ISSN 1472-2739 (on-line) 1472-2747 (printed)


*Algebraic & Geometric Topology*
Volume 1 (2001) 605{625
Published: 29 October 2001

# The Homflypt skein module of a connected sum of 3-manifolds

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**Abstract** If $M$ is an oriented 3-manifold, let $S(M)$ denote the Homflypt skein module of $M$. We show that $S(M_1 \# M_2)$ is isomorphic to $S(M_1)$ $S(M_2)$ modulo torsion. In fact, we show that $S(M_1 \# M_2)$ is isomorphic to $S(M_1)$ $S(M_2)$ if we are working over a certain localized ring. We show the similar result holds for relative skein modules. If $M$ contains a separating 2-sphere, we give conditions under which certain relative skein modules of $M$ vanish over speci ed localized rings.

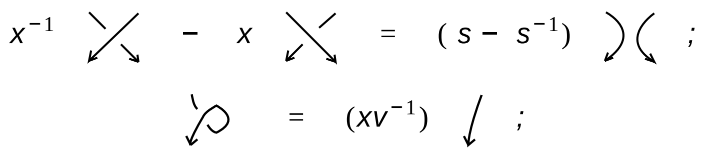
**AMS Classi cation** 57M25

**Keywords** Young diagrams, relative skein module, Hecke algebra

## 1 Introduction

We will be working with framed oriented links. By this we mean links equipped with a string orientation together with a nonzero normal vector eld up to homotopy. The links described by gures in this paper will be assigned the \blackboard" framing which points to the right when travelling along an oriented strand.

**De nition 1 The Homflypt skein module** Let $k$ be a commutative ring containing $x^{1}$; $v^{1}$; $s^{1}$; and $\frac{1}{s-s^{-1}}$. Let $M$ be an oriented 3-manifold. The Homflypt skein module of $M$ over $k$, denoted by $S_k(M)$, is the $k$-module freely generated by isotopy classes of framed oriented links in $M$ including the empty link, quotiented by the Homflypt skein relations given in the following gure.



$$L \sqcup \bigcirc = \frac{v^{-1} - v}{s - s^{-1}} L .$$

The last relation follows from the first two in the case $L$ is nonempty.

**Remark** (1) An embedding $f : M \to N$ of 3-manifolds induces a well defined homomorphism $f_* : S_k(M) \to S_k(N)$. (2) If $N$ is obtained by adding a 3-handle to $M$, the embedding $i : M \to N$ induces an isomorphism $i_* : S_k(M) \to S_k(N)$. (3) If $N$ is obtained by adding a 2-handle to $M$, the embedding $i : M \to N$ induces an epimorphism $i_* : S_k(M) \to S_k(N)$. (4) If $M_1 \sqcup M_2$ is the disjoint union of 3-manifolds $M_1$ and $M_2$, then $S_k(M_1 \sqcup M_2) = S_k(M_1) \otimes S_k(M_2)$.

Associated to a partition of $n$, $\lambda = (\lambda_1 \geq \ldots \geq \lambda_p \geq 1)$, $\lambda_1 + \cdots + \lambda_p = n$, is associated a Young diagram of size $|\lambda| = n$, which we denote also by $\lambda$. This diagram has $n$ cells indexed by $\{(i, j); 1 \leq i \leq p, 1 \leq j \leq \lambda_i\}$. If $c$ is the cell of index $(i, j)$ in a Young diagram $\lambda$, its content $cn(c)$ is defined by $cn(c) = j - i.$ Define

$$c_{\lambda,\mu} = v(s^{-1} - s) \sum_{c \in \lambda} s^{-2cn(c)} + v^{-1}(s - s^{-1}) \sum_{c \in \mu} s^{2cn(c)}$$

Let $I$ denote the submonoid of the multiplicative monoid of $\mathbb{Z}[v, s]$ generated by $v$, $s$, $s^{2n} - 1$ for all integers $n > 0$, and $c_{\lambda,\mu}$ for all pairs of Young diagrams $\lambda$, and $\mu$, with $|\lambda| = |\mu|$, and $|\lambda| \neq 0.$ Let $R$ be $\mathbb{Z}[v, s]$ localized at $I.$ [5, 7.2]

**Theorem 1**

$$S_{R[x,x^{-1}]}(M_1 \# M_2) = S_{R[x,x^{-1}]}(M_1) \otimes S_{R[x,x^{-1}]}(M_2).$$

**Remark** J. Przytycki has proved the analog of this result for the Kauffman bracket skein module [9]. Our proof follows the same general outline. We thank J. Przytycki for suggesting the problem of obtaining a similar result for the Homflypt skein module.

