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THE STRONG UNICITY CONSTANT FOR PROJECTIONS

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Abstract. Let $Y \subset l_\infty^n$ be a linear subspace and let $\mathcal{P}(l_\infty^n, Y)$ denote the set of linear projections. An estimation and calculation (in some particular cases) of the strong unicity constant for a minimal or cominimal projection $P_o \in \mathcal{P}(l_\infty^n, Y)$ will be presented.

1. Introduction. Let X be a normed space and let $Y \subset X$ be a linear subspace of X. The symbol $\mathcal{L}(X, Y)$ means the set of all linear continuous mappings from X to Y. A bounded linear operator P is called a *projection* if $Py = y$ for any $y \in Y$. Denote by $\mathcal{P}(X, Y)$ the set of all projections from X onto Y.

Definition 1.1. If $\mathcal{P}(X, Y) \neq \emptyset$ then a projection $P_o \in \mathcal{P}(X, Y)$ is called *minimal* iff

$$\|P_o\| = \lambda(Y, X) = \inf\{\|P\| : P \in \mathcal{P}(X, Y)\}. \tag{1.1}$$

Let Id be an identity on X.

Definition 1.2. If $\mathcal{P}(X, Y) \neq \emptyset$ then a projection $P_o \in \mathcal{P}(X, Y)$ is called *cominimal* iff

$$\|Id - P_o\| = \lambda_I(Y, X) = \inf\{\|Id - P\| : P \in \mathcal{P}(X, Y)\}. \tag{1.2}$$

The significance of this notion can be illustrated by the following well known inequality:

$$(1 + \|P\|)\operatorname{dist}(x, Y) \geq \|Id - P\|\operatorname{dist}(x, Y) \geq \|(Id - P)(x)\| \geq \operatorname{dist}(x, Y)$$

for every $x \in X \backslash Y$ and $P \in \mathcal{P}(X, Y)$.

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This means that if $\|P\|$ or $\|Id - P\|$ is small then Px is a "good" linear replacement of any $x \in X$ in Y. It is easily seen that

$$\|Id - P\| \geq 1 \text{ for every } P \in \mathcal{P}(X, Y).$$

It is also clear that if P_o is a cominimal projection then

$$\|Id - P_o\| = \text{dist}(Id, \mathcal{P}(X, Y)).$$

For more information concerning minimal and cominimal projections the reader is referred to [**67891112136789111213678911121367891112136789111213678911121367891111**
A more exhaustive list of references can be found in [**20**].

Lemma 1.3. (see, e.g., [**5**]). *Assume that X is a normed space and let $Y \subset X$ be a subspace of codimension k, $Y = \bigcap_{i=1}^{k} ker g^i$, where $g^i \in X^*$ are linearly independent. Let $P \in \mathcal{P}(X, Y)$. Then there exist $y^1, \dots, y^k \in X$ satisfying*

$$g^i(y^j) = \delta_{i,j}, \quad i, j = 1, \dots, k \tag{1.3}$$

such that

$$Px = x - \sum_{i=1}^{k} g^i(x) y^i \quad \text{for} \quad x \in X. \tag{1.4}$$

On the other hand, if $y^1, \dots, y^k \in X$ satisfy (1.3) then operator

$$P = Id - \sum_{i=1}^{k} g^i(\cdot) y^i$$

belongs to $\mathcal{P}(X, Y)$.

Definition 1.4. It is said that a projection $P \in \mathcal{P}(X, Y)$ is determined by $y^1, \dots, y^k \in X$ iff $y^1, \dots, y^k \in X$ satisfy (1.3) and (1.4).

Definition 1.5. Let Y_1, Y_2 be two linear subspaces of X. It is said that Y_1 is equivalent up to isometry to Y_2 iff there is a linear, surjective isometry T of X into itself such that $T(Y_1) = Y_2$.

Lemma 1.6. (see, e.g., [**20**]). *Let Y_1, Y_2 be two linear subspaces of X such that Y_1 is equivalent up to isometry to Y_2. Then $\lambda(Y_1, X) = \lambda(Y_2, X)$ and $\lambda_I(Y_1, X) = \lambda_I(Y_2, X)$.*

Now we recall some information about the strongly unique best approximation (theory of strong unicity has its origin in [**19**]).

Definition 1.7. It is said that $v_o \in V$ is a strongly unique best approximation (SUBA) for x in V iff there exists a constant $r > 0$ such that for every $v \in V$

$$\|x - v\| \geq \|x - v_o\| + r\|v - v_o\|. \tag{1.5}$$

The largest constant $r > 0$ satisfying (1.5) is called the strong unicity constant.

The notion of strong unicity leads to a simple proof of the Freud Theorem about the Lipschitz continuity of the best approximation mapping (see [**8**], p. 82).
Another application of the strong unicity is the estimate of the error of the algorithm for seeking for best approximation (see, e.g., [**8**], p. 98).
One can find further information of SUBA in [**1341019134101913410191341019134101 9**].

The aim of this paper is to estimate or calculate the strong unicity constant for minimal and cominimal projections in l^n_∞.
We now present some definitions and results which will be of use later.
Let $P \in \mathcal{P}(X, Y)$ and

$$\mathcal{L}_Y(X, Y) = \{L \in \mathcal{L}(X, Y) : L|_Y = 0\}.$$

Then $\mathcal{P}(X, Y) = P + \mathcal{L}_Y(X, Y)$ and

$$\lambda(Y, X) = dist(P, \mathcal{L}_Y(X, Y)).$$

Additionaly, $P_o \in \mathcal{P}(X, Y)$ is a minimal projection iff the operator 0 is an element of best approximation for P_o in $\mathcal{L}_Y(X, Y)$.
Analogously,

$$\lambda_I(Y, X) = dist(Id - P, \mathcal{L}_Y(X, Y)). \tag{1.6}$$

$P_o \in \mathcal{P}(X, Y)$ is a cominimal projection iff the operator 0 is is an element of best approximation for $Id - P_o$ in $\mathcal{L}_Y(X, Y)$.

Definition 1.8. It is said that a minimal (cominimal) projection $P_o \in \mathcal{P}(X, Y)$ is an element of best approximation iff the operator 0 is a strongly unique best approximation for P_o $(Id - P_o)$ in $\mathcal{L}_Y(X, Y)$.

Notice that

Remark 1.9. If a minimal projection $P_o \in \mathcal{P}(X, Y)$ is the strongly unique best approximation then there exists $r > 0$ such that for every projection $P \in \mathcal{P}(X, Y)$

$$\|P\| \geq \|P - P_o\| + r\|P - P_o\|. \tag{1.7}$$

If a cominimal projection $P_o \in \mathcal{P}(X,Y)$ is the strongly unique best approximation then there exists $r > 0$ such that for every projection $P \in \mathcal{P}(X,Y)$

$$\|Id - P\| \geq \|Id - P_o\| + r\|P - P_o\|. \tag{1.8}$$

The largest constant $r > 0$ satisfying (1.7) or (1.8) is called the strong unicity constant for projections.

