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THE DEGREE OF THE INVERSE OF A POLYNOMIAL AUTOMORPHISM

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Abstract. Let $F\colon \mathbb{C}^n \to \mathbb{C}^n$ be an invertible map for which both F and F^{-1} are polynomials. Then $\deg F^{-1} \leq (\deg F)^{n-1}$. This is a well-known result. The proof that we give here, at least for low n, does not depend on advanced algebraic geometry.

1. Introduction. In his 1939 paper [**4**] O.H. Keller introduced what is known as *Jacobian Conjecture*: prove or disprove that any polynomial mapping from $\mathbb{C}^n$ to itself with everywhere nonvanishing Jacobian determinant is necessarily invertible. The conjecture has attracted quite a number of mathematicians over the years, but it is still unanswered, even in dimension $n = 2$. A recent up-to-date report on the subject is van den Essen's book [**3**].

A polynomial mapping from $\mathbb{C}^n$ to itself which has a polynomial inverse is called a *polynomial automorphism*. Among the encouraging results in favour of the Jacobian conjecture there are the known facts that an injective polynomial mapping is necessarily an automorphism, together with a sharp estimate on the degree of the inverse. Both these results have been long known among algebraic geometers, with proofs that are rather inaccessible to outsiders. W. Rudin in [**7**] (1995) gave a proof of the invertibility result that only draws from basic complex analysis and algebra. Here we try to do the same for the degree estimate.

The degree of a polynomial mapping $F = (F_1, \ldots, F_n)$ of $\mathbb{C}^n$ into itself is defined as the largest of the degrees of the components $F_1, \ldots, F_n$. The estimate that we are going to prove is that if F is a polynomial automorphism

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of $\mathbb{C}^n$ then

$$\deg F^{-1} \leq (\deg F)^{n-1}. \tag{1}$$

The ingredients of our proof are some lemmas on the sets of zeros of holomorphic functions, straightforward consequences of Weierstrass' preparation theorem, and Bézout's classic theorem: if a system of m polynomial equations in m variables has a finite number of solutions, then this number is not larger than the product of the degrees of the polynomials (for a general proof see Łojasiewicz [**5**]). Actually, for the proof of estimate (1) in dimension n, only Bézout's theorem in dimension $n-1$ is needed. This means that when $n=2$ one can dispense with Bézout's theorem altogether and just use the Fundamental Theorem of Algebra. When $n=3$, we need Bézout's theorem in the plane, a case for which elementary proofs are known, using the concept of resultant of two polynomials (see, e.g., [**2**] and [**9**]).

Earlier proofs of the results, that we are aware of, can be found in Bass, Connell and Wright [**1**], Rusek and Winiarski [**8**], Płoski [**6**], Yu [**10**] (which is rather simple and also uses Bézout's theorem) and van den Essen [**3**].

2. Complex Analysis preliminaries. The following facts from Complex Analysis are easy, and something similar has probably already appeared in textbooks. We provide a proof for the convenience of the reader.

PROPOSITION 1. *Let $n \geq 2$, $m \geq 1$, Ω' be a nonempty open subset of $\mathbb{C}^{n-1}$, $a_0, \dots, a_m \colon \Omega' \to \mathbb{C}$ be holomorphic functions, with a_m not identically 0. Define*

$$f(z', z_n) := a_0(z') + a_1(z')z_n + \cdots + a_m(z')z_n^m \qquad \text{for } z' \in \mathbb{C}^{n-1},\ z_n \in \mathbb{C}. \tag{2}$$

Then there exist a nonempty open subset Ω'' of Ω' and holomorphic functions $\alpha_1, \dots, \alpha_M \colon \Omega'' \to \mathbb{C}$ and integers $m_1, \dots, m_M$ such that $\alpha_1(z'), \dots, \alpha_M(z')$ are pairwise distinct for any $z' \in \Omega''$ and f factorizes as

$$f(z', z_n) = a_m(z') \prod_{k=1}^{M} \big(z_n - \alpha_k(z')\big)^{m_k} \qquad \text{for all } z' \in \Omega'',\ z_n \in \mathbb{C}. \tag{3}$$

PROOF. We can assume that $a_m \neq 0$ on all of Ω' (otherwise remove from Ω' the set of zeros of a_m, which is closed with empty interior). Our claim is obvious if $m = 1$. Suppose it is true for all $r < m$ and let us prove it for $r = m$. The set of zeros of f is nonempty. Let $(\bar{z}', \bar{z}_n) \in \Omega' \times \mathbb{C}$ a zero of f with minimum order $k_1 \geq 1$ with respect to z_n. By Weierstrass preparation theorem, possibly after shrinking Ω', we can factorize f as

$$f(z', z_n) = p(z', z_n)h(z', z_n)\,, \tag{4}$$

where

$$p(z', z_n) = b_0(z') + \cdots + b_{k_1-1}(z') z_n^{k_1-1} + z_n^{k_1}, \tag{5}$$

$h, b_1, \ldots, b_{k_1-1}$ are holomorphic, and p and h share no zero. Now, for all z' the roots of the polynomial mapping $z_n \mapsto p(z', z_n)$ must all coincide, because otherwise there would be a zero of p, and hence of f, with multiplicity strictly less than k_1 with respect to z_n. Call this common root $\alpha_1(z')$. The function α_1 is holomorphic, and we can write

$$f(z', z_n) = \big(z_n - \alpha_1(z')\big)^{k_1} h(z', z_n). \tag{6}$$

The function h is seen now to be obtained by dividing the one-variable polynomial $z_n \mapsto f(z', z_n)$ k_1 times by the monomial $z_n - \alpha_1(z')$. Hence h is a function of the same form of f:

$$h(z', z_n) = b_0(z') + b_1(z') z_n + \cdots + b_{m-k_1}(z') z_n^{m-k_1}, \tag{7}$$

with b_i holomorphic and $b_{m-k_1} = a_m$. We can apply the induction hypothesis on h and get the result. □

