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ASYMPTOTIC STABILITY OF THE EXTENDED TJON–WU MODEL

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Abstract. We consider the generalized Tjon–Wu model. It describes the energy of a particle in an ideal gas colliding simultaneously with other particles of the same kind. Using the Zolotarev metric, the stability and asymptotic behaviour of stationary solutions are studied.

1. Introduction. The classical version of the Tjon–Wu model has the form

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} + u(t,x) &= (P_0u)(t,x) \\ u(0,x) &= u_0(x) \end{cases} \qquad \text{for} \quad t \geq 0, x \geq 0. \tag{1}$$

The operator P_0 is defined by

$$(P_0v)(x) = \int_x^{+\infty} \frac{dy}{y} \int_0^y v(y-z)v(z)dz, x \geq 0, \tag{2}$$

where v is the density function of an arbitrary distribution.

Problem (1), (2) was studied by Bobylev [**1**], Krook and Wu [**3**], as well as Tjon and Wu [**11**]. The problem stemmed from the theory of the Boltzmann equation and has a simple physical interpretation. Given arbitrary nonnegative t, the function $u(t,\cdot) : [0,+\infty) \to [0,+\infty)$ is the probability density function of the energy of a particle in an ideal gas. Due to the physical interpretation we assume that u satisfies the following conservation law with respect to energy

$$\int_0^{+\infty} xu(t,x)dx = 1.$$

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Assume that ξ_1 and ξ_2 are independent identically distributed random variables with the density function u. Further assume that η is a random variable uniformly distributed over the interval $[0,1]$ and independent of the vector (ξ_1, ξ_2). Then $P_0 u$ is the density function of the random variable

$$\eta \cdot (\xi_1 + \xi_2). \tag{3}$$

Physically, the random variables ξ_1 and ξ_2 represent the energy of two particles before the collision. Random variable (3) describes the energy of either particle after the impact. The random factor η defines the part of the total energy carried by the particle after the collision.

Most of the generalizations of classic problem (1), (2) refer to the variations of the operator P_0 with equations (1) kept intact. Lasota and Traple [**7**] proposed the operator P_h^+ of the form

$$(P_h^+ v)(x) = \int_x^{+\infty} h\left(\frac{x}{y}\right) \frac{dy}{y} \int_0^y v(y-z)v(z)dz, \tag{4}$$

where v is the density function of an arbitrary distribution. They replaced the uniform distribution with an arbitrary one represented by the density h. However, to keep the physical interpretation valid it was natural to assume that h is supported on $[0,+\infty)$.

In this paper we will study problem (1) for the distributions supported on the entire real line, which will be a mathematical improvement of the original physical phenomenon. Let $\mathbb{X} = [0,+\infty)$ or $\mathbb{X} = \mathbb{R}$. Define the Banach space

$$L_{1.1} = \left\{ v \in L^1(\mathbb{X}) : \|v\|_{1.1} < +\infty \right\},$$

where

$$\|v\|_{1.1} = \int_{\mathbb{X}} |v(x)|dx + \int_{\mathbb{X}} |x||v(x)|dx.$$

Fix $s > 1$ and denote

$$D_1 = \left\{ v \in L_{1.1}(\mathbb{X}) : \int_{\mathbb{X}} v(x)dx = \int_{\mathbb{X}} xv(x)dx = 1, v(x) \geq 0 \text{ for } x \in \mathbb{X} \right\}$$

and

$$D_{1.s} = \left\{ v \in D_1 : \int_{\mathbb{X}} |x|^s v(x)dx < +\infty \right\}.$$

In [**7, 8**] the authors proposed the following asymptotic stability result for problem (1), (4) for the densities supported on the positive half-line.

THEOREM 1.1. *Fix $s > 1$ and assume that a function $h : [0, +\infty) \to [0, +\infty)$ satisfies the following conditions*

$$\int_0^{+\infty} h(x)dx = 2\int_0^{+\infty} xh(x)dx = 1, \tag{5}$$

$$\int_0^{+\infty} x^s h(x)dx < \frac{1}{2}, \tag{6}$$

$$\sup\{xh(x) : x \geq 0\} < +\infty. \tag{7}$$

Then operator (4) *has a unique fixed point u_* in the set D_1. Moreover, for every initial function $u_0 \in D_1$ there exists a unique solution $u : [0, +\infty) \to D_1$ to equation* (1) *and*

$$\lim_{t\to+\infty} \|u(t) - u_*\|_{1.1} = 0. \tag{8}$$

We introduce some notations. Let $\mathcal{B}$ denote the σ-algebra of Borel subsets of $\mathbb{R}$ and $\mathcal{M}$ denote the family of all finite Borel measures on $\mathbb{R}$. By $\mathcal{M}_{prob} \subset \mathcal{M}$, denote the subfamily of probabilistic measures – distributions. Fix $\mu, \varphi \in \mathcal{M}_{prob}$. Let $\mu * \varphi$ denote the convolution of μ and φ given by

$$\int_{\mathbb{R}} f(z)(\mu * \varphi)(dz) = \int_{\mathbb{R}}\int_{\mathbb{R}} f(x+y)\mu(dx)\varphi(dy)$$

for every real Borel measurable function f such that the mapping $(x, y) \to f(x + y)$ is $\mu \otimes \varphi$-integrable. Analogously, let $\mu \circ \varphi$ denote the measure given by

$$\int_{\mathbb{R}} f(z)(\mu \circ \varphi)(dz) = \int_{\mathbb{R}}\int_{\mathbb{R}} f(xy)\mu(dx)\varphi(dy)$$

for every real Borel measurable function f such that the mapping $(x, y) \to f(xy)$ is $\mu \otimes \varphi$-integrable.

Define operators $P_{*n}, P_{\circ\varphi}, P^{\varphi}_{*n} : \mathcal{M}_{prob} \to \mathcal{M}_{prob}$. Let ξ_i $(i \in \mathbb{N})$ and η_j $(j \in \mathbb{N})$ be independent random variables with distributions μ and φ, respectively. Then

- operators P_{*n} $(n \in \mathbb{N})$ are defined by the following recurrence formulae

$$P_{*1}\mu = \mu, \qquad P_{*(n+1)}\mu = \mu * P_{*n}\mu. \tag{9}$$

  The measure $P_{*n}\mu$ is the distribution of the random variable $\xi_1 + \xi_2 + \ldots + \xi_n$;
- operators $P^n_{*\varphi}$ $(n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$ are defined by the formulae

$$P^0_{*\varphi}\mu = \mu, \qquad P^1_{*\varphi}\mu = \varphi * \mu, \qquad P^{n+1}_{*\varphi}\mu = \varphi * P^n_{*\varphi}\mu. \tag{10}$$

  The measure $P^n_{*\varphi}\mu$ is the distribution of the random variable $\eta_1 + \eta_2 + \ldots + \eta_n + \xi_1$;

- operator $P_{\circ\varphi}$ is defined by the formula

$$P_{\circ\varphi}\mu = \varphi \circ \mu. \tag{11}$$

The measure $P_{\circ\varphi}\mu$ is the distribution of the random variable $\eta_1 \cdot \xi_1$.