Let X be a normed space and let $V \subset X$ be a nonempty set. By $ext(V)$ we denote the set of its extreme points. For any $x \in X$

$$E(x) = \{f \in X^* : \|f\| = 1, f(x) = \|x\|\} \tag{1.9}$$

and if $S(X)$ denotes the unit sphere in X,

$$Ext(x) = \{f \in ext(S(X^*)) : f(x) = \|x\|\}. \tag{1.10}$$

Definition 1.10. ([**22**]). Let X be a normed space and let $V \subset X$ be a n-dimensional linear subspace. A set $I = \{\phi^1, \dots, \phi^k\} \in ext(S(X^*))$ is called *I-set* iff there exist positive numbers $\lambda^1, \dots, \lambda^k$ such that

$$\sum_{i=1}^{k} \lambda^i \phi^i|_V = 0. \tag{1.11}$$

If $I \subset E(x)$, then I is called an *I-set with respect to x.* An *I-set* I is said to be *minimal* if there is no proper subset of I which forms an I-set. A *minimal I-set* is called *regular* iff $k = n+1$ (by the Carathéodory theorem, $n+1$ is the largest possible number).

The importance of regular I-sets is illustrated by

Theorem 1.11. ([**22**]). *Let X be a real normed space and let V be an n-dimensional linear subspace. Let $x \in X \backslash V$, $v_o \in V$. If there exists a regular I-set for $x - v_o$, then v_o is the strongly unique best approximation for x in V.*

Theorem 1.12. ([**21**]). *Let X be a finite dimensional normed space. Then*

$$ext(S((\mathcal{L}(X,X))^*)) = ext(S(X^*)) \otimes ext(S(X)),$$

where $(x^* \otimes x)(L) = x^*(Lx)$ for $x \in X$, $x^* \in X^*$ and $L \in \mathcal{L}(X,X)$.

Let n, $k \in \mathbb{N}$, $n \geq 3$ and $n \geq k$. Let $X = l^n_\infty$, $Y = \bigcap_{i=1}^k \ker g^i$, where $g^i \in S(l_1^{(n)})$ are linearly independent. Let $P \in \mathcal{P}(X,Y)$. By Lemma 1.3, there exist $y^i \in X$, $i \in \{1, \dots, k\}$ such that $P = Id - \sum_{i=1}^k g^i(\cdot) y^i$. Then

Lemma 1.13. ([**16**]).

$$\|Id - P\| = \max_{j \in \{1,\dots,n\}} \left(\sum_{s=1}^{n} \left| \sum_{i=1}^{k} g^i_s y^i_j \right| \right). \tag{1.12}$$

THEOREM 1.14. ([**16**]). *Let* $g^1, g^2, \ldots, g^k \in S(X^*)$ $k \geq n$ *be linearly independent functionals such that* $g^i_j \geq 0$ *for every* $i \in \{1, 2, \ldots, k\}$, $j \in \{1, 2, \ldots, n\}$, $g^i_i > 0$, $g^i_j = 0$ *for every* $i, j \in \{1, 2, \ldots, k\}$, $i \neq j$. *Let* $P_o \in \mathcal{P}(X, Y)$ *and* $y^i \in X$ $(i \in \{1, 2, \ldots, k\})$ *determine* P_o *(see Def. 1.4). Then* $\|Id - P_o\| = 1$ *iff* $\mathrm{supp}(g^i) \cap \mathrm{supp}(g^j) = \emptyset$ *for every* $i \neq j$, *where*

$$\mathrm{supp}(g^i) = \{k : g^i_k \neq 0\}.$$

Moreover, if $g^i_j \neq 0$, *then for every* $t \in \{1, 2, \ldots, k\}$

$$y^t_j = \begin{cases} 0 & \text{for} \quad i \neq t \\ 1 & \text{for} \quad i = t. \end{cases} \tag{1.13}$$

2. The strong unicity constant. Let X be a real normed space and let $V \subset X$ be a N-*dimensional* linear subspace. Suppose that $x \in X$, $v_o \in V$. Let

$$I = \{\phi^1, \ldots, \phi^{N+1}\} \in ext(S(X^*)), \tag{2.1}$$

with positive constants $\lambda^1, \ldots, \lambda^{N+1}$ satisfying

$$\sum_{j=1}^{N+1} \lambda^j = 1, \tag{2.2}$$

be a regular *I-set with respect to* $x - v_o$ (see Def. 1.10).

LEMMA 2.1. *Let* $v \in V$. *If for every* $i \in \{1, \ldots, N+1\}$

$$\phi^i(v) = 0 \tag{2.3}$$

then $v = 0$.

PROOF. By the regularity of *I-sets*, every N elements of the *I-set* I are linearly independent in restriction to V. This proves the Lemma. □

LEMMA 2.2. *Let* $\sigma \in P_{N+1}$ *be a permutation of the set* $\{1, \ldots, N+1\}$. *For every subset of an* I*-set* I *of the form* $\phi^{\sigma(1)}, \ldots, \phi^{\sigma(N)}$ *there exists a basis* $v^1, \ldots, v^N$ *of the subspace* V *such that*

$$\phi^{\sigma(i)}(v^j) = \delta_{i,j}, \quad i, j = 1, \ldots, N, \tag{2.4}$$

and $\|v^i\| \geq 1$ *for every* $i \in \{1, \ldots, N\}$.

PROOF. Without loss of generality we assume that $\sigma(i) = i$ for every $i \in \{1, \ldots, N\}$.
By the regularity of the *I-set* I, the functionals $\phi^1|_V, \ldots, \phi^N|_V$ are linearly independent and form the basis of the subspace V^*. By the regularity of the *I-set* I, this implies the existence of vectors $v^1, \ldots, v^N$ satisfying (2.4). Now

we show that for every $i \in \{1,\dots,N\}$, $\|v^i\| \geq 1$. Suppose that there exists a vector $v^i \in V$ satisfying (2.4) and $\|v^i\| < 1$.
Then $v^i \neq 0$ and $\phi^i(\frac{v^i}{\|v^i\|}) = \frac{1}{\|v^i\|} > 1$, which contradicts assumption (2.1). □

Now we calculate the strong unicity constant r (see Def. 1.7) using functionals ϕ^i by (2.1). Since $\phi^i(x - v_o) = \|x - v_o\|$ for every $v \in V$, $v \neq v_o$ we get

$$\begin{aligned}\phi^i\left(\frac{v_o - v}{\|v_o - v\|}\right) &= \phi^i\left(\frac{v_o - x + x - v}{\|v - v_o\|}\right)\\ &= \frac{\phi^i(v_o - x)}{\|v - v_o\|} + \frac{\phi^i(x - v)}{\|v - v_o\|}\\ &\leq \frac{-\|x - v_o\| + \|x - v\|}{\|v - v_o\|}.\end{aligned}$$

So for every $i \in \{1,\dots,N+1\}$

$$\phi^i\left(\frac{v_o - v}{\|v - v_o\|}\right)\|v - v_o\| + \|x - v_o\| \leq \|x - v\|.$$

Put

$$r = \min\big\{\max\big\{\phi^i(\frac{v_o - v}{\|v_o - v\|}) : i \in \{1,\dots,n+1\}\big\} : v \in V\big\}. \tag{2.5}$$

Notice that for every $v \in V$

$$\|x - v\| \geq \|x - v_o\| + r\|v - v_o\|. \tag{2.6}$$

By the regularity of *I-set* (2.1) and Lemma 2.1, $r > 0$. It is easy to see that the constant r given by (2.5) is the strong unicity constant for $x - v_o$ (see Def. 1.7).