Proposition 2. *Let $f\colon \mathbb{C}^n \to \mathbb{C}$ be an n-variable polynomial whose gradient never vanishes. Then there exists $\bar{z}' \in \mathbb{C}^{n-1}$ such that all zeros of $z_n \mapsto f(\bar{z}', z_n)$ are simple.*

Proof. Let m be the degree of f with respect to z_n. If $m = 0$ the claim is trivially true. If $m > 0$ we can apply the previous Proposition, thus there exists a nonempty open set $\Omega' \subset \mathbb{C}^{n-1}$ and holomorphic, everywhere distinct functions $\alpha_1, \ldots, \alpha_M \colon \Omega' \to \mathbb{C}$, and integers $m_1, \ldots, m_M$, such that

$$f(z', z_n) = a_m(z') \prod_{k=1}^{M} \big(z_n - \alpha_k(z')\big)^{m_k} \tag{8}$$

over $\Omega' \times \mathbb{C}$; a_m is polynomial and does not vanish in Ω'. For no k can the exponent m_k be larger than 1, because otherwise the gradient of f would vanish at the points of the form $(z', \alpha_k(z'))$, as one can readily verify by differentiation. Hence for any $z' \in \Omega$ the one-variable polynomial $z_n \mapsto f(z', z_n)$ has m distinct roots. □

3. Estimate of the degree of the inverse.

Theorem 3. *Let $F\colon \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial automorphism. Then*

$$\deg F^{-1} \le (\deg F)^{n-1}. \tag{9}$$

Proof. Denote by $(z_1, \dots, z_n)$ the variables in $\mathbb{C}^n$, $F = (F_1, \dots, F_n)$, $F^{-1} = G = (G_1, \dots, G_n)$. Up to a linear change of coordinates, we can assume that the degree of G_1 with respect to z_n coincides with the full degree of G:

$$\deg G = \deg G_1 = \deg_{z_n} G_1 =: m \geq 1. \tag{10}$$

The gradient of G_1 never vanishes, because it is the first row of the Jacobian matrix of G. Hence we can apply the previous Proposition to G_1: there exists $\bar{z}' \in \mathbb{C}^{n-1}$ such that the number of distinct roots of the one-variable polynomial $z_n \mapsto G_1(\bar{z}', z_n)$ is the same as its degree:

$$m = \deg G = \deg_{z_n} G_1 = \#\big\{z_n \in \mathbb{C} \ : \ G_1(\bar{z}', z_n) = 0\big\}. \tag{11}$$

($\#A$ means the cardinality of the set A). Since G is bijective and $G(F(w)) = w$, we can write

$$\begin{aligned}
m &= \#\big\{z_n \in \mathbb{C} \ : \ G_1(\bar{z}', z_n) = 0\big\} \\
&= \#\big\{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} \ : \ z' = \bar{z}', \ G_1(\bar{z}', z_n) = 0\big\} \\
&= \#G\big(\big\{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} \ : \ z' = \bar{z}', \ G_1(\bar{z}', z_n) = 0\big\}\big) \\
&= \#\big\{w \in \mathbb{C}^n \ : \ \exists (z', z_n) \text{ s.t. } w = G(z', z_n), z' = \bar{z}', \ G_1(\bar{z}', z_n) = 0\big\} \\
&= \#\big\{w \in \mathbb{C}^n \ : \ \exists (z', z_n) \text{ s.t. } (z', z_n) = F(w), z' = \bar{z}', G_1(\bar{z}', z_n) = 0\big\} \\
&= \#\big\{w \in \mathbb{C}^n \ : \ \exists z_n \text{ s.t. } (F_1(w), \dots, F_{n-1}(w)) = \bar{z}', \\
&\qquad\qquad\qquad\qquad F_n(w) = z_n, G_1(F(w)) = 0\big\} \\
&= \#\big\{w \in \mathbb{C}^n \ : \ (F_1(w), \dots, F_{n-1}(w)) = \bar{z}', \ w_1 = 0\big\} \\
&= \#\big\{(w_2, \dots, w_n) \in \mathbb{C}^{n-1} \ : \ F_1(0, w_2, \dots, w_n) = \bar{z}_1, \dots, \\
&\qquad\qquad\qquad\qquad F_{n-1}(0, w_2, \dots, w_n) = \bar{z}_{n-1}\big\}.
\end{aligned}$$

This means that the degree of G is the same as the number of solutions of the following system of $n-1$ polynomial equations in the $n-1$ unknowns $w_2, \dots, w_n$:

$$\begin{cases} F_1(0, w_2, \dots, w_n) = \bar{z}_1 \\ \qquad\vdots \\ F_{n-1}(0, w_2, \dots, w_n) = \bar{z}_{n-1} \end{cases} \tag{12}$$

($\bar{z}_1, \dots, \bar{z}_{n-1}$ are fixed). Hence the number of solutions of this system is finite, and by Bézout's theorem

$$m \leq (\deg F_1)(\deg F_2) \cdots (\deg F_{n-1}) \leq (\deg F)^{n-1}. \tag{13}$$

□

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