Problem (1), stated originally in terms of probability densities, was translated into the language of the probabilistic measures in [**6, 8**]. Fix $N \in \mathbb{N}$. The authors studied the asymptotics and stability of solutions to the equation

$$\frac{d\mu}{dt} + \mu = P_N\mu \qquad \text{for} \quad t \geq 0. \tag{12}$$

Here $\mu : [0, +\infty) \to \mathcal{M}_{prob}$ is an unknown function. Given distributions φ_n $(n = 1, 2, \dots N)$, an operator $P_N : \mathcal{M}_{prob} \to \mathcal{M}_{prob}$ is defined by

$$P_N\mu = \sum_{n=1}^{N} p_n P_{\circ\varphi_n} P_{*n}\mu \tag{13}$$

for positive reals p_n $(n = 1, 2, \dots N)$ such that $\sum_{n=1}^{N} p_n = 1$.

Problem (12), (13) was also analyzed in the infinite case of $N = +\infty$ (see [**8**]). However, the investigators narrowed their study to the distributions supported on the positive half-line.

The object of this paper is to study the asymptotics and stability of the Cauchy problem

$$\frac{d\mu}{dt} + \mu = P\mu, \qquad\qquad \mu(0) = \mu_0 \tag{14}$$

on the subsets of the space of probability measures. We start with a generalized equation with the operator P given by

$$P\mu = P_\infty\mu = \sum_{n=1}^{+\infty} p_n P_{\circ\varphi_n} P_{*n}\mu \tag{15}$$

for positive reals p_n $(n \in \mathbb{N})$ such that $\sum_{n=1}^{+\infty} p_n = 1$. Certain progress is made in this paper. We analyze the asymptotic behaviour of the stationary solutions in a wider set of initial conditions. Previously the authors have assumed the existence of the second moment of the initial measure. In our investigation we assume the existence of the s-th $(1 < s \leq 2)$ moment of the initial measure and of the distributions φ_n $(n \in \mathbb{N})$, yet keeping main results valid.

The outline of the paper is as follows. In Section 2, we recall some known results concerning the properties of the Zolotarev probability metric. In Section 3, we study the topological properties of the Zolotarev metric. The connections between the weak convergence of measures and the convergence in the Zolotarev metric are analyzed. Then we show that certain subsets of $\mathcal{M}_{prob}$ equipped with the Zolotarev metric are complete. In Section 4, we deal with the operators $P_{*n}, P_{\circ\varphi}, P^n_{*\varphi}$ and prove some inequalities involving these operators. In Section 5, we formulate and prove some results concerning the asymptotic stability of solutions to equation (14) with the operator P_∞ given by (15). These results are used to prove a generalization of Theorem 1.1.

2. Zolotarev probability metrics. Denote

- $\mathcal{B} = \mathcal{B}(\mathbb{R})$ – the σ-field of Borel subsets of $\mathbb{R}$;
- $\mathcal{M} = \mathcal{M}(\mathbb{R})$ – the family of finite Borel measures on $\mathbb{R}$;
- $\mathcal{M}_{sig} = \mathcal{M}_{sig}(\mathbb{R})$ – the vector space of signed measures on $\mathbb{R}$

$$\mathcal{M}_{sig} = \{\mu_1 - \mu_2 : \mu_1, \mu_2 \in \mathcal{M}\};$$

- $\mathcal{M}_{prob} = \mathcal{M}_{prob}(\mathbb{R})$ – the family of probability measures on $\mathbb{R}$.

We will use the notation

$$\langle f, \mu\rangle = \langle f(\cdot), \mu\rangle = \int_{\mathbb{R}} f(x)\mu(dx), \qquad \mu \in \mathcal{M}_{sig}.$$

For $\mu \in \mathcal{M}_{sig}$ and $k \geq 0$, let $m_k(\mu) = \langle(\cdot)^k, \mu\rangle$ and $|m|_k(\mu) = \langle|\cdot|^k, |\mu|\rangle$ denote the k-th moment and the k-th absolute moment, respectively. Here $|\mu|$ stands for the total variation of a measure μ.

Fix $s > 0$. We will use the following decomposition of s into integer and fractional part

$$s = l + \alpha, \qquad \text{where} \quad l \in \mathbb{N}_0 \quad \text{and} \quad \alpha \in (0, 1]. \tag{16}$$

Fix $r \in \mathbb{R}$ and $(l+1)$-tuple $\mathbf{r} = (r_1, \ldots, r_l; r_s) \in \mathbb{R}^l \times \mathbb{R}_+$. Denote

- $\mathcal{M}_s = \{\mu \in \mathcal{M}_{prob} : |m|_s(\mu) < +\infty\}$;
- $\mathcal{M}^r_1 = \{\mu \in \mathcal{M}_{prob} : m_1(\mu) = r\}$;
- $\mathcal{M}_{sig,s} = \{\mu \in \mathcal{M}_{sig} : |m|_s(\mu) < +\infty\}$;
- $\mathcal{M}^0_{sig,s} = \{\mu \in \mathcal{M}_{sig} : m_i(\mu) = 0, i = 0, 1, \ldots, l, \quad |m|_s(\mu) < +\infty\}$;
- $\mathcal{M}_{\mathbf{r}} = \{\mu \in \mathcal{M}_{prob} : m_i(\mu) = r_i, i = 1, 2, \ldots, l, \quad |m|_s(\mu) \leq r_s\}$.

Throughout the paper, we will assume the set $\mathcal{M}_{\mathbf{r}}$ to be nonempty.