For $\lambda^1,\dots,\lambda^{n+1}$ satisfying (2.2), let

$$\lambda_{min} := \min\{\lambda^j : j \in \{1,\dots,N+1\}\}. \tag{2.7}$$

Let $k \in \{1,\dots,N+1\}$. Now for functionals ϕ^i $(i \in \{1,\dots,k-1,k+1,\dots,N+1\})$ we find vectors $v^i(k)$ by Lemma 2.2. Let

$$l(k) = \min\big\{\phi^i(\frac{v^i(k)}{\|v^i(k)\|}) : i \in \{1,\dots,k-1,k+1,\dots,N+1\}\big\}, \tag{2.8}$$

$$l := \max\{l(k) : k \in \{1,\dots,N+1\}\}. \tag{2.9}$$

Now we may state

Theorem 2.3.

$$r \geq l \cdot \frac{\lambda_{min}}{2 - \lambda_{min}}.$$

Proof. Fix $v \in S_V$. Without loss of generality we assume that $l = l(N+1)$. First assume that

$$\phi^{N+1}(v) = \max_{i \in \{1,\dots,N+1\}} \{\phi^i(v)\}. \tag{2.10}$$

By Lemma 2.2, we find vectors $v^i \in V$ satisfying (2.4) for functionals $\phi^1, \dots, \phi^N$. Notice that $\frac{v^i}{\|v^i\|}$ form a basis of the subspace V. So there exist numbers $a_i(v) \in R$ such that

$$v = \sum_{i=1}^{N} a_i(v) \frac{v^i}{\|v^i\|}. \tag{2.11}$$

Moreover, for every $i \in \{1, \dots, N\}$, $\phi^i(\frac{v^i}{\|v^i\|}) > 0$. Notice that

$$1 = \|v\| \leq \sum_{i=1}^{N} |a_i(v)|. \tag{2.12}$$

Since $\phi^i(v^j) = \delta_{i,j}$, then for $i \in \{1, \dots, N\}$

$$\phi^{N+1}(v) \geq \phi_i(v) = a_i(v)\ \phi^i\left(\frac{v^i}{\|v^i\|}\right). \tag{2.13}$$

By (1.11),

$$\begin{aligned}\lambda^{N+1}\phi^{N+1}(v) = & \sum_{i=1}^{N} \lambda^i(-\phi^i(v)) \\ = & \sum_{i=1}^{N} \lambda^i(-a_i(v))\phi^i\left(\frac{v^i}{\|v^i\|}\right).\end{aligned}$$

By (2.2),

$$\phi^{N+1}(v) = \sum_{i=1}^{N} \lambda^i\left(\phi^{N+1}(v) - a_i(v)\phi^i\frac{v^i}{\|v^i\|}\right).$$

The coordinates $a_i(v)$ may be positive, negative or equal to 0, but by (2.13), the number $(\phi^{N+1}(v) - a_i(v)\phi^i\frac{v^i}{\|v^i\|})$ is not negative for every $i \in \{1, \dots, N\}$. Taking everything into consideration, we get

$$\begin{aligned}\sum_{i=1}^{N}\left(\lambda^i|a_i(v)|\phi^i(\frac{v^i}{\|v^i\|})\right) \leq & \sum_{i=1}^{N}\lambda^i\left(|a_i(v)|\phi^i(\frac{v^i}{\|v^i\|}) - \phi^{N+1}(v)\right) + \sum_{i=1}^{N}\lambda_i\phi^{N+1}(v) \\ \leq & \sum_{i=1}^{N}\lambda^i\left|a_i(v)\phi^i(\frac{v^i}{\|v^i\|}) - \phi^{N+1}(v)\right| + (1-\lambda^{N+1})\phi^{N+1}(v)\end{aligned}$$

$$= \sum_{i=1}^{N} \lambda^i \Big(\phi^{N+1}(v) - a_i(v)\phi^i(\frac{v^i}{\|v^i\|}) \Big) + (1-\lambda^{N+1})\phi^{N+1}(v)$$

$$= (2-\lambda^{N+1})\phi^{N+1}(v) \leq (2-\lambda_{min})\phi^{N+1}(v).$$

By (2.8),

$$\phi^i(\frac{v^i}{\|v^i\|}) \geq l(N+1) = l$$

for $i \in \{1, \dots, n\}$. By (2.12)

$$\phi^{N+1}(v) \geq \frac{\lambda_{min}}{2-\lambda_{min}} \cdot l(N+1). \tag{2.14}$$

Taking the infimum on the left side in (2.14), by (2.9), we get the result.
Now suppose that

$$\phi^{N+1}(v) < \max_{i \in \{1,\dots,N+1\}} \{\phi^i(v)\}.$$

Without loss of generality we may assume that

$$\phi^1(v) = \max_{i \in \{1,\dots,N+1\}} \{\phi^i(v)\}.$$

$$\lambda^1\phi^1(v) = \sum_{i=2}^{N+1} \lambda^i(-\phi^i(v)).$$

Analogously, by (1.11) and (2.2),

$$\phi^1(v) = \sum_{i=2}^{N+1} \lambda^i(\phi^1(v) - \phi^i(v)).$$

$\phi^1(v) \geq \phi^i(v)$ so

$$\phi^1(v) \geq \sum_{i=2}^{N} \lambda^i(\phi^1(v) - \phi^i(v)). \tag{2.15}$$

By Lemma 2.2, we take the same vectors $v^i \in V$ as above for functionals $\phi^1, \dots, \phi^N$, so (2.11) is satisfied. Analogously the numbers

$$\big(\phi^1(v) - a_i(v)\phi^i\frac{v^i}{\|v^i\|}\big), \quad (\phi^1(v) - \phi^{N+1}(v)) \tag{2.16}$$

are not negative for every $i \in \{2, \dots, N\}$.
By (2.15), (2.2), (2.16), reasoning in the same way as in the previous situation, we get

$$(2 - 2\lambda_{min})\phi^1(v) \geq \sum_{i=2}^{N} \lambda^i |a_i(v)| \phi^i(\frac{v^i}{\|v^i\|}).$$

Hence

$$(2-\lambda_{min})\phi^1(v) \geq \sum_{i=2}^{N} \lambda^i |a_i(v)| \phi^i\big(\frac{v^i}{\|v^i\|}\big) + \lambda_{min}\phi^1(v),$$

where

$$\phi^1(v) = a_1(v)\phi^1\big(\frac{v^1}{\|v^1\|}\big) \geq |a_1(v)| l(N+1).$$

Hence by (2.12), (2.9)

$$(2-\lambda_{min})\phi^1(v) \geq \lambda_{min} \cdot l.$$

Taking the infimum and applying (2.9), we get the result. □

3. The strong unicity constant for minimal and cominimal projections.

DEFINITION 3.1. Let X be a normed space, $Y \subset X$ a linear subspace and $P_o \in \mathcal{P}(X,Y)$ a cominimal projection. It is said that P_o is determined by *I-set* iff there exists a regular *I-set* with respect to $Id - P_o$ (see Def. 1.10 and Theorem 1.11).

Let n, $k \in \mathbb{N}$, $n \geq 3$, $n \geq k$.
Let $X = l_\infty^n$ and $Y = \bigcap_{i=1}^k \ker g^i$, where $g^i \in S(X^*)$ are linearly independent. Let P_o, $P \in \mathcal{P}(X,Y)$, $P = Id - \sum_{i=1}^k g^i(\cdot)y^i$, $P = Id - \sum_{i=1}^k g^i(\cdot)\widetilde{y}^i$, where $\widetilde{y}^i$, $y^i \in X$, $i \in \{1,\dots,k\}$.
Then

LEMMA 3.2.

$$\|P_o - P\| = \max_{i\in\{1,\dots,n\}} \Big\{ \sum_{s=1}^{n} \big| \sum_{j=1}^{k} g_s^j (y_i^j - \widetilde{y}_i^j)\big| \Big\}. \tag{3.1}$$

PROOF. Put $x \in S_X$. Then

$$\|(P_o)(x)\| = \max_{i\in\{1,\dots,n\}} \Big\{ \big| \sum_{j=1}^{k} g^j(x)(y_i^j - \widetilde{y}_i^j)\big| \Big\} \leq \max_{i\in\{1,\dots,n\}} \Big\{ \sum_{s=1}^{n} \big| \sum_{j=1}^{k} g_s^j (y_i^j - \widetilde{y}_i^j)\big| \Big\}.$$

Setting $x = (x_1, x_2, \dots, x_n)$ such that

$$x_s = \begin{cases} sgn \sum_{j=1}^{k} g_s^j(y_i^j - \widetilde{y}_i^j) & \text{if} \quad \sum_{j=1}^{k} g_s^j(y_i^j - \widetilde{y}_i^j) \neq 0 \\ 0 & \text{if} \quad \sum_{j=1}^{k} g_s^j(y_i^j - \widetilde{y}_i^j) = 0 \end{cases}$$

for $s = \{1, 2, \dots, n\}$, we get (3.1).

