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LOCAL COHOMOLOGY AND MATLIS DUALITY

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Abstract. Relations between (set-theoretic) complete intersections and local cohomology are studied; it is explained in what sense Matlis duals of certain local cohomology modules carry enough information to decide whether the given ideal is a complete intersection or not. Finally, we present some related results on associated primes of Matlis duals of local cohomology modules.

1. The situation – notation and basic definitions. Let I be an ideal of a (always commutative, noetherian) ring R. For every R-module M one sets

$$\Gamma_I(M) := \{m \in M | I^n \cdot m = 0 \forall n \gg 0\}$$

(that is, $\Gamma_I(M)$ is the largest submodule of M whose support is contained in $V(I)$). The (right) derived functors of the (left exact) functor Γ_I are called *local cohomology functors* H^i_I *with support in* I (for $i \in \mathbb{N}$). One can show that these functors are affine versions of Serre cohomology on sheaves. [**3**] and [**1**] are general references for local cohomology.

Usually, we will assume in addition that R is local with maximal ideal m. In this case we denote by $E := E_R(R/m)$ a fixed R-injective hull of the R-module R/m and by D the (contravariant) functor $\mathrm{Hom}_R(\cdot, E)$. Some of the following ideas are contained in the first author's Habilitationsschrift [**7**].

Definition 1.1. I is a *set-theoretic complete intersection* iff it can be generated by $\mathrm{height}(I)$ many elements up to radical, i.e., iff $\mathrm{ara}(I) = \mathrm{height}(I)$, where $\mathrm{ara}(I)$ is the minimal number of generators of I up to radical.

Remark 1.2. *It is an easy consequence of Krull's principal ideal theorem that there is always the inequality* $\mathrm{ara}(I) \geq \mathrm{height}(I)$.

From now on we will use "complete intersection" for "set-theoretic complete intersection."

Example 1.3. Let k be a field, $d \in \mathbb{N}$, $d \geq 3$ and let $\mathbb{P}_k^n$ denote projective n-space over k. Furthermore, let C_d be the (projective, smooth) curve which is the image of

$$\mathbb{P}_k^1 \to \mathbb{P}_k^3, (u : v) \mapsto (u^d : u^{d-1}v : uv^{d-1} : v^d).$$

It is well-known that C_3 is a complete intersection. In addition, in the case of $\mathrm{char}(k) > 0$ Hartshorne ([**4**, Theorem*]) resp. Bresinsky, second author and Renschuch ([**2**]) have shown that C_d is a complete intersection (for every $d \geq 3$). The case $\mathrm{char}(k) = 0$ is open, even for $d = 4$. The curve C_4 is the famous *Macaulay curve*.

2. Results. The following (first) remark is an easy consequence of the fact that one may use Čech cohomology to compute Serre cohomology over an affine scheme:

Remarks 2.1. *(i) I is a complete intersection $\Rightarrow$ $H_I^l(R) = 0$ for every $l > \mathrm{height}(I)$. More generally, for every ideal I, one has $H_I^l(R) = 0$ for every $l > \mathrm{ara}(I)$.*

(ii) The reversed statement of (i) does not hold in general, here is an example (later we will refer to this example again): Let $R = k[[x, y, z, w]]$ be a formal power series ring over a base field k in four variables. Set $f := xw - yz$, $g_1 := y^3 - x^2z$, $g_2 := z^3 - u^2w$. It is easy to see that $I := \sqrt{(f, g_1, g_2)R}$ is the height two prime ideal of R which corresponds to the curve C_4. In particular, $I/fR \subseteq R/fR$ has height one. We claim that both $H_{I/fR}^l(R/fR) = 0$ for every $l > 1$ and $\mathrm{ara}(I/fR) \geq 2$ hold (in particular, I/fR is not a complete intersection, i.e., it is an example, where the reversed statement from (i) does not hold):

Proof of (ii). Let $y_0, \ldots, y_3$ be new variables and set $S := k[[y_0, y_1, y_2, y_3]]$. Denote by R_1 the three-dimensional subring $R_1 := k[[y_0y_1, y_0y_2, y_1y_3, y_2y_3]]$ of S. The ring homomorphism

$$R \to R_1, x \mapsto y_0y_1, y \mapsto y_0y_2, z \mapsto y_1y_3, w \mapsto y_2y_3$$