Let $\mathcal{F}_s$ denote the family of l-times differentiable functions satisfying the inequality

$$\left|f^{(l)}(x) - f^{(l)}(y)\right| \leq |x - y|^\alpha \qquad \text{for every } x, y \in \mathbb{R}.$$

Now we can equip the vector space $\mathcal{M}^0_{sig,s}$ with the norm based on the family $\mathcal{F}_s$

$$\|\mu\|_s = \sup\{|\langle f, \mu\rangle| : f \in \mathcal{F}_s\}, \qquad \mu \in \mathcal{M}^0_{sig,s}. \tag{17}$$

The norm $\|\cdot\|_s$ generates the following metric ζ_s, introduced and studied by Zolotarev [**12, 13**]

$$\zeta_s(\mu_1, \mu_2) = \|\mu_1 - \mu_2\|_s, \qquad \mu_1, \mu_2 \in \mathcal{M}_{\mathbf{r}}. \tag{18}$$

Let $c \in \mathbb{R}$ and let δ_c denote the Dirac measure $\delta_c(\{c\}) = 1$. The Zolotarev metric ζ_s is the ideal metric of the order s. It means that for arbitrary $\mu_1, \mu_2 \in \mathcal{M}_{\mathbf{r}}$ the following inequalities are satisfied

$$\zeta_s(\mu_1 * \varphi, \mu_2 * \varphi) \le \zeta_s(\mu_1, \mu_2) \qquad \text{for every } \varphi \in \mathcal{M}_{\mathbf{r}}, \tag{19}$$

$$\zeta_s(\delta_c \circ \mu_1, \delta_c \circ \mu_2) \le |c|^s \zeta_s(\mu_1, \mu_2) \qquad \text{for every } c \in \mathbb{R}. \tag{20}$$

3. Topological properties of the Zolotarev metric. We will examine the connections between the convergence in the Zolotarev metric and the weak convergence of distributions. Recall that a sequence of probability measures $(\mu_n)_{n=1}^{+\infty}$ converges weakly to a distribution μ if

$$\lim_{n \to +\infty} \langle f, \mu_n \rangle = \langle f, \mu \rangle \qquad \text{for every } f \in \mathcal{C}_b,$$

where $\mathcal{C}_b$ is the family of bounded continuous functions on $\mathbb{R}$. The weak convergence will be denoted by $\mu_n \rightharpoonup \mu$.

Theorem 3.1. *Fix $s>0$ with decomposition* (16). *Let $\mathbf{r} = (r_1, \dots, r_l; r_s) \in \mathbb{R}^l \times \mathbb{R}_+$ and assume that $\mu, \mu_n \in \mathcal{M}_{\mathbf{r}}$ ($n \in \mathbb{N}$). If $\lim_{n \to +\infty} \zeta_s(\mu_n, \mu) = 0$ then $\mu_n \rightharpoonup \mu$. Conversely, if $\mu_n \rightharpoonup \mu$ and*

$$\lim_{n \to \infty} |m|_s(\mu_n) = |m|_s(\mu), \tag{21}$$

then $\lim_{n \to +\infty} \zeta_s(\mu_n, \mu) = 0$.

Proof. [Outlined] Proof of the first part of the theorem is based on the following well-known inequality (see [**14**], p. 123, Theorem 1.5.8)

$$\pi^{s+1}(\mu_1, \mu_2) \le c_s \cdot \zeta_s(\mu_1, \mu_2), \tag{22}$$

where the constant c_s depends on s only. Here π is the Levy–Prokhorov metric defined by

$$\pi(\mu_1, \mu_2) = \inf \left\{ \varepsilon > 0 : \mu_1(A) \le \mu_2(A^\varepsilon) + \varepsilon, A \ne \emptyset, \text{closed} \right\},$$

where

$$A^\varepsilon = \left\{ x : \text{dist } (A, x) < \varepsilon \right\}, \qquad \text{dist } (A, x) = \inf \left\{ d(x, y) : y \in A \right\}.$$

Now it suffices to use the fact that the Levy–Prokhorov metric metrizes the weak convergence.

Proof of the second part of the theorem is based on the inequality (see [**15**], p. 749, Theorem 3)

$$\zeta_s(\mu_1, \mu_2) \le \frac{\Gamma(1+\alpha)}{\Gamma(1+s)} \left\{ 2l\kappa_s(\mu_1, \mu_2) + \left[2\kappa_s(\mu_1, \mu_2) \right]^\alpha b_s^{1-\alpha} \right\}, \tag{23}$$

where $b_s = \min(|m|_s(\mu_1), |m|_s(\mu_2))$ and κ_s is the difference pseudomoment metric defined by

$$\kappa_s(\mu_1, \mu_2) = s \int_{\mathbb{R}} |x|^{s-1} |F_1(x) - F_2(x)| dx.$$

To finish the proof, we use the fact that the metric κ_s metrizes the weak convergence plus the convergence of the s-th absolute moments (see [**10**], p. 301). $\square$

The following lemma deals with the properties of the weak convergence of distributions in the set $\mathcal{M}_{\mathbf{r}}$.

LEMMA 3.2. *Fix $s > 0$ with decomposition* (16) *and let $\mathbf{r} = (r_1, \dots, r_l; r_s) \in \mathbb{R}^l \times \mathbb{R}_+$. If $\mu_n \in \mathcal{M}_{\mathbf{r}}$ $(n \in \mathbb{N})$, $\mu \in \mathcal{M}_{prob}$ and $\mu_n \rightharpoonup \mu$, then $\mu \in \mathcal{M}_{\mathbf{r}}$.*

PROOF. Assume that the sequence $(\mu_n)_{n=1}^{+\infty}$ converges weakly to a measure $\mu \in \mathcal{M}_{prob}$. For an arbitrary $\delta > 0$, define

$$g_\delta(x) = \min\{\delta, |x|^s\}.$$

The functions g_δ, $\delta > 0$, are continuous and bounded. Moreover, they form a family increasing with respect to the parameter δ. Since $\mu_n \in \mathcal{M}_{\mathbf{r}}$ there is

$$|\langle g_\delta, \mu_n \rangle| \le \langle |\cdot|^s, \mu_n \rangle \le r_s \qquad \text{for } n \in \mathbb{N} \qquad \text{and arbitrary } \delta > 0.$$

Passing to the limit as $n \to +\infty$, by the weak convergence of $(\mu_n)_{n=1}^{+\infty}$, we obtain

$$|\langle g_\delta, \mu \rangle| \le r_s \qquad \text{for arbitrary } \delta > 0.$$

Then passing to the limit as $\delta \to +\infty$, by virtue of the Lebesgue Monotone Convergence Theorem, we get

$$|\langle |\cdot|^s, \mu \rangle| \le r_s. \tag{24}$$

Fix $1 \le k \le l$. For an arbitrary $\delta > 0$, define

$$h_\delta(x) = \begin{cases} x^k, & |x| \le \delta \\ \delta^k, & x > \delta \\ (-\delta)^k, & x < -\delta \end{cases}.$$

We will use the following quite obvious inequality

$$\langle |\cdot|^k I_{|x| \ge \delta}, \mu_n \rangle \le \delta^{k-s} r_s. \tag{25}$$