□

Now, unless stated otherwise, we assume that $k = 2$. Let $g^1, g^2 \in S(X^*)$ be linearly independent functionals such that

$$g^1 = (g_1^1, 0, g_3^1, \dots, g_n^1) \tag{3.2}$$

$$g^2 = (0, g_2^2, g_3^2, \dots, g_n^2), \tag{3.3}$$

$$g_1^1, g_2^2 > 0,\ g_j^1,\ g_j^2 \geq 0 \text{ and } g_j^1 + g_j^2 > 0 \quad \text{for} \quad j \in \{1, \dots, n\}. \tag{3.4}$$

Suppose that

$$\det \begin{bmatrix} g_i^1 & g_j^1 \\ g_i^2 & g_j^2 \end{bmatrix} \neq 0 \tag{3.5}$$

for every $i, j \in \{1, 2, \dots, n\}$, $i \neq j$. Moreover, we assume that

$$\frac{g_3^1}{g_3^2} < \frac{g_4^1}{g_4^2} < \dots < \frac{g_n^1}{g_n^2}. \tag{3.6}$$

Hence $Y = \ker g^1 \cap \ker g^2$ is a subspace of codimension 2 in R^n.
Let $y^1, y^2 \in R^n$ satisfy (1.3), $P_o \in \mathcal{P}(X, Y)$ be projection determined by y^1, y^2 (see Def. 1.4), which means that

$$(Id - P_o)(x) = g^1(x)y^1 + g^2(x)y^2.$$

First assume that $n = 3$.

Lemma 3.3. *Let $P \in \mathcal{P}(X, Y)$ and let $P_o \in \mathcal{P}(X, Y)$ be a cominimal projection determined by an I-set*

$$\phi^1 = e_1 \otimes (1, -1, 1), \quad \phi^2 = e_2 \otimes (-1, 1, 1), \quad \phi^3 = e_3 \otimes (1, 1, 1). \tag{3.7}$$

Then

$$\|P_o - P\| \leq \max\big\{|\phi^1(P_o - P)|, |\phi^2(P_o - P)|, |\phi^3(P_o - P)|\big\} \cdot \max\Big\{\frac{g_1^1}{g_3^1}, \frac{g_2^2}{g_3^2}, 1\Big\} \cdot \|w^3\|. \tag{3.8}$$

Proof. Notice that by Theorem 2.5 in [**16**], if g^1, g^2 satisfy (3.2)–(3.4), then the functionals ϕ^1, ϕ^2, ϕ^3 by (3.7) form a regular I-set. By Theorem 3.2 and Theorem 3.9 in [**16**], P_o determined by I-set (3.7) is cominimal.
For every projection $P \in \mathcal{P}(X, Y)$, $P_o - P \in \mathcal{L}_Y(X, Y)$ and *dim* $\mathcal{L}_Y(X, Y) = 2(3 - 2) = 2$.
Moreover, the operators $\{g^1(\cdot)w^3, g^2(\cdot)w^3\}$, where $w^3 = \big(\frac{-g_3^1}{g_1^1}, \frac{-g_3^2}{g_2^2}, 1\big) \in X$, form a basis of the space $\mathcal{L}_Y(X, Y)$. Hence

$$(P_o - P)(x) = \alpha g^1(x) + \beta g^2(x),$$

for some $\alpha, \beta \in R$. By Lemma 3.2,

$$\|P_o - P\| = |g^1(x)\alpha + g^2(x)\beta| \, \|w^3\|,$$

where $x = \pm(1,-1,1)$ or $x = \pm(-1,1,1)$ or $x = \pm(1,1,1)$. Hence

$$\|P_o - P\| = \max\left\{\frac{g_1^1}{g_3^1}|\phi^1(P_o - P)|, \frac{g_2^2}{g_3^2}|\phi^2(P_o - P)|, |\phi^3(P_o - P))|\|w^3\|\right\}.$$

Finally

$$\|P_o - P\| \le \max\big\{|\phi^1(P_o - P)|, |\phi^2(P_o - P)|, |\phi^3(P_o - P)|\big\} \max\big\{\frac{g_1^1}{g_3^1}, \frac{g_2^2}{g_3^2}, 1\big\}\|w^3\|.$$

□

Keeping the assumption of Lemma 3.3 we get

THEOREM 3.4. *Let λ^1, λ^2, $\lambda^3 > 0$ be the constants (see Def. 1.10) for I-set (3.7). Put*

$$\lambda_{min} = min\{\lambda^i : i = 1,2,3\},$$
$$\lambda_{max} = max\{\lambda^i : i = 1,2,3\},$$
$$w^3 = \big(\frac{-g_3^1}{g_1^1}, \frac{-g_3^2}{g_2^2}, 1\big).$$

Then

$$r \ge \frac{\lambda_{min}}{\lambda_{max}} \cdot \frac{\min\left\{\frac{g_3^1}{g_1^1}, \frac{g_3^2}{g_2^2}, 1\right\}}{\|w^3\|}. \tag{3.9}$$

PROOF. By (2.5), it immediately follows that for every $v \in S(\mathcal{L}_Y(X,Y))$ it is sufficient to estimate from the number

$$\max\{\phi^i(v) : i \in \{1,2,3\}\}$$

from below. Since $v = P - P_o$ for some $P \in \mathcal{P}(X,Y)$, $(P \neq P_o)$ by Lemma 3.3, we get

$$\max\big\{|\phi^1(v)|, |\phi^2(v)|, |\phi^3(v)|\big\} \ge \frac{\min\left\{\frac{g_3^1}{g_1^1}, \frac{g_3^2}{g_2^2}, 1\right\}}{\|w^3\|}. \tag{3.10}$$

Without loss of generality we may assume that

$$\phi^1(v) = \max\{\phi^1(v), \phi^2(v), \phi^3(v)\}.$$

Since ϕ^1, ϕ^2, ϕ^3 form a regular *I-set*, there is $\phi^1(v) > 0$.
If $\phi^1(v) < \max\{|\phi^1(v)|, |\phi^2(v)|, |\phi^3(v)|\}$, then by (1.11)

$$\lambda^1\phi^1(v) = -\lambda^2\phi^2(v) - \lambda^3\phi^3(v).$$

Hence

$$\phi^1(v) = \frac{\lambda^2}{\lambda^1}(-\phi^2(v)) + \frac{\lambda^3}{\lambda^1}(-\phi^3(v)).$$

It is easily seen that for $i=2$ or $i=3$

$$-\phi^i(v) = \max\left\{|\phi^1(v)|, |\phi^2(v)|, |\phi^3(v)|\right\} > 0.$$

Hence

$$\phi^1(v) \geq -\frac{\lambda_{min}}{\lambda_{max}}\phi^i(v)$$

$$\geq \frac{\lambda_{min}}{\lambda_{max}} \frac{\min\left\{\frac{g_3^1}{g_1^1}, \frac{g_3^2}{g_2^2}, 1\right\}}{\|w^3\|},$$

and by (2.5), we get the result.
If $\phi^1(v) = max\{|\phi^1(v)|, |\phi^2(v)|, |\phi^3(v)|\}$, then the theorem immediately follows from Lemma 3.3. □

Remark 3.5. The previous estimate is satisfied for $n=3$ only because of the form of vectors x building functionals ϕ^1, ϕ^2, ϕ^3.