clearly induces an isomorphism

$$R/fR \cong R_1 (\subseteq S).$$

Now consider the k-linear map

$$k[y_0, y_1, y_2, y_3] \xrightarrow{\varphi} R_1$$

that sends a term $y_0^{\alpha_0} y_1^{\alpha_1} y_2^{\alpha_2} y_3^{\alpha_3}$ to $y_0^{\alpha_0} y_1^{\alpha_1} y_2^{\alpha_2} y_3^{\alpha_3} \in R_1$ if $\alpha_0 + \alpha_3 = \alpha_1 + \alpha_2$ holds, and to zero otherwise. Note that φ is well-defined by construction and naturally induces a map

$$S = k[[y_0, y_1, y_2, y_3]] \xrightarrow{\tilde{\varphi}} R_1.$$

Now it is easy to see that $\tilde{\varphi}$ is R_1-linear and makes R_1 into a direct summand in S (as an R_1-submodule). Thus $H_I^2(R/fR)$ is isomorphic to a direct summand of $H_{IS}^2(S)$. We have

$$IS = (g_1, g_2)S = ((y_0 y_2^3 - y_1^3 y_3) \cdot y_0^2, (y_0 y_2^3 - y_1^3 y_3) \cdot (-y_3^2))S$$

and

$$\sqrt{IS} = (y_0 y_2^3 - y_1^3 y_3)S.$$

This implies $H_{IS}^2(S) = 0$ and thus, by what we have seen above, $H_I^2(R/fR) = 0$. Now we show $\operatorname{ara}(I(R/fR)) = 2$: We assume $\operatorname{ara}(I(R/fR)) \neq 2$; then we clearly have $\operatorname{ara}(I(R/fR)) = 1$. Let $h \in R$ be such that

$$I(R/fR) = \sqrt{h(R/fR)}$$

holds. This implies

$$\sqrt{IS} = \sqrt{hS}\ .$$

We have seen before that

$$\sqrt{IS} = (y_0 y_2^3 - y_1^3 y_3)S$$

holds. S is a unique factorization domain and so there exist $N \geq 1$ and $s \in S$ such that

$$h = (y_0 y_2^3 - y_1^3 y_3)^N \cdot s \text{ and } (y_0 y_2^3 - y_1^3 y_3) \not| s$$

hold. From $h \in R_1 \subseteq S$ it follows that all terms $y_0^{\alpha_0} y_1^{\alpha_1} y_2^{\alpha_2} y_3^{\alpha_3}$ in $h \in S$ have the property $\alpha_0 + \alpha_3 = \alpha_1 + \alpha_2$; on the other hand, all terms $y_0^{\alpha_0} y_1^{\alpha_1} y_2^{\alpha_2} y_3^{\alpha_3}$ of $(y_0 y_2^3 - y_1^3 y_3)^N$ have the property $(\alpha_0 + \alpha_3) - (\alpha_1 + \alpha_2) = -2N$. So we can assume that all terms $y_0^{\alpha_0} y_1^{\alpha_1} y_2^{\alpha_2} y_3^{\alpha_3}$ of s have the property $(\alpha_0 + \alpha_3) - (\alpha_1 + \alpha_2) = 2N$. But then s cannot be a unit in S and so

$$(y_0 y_2^3 - y_1^3 y_3)S = \sqrt{hS} = (y_0 y_2^3 - y_1^3 y_3)S \cap \sqrt{sS}$$

clearly leads to a contradiction. □

Thus the implication from Remark 2.1 (i) is not an equivalence, in general; the next result answers the question what additional condition is required to get equivalence:

THEOREM 2.2. *Set* $h := \operatorname{height}(I)$ *and let* $f_1, \ldots f_h \in I$ *be an* R*-regular sequence. The following statements are equivalent:*

(i) $\sqrt{(f_1, \ldots, f_h)R} = \sqrt{I}$*; in particular,* I *is a complete intersection.*

(ii) $H_I^l(R) = 0$ *for every* $l > h$ *and* $f_1, \ldots, f_h$ *is a* $D(H_I^h(R))$*-regular sequence.*

(iii) $H_I^l(R) = 0$ *for every* $l > h$ *and* $f_1, \ldots, f_h$ *is a* $D(H_I^h(R))$*-quasiregular sequence.*

The (technical) proof can be found in [**7**, Corollary 1.1.4]. Note that a sequence $\underline{x} = x_1, \dots, x_n \in R$ is called M-*quasiregular* for a given R-module M if multiplication by every x_i is injective on $M/(x_1, \dots, x_{i-1}M)$ for every $i = 1, \dots, n$.

Due to the importance of $D(H^h_I(R))$-regular sequences (in the above situation) contained in I, one might be tempted to define a notion of depth in he following sense:

Definition 2.3. For every ideal I of R and every $l \in \mathbb{N}$ let $\operatorname{depth}(I, D(H^l_I(R)))$ be the maximal length of a $D(H^l_I(R))$-regular sequence inside I.