Fix $\varepsilon > 0$. We can choose $\delta > 0$ such that $\delta^{k-s} r_s < \varepsilon$. For such δ, from (25) we obtain

$$|\langle r_k - h_\delta, \mu_n \rangle| \le \varepsilon.$$

The functions $r_k - h_\delta$, $\delta > 0$, are continuous and bounded. Thus, by the weak convergence of $(\mu_n)_{n=1}^{+\infty}$ as $n \to +\infty$, we get

$$|\langle r_k - h_\delta, \mu \rangle| \le \varepsilon \qquad \text{for arbitrary } \delta > 0.$$

Moreover, the functions $|r_k - h_\delta|$, $\delta > 0$, are uniformly bounded from above by the μ-integrable function $r_k + |\cdot|^k$. Hence, passing to the limit as $\delta \to +\infty$, we obtain

$$\left|\langle r_k - (\cdot)^k, \mu\rangle\right| \le \varepsilon.$$

Since $\varepsilon > 0$ and $1 \le k \le l$ are arbitrary, we conclude that $m_k(\mu) = r_k$ for $k = 1, 2, \ldots, l$. This and (24) complete the proof. □

Before analysing the completeness of the metric space $(\mathcal{M}_{\mathbf{r}}, \zeta_s)$ we shall need some prerequisites.

Observation 3.3. *Fix $s \in (0,1]$ and let $f \in \mathcal{F}_s$. For an arbitrary constant c, the functions g_r and g_l defined by*

$$g_r(x) = \begin{cases} f(x) & \text{if} \quad x \le c \\ f(c) & \text{if} \quad x > c \end{cases} \qquad \text{and} \quad g_l(x) = \begin{cases} f(c) & \text{if} \quad x < c \\ f(x) & \text{if} \quad x \ge c \end{cases}$$

belong to $\mathcal{F}_s$.

Observation 3.4. *Fix $s \in (0,1]$, $x_0 \in \mathbb{R}$ and bounded functions $f_1, f_2 \in \mathcal{F}_s$ such that $f_1(x_0) = f_2(x_0)$. Then there exists a nonnegative constant d such that the function f defined by*

$$f(x) = \begin{cases} f_1(x) & \text{if} \quad x \le x_0 \\ f_1(x_0) & \text{if} \quad x_0 < x < x_0 + d \\ f_2(x-d) & \text{if} \quad x_0 + d \le x \end{cases}$$

belongs to the family $\mathcal{F}_s$.

Using the above observations, we can prove the following lemma.

Lemma 3.5. *Fix $s \in (0,2]$ with decomposition (16). For every function $f \in \mathcal{F}_s$, there exist a constant $c \in \mathbb{R}$ and sequence of continuous functions $(f_i)_{i=1}^{+\infty}$ such that $f_i \in \mathcal{F}_s$ $(i \in \mathbb{N})$ and*

$$\lim_{i\to+\infty} f_i(x) = f(x) \qquad \text{for every } x \in \mathbb{R}, \tag{26}$$

$$f_i(x) \in \mathcal{C}_b \qquad \text{for every } i \in \mathbb{N}, \tag{27}$$

$$|f_i(x)| \le c(1 + |x|^s) \qquad \text{for every } x \in \mathbb{R} \text{ and every } i \in \mathbb{N}. \tag{28}$$

Proof. We will give a constructive proof in two cases: first for $s \in (0,1]$, then for $s \in (1,2]$.

Let $s \in (0,1]$ and fix $f \in \mathcal{F}_s$ (then, according to (16), $\alpha = s$). For every $i \in \mathbb{N}$ there exist a positive constant b_i and monotonous functions $\lambda_i, \omega_i \in \mathcal{F}_s$ such that

$$\lambda_i(-i) = f(-i), \quad \lambda_i(x) = 0 \text{ for } x \le -i - b_i$$

and

$$\omega_i(i) = f(i), \quad \omega_i(x) = 0 \text{ for } x \ge i + b_i.$$

Due to Observation 3.3 and Observation 3.4, we can choose a constant $a_i > i$ such that the function

$$f_i(x) = \begin{cases} f(x) & \text{if } |x| \le i \\ f(i\text{sgn } x) & \text{if } i < |x| \le a_i \\ \omega_i(x - a_i + i) & \text{if } a_i < x \le a_i + b_i \\ \lambda_i(x + a_i - i) & \text{if } -a_i - b_i \le x < -a_i \end{cases}$$

belongs to the family $\mathcal{F}_s$ for every $i \in \mathbb{N}$. It is easy to verify that for every $i \in \mathbb{N}$ the function f_i satisfies the conditions (26)–(28) and has a compact support.

Further, let $s \in (1,2]$ and fix $f \in \mathcal{F}_s$ (here $\alpha = s-1$). The derivative $g = f'$ belongs to $\mathcal{F}_{s-1}$ and there exists a sequence $(g_i)_{i=1}^{+\infty}$, $g_i \in \mathcal{F}_{s-1}$, of functions with compact supports and satisfying the conditions (26)–(28). Define the sequence $(f_i)_{i=1}^{+\infty}$ by

$$f_i(x) = f(0) + \int_0^x g_i(y)dy \qquad \text{for } x \in \mathbb{R} \text{ and } i \in \mathbb{N}.$$

We conclude that $f_i' = g_i \in \mathcal{F}_{s-1}$ for every $i \in \mathbb{N}$. Hence $f_i \in \mathcal{F}_s$, $i \in \mathbb{N}$. It is easy to verify that the functions f_i satisfy the conditions (26)–(28). This completes the proof. $\square$

The following theorem concerning completeness of the metric space $(\mathcal{M}_{\mathbf{r}}, \zeta_s)$ holds.

THEOREM 3.6. *Fix $s \in (0,2]$. For every $\mathbf{r} = (r_1; r_s) \in \mathbb{R} \times \mathbb{R}_+$, the metric space $(\mathcal{M}_{\mathbf{r}}, \zeta_s)$ is a complete metric space.*

PROOF. Assume that $(\mu_n)_{n=1}^{+\infty}$ is the Cauchy sequence in the space $(\mathcal{M}_{\mathbf{r}}, \zeta_s)$. Inequality (22) implies that $(\mu_n)_{n=1}^{+\infty}$ is also the Cauchy sequence in the complete metric space $(\mathcal{M}_{prob}, \pi)$. Hence there exists a measure $\mu \in \mathcal{M}_{prob}$ such that $\mu_n \rightharpoonup \mu$. By Lemma 3.2, we know that $\mu \in \mathcal{M}_{\mathbf{r}}$. We only need to show that $\lim_{n\to+\infty} \zeta_s(\mu_n, \mu) = 0$.