Now estimate a strong unicity constant for projections (see Remark 1.9) in the case of $n \geq 3$.

Let $s \in \{3,\ldots,n\}$, $p \in \{1,2,s\}$ and $k \in \{3,\ldots,n\}$, $k \neq s$. Let

$$\phi^p = e_p \otimes x^p, \quad \phi_1^k = e_k \otimes x^k, \quad \phi_2^k = e_k \otimes z^k \tag{3.11}$$

$e_t(x) = x_t$ for $x \in R^n$ and $t \in \{1,\ldots,n\}$.

Remark 3.6. Let I be *I-set* of form (3.11). Suppose that this *I-set* determines $Id - P_o$ (see Def. 1.10) with λ^1, λ^2, λ^s, λ_1^k, λ_2^k, ($k \in \{3,\ldots,n\}$, $k \neq s$) such that

$$\lambda^1 + \lambda^2 + \lambda^s + \sum_{k=3,k\neq s}^{n} (\lambda_1^k + \lambda_2^k) = 1. \tag{3.12}$$

Recall that the functional $0 \in V = \mathcal{L}_Y(X,Y)$ is the strongly unique best approximation for $Id - P_o$, $dim\, \mathcal{L}_Y(X,Y) = 2(n-2)$ and a basis of $\mathcal{L}_Y(X,Y)$ is the set $\{g^1(\cdot)w^k, g^2(\cdot)w^k\}$, $k \in \{3,\ldots,n\}$ ($w^k = (\frac{-g_k^1}{g_1^1}, \frac{-g_k^2}{g_2^2}, 0, \ldots, 0, 1, 0, \ldots, 0) \in R^n$, 1 is equal to k-th coordinate).
To estimate a strong unicity constant, we calculate or estimate from above the norm of v by Lemma 2.2.
Any operator $v \in \mathcal{L}_Y(X,Y)$ is of the form

$$v(\cdot) = \sum_{k=3}^{n} \alpha^k g^1(\cdot)w^k + \beta^k g^2(\cdot)w^k. \tag{3.13}$$

Hence

$$\|v\| \leq \sum_{k=3}^{n} \big(|\alpha^k| + |\beta^k|\big)\|w^k\|. \tag{3.14}$$

If v satisfies (2.4), then from (3.13) we calculate the numbers $\{\alpha^k, \beta^k\}$.

REMARK 3.7. Notice that for *I-set* (3.11)

$$\phi^1(v) = \sum_{k=3}^{n} \big(-\frac{g^1_k}{g^1_1}\big)\big(\alpha^k g^1(x^1) + \beta^k g^2(x^1)\big),$$

$$\phi^2(v) = \sum_{k=3}^{n} \big(-\frac{g^2_k}{g^2_2}\big)\big(\alpha^k g^1(x^2) + \beta^k g^2(x^2)\big),$$

$$\phi^s(v) = \alpha^s g^1(x^s) + \beta^s g^2(x^s), \tag{3.15}$$

$$\phi^k_1(v) = \alpha^k g^1(x^k) + \beta^k g^2(x^k),$$

$$\phi^k_2(v) = \alpha^k g^1(z^k) + \beta^k g^2(z^k).$$

By (2.4), (3.15) is a Cramer system of equations.

Now we will show how to estimate the strong unicity constant r satisfying (1.8) in case of a cominimal projection determined by *I-set*. The main technical problem is in calculating or estimating the number $l(k)$ (see (2.8)) for some k or, which gives better accuracy, the number l (see (2.9)).

THEOREM 3.8. *Let $n = 4$ and $Y = \ker g^1 \cap \ker g^2 \subset X$, where $g^1, g^2 \in S(X^*)$ are linearly independent fuctionals satisfying (3.2)–(3.4) and (3.6). Let $P_o \in \mathcal{P}(X, Y)$ be a cominimal projection determined by an I-set (see Theorem 2.5 and Theorem 3.2 in* [**16**]*):*

$$\phi^1 = e_1 \otimes (1, -1, 1, 1), \quad \phi^2 = e_2 \otimes (-1, 1, 1, 1),$$

$$\phi^3 = e_3 \otimes (1, 1, 1, 1), \tag{3.16}$$

$$\phi^4_1 = e_4 \otimes (1, 1, 1, 1), \quad \phi^4_2 = e_4 \otimes (1, -1, 1, 1).$$

Then

$$r \geq \frac{\lambda_{min}}{2 - \lambda_{min}} \frac{\min_{l=1,2}\{g^l_l\} \min\big\{1, \frac{g^l_k}{g^l_l} : l = 1, 2, k = 3, 4\big\}}{\max_{k=3,4}\{\|w^k\|\}}. \tag{3.17}$$

Proof. Using the form of the *I-set* determining the cominimal projection P_o we will estimate the number $l(2)$ (see (2.8)).
First we will calculate vectors $v \in \mathcal{L}_Y(X,Y)$ using Lemma 2.2.
Recall that if $v \in \mathcal{L}_Y(X,Y)$ then v satisfies (3.13). Then

$$\begin{cases} \phi^1(v) = \sum_{k=3}^{4} \big(-\frac{g_k^1}{g_1^1}\big)\big(\alpha^k g^1(x^1) + \beta^k g^2(x^1)\big) \\ \phi^3(v) = \alpha^3 g^1(x^3) + \beta^3 g^2(x^3) \\ \phi_1^4(v) = \alpha^4 g^1(x^4) + \beta^4 g^2(x^4) \\ \phi_2^4(v) = \alpha^4 g^1(z^4) + \beta^4 g^2(z^4), \end{cases} \tag{3.18}$$

where $x^1 = z^4 = (1,-1,1,1)$, $x^2 = (-1,1,1,1)$, $x^3 = x^4 = (1,1,1,1)$.
By the fact that (3.16) forms an *I-set* (see Theorem 2.5 in [**16**]), (3.18) is a Cramer system of equations.
Let $v^2 = v^2(2) \in \mathcal{L}_Y(X,Y)$ satisfy (see Lemat 2.2):

$$\begin{cases} \phi^1(v) = 0 \\ \phi^3(v) = 1 \\ \phi_1^4(v) = 0 \\ \phi_2^4(v) = 0. \end{cases} \tag{3.19}$$

Then

$$\alpha^3 = 1 - \frac{1}{2g_2^2}, \quad \beta^3 = \frac{1}{2g_2^2},$$
$$\alpha^4 = 0, \quad \beta^4 = 0,$$

and

$$\|v^2\| \le (|\alpha^3| + |\beta^3|)\|w_3\| = \max\big\{\frac{1}{g_2^2} - 1, 1\big\}\|w_3\|. \tag{3.20}$$