Let $I'$ denote the submonoid of the multiplicative monoid of $R$ generated by $v^4 - s^{2n}$, for all $n.$ Let $R'$ be $R$ localized at $I'.$ It follows from [4], $S_{R'[x,x^{-1}]}(S^1 \times S^2)$ is the free $R'[x, x^{-1}]$-module generated by the empty link.

**Corollary 1** $S_{R'[x,x^{-1}]}(\#^m S^1 \times S^2)$ *is a free module generated by the empty link.*

**Remark** This allows us to define a "Homflypt rational function" $f$ in $R'$ for oriented framed links in $\#^m S^1 \times S^2$. If $L$ is such a link, one defines $f(L)$ by $L = f(L)\emptyset \in S_{R'}(\#^m S^1 \times S^2)$. A specific example is given in section 5.

Let $I = R$ with $x = v$; then $S_I(M)$ is a version of the Homflypt skein module for unframed links. The next two corollaries follows from the universal coefficient property for skein modules which has been described by J. Przytycki [9] for the Kauffman bracket skein module. The proof given there holds generally for essentially any skein module.

**Corollary 2** $S_I(M_1 \# M_2) = S_I(M_1) \otimes S_I(M_2).$

Let $I' = R'$ with $x = v.$

**Corollary 3** $S_{I'}(\#^m S^1 \times S^2)$ *is a free* $I'$*-module generated by the empty link.*

**Definition 2 The relative Homflypt skein module** Let $X = \{x_1, x_2, \cdots, x_n\}$ be a finite set of input framed points in $\partial M$, and let $Y = \{y_1, y_2, \cdots, y_n\}$ be a finite set of output framed points in the boundary $\partial M$. Define the relative skein module $S_k(M, X, Y)$ to be the $k$-module generated by relative framed oriented links in $(M, \partial M)$ such that $L \cap \partial M = \partial L = \{x_i, y_i\}$ with the induced framing, considered up to an ambient isotopy fixing $\partial M$, quotiented by the Homflypt skein relations.

Let $S(M)$ denote $S_{R[x,x^{-1}]}(M,)$ and let $S(M, X, Y)$ denote $S_{R[x,x^{-1}]}(M, X, Y).$ We have the following version of Theorem 1 for relative skein modules. At this point we must work over the field of fractions of $\mathbb{Z}[x, v, s]$ which we denote by $F.$ This is because we do not know whether the relative skein module of a handlebody is free. We conjecture that it is free. In the proof of Theorem 1, we use the absolute case first obtained by Przytycki [8]. We state Theorem 2 over $F,$ but conjecture it over $R'[x, x^{-1}].$

**Theorem 2** *Let* $M_1$ *and* $M_2$ *be connected oriented 3-manifolds. Let* $X_i = \{x_1, x_2, \cdots, x_n\}$ *be a finite set of input framed points in* $\partial M_i$*, and let* $Y_i = \{y_1, y_2, \cdots, y_n\}$ *be a finite set of output framed points in the boundary* $\partial M_i.$ *Let* $X = X_1 \cup X_2,$ *and* $Y = Y_1 \cup Y_2$*, then*

$$S_F(M_1 \# M_2, X, Y) = S_F(M_1, X_1, Y_1) \otimes S_F(M_2, X_2, Y_2).$$

We also have the following related result. Let $I_r$ denote the submonoid of the multiplicative monoid of $\mathbb{Z}[x, v, s]$ generated by $x,\ v,\ s,\ s^{2n} - 1$ for all integers

$n > 0$; and $x^r - 1 - c$ $_{;}$ for all pairs of Young diagrams , and ; such that $j\ j - j\ j = r$, and $j\ j$ and $j\ j$ are not both zero. Let $k_r$ be $\mathbb{Z}[x; v; s]$ localized at $I_r$: Note $k_0 = R[x; x^{-1}]$:

**Theorem 3** *Suppose $M$ is connected and contains a 2-sphere ; such that $M -$ has two connected components. Let $M^0$ be one of these components. If $jX \setminus M^0 j - jY \setminus M^0 j = r \neq 0$; then $S_{k_r}(M; X; Y) = 0$:*

In section 2, we prove that there is an epimorphism from $S(H_{m_1})$ $S(H_{m_2})$ to $S(H_{m_1} \# H_{m_2})$: Here and below, we let $H_m$ denote a handlebody of genus $m$: In section 3, we prove Theorem 1 in the case of handlebodies. We prove Theorem 1 in the general case in section 4. Section 5 describes the class of a certain link in the $S^1$ $S^2 \# S^1$ $S^2$: Section 6 gives a proof of a lemma needed in section 2. In section 7, we discuss the proofs of Theorems 2 and 3.