Let $\varepsilon > 0$ be arbitrary. The sequence $(\mu_n)_{n=1}^{+\infty}$ satisfies the Cauchy condition in metric ζ_s, hence there exists $N \in \mathbb{N}$ such that for every $n, m \ge N$ and every $f \in \mathcal{F}_s$, the following inequality holds

$$|\langle f, \mu_n - \mu_m \rangle| \le \varepsilon.$$

Fix an arbitrary $f \in \mathcal{F}_s$ and let $(f_i)_{i=1}^{+\infty}$ be a sequence of functions chosen as in Lemma 3.5. Since $f_i \in \mathcal{F}_s$ ($i \in \mathbb{N}$), for every $n, m \ge N$, there is

$$|\langle f_i, \mu_n - \mu_m \rangle| \le \varepsilon \qquad \text{for every } i \in \mathbb{N}.$$

By the weak convergence of $(\mu_m)_{m=1}^{+\infty}$ as $m \to +\infty$, we get

$$|\langle f_i, \mu_n - \mu \rangle| \le \varepsilon \qquad \text{for every } i \in \mathbb{N} \text{ and } n \ge N. \tag{29}$$

Next fix $n \geq N$ in (29). Using condition (28), we can pass to the limit as $i \to +\infty$. By (26), we obtain

$$|\langle f, \mu_n - \mu \rangle| \leq \varepsilon \qquad \text{for } n \geq N. \tag{30}$$

The function $f \in \mathcal{F}_s$ was arbitrary and N was chosen independently of f. Hence inequality (30) gives the convergence in the Zolotarev metric, which completes the proof. □

4. Nonlinear Markov operators. In this chapter we analyze properties of the operators P_{*n}, $P_{\circ\varphi}$ and $P^n_{*\varphi}$ given by (9), (10) and (11), respectively. Fix $s > 0$ with decomposition (16) and let $\mathbf{r} = (r_1, \ldots, r_l; r_s) \in \mathbb{R}^l \times \mathbb{R}_+$. It is easy to verify that the following inequality holds (see [**8**]).

Lemma 4.1. *Assume that $\mu_1, \mu_2 \in \mathcal{M}_{\mathbf{r}}$ and $\varphi \in \mathcal{M}_s$. Then*

$$\zeta_s(P_{\circ\varphi}\mu_1, P_{\circ\varphi}\mu_2) \leq |m|_s(\varphi)\zeta_s(\mu_1, \mu_2). \tag{31}$$

Using the triangle inequality and the fact that ζ_s is the ideal metric, it is easy to verify that the operators P_{*n} and $P^n_{*\varphi}$ are Lipschitzean operators with respect to the Zolotarev metric ζ_s.

Lemma 4.2. *Assume that $\mu_1, \mu_2 \in \mathcal{M}_{\mathbf{r}}$ and $\nu \in \mathcal{M}_s$. Then*

$$\zeta_s(P_{*n}\mu_1, P_{*n}\mu_2) \leq n\zeta_s(\mu_1, \mu_2) \tag{32}$$

and

$$\zeta_s(P^n_{*\nu}\mu_1, P^n_{*\nu}\mu_2) \leq \zeta_s(\mu_1, \mu_2) \tag{33}$$

for $n \in \mathbb{N}$.

The following relations concerning moments of $P_{\circ\varphi}\mu$, $P_{*n}\mu$ and $P^n_{*\varphi}\mu$ hold.

Lemma 4.3. *Assume that $\mu, \varphi \in \mathcal{M}_1$. Then*

$$m_1(P_{\circ\varphi}\mu) = m_1(\varphi)m_1(\mu), \tag{34}$$

$$m_1(P_{*n}\mu) = nm_1(\mu) \tag{35}$$

for $n \in \mathbb{N}$. If $\mu, \varphi \in \mathcal{M}_s$, then

$$|m|_s(P_{\circ\varphi}\mu) = |m|_s(\varphi)|m|_s(\mu). \tag{36}$$

Moreover, if $s \in (1, 2]$, then

$$|m|_s(P_{*n}\mu) \leq n|m|_s(\mu) + \frac{s}{2}n(n-1)|m|_1^s(\mu) \tag{37}$$

for $n \in \mathbb{N}$.

PROOF. Formulae (34)–(36) follow directly from the definitions of the operators $P_{\circ\varphi}$, P_{*n}, $P^n_{*\varphi}$ and properties of moments. To obtain inequality (37), we use the formula

$$|x+y|^s \le |x|^s + s|x||y|^{s-1} + |y|^s, \qquad s > 1.$$

Hence, for arbitrary probability measures μ_1 and μ_2 there is

$$\begin{aligned} |m|_s(\mu_1 * \mu_2) &\le |m|_s(\mu_1) + s|m|_1(\mu_1)|m|_{s-1}(\mu_2) + |m|_s(\mu_2) \\ &\le |m|_s(\mu_1) + s|m|_1(\mu_1)|m|_1^{s-1}(\mu_2) + |m|_s(\mu_2). \end{aligned}$$

Using the definition of the operator P_{*n}, formula (37) follows by the mathematical induction. □

5. A Generalized Tjon-Wu Equation. In this section, we give a generalization of the results by Lasota [**5**] and Lasota and Traple [**8**] concerning the stability of integro-differential equations in vector spaces. Let E be a real vector space with a norm $\|\cdot\|$ and let $D \subset E$ be a nonempty and convex set. Assume that the metric space (D, ρ), where $\rho(x, y) = \|x - y\|$, is complete. We consider the Cauchy problem of the form

$$\begin{cases} \frac{du}{dt} + u = Pu \\ u(0) = u_0 \end{cases}, \tag{38}$$

where $P : D \to D$. A mapping $u : [0, +\infty) \to D$ is called a solution to problem (38) if the strong derivative $\frac{du}{dt}$ exists in the space $(E, \|\cdot\|)$ for every $t \ge 0$ and u satisfies (38).

The following well-known consequence of the results of Crandall [**2**], Lasota [**5**] and the Banach contraction principle holds.

THEOREM 5.1. *Assume that the operator $P : D \to D$ satisfies the Lipschitz condition*

$$\rho(Pv_1, Pv_2) \le L\rho(v_1, v_2), \qquad 0 \le L < +\infty, \qquad v_1, v_2 \in D.$$

Then for every initial point $u_0 \in D$ there exists a unique solution u to problem (38).

If additionally $L < 1$, then the operator P has a unique stationary point u_*. Moreover, for every initial point $u_0 \in D$ the unique solution u to problem (38) *satisfies the inequality*

$$\rho(u(t), u_*) \le c \cdot e^{-(1-L)t} \qquad \textit{for every } t \ge 0,$$

where $c = \frac{1}{1-L}\rho(u_0, P_0)$.

