamma(-\log x), \quad f_2(x) = 2x^\gamma(-\log x).$$

Define the set

$$K := \left\{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, f_1(x) \leq y \leq f_2(x)\right\}.$$

Then the Markov inequality is fulfilled for K*:*
there exist two positive real numbers M, β *such that for all positive integers* n

$$\sup\left\{\left|\frac{\partial P}{\partial x}(x, y)\right| + \left|\frac{\partial P}{\partial y}(x, y)\right| : (x, y) \in K\right\} \leq M n^\beta \sup\{|P(x, y)| : (x, y) \in K\},$$

where $P(x, y)$ *is any polynomial of degree* n *(with real coefficients).*

Let us observe that for $\varepsilon \leq 0$

$$\lim_{t \to 0} t^\varepsilon(-\log t) = +\infty$$

and for $\varepsilon > 0$

$$\lim_{t \to 0} t^\varepsilon(-\log t) = 0.$$

Therefore, there exists no polynomial map $\mathbb{R} \ni t \to \psi(t) = (x(t), y(t)) \in \mathbb{R}^2$ such that $\psi(t) \in K$ for all $0 \leq t \leq 1$ and $\psi(0) = (0; 0)$ (K is not polynomially cuspidal at $(0; 0)$). Hence the theorems from **[5]** or **[6]** cannot be used, but the proof from **[3]** can be easily adapted.

Proof. We begin by recalling the notion of the extremal function. Let K_0 be a compact subset of $\mathbb{C}$. The extremal function of Leja is defined by

$$\Phi_{K_0}(z) := \sup\left\{|p(z)|^{\frac{1}{\deg p}}\right\}, \quad z \in \mathbb{C},$$

the supremum being taken over all polynomials $p : \mathbb{C} \to \mathbb{C}$ (of degree at least 1) with $||p||_{K_0} \leq 1$ ($||p||_{K_0}$ denotes $\sup |p|(K_0)$). It is known that for a line segment $[a; b] \subset \mathbb{R}$

$$\Phi_{[a;b]}(z) = |v(z)|, \quad z \in \mathbb{C},$$

where

$$v(z) = \frac{b + a - 2z + 2\sqrt{(b - z)(a - z)}}{b - a},$$

with the branch of the root properly chosen (so that $|v(z)| \geq 1$ for all complex z).

It is easy to check that for $0 < a < b$

$$\Phi_{[a;b]}(0) = \left|\frac{(\sqrt{a} + \sqrt{b})^2}{(\sqrt{b} - \sqrt{a})(\sqrt{b} + \sqrt{a})}\right| = \left|\frac{1 + \sqrt{\frac{a}{b}}}{1 - \sqrt{\frac{a}{b}}}\right|.$$

It follows immediately from the definition of Φ_{K_0} that

$$|p(z)| \leq ||p||_{K_0} \left(\Phi_{K_0}(z)\right)^{\deg p}$$

for each polynomial p ($z \in \mathbb{C}$). The above-mentioned inequality is known as the Bernstein–Walsh inequality.

We will also use the classical Markov inequality for the line segment $[a;b]\subset\mathbb{R}$

$$|p'(x)| \leq \frac{2n^2}{b-a}||p||_{[a;b]}, \quad a \leq x \leq b,$$

where p is any polynomial of degree at most n.

Let us also recall the following property of the function h which is convex on a line segment $[0;l_0]$, $l_0 > 0$ (and fulfils the conditions: $h(0) = 0$, h' exists on $[0;l_0]$):

$$h(w_1 + w_2) \geq h(w_1) + h(w_2), \quad w_1 \geq 0,\ w_2 \geq 0,\ w_1 + w_2 \leq l_0.$$

The proof is standard. The function

$$\varphi(x) := h(x + w_1) - h(x) - h(w_1), \quad 0 \leq x \leq l_0 - w_1,$$

has the derivative $\varphi'(x)$ which is nonnegative, because the derivative h' of the convex function h is increasing. From this we conclude that φ is increasing on $[0; l_0 - w_1]$. Hence

$$\varphi(w_2) \geq \varphi(0) = 0,$$

which is the desired conclusion.