# 2 Epimorphism for Handlebodies

## 2.1 The $n$th Hecke algebra of Type A

We will use the related work of C. Blanchet [2], A. Aiston and H. Morton [1] on the $n$th Hecke algebra of Type A. This is summarized in section 3 of [4] whose conventions we follow. For the convenience of the reader, we give the basic de nitions in this subsection.

Note that $s^{2n} - 1$ is invertible in $R$ for integers $n > 0$: It follows that the quantum integers $[n] = \dfrac{s^n - s^{-n}}{s - s^{-1}}$ for $n > 0$ are invertible in $k$. Let $[n]! = \prod_{j=1}^{n} [j]$, so $[n]!$ is invertible for $n > 0$.

**The Hecke category** The $k$-linear Hecke category $H$ is de ned as follows. An object in this category is a disc $D^2$ equipped with a set of framed points. If $= (D^2; l)$ and $= (D^2; l^0)$ are two objects, the module $Hom_H( ; )$ is $S(D^2$ $[0; 1]; l$ $1; l^0$ $0)$. The notation $H( ; )$ and $H$ will be used for $Hom_H( ; )$ and $H( ; )$ respectively. The composition of morphisms are by stacking the rst one on the top of the second one.

Let denote the monoid structure on $H$ given by embedding two disks $D^2$ side by side into one disk. For a Young diagram , by assigning each cell of a point equipped with the horizontal (to the left) framing, we obtain an object of the category $H$ denoted by $\Box$ . When is the Young diagram with a single row of $n$ cells, $H_\lambda$ will be denoted by $H_n$, which is the $n$th Hecke algebra of type A [7], [10].

For each permutation $\pi \in S_n$, a positive permutation braid, $\omega_\pi$, is a braid which realizes the permutation $\pi$ with all crossings positive [6]. Let $\sigma_i \in H_n;\ i = 1, \ldots, n-1$, be the positive permutation corresponding to the transposition $(i\ i+1)$. As in [1], define

$$f_n = \frac{1}{[n]!} s^{-\frac{n(n-1)}{2}} \sum_{\pi \in S_n} (xs^{-1})^{-l(\pi)} \omega_\pi$$

and

$$g_n = \frac{1}{[n]!} s^{\frac{n(n-1)}{2}} \sum_{\pi \in S_n} (-xs)^{-l(\pi)} \omega_\pi$$

Here $l(\pi)$ is the length of $\pi$.

**Idempotents in the Hecke Algebra** [1] For a Young diagram $\lambda$ of size $n$, let $F_\lambda$ be the element in $H_\lambda$ formed with one copy of $[\lambda_i]! f_{\lambda_i}$ along the row $i$, for $i = 1, \ldots, p$. We let $\lambda^\vee$ denote the Young diagram whose rows are the columns of $\lambda$. Let $G_\lambda$ be the element in $H_\lambda$ formed with one copy of $[\lambda^\vee_j]! g_{\lambda^\vee_j}$ along the column $j$, for $j = 1, \ldots, q$. Let $\tilde{y}_\lambda = F_\lambda G_\lambda$, then $\tilde{y}_\lambda$ is a quasi-idempotent. Let $y_\lambda$ be the normalized idempotent from $\tilde{y}_\lambda$.

**A Basis for the $n$th Hecke Algebra $H_n$** A standard tableau $t$ with the shape of a Young diagram $\lambda = \lambda(t)$ is a labeling of the cells, with the integers 1 to $n$ increasing along the rows and the columns. Let $t'$ be the tableau obtained by deleting the cell numbered by $n$. Note the cell numbered by $n$ in a standard tableau is an *extreme cell*. C. Blanchet defines $\alpha_t \in H(n; \lambda)$ and $\beta_t \in H(\lambda; n)$ inductively by

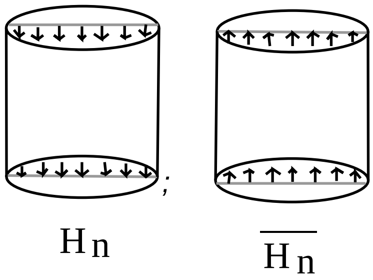
$$\alpha_1 = \beta_1 = 1_1;$$
$$\alpha_t = (\alpha_{t'} \otimes 1_1)\rho_t y_\lambda;$$
$$\beta_t = y_\lambda \rho_t^{-1}(\beta_{t'} \otimes 1_1).$$

Here $\rho_t \in H(\lambda(t') \otimes 1; \lambda)$ is the isomorphism given by an arc joining the added point to its place in $\lambda$ in the standard way.