In what follows we will use Theorem 5.1 to analyse the asymptotics and stability of solutions to operator-differential equations supported on the real line. The following theorems are generalizations of the results contained in [**6**] and [**8**]. Let $\varphi_n \in \mathcal{M}_{prob}$ ($n \in \mathbb{N}$) and let p_n be a sequence of nonnegative real numbers such that

$$\sum_{n=1}^{+\infty} p_n = 1. \tag{39}$$

We consider the generalized Tjon–Wu problem of the form

$$\begin{cases} \frac{d\mu}{dt} + \mu = P\mu \\ \mu(0) = \mu_0 \end{cases}, \tag{40}$$

where $\mu(t) \in \mathcal{M}_{prob}$ and $t \geq 0$. The operator $P : \mathcal{M}_{prob} \to \mathcal{M}_{prob}$ is defined by

$$P\mu = P_\infty\mu = \sum_{n=1}^{+\infty} p_n P_n \mu, \qquad \text{where} \quad P_n\mu = P_{\circ\varphi_n} P_{*n}\mu. \tag{41}$$

In the following, we will assume that the initial measure μ_0 is not a Dirac measure with mass at the point 0.

Problem (40), (41) has been already studied by several authors. However, in their publications, some additional assumptions have been made. For example, Lasota [**6**] studied a simplified version of the operator P_∞. In [**8**] Lasota and Traple considered the distributions supported on the positive half-line only.

Denote

$$J = \sum_{n=1}^{+\infty} n p_n m_1(\varphi_n)$$

and

$$L_s = \sum_{n=1}^{+\infty} n p_n |m|_s(\varphi_n), \qquad s > 0. \tag{42}$$

The following lemma gives sufficient conditions for the operator P_∞ to satisfy the energy conservation law.

Lemma 5.2. *Fix $s \in (1, 2]$. Assume that nonnegative real numbers p_n ($n \in \mathbb{N}$) satisfy* (39) *and probability measures $\varphi_n \in \mathcal{M}_1$ ($n \in \mathbb{N}$) are such that*

$$J = 1. \tag{43}$$

Then for an arbitrary distribution $\mu \in \mathcal{M}_1$ the following identity holds

$$m_1(P_\infty\mu) = m_1(\mu). \tag{44}$$

If $\varphi_n \in \mathcal{M}_s$ ($n \in \mathbb{N}$) and inequality

$$L_s < 1 \tag{45}$$

is satisfied, then for every $r_1 \in \mathbb{R}$ there exists $\tilde{r} \in \mathbb{R}_+$ such that for every $r \geq \tilde{r}$ the operator P_∞ maps the set $\mathcal{M}_{(r_1;r)}$ into itself.

PROOF. Evidently, $P_\infty(\mathcal{M}_{prob}) \subset \mathcal{M}_{prob}$. Take an arbitrary $\mu \in \mathcal{M}_1$. By the properties of the expectation and using Lemma 4.3, we obtain

$$m_1(P_\infty\mu) = Jm_1(\mu).$$

This and (43) yield (44). Next let $r > 0$ be such that $|m|_s(\mu) \leq r$. By definition of the operator P_∞ and by (36), there is

$$|m|_s(P_\infty\mu) = \sum_{n=1}^{+\infty} p_n|m|_s(\varphi_n)|m|_s(P_{*n}\mu).$$

Hence, using (37) we obtain

$$|m|_s(P_\infty\mu) \leq rL_s + \frac{s}{2}|m|_1^s(\mu)\sum_{n=1}^{+\infty} n(n-1)p_n|m|_s(\varphi_n). \tag{46}$$

Take

$$\tilde{r} = \left(1 + \frac{A_s}{1-L_s}\right)|m|_1^s(\mu), \qquad A_s = \frac{s}{2}\sum_{i=n}^{+\infty} n(n-1)p_n|m|_s(\varphi_n). \tag{47}$$

By (46) and (47), we get $|m|_s(P_\infty\mu) \leq r$ for every $r \geq \tilde{r}$. This and (44) yield $P_\infty(\mathcal{M}_{(r_1;r)}) \subset \mathcal{M}_{(r_1;r)}$ for every $r \geq \tilde{r}$. □

Now we are going to compute the Lipschitz constant for the operator P_∞ in the space $(\mathcal{M}_{\mathbf{r}}, \zeta_s)$.

LEMMA 5.3. *Fix $s \in [1,2]$ and let $\mathbf{r} = (r_1; r_s) \in \mathbb{R} \times \mathbb{R}_+$. Assume that nonnegative real numbers p_n $(n \in \mathbb{N})$ satisfy* (39) *and that $\varphi_n \in \mathcal{M}_s$ $(n \in \mathbb{N})$. Then, in the metric space $(\mathcal{M}_{\mathbf{r}}, \zeta_s)$, the operator P_∞ satisfies the Lipschitz condition*

$$\zeta_s(P_\infty\mu_1, P_\infty\mu_2) \leq L_s\zeta_s(\mu_1,\mu_2). \tag{48}$$

PROOF. For arbitrary distributions $\mu_1, \mu_2 \in \mathcal{M}_{(r_1;r_s)}$, by (39), (41) and by the definition of the Zolotarev metric ζ_s, it follows that

$$\zeta_s(P_\infty\mu_1, P_\infty\mu_2) \leq \sum_{n=1}^{+\infty} p_n\zeta_s(P_n\mu_1, P_n\mu_2). \tag{49}$$

Next by (32), (31) and (33), we get

$$\zeta_s(P_n\mu_1, P_n\mu_2) \leq n|m|_s(\varphi_n)\zeta_s(\mu_1,\mu_2). \tag{50}$$

Combining inequalities (49) and (50), we obtain (48). □

The main result of this chapter is the following theorem describing the asymptotics and stability of the solutions the problem (40), (41).

THEOREM 5.4. *Fix $s \in (1, 2]$ and let p_n $(n \in \mathbb{N})$ be nonnegative real numbers satisfying* (39). *Assume that distributions $\varphi_n \in \mathcal{M}_1$ $(n \in \mathbb{N})$ satisfy relation* (43) *and the condition*

$$L_1 = \sum_{n=1}^{+\infty} np_n |m|_1(\varphi_n) < +\infty. \tag{51}$$

Fix $r_1 \in \mathbb{R}$. Then for every initial measure $\mu_0 \in \mathcal{M}_1^{r_1}$ there exists in the set $\mathcal{M}_1^{r_1}$ a unique solution μ to problem (40), (41).