The properties of the function

$$f(x) = Cx^{\gamma}(-\log x), \quad 0 < x \leq 1,\ f(0) := 0$$

($C \in \mathbb{R}, \gamma \in \mathbb{R}, C > 0, \gamma \geq 2$) will also be useful in our proof. We leave it to the reader to verify that the function f fulfils the following conditions:

(1) f is increasing for $0 \leq x \leq \exp\left(-\frac{1}{\gamma}\right)$.

(2) f is convex for $0 \leq x \leq \exp\left(-\frac{1}{\gamma} - \frac{1}{\gamma-1}\right)$.

(3) $|f'(x)| \leq C\left(1 + \frac{1}{\gamma-1}\right)\exp\left(\frac{1}{\gamma} - 2\right)$ if $0 \leq x \leq \exp\left(-\frac{1}{\gamma}\right)$.

Let us observe that for $\gamma \geq 2$

$$\exp\left(-\frac{1}{\gamma}\right) > \exp\left(-\frac{1}{\gamma} - \frac{1}{\gamma-1}\right) \geq \exp\left(-\frac{3}{2}\right) > \frac{1}{5}$$

and

$$\left(1 + \frac{1}{\gamma-1}\right)\exp\left(\frac{1}{\gamma} - 2\right) \leq 2\exp\left(-\frac{3}{2}\right) < \frac{1}{2}.$$

Define (for each positive integer n) the subset of K:

$$K(n) := \left\{(x,y) \in \mathbb{R}^2 : 0 \leq x < \lambda\exp(-d_n),\ f_1(x) \leq y \leq f_2(x)\right\},$$

where

$$\lambda := 1 - \left(\frac{1}{2}\right)^{\frac{1}{\gamma}} = 1 - \left(\frac{1}{2}\right)^{\frac{l}{k}} > 0$$

and $d_n := 2l \log\left(9n^2\right)$. Hence

$$\exp\left(-d_n\right) = \frac{1}{\left(9n^2\right)^{2l}}, \quad (1-\lambda)^{\frac{k}{l}} = \frac{1}{2}.$$

We fix a polynomial (of degree n) with real coefficients: $P = P(x,y)$ ($(x,y) \in \mathbb{R}^2$). Without loss of generality we can assume that $||P||_K \leq 1$ ($||P||_K$ denotes the supremum norm on K). Let

$$Q(x,y) := \left|\frac{\partial P}{\partial x}(x,y)\right| + \left|\frac{\partial P}{\partial y}(x,y)\right|.$$

We have to estimate $Q(x_0, y_0)$, where $(x_0, y_0) \in K$. We first consider the case $(x_0, y_0) \notin K(n)$. An easy computation shows that

$$f_2(\lambda \exp(-d_n)) - f_1(\lambda \exp(-d_n)) > \frac{1}{An^{4k}},$$

where A is a real positive constant (depending on k and l). From this (and from the conditions fulfilled by the function $f(x) = Cx^\gamma(-\log x), C > 0, \gamma \geq 2$) we conclude that the set K contains two segments (a vertical one and a horizontal one) passing through (x_0, y_0), whose length is at least $\frac{1}{An^{4k}}$. By the classical Markov inequality for the line segment in $\mathbb{R}$, we get

$$Q(x_0, y_0) \leq 2\left(2n^2 An^{4k}\right) = 4An^{4k+2}.$$

We now turn to the case $(x_0, y_0) \in K(n)$. Consider the polynomial of one real variable:

$$H(t) := \frac{\partial P}{\partial y}\left(x_0 + t^l, y_0 + d_n t^k\right),$$

where $t \geq 0$. Of course the degree of H is not greater than nk. We first observe that the points

$$(x(t), y(t)) = \left(x_0 + t^l, y_0 + d_n t^k\right)$$

belong to $K \backslash K(n)$ for $(9n^2)^{-2} \leq t \leq (9n^2)^{-1}$ (for these values of the parameter t we have $0 \leq x(t) < \frac{1}{81} + \frac{1}{9} < \frac{1}{5}$). Of course $(x(t), y(t)) \notin K(n)$, because

$$x(t) = x_0 + t^l \geq t^l \geq \left(9n^2\right)^{-2l} = \exp\left(-d_n\right) > \lambda \exp\left(-d_n\right).$$