Analogously, for $v^1 = v^1(2) \in \mathcal{L}_Y(X,Y)$ satisfying

$$\begin{cases} \phi^1(v) = 1 \\ \phi^3(v) = 0 \\ \phi_1^4(v) = 0 \\ \phi_2^4(v) = 0, \end{cases} \tag{3.21}$$

$$\alpha^3 = -\frac{g_1^1}{2g_3^1 g_2^2}, \quad \beta^3 = \frac{g_1^1}{2g_3^1 g_2^2},$$
$$\alpha^4 = 0, \quad \beta^4 = 0,$$

$$\|v^1\| \le \frac{g_1^1}{g_3^1 g_2^2}\|w_3\|. \tag{3.22}$$

For $v^3 = v^3(2) \in \mathcal{L}_Y(X,Y)$, which is given by

$$(3.23)\qquad \begin{cases} \phi^1(v) = 0 \\ \phi^3(v) = 0 \\ \phi^4_1(v) = 1 \\ \phi^4_2(v) = 0, \end{cases}$$

there is

$$(3.24)\qquad \|v^3\| \le \max\Big\{\frac{1}{g_2^2} - 1, 1\Big\}\|w_4\|,$$

and for $v^4 = v^4(2) \in \mathcal{L}_Y(X,Y)$ being the solution of

$$(3.25)\qquad \begin{cases} \phi^1(v) = 0 \\ \phi^3(v) = 0 \\ \phi^4_1(v) = 0 \\ \phi^4_2(v) = 1, \end{cases}$$

we get

$$(3.26)\qquad \|v^4\| \le \frac{g_1^1}{g_4^1 g_2^2}\|w_3\|.$$

Hence if $v \in \mathcal{L}_Y(X,Y)$ is given by Lemma 2.2, then v meets to one of the equalities: (3.19), (3.21), (3.23) or (3.25). Hence

$$(3.27)\qquad \|v\| \le \frac{\max_{k=3,4}\{\|w^k\|\}}{g_2^2 \min\big\{1, \frac{g_k^l}{g_l^l} : l = 1,2, k = 3,4\big\}}$$

and consequently (see (2.8))

$$(3.28)\qquad \begin{aligned} l(2) &\ge \frac{g_2^2 \min\big\{1, \frac{g_k^l}{g_l^l} : l = 1,2, k = 3,4\big\}}{\max_{k=3,4}\{\|w^k\|\}} \\ &\ge \frac{\min_{l=1,2}\{g_l^l\} \min\big\{1, \frac{g_k^l}{g_l^l} : l = 1,2, k = 3,4\big\}}{\max_{k=3,4}\{\|w^k\|\}}. \end{aligned}$$

The result easily follows from Theorem 2.3. □

Now the estimate of the strong unicity constant r satisfying (1.7) for minimal projections will be presented. It concerns a minimal projection determined by the *I-set* from [**14**].

THEOREM 3.9. *Let* $n = 4$ *and* $Y = \ker g^1 \cap \ker g^2 \subset X$*, where* $g^1, g^2 \in S(X^*)$ *are linearly independent functionals satisfying (3.2)–(3.4).*
Let $P_o \in \mathcal{P}(X,Y)$ *be a minimal projection determined by the I-set* (*see* [**14**])

$$\phi^1 = e_2 \otimes (1,1,-1,-1),$$
$$\phi_1^3 = e_3 \otimes (-1,-1,1,-1), \quad \phi_2^3 = e_3 \otimes (-1,1,1,-1),$$
$$\phi_1^4 = e_4 \otimes (-1,-1,-1,1) \quad \phi_2^4 = e_4 \otimes (-1,1,-1,1).$$

Then

$$r \geq \frac{\lambda_{min}}{2-\lambda_{min}} \frac{1}{\Theta}, \tag{3.29}$$

where

$$\Theta = \max\left\{ \frac{1}{2g_i^i}\big(1 + \big|\frac{1-2g_k^i}{1-2g_l^j}\big|\big) : \quad i,j = 1,2, \ k,l = 3,4 \right\} \max\big\{\|w_3\|, \|w_4\|\big\}.$$

PROOF. Notice that in [**14**] one can find the proof of the fact that the above *I-set* determines a minimal projection. Hence, by Theorem 3.8, it is sufficient to calculate or estimate the number Θ. For convience, the constant $l(1)$ (see (2.8)) will be estimated. The idea of the proof is the same as in Theorem 3.8. Let $v^1 = v^1(1) \in \mathcal{L}_Y(X,Y)$ satisfy the system of equations (see Lemma 2.2)

$$\begin{cases} \phi_1^3(v) = 1 \\ \phi_2^3(v) = 0 \\ \phi_1^4(v) = 0 \\ \phi_2^4(v) = 0. \end{cases} \tag{3.30}$$

Hence

$$\alpha^3 = \frac{2g_4^2 - 1}{2g_2^2(1-2g_3^1)}, \quad \beta^3 = \frac{-1}{2g_2^2},$$
$$\alpha^4 = 0, \quad \beta^4 = 0.$$

Thus

$$\|v^1\| \leq (|\alpha^3| + |\beta^3|)\|w_3\| \leq \Theta. \tag{3.31}$$

For $v^2 = v^2(1) \in \mathcal{L}_Y(X,Y)$, which is the solution of

$$\begin{cases} \phi_1^3(v) = 0 \\ \phi_2^3(v) = 1 \\ \phi_1^4(v) = 0 \\ \phi_2^4(v) = 0, \end{cases} \tag{3.32}$$

we get

$$\alpha^3 = \frac{-(1-2g_3^2)}{2g_2^2(1-2g_3^1)}, \quad \beta^3 = \frac{1}{2g_2^2},$$

$$\alpha^4 = 0, \quad \beta^4 = 0.$$

For $v^3 = v^3(1)$ and $v^4 = v^4(1)$, we proceed in the same way. □

REMARK 3.10. Notice that all the above estimates of the strong unicity constant r satisfying (1.7) or (1.8) depend on the number λ_{min}. By assumption (2.2),

$$\lambda_{min} < \frac{1}{N+1},$$

where N is the dimension of the space $V = \mathcal{L}_Y(X, Y)$, so $N = 2n - 4$.

Let $n \in \mathbb{N}$, $n \geq 3$ and $X = l_\infty^n$. Let $g^1, g^2 \in S(X^*)$ be linearly independent functionals satifying (3.2)–(3.4), (3.6); put $Y = \ker g^1 \cap \ker g^2$.

EXAMPLE 3.11. 1. Fix $n = 3$, $g^1 = \left(\frac{1}{3}, 0, \frac{2}{3}\right)$, $g^2 = \left(0, \frac{3}{4}, \frac{1}{4}\right)$. A cominimal projection P_o is determined by *I-set* (3.7), (see Theorem 2.5 and Theorem 3.9 in [**16**]). By Theorem 3.4, $r \geq 0,012346$, where $\lambda_{min} = \lambda^1 \approx 0,05556$, $\lambda_{max} = \lambda^2 = \frac{3}{4}$.
2. Put $n = 4$, $g^1 = \left(\frac{1}{3}, 0, \frac{1}{3}, \frac{1}{3}\right)$, $g^2 = \left(0, \frac{5}{12}, \frac{4}{12}, \frac{3}{12}\right)$. By Theorem 2.5 and Theorem 3.2 in [**16**], a cominimal projection is determined by *I-set* from the thesis of Theorem 3.8. Using the estimate from Theorem 3.8, we get $r \geq 0,004839$.
3. Let $n = 5$, $g^1 = \left(\frac{11}{51}, 0, \frac{1}{51}, \frac{24}{51}, \frac{15}{51}\right)$, $g^2 = \left(0, \frac{11}{81}, \frac{10}{81}, \frac{42}{81}, \frac{18}{81}\right)$. Analogously as in Theorem 2.5 in [**16**], one can check that the system