But this notion is not well-behaved in the sense that, in general, not all maximal regular sequences have the same length; here is a concrete example:

Again, like in Remark 2.1 (ii), let $I \subseteq k[[x, y, z, w]]$ be the ideal corresponding to the curve C_4; assume $\operatorname{char}(k) > 0$. As we mentioned before, I is a complete intersection. Therefore, because of Theorem 2.2, $\operatorname{depth}(I, D(H^2_I(R))) = 2$. On the other hand, one can show that $f := xw - yz \in I$ is a regular sequence (of length one) on $D(H^2_I(R))$ (this follows e.g. from calculations in Remark 2.1 (ii)). But there is no $h \in I$ such that $\sqrt{(f, h)R} = \sqrt{I}$, because I/fR is not a a complete intersection. Therefore, again because of Theorem 2.2, the sequence consisting solely of f is already maximal.

Nevertheless, Theorem 2.2 suggests to study $D(H^h_I(R))$-regular sequences contained in I; this problem is related to the study of the set $\operatorname{Ass}_R(D(H^h_I(R)))$ of associated primes of $D(H^h_I(R))$.

The general idea that associated primes of $D(H^h_I(R))$ tend to be "small" becomes concrete in the following special

Example 2.4. Let $R = k[[X, Y]]$ be a formal power series ring over a field k in two variables and set $I := XR$. Čech cohomology shows $H^1_I(R) = k[[Y]][X^{-1}]$. A tedious calculation based on this description shows $D(H^1_I(R)) = k[Y^{-1}][[X]]$. Note that, for any ring S, an expression like $S[X^{-1}]$ stands for the direct sum over all $S \cdot X^{-l}$ for $l \leq -1$. Also note that $k[Y^{-1}][[X]]$ is bigger than $k[[X]][Y^{-1}]$.

Using the above description of $D(H^1_I(R))$ we consider the element $Y^{-1}X + Y^{-4}X^2 + Y^{-9}X^3 + \cdots \in D(H^1_I(R))$. It is not too difficult to see that its annihilator in R is zero; in particular, $\{0\} \in \operatorname{Ass}_R(D(H^1_I(R)))$.

A generalization of the preceding example is

Theorem 2.5. *Let $i \in \mathbb{N}^+$. For an arbitrary sequence $\underline{x} = x_1, \dots, x_i$ of elements of R one has*

$$\{p \in \operatorname{Spec}(R) | \underline{x} \text{ is part of a s. o. p. of } R/p\} \subseteq \operatorname{Ass}_R(D(H^i_{(\underline{x})R}(R))). \tag{1}$$

A proof can be found in [**7**, Theorem 3.1.3]. On the other hand, it was shown also in [**7**, Remark 1.2.1] that

REMARK 2.6. *In the above situation*

$$\mathrm{Ass}_R(D(H^i_{(\underline{x})R}(R))) \subseteq \{p \in \mathrm{Spec}(R) | H^i_{(\underline{x})R}(R/p) \neq 0\} \tag{2}$$

holds.

But while one can show that, in general, (1) is not an equality, it is conjectured that (2) is an equality; this is conjecture (*) from [**5, 7, 8**].

THEOREM 2.7. *The following statements are equivalent:*

(i) Conjecture (*) holds, i. e. for every noetherian local ring (R, m), every $i > 0$ and every sequence $x_1, \dots, x_i$ of elements of R the equality

$$\mathrm{Ass}_R(D(H^i_{(x_1,\dots,x_i)R}(R))) = \{p \in \mathrm{Spec}(R) | H^i_{(x_1,\dots,x_i)R}(R/p) \neq 0\}$$

holds.

(ii) For every noetherian local ring (R, m), every $i > 0$ and every sequence $x_1, \dots, x_i$ of elements of R the set

$$Y := \mathrm{Ass}_R(D(H^i_{(x_1,\dots,x_i)}(R)))$$

is stable under generalization, i.e., the implication

$$p_0, p_1 \in \mathrm{Spec}(R), p_0 \subseteq p_1, p_1 \in Y \Longrightarrow p_0 \in Y$$

holds.

(iii) For every noetherian local domain (R, m), every $i > 0$ and every sequence $x_1, \dots, x_i$ of elements of R the implication

$$H^i_{(x_1,\dots,x_i)}(R) \neq 0 \Longrightarrow \{0\} \in \mathrm{Ass}_R(D(H^i_{(x_1,\dots,x_i)R}(R)))$$

holds.

