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A REMARK ON THE MOORE THEOREM

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Abstract. This paper contains a simple generalization of the classical Moore theorem. In this generalization one considers the triods with one "exotic" ray without changing the statement.

1. Introduction. The classical Moore theorem describes a certain nice property of the plane $\mathbb{R}^2$. It was generalized by Young [**7**] to the case of $\mathbb{R}^n$, but in our paper we will consider the two-dimensional case only. Before recalling the Moore theorem, we recall the definition of the triod. The definition presented below is an exact copy of the original Moore definition [**4**].

Definition 1. If O, A_1, A_2 and A_3 are four distinct points, and for each n $(1 \leq n \leq 3)$, r_n is an irreducible continuum from A_n to O and no two of the continua r_1, r_2 and r_3 have any point in common except O, then the continuum $r_1 \cup r_2 \cup r_3$ is a *triod*, the point O is the *emanation point*, and the continua r_1, r_2 and r_3 are the *rays* of this triod.

Now we are in a position to formulate the Moore theorem ([**3434**][1], [**5757**]).

Theorem 2. *In $\mathbb{R}^2$, each family of pairwise disjoint triods is at most countable.*

The proofs of this theorem can be found ([**3457345734573457**]).

In our paper, we present a slight and simple generalization of this theorem, which in fact consists in a slight modification of the definition of the triod. It appears that the Moore theorem remains true if one understands the notion of triod in a more general sense.

[1]In fact, the original version is a little more general.

2. The main theorem. We start with a new definition of the triod.

DEFINITION 3. Let O, A_1, A_2 be three distinct points, t_i an irreducible continua from A_i to O for $i \in \{1, 2\}$, t_3 a connected set containing at least two points and $t_1 \cap t_2 = t_1 \cap \overline{t_3} = t_2 \cap \overline{t_3} = \{O\}$. A *generalized triod* is a the set $t = t_1 \cup t_2 \cup t_3$.

As in the original definition, the point O will be called the emanation point, and the sets t_1, t_2, t_3 will be said the rays of the generalized triod. The union $t_1 \cup t_2$ will be called *the hat* of the generalized triod.

We see that the only difference is that we do not require t_3 to be compact.

It will be convenient to say that the rays t_1 and t_2 are *simple rays* and the ray t_3 is an *exotic ray.*

Now the following theorem holds.

THEOREM 4. *(The generalized Moore theorem) Each family of generalized triods in* $\mathbb{R}^2$ *with pairwise disjoint hats is at most countable.*

Before proving this theorem, let us observe that a generalized triod does not have to be closed and the generalized triod does not have to be continuum even after closure. The closure of a bounded generalized triod is a triod, but the family of closures of pairwise disjoint bounded generalized triods does not have to be the family of pairwise disjoint triods any longer (Example 5); hence, one cannot apply the classical Moore theorem in order to prove the generalized version (even for bounded generalized triods).

EXAMPLE 5. $t := [-1, 0] \times \{1\} \cup \{0\} \times [1, 2] \cup \{(x, \sin\frac{1}{x}) : x \in (0, 1]\}$, $s := [-1, 0] \times \{0\} \cup \{0\} \times [-\frac{1}{2}, \frac{1}{2}]$. We see that $t \cap s = \varnothing$ and $\overline{t} \cap \overline{s} \neq \varnothing$.

PROOF OF THE GENERALIZED MOORE THEOREM. In this proof, for convenience, we will use the term *triod* instead of *generalized triod.* Let us suppose that there exists an uncountable family $\Im$ of triods with pairwise disjoint hats.

For each triod there exists $\delta > 0$ such that the ball with the center at its emanation point and the radius δ does not contain any of the rays of the triod. Then there exist a number $d > 0$ and an uncountable subfamily $\Im_1$ of the family $\Im$ such that any triod in $\Im_1$ any its rays is not contained in the ball with the center at its emanation point and the radius d.

Since the hats of triods considered are pairwise disjoint, then the set of emanation points of the triods in $\Im_1$ is uncountable, hence there exists[2] a ball K with the radius $\frac{d}{3}$ in which there lies an uncountable subset of the set of emanation points of the triods in $\Im_1$.

With each triod $t = t_1 \cup t_2 \cup t_3$ (where, as above, rays t_1, t_2 are simple and t_3 is exotic) in $\Im_1$ and with the emmanation point in K, we associate a

[2]Since each uncountable set has a condensation point (see **[2]**, p. 178).

new triod $q(t) = q_1(t) \cup q_2(t) \cup q_3(t)$ contained in t, with the same emanation point O, where $q_1(t)$ and $q_2(t)$ are the irreducible continua from O to ∂K [3] and $q_3(t) = t_3$. We denote this new family of triods we denote by $\Im_2$. For each triod q in $\Im_2$, each of the intersections $q_1 \cap \partial K$ and $q_2 \cap \partial K$ are not empty. Let us select the points, say A_q and B_q, respectively from these sets.