Note that $\beta_t \alpha_{t'} = 0$ if $t' \neq t$, and $\beta_t \alpha_t = y_{\lambda(t)}$.

**Theorem 4** (Blanchet) *The family $\alpha_t \beta_{t'}$ for all standard tableaux $t, t'$ such that $\lambda(t) = \lambda(t')$ for all Young diagrams $\lambda$ with $|\lambda| = n$ forms a basis for $H_n$.*

Let $\overline{H_n}$ denote $H_n$ with the reversed string orientation.

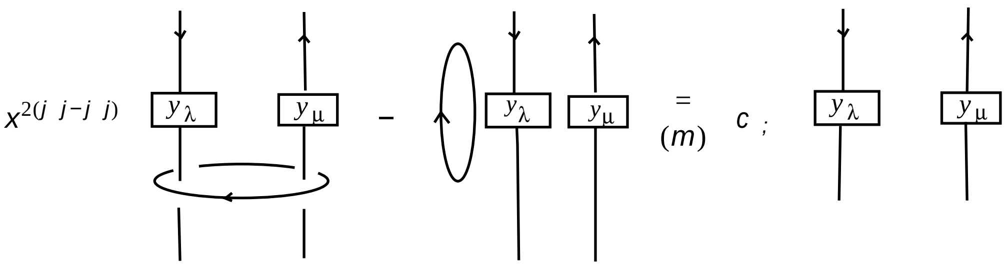



## 2.2 The Epimorphism on the Handlebodies

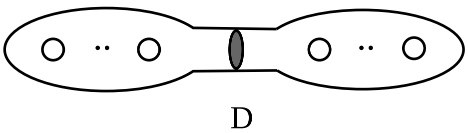
If $X_m$ is a set of $m$ distinguished framed points in $D^2 \times \{1\}$ and $Y_m$ be a set of $m$ distinguished framed points in $D^2 \times \{0\}$, Let $\underset{(m)}{=}$ denote equality in $S(D^2 \times I; X, Y)$ modulo the submodule $L(m)$ generated by links which intersects $D^2 \times \{\frac{1}{2}\}$ in less than $m$ points.

In section 6, we derive:

**Lemma 2.1** *Let $\lambda$, $\mu$ be two Young diagrams, and $m = |\lambda| + |\mu|$:*




Let $H_m$ be a handlebody of genus $m$. Let $D$ be a separating meridian disc of $H_m$, let $\gamma = \partial D$. Let $(H_m)_\gamma$ be the manifold obtained by adding a 2-handle to $H_m$ along $\gamma$.



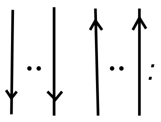

Let $V_D = [-1, 1] \times D$ be the regular neighborhood of $D$ in $H_m$, $V_D$ can be projected into a disc $D_p = [-1, 1] \times [0, 1]$.

**Lemma 2.2** (The Epimorphism Lemma) *The embedding* $i$ : $H_m - D$ ! $(H_m)_Y$ *induces an epimorphism:*

$$i \ : S(H_m - D) \twoheadrightarrow S((H_m)_Y):$$

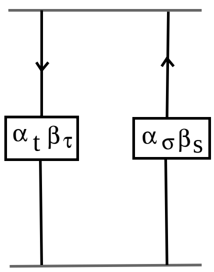
**Proof** Let $z_n$ be a link in $H_m$ in general position with $D$ and cutting $D$ $2n$ times, let $z_n^0 = z_n \setminus V_D$, i.e.,

$z_n^0$ =



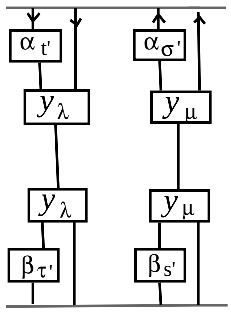

Note $z_n^0$ 2 $H_n$ $\overline{H_n}$. Using the basis elements $_t$ of $H_n$ given in the previous theorem, $z_n^0$ can be written as a linear combination of the elements $_t$ $_s$, where $_s$ is $_s$ with the reversed orientation. A diagram of $_t$ $_s$ is given by the following:

$_t$ $_s$ =

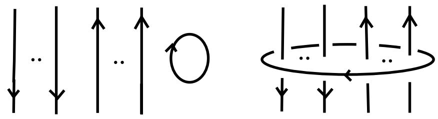



By the inductive de nition of $_t$; ; ; $_s$, an alternative diagram of $_t$ $_s$ is given by:

$_t$ $_s$ =

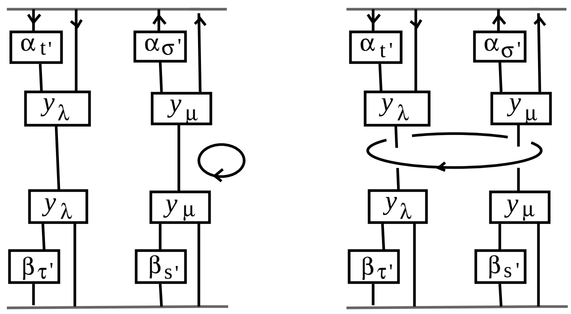


We will consider the sliding relation given by:



$z_n \cup O$ $\qquad$ $p(z_n)$ (I)

From the above observation, we will be interested in the following relation:




(II)

From Relation II, and Lemma 2.1, as $| \ | = | \ |$, in $S(D^2 \ I)_\gamma$ we have

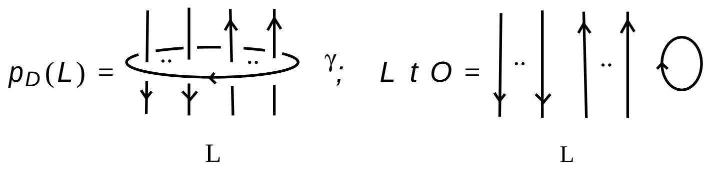
$$c_{\ ;\ } (\ _t \qquad _s) \in L_{| \ |+| \ |}.$$

As $c_{\ ;\ }$ is invertible in $R$, we have that $\ _t \qquad _s \in L(| \ | + | \ |)$. By induction, we can eliminate all elements of $(H_m)_\gamma$ which cut the 2-disk $D_\gamma$ non-trivially. Thus $i$ is an epimorphism. □

## 3 Isomorphism for handlebodies

Recall that $(H_m)_\gamma$ is obtained by adding a 2-handle to $H_m$ along $\gamma$. From [4] section 2, we have $S((H_m)_\gamma) = S(H_m)/R$, where $R$ is the submodule of $S(H_m)$ given by the collection $\{\ '(z) - \ ''(z) \mid z \in S(H_m; A; B)\}$. Here $A; B$ are two points on $\gamma$, which decompose $\gamma$ into two intervals $\gamma'$ and $\gamma''$, $z$ is any element of the relative skein module $S(H_m; A; B)$ with $A$ an input point and $B$ an output point, and $\ '(z)$ and $\ ''(z)$ are given by capping o with $\gamma'$ and $\gamma''$, respectively, and pushing the resulting links back into $H_m$.

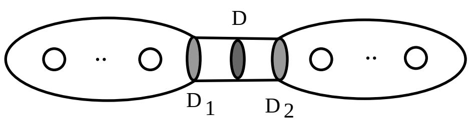
Let $I_0$ be the submodule of $S(H_m)$ given by the collection $\{p_D(L) - L \sqcup O \mid L \in S(H_m)\}$, where $O$ is the unknot. Locally, we have the following diagram description.



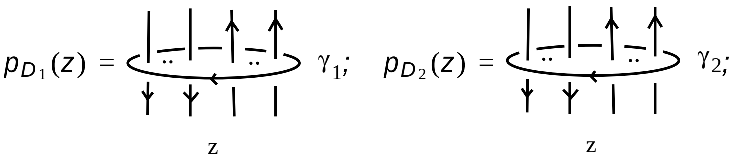

**Lemma 3.1** $R = I_0$.

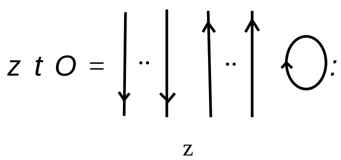
**Proof** First note $R \supseteq I_0$. We need only show that $R \subseteq I_0$. Let be the projection map $: S(H_m) \to S(H_m)/I_0$. We will show that $(R) = 0$ in $S(H_m)/I_0$, i.e. $R \subseteq I_0$. We show this by proving now that $({}'(z)) = ({}''(z))$ for any $z \in S(H_m; A; B)$.