(iv) For every noetherian local ring (R, m), every finitely generated R-module M, every $i > 0$ and every sequence $x_1, \dots, x_i$ of elements of R the equality

$$\mathrm{Ass}_R(D(H^i_{(x_1,\dots,x_i)R}(M))) = \{p \in \mathrm{Supp}_R(M) | H^i_{(x_1,\dots,x_i)R}(M/pM) \neq 0\} \tag{3}$$

holds.

PROOF. First we show that (i) – (iii) are equivalent.
(i) $\Longrightarrow$ (ii): In the given situation we have

$$\mathrm{Hom}_R(R/p_1, D(H^i_{(x_1,\dots,x_i)R}(R))) \neq 0;$$

this implies

$$\begin{aligned} 0 &\neq \mathrm{Hom}_R(R/p_0, D(H^i_{(x_1,\dots,x_i)R}(R))) \\ &= \mathrm{Hom}_R(H^i_{(x_1,\dots,x_i)R}(R) \otimes_R (R/p_0), E_R(R/m)) \\ &= D(H^i_{(x_1,\dots,x_i)R}(R/p_0)). \end{aligned}$$

Thus conjecture (*) implies that p_0 is associated to $D(H^i_{(x_1,\ldots,x_i)R}(R))$.
(ii) $\Longrightarrow$ (iii): We assume that $H^i_{(x_1,\ldots,x_i)R}(R) \neq 0$. This implies

$$D(H^i_{(x_1,\ldots,x_i)R}(R)) \neq 0$$

and hence $\mathrm{Ass}_R(D(H^i_{(x_1,\ldots,x_i)R}(R))) \neq \emptyset$; now (ii) shows

$$\{0\} \in \mathrm{Ass}_R(D(H^i_{(x_1,\ldots,x_i)R}(R))).$$

(iii) $\Longrightarrow$ (i): We know that $\subseteq$ holds always; we take a prime ideal p of R such that $H^i_{(x_1,\ldots,x_i)R}(R/p) \neq 0$ and we have to show $p \in \mathrm{Ass}_R(D(H^i_{(x_1,\ldots,x_i)R}(R)))$: We apply (iii) to the domain R/p and get an R-linear injection

$$\begin{aligned} R/p &\to D(H^i_{(x_1,\ldots,x_i)(R/p)}(R/p)) \\ &= \mathrm{Hom}_R(H^i_{(x_1,\ldots,x_i)R}(R/p), E_R(R/m)) \\ &= \mathrm{Hom}_R(H^i_{(x_1,\ldots,x_i)R}(R) \otimes_R R/p, E_R(R/p)) \\ &= \mathrm{Hom}_R(R/p, D(H^i_{(x_1,\ldots,x_i)R}(R))) \\ &\subseteq D(H^i_{(x_1,\ldots,x_i)R}(R)). \end{aligned}$$

Note that we used $H^i_{(x_1,\ldots,x_i)(R/p)}(R/p) = H^i_{(x_1,\ldots,x_i)R}(R/p)$ and the fact that $\mathrm{Hom}_R(R/p, E_R(R/m))$ is an R/p-injective hull of R/m. Now it is clearly sufficient to show that (i) implies (iv): "$\subseteq$": Every element p of the left-hand side of identity (3) must contain $\mathrm{Ann}_R(M)$ and hence is an element of $\mathrm{Supp}_R(M)$; furthermore, it satisfies

$$\begin{aligned} 0 &\neq \mathrm{Hom}_R(R/p, D(H^i_{(x_1,\ldots,x_i)R}(M))) \\ &= \mathrm{Hom}_R(R/p \otimes_R H^i_{(x_1,\ldots,x_i)R}(M), E_R(R/m)) \\ &= D(H^i_{(x_1,\ldots,x_i)R}(M/pM)). \end{aligned}$$

"$\supseteq$": Let p be an element of the support of M such that $H^i_{(x_1,\ldots,x_i)R}(M/pM)$ is not zero. We set $\overline{R} := R/\mathrm{Ann}_R(M)$, M is an $\overline{R}$-module. $p \supseteq \mathrm{Ann}_R(M)$, we set $\overline{p} := p/\mathrm{Ann}_R(M)$. Clearly, our hypothesis implies that $H^i_{(x_1,\ldots,x_i)\overline{R}}(\overline{R}) \neq 0$. We apply (i) to $\overline{R}$ and deduce

$$\overline{p} \in \mathrm{Ass}_{\overline{R}}(D(H^i_{(x_1,\ldots,x_i)\overline{R}}(\overline{R}))).$$