$$\phi^1 = e_1 \otimes (1, -1, -1, 1, 1), \quad \phi^2 = e_2 \otimes (-1, 1, 1, 1, -1),$$
$$\phi_1^3 = e_3 \otimes (-1, 1, 1, 1, 1), \quad \phi_2^3 = e_3 \otimes (-1, 1, 1, 1, -1),$$
$$\phi^4 = e_4 \otimes (1, 1, 1, 1, 1),$$
$$\phi_1^5 = e_5 \otimes (1, 1, 1, 1, 1), \quad \phi_2^5 = e_5 \otimes (1, -1, 1, 1, 1)$$

form a regular *I-set* (Def. 1.10), which determines a cominimal projection (see Def. 3.1 and Theorem 1.11). By Theorem 2.3 and by the simple calculation, we get $l \geq l(1) \approx 0,024897$ (see (2.8), (2.9)) and $r \geq 0,00045$.
4. Let $n = 7$, $g^1 = \left(\frac{1}{2}, 0, \frac{1}{10}, \frac{9}{200}, \frac{1}{200}, \frac{1}{4}, \frac{1}{10}\right)$, $g^2 = \left(0, \frac{23}{50}, \frac{1}{4}, \frac{1}{10}, \frac{1}{100}, \frac{43}{250}, \frac{2}{250}\right)$. Reasoning in the same way as in Theorem 2.5 in [**16**], we can check that

$$\phi^1 = e_1 \otimes (1, -1, 1, 1, 1, 1, 1), \quad \phi^2 = e_2 \otimes (-1, 1, 1, 1, 1, 1, -1),$$
$$\phi_1^3 = e_3 \otimes (1, 1, 1, 1, 1, 1, 1), \quad \phi_2^3 = e_3 \otimes (-1, 1, 1, 1, 1, 1, 1),$$
$$\phi_1^4 = e_4 \otimes x^4, \quad \phi_2^4 = e_4 \otimes z^4,$$
$$\phi_1^5 = e_5 \otimes x^5, \quad \phi_2^5 = e_5 \otimes z^5,$$
$$\phi^6 = e_6 \otimes (1, 1, 1, 1, 1, 1, 1),$$
$$\phi_1^7 = e_7 \otimes (1, 1, 1, 1, 1, 1, 1), \quad \phi_2^7 = e_7 \otimes (1, -1, 1, 1, 1, 1, 1),$$

where $x^3 = x^4 = x^5$, $z^3 = z^4 = z^5$ form a regular *I-set* which determines a cominimal projection. By Theorem 2.3 (by estimate of $l(1)$), we get $r \geq 0,00023$.

Example 3.12. Let $n = 4$, $g^1 = \left(\frac{1}{3}, 0, \frac{1}{3}, \frac{1}{3}\right)$, $g^2 = \left(0, \frac{5}{12}, \frac{4}{12}, \frac{3}{12}\right)$. Functionals g^1, g^2 satisfy the assumptions of Theorem 3.9, so there exists a minimal projection $P_o \in \mathcal{P}(X, Y)$, where $Y = \ker g^1 \cap \ker g^2 \subset X$. Additionaly one can check (see [**14**]) that the *I-set* which determines the minimal projections P_o is of the form

$$\phi^1 = e_2 \otimes (1, 1, -1, -1),$$

$$\phi_1^3 = e_3 \otimes (-1, -1, 1, -1), \quad \phi_2^3 = e_3 \otimes (-1, 1, 1, -1),$$

$$\phi_1^4 = e_4 \otimes (-1, -1, -1, 1), \quad \phi_2^4 = e_4 \otimes (-1, 1, -1, 1).$$

By Theorem 3.9, we get $r \geq \frac{5}{552}$, where $\Theta = \frac{18}{5}$, $\lambda_{min} = \frac{1}{15}$.

Let n, $k \in \mathbb{N}$, $n \geq 3$, $n \geq k$.
Let $X = l_\infty^n$ and $Y = \bigcap_{i=1}^k \ker g^i$, where $g^i \in S(X^*)$ satisfy the following conditions:
$g_j^i \geq 0$ for every $i \in \{1, 2, \ldots, k\}$, $j \in \{1, 2, \ldots, n\}$, $g_i^i > 0$, $g_j^i = 0$ for $i \in \{1, 2, \ldots, k\}$, $i \neq j$ $\operatorname{supp}(g^i) \cap \operatorname{supp}(g^j) = \emptyset$, for every $i \neq j$, where

$$\operatorname{supp}(g^i) = \{k : g_k^i \neq 0\}.$$

Let $P_o \in \mathcal{P}(X, Y)$ be a cominimal projection. Then by Theorem 1.14, $\|Id - P_o\| = 1$ and P_o is determined by $y^j \in X$ satisfying (1.3) such that if $g_j^i \neq 0$ then for every $t \in \{1, \ldots, k\}$, (see Lemma 1.3) the assumption (1.13) is satisfied. Then the following is true.

Theorem 3.13. *If*

$$\bigcup_{i=1}^{k} \operatorname{supp}(g^i) = \{1, \ldots, n\} \tag{3.33}$$

then

$$r = \min\left\{\frac{g_j^i}{1 - g_j^i} : g_j^i \in (0, 1), i \in \{1, 2, \ldots, k\}, j \in \{1, 2, \ldots, n\}\right\}.$$

Proof. We will work with inequality (2.6). Let $P \in \mathcal{P}(X, Y)$ be a projection determined by vectors $\widetilde{y}^1, \widetilde{y}^2 \in R^n$ (see Def. 1.4). By Lemma 1.13 and by the form of functionals g^1, g^2, we get

$$\|Id - P\| = \max_{j \in \{1, \ldots, n\}} \left\{\sum_{i=1}^{k} |\widetilde{y}_j^i|\right\}, \tag{3.34}$$

$$(3.35)\qquad \|P-P_o\| = \max_{j\in\{1,\dots,n\}} \Big\{ \sum_{i=1}^{k} |\widetilde{y}^i_j - y^i_j| \Big\}.$$

Without loss of generality (see Lemma 1.6), combining (1.13) and (3.33), we can assume that

$$(3.36)\qquad \|P-P_o\| = \Big\{ |\widetilde{y}^1_1 - 1| + \sum_{i=2}^{k} |\widetilde{y}^i_1| \Big\}.$$

Suppose that $\widetilde{y}^1_1 < 1$.
By (1.3) and by the fact that for $i \in \{1,\dots,n\}$

$$\|g^i\| = \sum_{j=1}^{n} g^i_j = 1,$$

we get

$$\widetilde{y}^1_1 - 1 = \frac{1}{g^1_1} \sum_{j=k+1}^{n} g^1_j (1 - \widetilde{y}^1_j).$$

Since $\widetilde{y}^1_1 < 1$,

$$|1 - \widetilde{y}^1_1| = \frac{1}{g^1_1} \sum_{j=k+1}^{n} g^1_j (\widetilde{y}^1_j - 1).$$

For $i \in \{2,\dots,n\}$,

$$\widetilde{y}^i_1 = -\frac{1}{g^1_1} \sum_{j=k+1}^{n} g^1_j \widetilde{y}^i_j.$$

Hence

$$|\widetilde{y}^1_1 - 1| + \sum_{i=2}^{k} |\widetilde{y}^i_1| = \frac{1}{g^1_1} \sum_{j=k+1}^{n} g^1_j \Big((\widetilde{y}^1_j - 1) + \sum_{i=2}^{k} |\widetilde{y}^i_j| \Big) \le \|g^i\|.$$