We have to prove that $(x(t), y(t)) \in K$. Suppose, contrary to our claim, that $(x(t), y(t)) \notin K$. Then either $y(t) > f_2(x(t))$ or $y(t) < f_1(x(t))$. Let us consider the possibility: $y(t) > f_2(x(t))$. Take the interval

$$J := \left[0; \frac{1}{(9n^2)^l}\right].$$

It is easy to check that for all $u \in J$

$$d_n u^{\frac{k}{l}} \le f_2(u) = 2u^{\frac{k}{l}}(-\log u).$$

Of course $x(t) - x_0 = t^l \in J$. Therefore

$$d_n t^k = d_n (x(t) - x_0)^{\frac{k}{l}} \le f_2(x(t) - x_0).$$

The function $f_2(u)$ is convex and differentiable for $0 \le u \le \frac{1}{5}$ (the condition $f_2(0) = 0$ is also fulfilled). Hence

$$f_2(x(t)) \ge f_2(x(t) - x_0) + f_2(x_0) \ge d_n t^k + y_0 = y(t),$$

a contradiction.

Consider now the possibility: $y(t) < f_1(x(t))$. We have $x(t) \ge \exp(-d_n)$ and $x_0 < \lambda \exp(-d_n)$. It follows that

$$\frac{x(t) - x_0}{x(t)} = 1 - \frac{x_0}{x(t)} \ge 1 - \lambda.$$

From this we deduce that

$$y(t) = y_0 + d_n t^k = y_0 + d_n (x(t) - x_0)^{\frac{k}{l}} \ge d_n ((1-\lambda)x(t))^{\frac{k}{l}} = \frac{d_n}{2}(x(t))^{\frac{k}{l}}.$$

This gives

$$f_1(x(t)) = \frac{1}{2}(x(t))^{\frac{k}{l}}(-\log x(t)) > y(t) \ge \frac{d_n}{2}(x(t))^{\frac{k}{l}}.$$

We thus get $x(t) < \exp(-d_n)$, which is impossible.

We are now in a position to prove the Markov inequality in the case: $(x_0, y_0) \in K(n)$. We apply the Bernstein–Walsh inequality ($p = H$, $z = 0$, $K_0 = \left[(9n^2)^{-2}; (9n^2)^{-1}\right]$) and get

$$\begin{aligned} \left|\frac{\partial P}{\partial y}(x_0, y_0)\right| &= |H(0)| \le ||H||_{K_0} (\Phi_{K_0}(0))^{\deg H} \\ &\le \left|\left|\frac{\partial P}{\partial y}\right|\right|_{K \setminus K(n)} \left(\frac{1 + \frac{1}{3n}}{1 - \frac{1}{3n}}\right)^{nk}. \end{aligned}$$

From what has already been proved,

$$\left|\left|\frac{\partial P}{\partial y}\right|\right|_{K \setminus K(n)} \le 2An^{4k+2}.$$

It is easy to check that the function $h(t) = \log \frac{1+t}{1-t}$ $(0 \leq t < 1)$ is convex. From this it follows that for $0 < t \leq \frac{1}{3}$

$$h(t) \leq 3t \log 2 < 3t.$$

Hence

$$\left(\frac{1+\frac{1}{3n}}{1-\frac{1}{3n}}\right)^{nk} = \exp\left(nk \log\left(\frac{1+\frac{1}{3n}}{1-\frac{1}{3n}}\right)\right) < \exp\left(\frac{nk}{n}\right) = e^k.$$

It is now obvious that

$$\left|\frac{\partial P}{\partial y}(x_0, y_0)\right| \leq 2Ae^k n^{4k+2}.$$

The same conclusion can be drawn for $\frac{\partial P}{\partial x}$:

$$\left|\frac{\partial P}{\partial x}(x_0, y_0)\right| \leq 2Ae^k n^{4k+2}.$$

We thus get

$$Q(x_0, y_0) \leq 4Ae^k n^{4k+2}.$$

This completes the proof of the theorem (we obtain the constants $M = 4Ae^k$, $\beta = 4k + 2$). □

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