Hence there is an R-linear injection

$$0 \to R/p = \overline{R}/\overline{p} \to D(H^i_{(x_1,\ldots,x_i)\overline{R}}(\overline{R})),$$

which induces an R-linear injection

$$\begin{aligned} 0 &\to \operatorname{Hom}_R(M, R/p) \\ &\to \operatorname{Hom}_R(M, D(H^i_{(x_1,\ldots,x_i)\overline{R}}(\overline{R}))) \\ &= \operatorname{Hom}_{\overline{R}}(M, D(H^i_{(x_1,\ldots,x_i)\overline{R}}(\overline{R}))) \\ &= D(H^i_{(x_1,\ldots,x_i)\overline{R}}(M)) \\ &= D(H^i_{(x_1,\ldots,x_i)R}(M)). \end{aligned}$$

Note that for the second equality we have used Hom-Tensor adjointness and for the last equality the facts that M is an $\overline{R}$-module and that $\operatorname{Hom}_R(\overline{R}, E_R(R/m))$ is an $\overline{R}$-injective hull of R/m; It is sufficient to show $p \in \operatorname{Ass}_R(\operatorname{Hom}_R(M, R/p))$; but M is finite and so we have

$$(\operatorname{Hom}_R(M, R/p))_p = \operatorname{Hom}_{R_p}(M_p, R_p/pR_p) \neq 0,$$

which shows that pR_p is associated to the R_p-module $(\operatorname{Hom}_R(M, R/p))_p$. Thus $p \in \operatorname{Ass}_R(\operatorname{Hom}_R(M, R/p))$. □

Even in the following special case it seems to be very difficult to completely calculate the set of associated primes:

EXAMPLE 2.8. Let $R = k[[x_1, \ldots, x_n]]$ be a formal power series ring in $n \geq 2$ (to avoid trivial cases) variables and set $I := (x_1, \ldots, x_i)$ for some $1 \leq i \leq n$. Let m denote the maximal ideal of R.

- Case $i = n$: It is easy to see that (*) holds.
- Case $i = n - 1$: Conjecture (*) holds (this follows e.g. from Theorem 2.10 below).
- Case $i = n - 2$: It is easy to see (e.g. from Remark 2.6) that for every prime ideal p of R one has $p \in \operatorname{Ass}_R(D(H^{n-2}_I(R))) \Rightarrow \operatorname{height}(p) \leq 2$; in addition, for every height two prime ideal p of R, one has

  $$p \in \operatorname{Ass}_R(D(H^{n-2}_I(R))) \iff I + p \text{ is } m\text{-primary}.$$

  For every prime element $p \in I \setminus mI$ one has

  $$pR \notin \operatorname{Ass}_R(D(H^{n-2}_I(R)))$$

  (this can be deduced from $H^{n-2}_I(R/pR) = 0$). But note that, in general, there are prime elements $p \in I$ such that $pR \in \operatorname{Ass}_R(D(H^{n-2}_I(R)))$. For example, take $n = 5, h = 3, k = \mathbb{Q}$ and $p := -X_2X_4^2 + X_3X_4X_5 - X_1X_5^2 + 4X_1X_2 - X_3^2 \in I$. Then pR is (even maximal) in $\operatorname{Ass}_R(D(H^3_I(R)))$ (this is explained and proven in [**7**, Remarks 4.3.2]).

  Finally, if $p \in R$ is a prime element that is not contained in I, one has

  $$pR \in \operatorname{Ass}_R(D(H^{n-2}_I(R)))$$

  (this follows from Theorem 2.5).

Finally, there are two results on the set $\mathrm{Ass}_R(D(H_J^{\dim(R)}(R)))$ for one-dimensional ideals $J \subseteq R$:

THEOREM 2.9. *Let $J \subseteq R$ be an ideal of the local ring R such that $\dim(R/J) = 1$ and $H_J^{\dim(R)}(R) = 0$. Then*

$$\mathrm{Assh}(D(H_J^{\dim(R)-1}(R))) = \mathrm{Assh}(R)$$

(here Assh *stands for those associated primes of highest dimension) holds.*

THEOREM 2.10. *Let $J \subseteq R$ be an ideal of the local complete ring R such that $\dim(R/J) = 1$ and $H_J^{\dim(R)}(R) = 0$. Then*

$$\mathrm{Ass}_R(D(H_J^{\dim(R)-1}(R)))$$

$$= \{P \in \mathrm{Spec}(R) | \dim(R/P) = \dim(R) - 1, \dim(R/(P+J)) = 0\} \cup \mathrm{Assh}(R)$$

holds.

Proofs of the preceding two results can be found in [**7**, Theorems 3.2.6 and 3.2.7].

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