Moreover,

$$\frac{g^1_1}{1-g^1_1}\|P-P_o\| + 1$$

$$\le \frac{1}{1-g^1_1} \sum_{j=k+1}^{n} g^1_j \Big((\widetilde{y}^1_j - 1) + \sum_{i=2}^{k} |\widetilde{y}^i_j| + 1 - g^1_1 \Big)$$

$$= \frac{1}{1-g^1_1} \sum_{j=k+1}^{n} g^1_j \Big(\widetilde{y}^1_j + \sum_{i=2}^{k} |\widetilde{y}^i_j| \Big) \le \dots$$

(Since $\|Id - P_o\| = 1$ and $\|g^i\| = 1$ for $i \in \{2, \dots, n\}$ then $1 - g_1^1 = \sum_{j=k+1}^{n} g_j^1$.)

$$\dots \leq \frac{1}{1 - g_1^1} \sum_{j=k+1}^{n} g_j^1 \sum_{i=1}^{k} |\widetilde{y}_j^i| \leq \|Id - P\|.$$

Notice that if the coordinates $\widetilde{y}_j^i$ are all positive or all negative for $i \in \{2, \dots, k\}$, $j \in \{k+1, \dots, n\}$ and $\frac{g_1^1}{1-g_1^1} = \min\big\{\frac{g_j^i}{1-g_j^i} : g_j^i \in (0,1), i \in \{1, 2, \dots, k\}, j \in \{1, 2, \dots, n\}\big\}$, then the above inequalities change into equalities which gives the results.

If $\widetilde{y}_1^1 \geq 1$ we get that $\|P - P_o\| = 1 + \|Id - P_o\|$. □

If (3.33) is not satisfied, then a cominimal projection P_o need not be strongly unique.

Example 3.14. Let $n, k \in \mathbb{N}$, $n \geq 1$, $k = 1$ and $X = l_\infty^{n+1}$. Assume that $g \in S(X^*)$ is of the form

$$g = (0, g_2, \dots, g_{n+1}),$$

where $g_2 > 0$. Let $Y = \ker g \subset X$ and $P_o \in \mathcal{P}(X, Y)$ be a cominimal projection. By Theorem 1.14, we get $\|Id - P_o\| = 1$.
Let $P \in \mathcal{P}(X, Y)$ be a projection determined by a vector $y = (y_1, 1, \dots, 1) \in R^{n+1}$, where $y_1 > 1$ (see Def. 1.4). Notice that by Lemma 3.2, $\|P - P_o\| = 1$. Hence the projection P_o is not strongly unique.

Remark 3.15. In the case of a subspace Y of $X = l_\infty^n$ for which $\|Id - P_o\| = 1$, the constant r could be larger then in the case of a subspace for which $\|Id - P_o\| > 1$, but r also depends on n. It follows from the equality

$$r = \min\Big\{\frac{g_j^i}{1 - g_j^i} : g_j^i \in (0,1), i \in \{1, 2, \dots, k\}, j \in \{1, 2, \dots, n\}\Big\} = \frac{g_j^i}{1 - g_j^i},$$

where

$$g_j^i = \min\big\{g_j^i \in (0,1)\big\} \leq \frac{1}{n-1}.$$

Hence

$$r \leq \frac{1}{n-2}.$$

Example 3.16. 1. Let $n = 3$, $g^1 = \left(\frac{1}{3}, 0, \frac{2}{3}\right)$, $g^2 = (0, 1, 0)$. Then by Theorem 3.13, $r = \frac{1}{2}$.
2. Let $n = 4$, $g^1 = (\frac{1}{3}, 0, \frac{2}{3}, 0)$, $g^2 = (0, \frac{1}{2}, 0, \frac{1}{2})$. Then $r = \frac{1}{2}$.
3. Let $n \geq 3$ and $g^1 = (\frac{1}{n-1}, 0, \frac{1}{n-1}, \ldots, \frac{1}{n-1})$, $g^2 = (0, 1, 0 \ldots, 0)$. Then $r = \frac{1}{n-2}$.

References

1. Ault D.A., Deutsch F.R., Morris P.D., Olsen J.E., *Interpolating subspaces in approximation theory*, J. Approx. Theory, **3** (1970), 164–182.
2. Baronti M., Lewicki G., *Strongly unique minimal projections onto hyperlanes*, J. Approx. Theory, **78, 1** (1994), 1–18.
3. Bartelt M.W., Henry M.S., *Continuity of the strong unicity constant on $C(X)$ for changing X*, J. Approx. Theory, **28** (1980), 85–97.
4. Bartelt M.W., McLaughlin H.W., *Characterizations of strong unicity in approximation theory*, J. Approx. Theory, **9** (1973), 255–266.
5. Blatter J., Cheney E.W., *Minimal projections onto hyperplanes in sequence spaces*, Ann. Mat. Pura Appl., **101** (1974), 215–227.
6. Chalmers B.L., Metcalf F.T., *A characterization and equations for minimal projections and extensions*, J. Operator Theory, **32** (1994), 31–46.
7. Chalmers B.L., Metcalf F.T., *The determination of minimal projections and extensions in L_1*, Trans. Amer. Math. Soc., **329** (1992), 289–305.
8. Cheney E.W., *Introduction to Approximation Theory*, Mc Grow Hill, New York, 1966.
9. Cheney E.W., Franchetti C., *Minimal projections in L_1 Spaces*, Duke Math. J., **43**, No. **3** (1976), 501–510, MR 54 #11044.
10. Dunham C.B., *A uniform constant of strong uniqueness on an interval*, J. Approx. Theory, **28** (1980), 207–211.
11. Fisher S.D., Morris P.D., Wulbert D.E., *Unique minimality of Fourier projections*, Trans. Amer. Math. Soc., **265** (1981), 235–246.
12. Franchetti C., *Projections onto hyperplanes in Banach spaces*, J. Approx. Theory, **38** (1983), 319–333.
13. König H., Tomczak-Jaegermann N., *Norms of minimal projections*, J. Funct. Anal., **119** (1994), 253–280.
14. Lewicki G., *Minimal projections onto two dimensional subspaces of l^4_∞*, J. Approx. Theory, **88** (1997), 92–108.
15. Lewicki G., *Strong unicity criterion in some space of operators*, Comment. Math. Univ. Carolinae, **34** (1993), 81–87.
16. Lipieta A., *Cominimal projections in l^n_∞*, J. Approx. Theory, **96** (1999), 86–100.
17. Lokot' V.V., *The constant of strongly unique minimal projections onto hyperplanes in l^n_∞*, Math. Zamet., **72, 5** (2002), 723–728 (in Russian).
18. Martinov O., *Constants of strong unicity of minimal projections onto two dimensional subspaces of l^4_∞*, J. Approx. Theory, **118** (2002), 175–187.
19. Newman D.J., Shapiro H.S., *Some theorems on Czebyszew approximation*, Duke Math. J., **30** (1963), 673–681.
20. Odyniec W., Lewicki G., *Minimal projections in Banach spaces*, Lecture Notes in Math., Vol. **1449**, Springer-Verlag, Berlin, Heidelberg, New York, 1990.

21. Ruess W.M., Stegall C., *Extreme points in duals of operator spaces*, Math. Ann., **261** (1982), 535–546.
22. Sudolski J., Wójcik A., *Some remarks on strong uniqueness of best approximation*, J. Approx. Theory and its Appl., **6**, No. **2** (1990), 44–78.

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