If additionally $\varphi_n \in \mathcal{M}_s$ $(n \in \mathbb{N})$, $\mu_0 \in \mathcal{M}_1^{r_1} \cap \mathcal{M}_s$ and condition (45) *is satisfied, then there exists a constant $r_s = r_s(\mu_0) > 0$ such that $\mu(t) \in \mathcal{M}_{(r_1;r_s)}$ for every $t \geq 0$. Moreover, the operator P_∞ admits a unique stationary measure μ_* in the set $\mathcal{M}_1^{r_1} \cap \mathcal{M}_s$. Furthermore, for $\mu_0 \in \mathcal{M}_1^{r_1} \cap \mathcal{M}_s$ the solution μ satisfies the inequality*

$$\zeta_s(\mu(t), \mu_*) \leq ce^{-(1-L_s)t} \qquad \textit{for every } t \geq 0, \tag{52}$$

where $c = \frac{1}{1-L_s}\zeta_s(\mu_0, P_\infty \mu_0)$.

PROOF. Assume that $\mu_0 \in \mathcal{M}_1^{r_1}$ is the initial measure for problem (40), (41). The set $\mathcal{M}_1^{r_1}$ is convex and (by Theorem 2.1 of [**6**]) the space $(\mathcal{M}_1^{r_1}, \zeta_1)$ is complete. Using (51) and (43), we obtain that $P_\infty(\mathcal{M}_1^{r_1}) \subset \mathcal{M}_1^{r_1}$. By virtue of Lemma 5.3, the operator P_∞ satisfies inequality (48) for $s = 1$ in the space $(\mathcal{M}_1^{r_1}, \zeta_1)$. Hence, by virtue of Theorem 5.1, problem (40), (41) has a unique solution μ and $\mu \in \mathcal{M}_1^{r_1}$.

Next assume that $\mu_0 \in \mathcal{M}_1^{r_1} \cap \mathcal{M}_s$ is the initial measure for problem (40), (41). Since condition (45) is satisfied, by virtue of Lemma 5.2, we can choose a constant $\bar{r} > 0$ such that

$$P_\infty(\mathcal{M}_{(r_1;\bar{r})}) \subset \mathcal{M}_{(r_1;\bar{r})}. \tag{53}$$

Define $r_s = \max(\bar{r}, |m|_s(\mu_0))$. The set $\mathcal{M}_{(r_1;r_s)}$ is a convex subset of the vector space $E = \mathcal{M}_{sig,s}$ and by Theorem 3.6, the space $(\mathcal{M}_{(r_1;r_s)}, \zeta_s)$ is complete. By virtue of Lemma 5.3, the operator P_∞ satisfies inequality (48) in the space $(\mathcal{M}_{(r_1;r_s)}, \zeta_s)$. Hence, by virtue of Theorem 5.1, problem (40), (41) admits a unique solution μ and $\mu \in \mathcal{M}_{(r_1;r_s)}$. Moreover, there exists a unique stationary measure μ_* for the operator P_∞, $\mu_* \in \mathcal{M}_{(r_1;r_s)}$ and inequality (52) is satisfied.

The constant r_s can be arbitrary large; thus μ_* is the unique stationary solution to problem (40), (41) in the entire set $\mathcal{M}_1^{r_1} \cap \mathcal{M}_s$. $\square$

The following proposition makes use of the stability results given by Theorem 5.4 and is a generalization of Theorem 1.1. We study the solutions to the problem

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} + u(t,x) &= (P_h u)(t,x) \\ u(0,x) &= u_0(x) \end{cases} \qquad \text{for} \quad t \geq 0, \tag{54}$$

where

$$(P_h v)(x) = \int_{\mathbb{R}} h\left(\frac{x}{y}\right)\frac{dy}{|y|}\int_{\mathbb{R}} v(y-z)v(z)dz, \tag{55}$$

with v the density function of an arbitrary distribution.

We will need a few prerequisites. We begin with a characterization of the convolutions of distribution possessing atoms (see [**9**]).

LEMMA 5.5. *Assume that a measure $\mu \in \mathcal{M}_{prob}$ has its greatest atom at the point $c \in \mathbb{R}$. Then*

$$(\mu * \mu)(\{x\}) \le \mu(\{c\}) \qquad \text{for every } x \in \mathbb{R}. \tag{56}$$

Moreover, if supp μ *is not a finite set, then*

$$(\mu * \mu)(\{x\}) < \mu(\{c\}) \qquad \text{for every } x \in \mathbb{R}. \tag{57}$$

Next, using Lemma 5.5, we will prove that if operator (55) defined in terms of the distributions admits a stationary measure then this measure has a continuous distribution function.

LEMMA 5.6. *Let $\varphi \in \mathcal{M}_{prob}$ be such that*

$$\varphi(\{0\}) = 0. \tag{58}$$

Let $\mu \in \mathcal{M}_{prob}$ satisfy the identity

$$\hat{\mu}(t) = \int_{\mathbb{R}} \hat{\mu}^2(tr)\varphi(dr), \tag{59}$$

where $\hat{\mu}$ denotes the characteristic function of the measure μ. If supp μ *is not a finite set, then the distribution function of μ is continuous on $\mathbb{R}$.*

PROOF. Fix $\mu \in \mathcal{M}_{prob}$. Assume that μ has atoms and let $\mu(\{c\})$ be its greatest atom. Applying the well-known identity for characteristic functions

$$\mu(\{c\}) = \lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T} e^{-cti}\hat{\mu}(t)dt$$

and conditions (58) and (59), we obtain the identity

$$\mu(\{c\}) = \int_{\mathbb{R}\setminus\{0\}} (\mu * \mu)\left(\left\{\frac{c}{r}\right\}\right)\varphi(dr). \tag{60}$$

Using the fact that supp μ is not a finite set, by virtue of (57), we get

$$\mu(\{c\}) > (\mu * \mu)(\{x\}) \qquad \text{for every } x \in \mathbb{R}.$$

This and (60) yield a contradiction

$$\mu(\{c\}) < \int_{\mathbb{R}\setminus\{0\}} \mu(\{c\})\varphi(dr) = \mu(\{c\}),$$

which completes the proof. $\square$

The following result concerning the strong stability of the solutions to problem (54), (55) for the densities supported on the whole real line holds.