Since $A_q \neq B_q$, then there exists $e > 0$ and an uncountable subfamily $\Im_3$ of the family $\Im_2$ such that

$$\forall q \in \Im_3 \ \mathrm{dist}(A_q, B_q) > e. \tag{1}$$

Since A_q and B_q are not in $\overline{q_3}$, then there exists $\varepsilon \in (o, e)$ and an uncountable subfamily $\Im_4$ of the family $\Im_3$ such that

$$\forall q \in \Im_4 \ \min\{\mathrm{dist}(A_q, \overline{q_3}), \mathrm{dist}(B_q, \overline{q_3})\} > \varepsilon. \tag{2}$$

It is obvious that on the boundary ∂K there exist an arc, say a, of a length ε and an uncountable subfamily $\Im_5$ of $\Im_4$ such that for each triod q in $\Im_5$ the point $A_q \in a$. It follows from (1) that for each triod q in $\Im_5$ the point $B_q \notin a$. Then there exist an arc b on ∂K of a length ε, disjoint with a, and an uncountable subfamily $\Im_6$ of $\Im_5$ such that for each triod $q \in \Im_6$ the point $B_q \in b$.

Now let us consider three triods: $p = p_1 \cup p_2 \cup p_3$, $r = r_1 \cup r_2 \cup r_3$, $s = s_1 \cup s_2 \cup s_3$ in the family $\Im_6$. The points A_p, A_r, A_s are pairwise different; thus we may assume that A_p lies between A_r and A_s on the arc a.

Since the sets: arc $A_pA_rB_p$ and $p_1 \cup p_2$ are continua and their intersection is not connected (since A_p and B_p are in this intersection but not A_r). Then the second Janiszewski theorem[4] implies that their union separates the plane. There exists a ball K_r with the center at A_r, disjoint from $p_1 \cup p_2$. Then there exists a point A'_r which is in r_1 and K_r. If B_r is not in arc $A_pA_rB_p$, then the irreducible continuum between the points A'_r and B_r contained in $r_1 \cup r_2$ intersects $p_1 \cup p_2$. But this is impossible, since the hats of triods are pairwise disjoint. Hence B_r lies in arc $A_pA_rB_p$ similarly as the point B_s lies on the arc $A_pA_sB_p$.

Let us observe now that the sets: $A_rA_pA_s \cup (s_1 \cup s_2)$ and $B_sB_pB_r \cup (r_1 \cup r_2)$ are continua. Their common part intersects disjoint arc a as well as b and is contained in $a \cup b$, hence is not connected. Then the second Janiszewski theorem implies that their union, say M, separates the plane, and we see that the emanation point of the triod p belongs to the bounded connected component, say N, of $\mathbb{R}^2 \backslash M$. We can now take a ball $K' \subset N$ with the center at the emanation point of the triod p. Let us observe that there are points of the

[3] Their existence follows from the Brouwer Reduction Theorem (see **[6]**, p. 43, or **[2]**, p. 172).

[4] See **[1]**, or **[2]**, p. 277.

triod p_3 both in the ball K' and outside $\overline{K}$. Hence M separates these points in p_3. But this is impossible, since from our assumptions and from (2) it follows that p_3 is disjoint from M. This ends the proof of the theorem. □

Let us remark that the generalization of the Moore theorem presented above is not true if one considers the triods with two exotic rays. Indeed, let us consider the following example.

DEFINITION 6. Let O and A_1 be two distinct points, t_1 be an irreducible continuum from A_1 to O, let t_2 and t_3 be two connected and at least two point sets such that, $t_1 \cap \overline{t_2} = t_1 \cap \overline{t_3} = \overline{t_2} \cap \overline{t_3} = \{O\}$. The triod-like set is now the set $t = t_1 \cup t_2 \cup t_3$.

This definition admits an uncountable family of pairwise disjoint triod-like sets.

EXAMPLE 7. Let us set

$$t = [-1, 0] \times \{0\} \cup \left\{(x, \sin\frac{1}{x} + 1) : x \in (0, 1]\right\} \cup \left\{(x, \sin\frac{1}{x} - 1) : x \in (0, 1]\right\}.$$

Then the set $\{t + (0, c) : c \in [0, 2)\}$ is an uncountable family of pairwise disjoint triod-like sets.

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