THEOREM 5.7. *Let $s \in (1,2]$ and $c > 0$. Assume that a function $h : \mathbb{R} \to [0,+\infty)$ satisfies the following conditions*

$$\int_{\mathbb{R}} h(x)dx = 2\int_{\mathbb{R}} xh(x)dx = 1, \tag{61}$$

$$\int_{\mathbb{R}} |x|^s h(x)dx < \frac{1}{2}, \tag{62}$$

$$|x|h(x) \le c \quad \text{for every } x \in \mathbb{R}. \tag{63}$$

Then there exists a stationary function u_* of the operator P_h in the set $D_{1.s}$. Additionally, for every initial density $u_0 \in D_{1.s}$ there exists a unique solution $u : [0,+\infty) \to D_{1.s}$ to problem (54), (55) *and*

$$\lim_{t\to+\infty} \|u(t) - u_*\|_{1.1} = 0. \tag{64}$$

PROOF. Using assumption (61), it is easy to see that $P_h(D_1) \subset D_1$. From (61) it follows that the operator P_h satisfies the Lipschitz condition in the set D_1

$$\|P_h v_1 - P_h v_2\|_{1.1} \le 2d \|v_1 - v_2\|_{1.1}, \qquad \text{where } d = \int_{\mathbb{R}} |x|h(x)dx.$$

Since D_1 is a closed and convex subset of the Banach space $L_{1.1}$, by Theorem 5.1, equation (54) admits a unique solution $u \in D_1$ for every $u_0 \in D_1$.

Fix $u_0 \in D_{1.s}$ and consider the integral form of equation (54):

$$u(t,x) = e^{-t}u_0(x) + \int_0^t e^{-(t-s)} P_h u(s,x)ds, \qquad t \ge 0. \tag{65}$$

Define the measures

$$\psi_0(A) = \int_A u_0(x)dx, \qquad \psi(t)(A) = \int_A u(t,x)dx, \quad t > 0$$

and

$$\varphi(A) = \int_A h(x)dx$$

for $A \in \mathcal{B}(\mathbb{R})$. The function $\psi : [0,+\infty) \to \mathcal{M}_1^1$ is a solution to problem (40), (41) with the initial measure $\mu_0 = \psi_0$ and for the operator $P = P_2 = P_{\circ\varphi}P_{*2}$. Condition (61) gives

$$J = 2m_1(\varphi) = 1 \qquad \text{and} \qquad L_1 = 2|m|_1(\varphi) < +\infty.$$

By (61) and (62), the conditions of Theorem 5.4 are satisfied. Therefore, ψ is the unique solution to problem (40), (41) and there exists a constant $r_s > 0$ such that $\psi(t) \in \mathcal{M}_{(r_1;r_s)}$ for every $t \ge 0$. This implies that $u(t) \in D_{1.s}$ for every $t \geq 0$. Moreover, the operator P admits a unique stationary measure ψ_* in the set $\mathcal{M}_1^1 \cap \mathcal{M}_s$ and relation (52) holds.

We will prove that the trajectory $\mathcal{U} = \{u(t) : t \geq 0\}$ is a relatively weakly compact set in the space L_1. Fix $\varepsilon > 0$. Using the Markov inequality and the fact that $m_s(\psi(t)) \leq r_s$ for every $t \geq 0$, we conclude that there exists a constant $a > 0$ such that

$$\psi(t)(\{|x| \geq a\}) = \int_{|x|\geq a} u(t,x)dx \leq \varepsilon, \qquad t \geq 0. \tag{66}$$

The measure φ is absolutely continuous. Therefore, the stationary measure ψ_* is not a Dirac measure with mass at $x = 1$. From this fact and the relation $m_1(\psi_*) = 1$ it follows that there exists $x_0 > 1$ such that $x_0 \in \operatorname{supp} \psi_*$. Similarly, the identity $m_1(\varphi) = \frac{1}{2}$ yields the existence of a $y_0 > \frac{1}{2}$ such that $y_0 \in \operatorname{supp} \varphi$. From the identity $\psi_* = \varphi \circ (\psi_* * \psi_*)$, we obtain $2x_0y_0 \in \operatorname{supp} \psi_*$. Repeating this procedure, we obtain $(2y_0)^n x_0 \in \operatorname{supp} \psi_*$ for every $n \in \mathbb{N}$. Hence the support of the measure ψ_* is not a finite set and by Lemma 5.6, we obtain $\psi_*(\{c\}) = 0$ for every $c \in \mathbb{R}$. Therefore, we can choose $b > 0$ such that $\psi_*([-b, b]) < \frac{\varepsilon}{2}$. Using this and the weak convergence $\psi(t) \rightharpoonup \psi_*$ as $t \to +\infty$, we obtain the existence of $t_0 > 0$ such that

$$\int_{-b}^{b} u(t,x)dx = \psi(t)([-b,b]) < \varepsilon \quad \text{for every } t \geq t_0. \tag{67}$$

From (65), (63) and (55), for $A \subset Y = \{x : b < |x| < a_0\}$, we obtain

$$\int_A u(t,x)dx \leq \int_A u_0(x)dx + |A|\frac{c}{b}, \qquad t \geq 0,$$

where $|A|$ denotes the Lebesgue measure of A. Hence there exists $\delta > 0$ such that

$$A \subset Y, \quad |A| < \delta \qquad \Rightarrow \qquad \int_A u(t,x)dx < \varepsilon \quad \text{for every } t \geq 0. \tag{68}$$

Therefore, by (66), (67) and (68), the trajectory $\mathcal{U}$ is a relatively weakly compact set in L_1.

By (65), the set $\mathcal{U}$ is contained in the closed convex hull of the set $\{u_0\} \cup P_h(\mathcal{U})$. By virtue of Theorem 4.1 in [**4**], the operator P_h maps weakly compact sets into strongly compact sets. Hence the set $\mathcal{U}$ is contained in a convex strongly compact set in L_1. The compactness of the trajectory $\mathcal{U}$ in the set L_1 and the weak convergence $\psi(t) \rightharpoonup \psi_*$ as $t \to +\infty$ ($\psi_* \in \mathcal{M}_1^1 \cap \mathcal{M}_s$) imply that the stationary measure ψ_* is absolutely continuous. Let u_* be the density function of the measure ψ_*

$$\psi_*(A) = \int_A u_*(x)dx \qquad \text{for } A \in \mathcal{B}(\mathbb{R}).$$

By virtue of Theorem 5.4, the function u_* belongs to the family $D_{1.s}$. Using again the L_1-compactness of the trajectory $\mathcal{U}$ and the uniqueness of the weak limit, we obtain the L_1-convergence of $u(t,\cdot)$ to $u_*(\cdot)$ as $t \to +\infty$

$$\lim_{t\to+\infty} \int_{\mathbb{R}} |u(t,x) - u_*(x)|\, dx = 0. \tag{69}$$

To finish the proof we only need to show that

$$\lim_{t\to+\infty} \int_{\mathbb{R}} |x|\, |u(t,x) - u_*(x)|\, dx = 0. \tag{70}$$

The conclusion follows immediately from the L_1-convergence of the densities $u(t,x)$ to the stationary density u_* and the fact that there exists a constant $r_s > 0$ such that $\int_{\mathbb{R}} |x|^s u(t,x)dx < r_s$ for every $t \geq 0$. This completes the proof. $\square$

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