Recall that $V_D = [-1, 1] \times D$ is the regular neighborhood of $D$ in $H_m$. Let $D_1 = \{-1\} \times D$ and $D_2 = \{1\} \times D$. Let $\gamma_1 = \partial D_1$ and $\gamma_2 = \partial D_1$, note $\gamma_1$ and $\gamma_2$ are parallel to $\gamma$.




Let $I_1 = \{p_{D_1}(z) - z \sqcup O \mid z \in S(H_m; A; B)\}$, $I_2 = \{p_{D_2}(z) - z \sqcup O \mid z \in S(H_m; A; B)\}$, where locally




Let ${}_{A;B}$ be the projection map ${}_{A;B} : S(H_m; A; B) \to S(H_m; A; B)/(I_1 + I_2)$. Note that ${}'(I_i) = I_0$ and ${}''(I_i) = I_0$ for $i = 1, 2$.

Let $z \in S(H_m - (D_1 \cup D_2); A; B)$, then ${}'(z) = {}''(z)$ in $S(H_m - D_1 - D_2)$, since $V_D = [-1, 1] \times D$ is a 3-disc and closing a relative link along $\gamma$ by $\gamma'$ and $\gamma''$ in $V_D$ gives isotopic links. Let $: S(H_m - (D_1 \cup D_2); A; B) \to S(H_m - D_1 - D_2)$ denote the map which sends $z$ to ${}'(z) = {}''(z)$.

In general, let $z \in S(H_m; A, B)$. Now consider the following commutative diagram,

$$\begin{array}{ccccc}
S(H_m - (D_1 \cup D_2); A, B) & \xrightarrow{j_1} & S(H_m; A, B) & \xrightarrow{\ _{A,B}} & S(H_m; A, B)/(I_1 + I_2) \\
\Big\downarrow & & \Big\downarrow {}' & & \Big\downarrow {}' \\
S(H_m - (D_1 \cup D_2)) & \xrightarrow{j_2} & S(H_m) & \longrightarrow & S(H_m)/I_0 \\
\Big\uparrow & & \Big\uparrow {}'' & & \Big\uparrow {}'' \\
S(H_m - (D_1 \cup D_2); A, B) & \xrightarrow{j_1} & S(H_m; A, B) & \xrightarrow{\ _{A,B}} & S(H_m; A, B)/(I_1 + I_2)
\end{array}$$

Here $j_1$ and $j_2$ are induced by inclusion maps. Also $'$, and $''$ are induced by $'$, and $''$ respectively. By an argument similar to the proof of Lemma 2.2, the composition map $_{A,B}\, j_1 : S(H_m - (D_1 \cup D_2); A, B) \to S(H_m; A, B)/(I_1 + I_2)$ is an epimorphism.

Take $z \in S(H_m; A, B)$, then $_{A,B}(z) \in S(H_m; A, B)/(I_1 + I_2)$. As $_{A,B}\, j_1$ is an epimorphism, there exists $z' \in S(H_m - (D_1 \cup D_2); A, B)$ such that $_{A,B}\, j_1(z') = {}_{A,B}(z)$. By the commutativity of the diagram, $j_2(\ (z')) = (\ '(z))$ and $j_2(\ (z')) = (\ ''(z))$. Thus $(\ '(z)) = (\ ''(z))$. □

**Corollary 4** *The embedding $H_m \to (H_m)_Y$ induces an isomorphism*

$$S(H_m)/I_0 = S((H_m)_Y).$$

Now we want to show that the embedding $H_m - D \to (H_m)_Y$ induces an isomorphism

$$S(H_m - D) = S((H_m)_Y).$$

**Lemma 3.2**

$$S(H_m - D) \cap I_0 = 0.$$

**Proof** Przytycki [8] calculated the unframed Homflypt skein module of a handlebody. It follows from this, the universal coefficient property of skein modules and an argument of Morton in [6] section (6.2) that $S(H_m)$ is free. As $S(H_m - D)$ is free, the map $S(H_m - D) \to S_F(H_m - D)$, induced by $R[x, x^{-1}] \to F$ is injective. Let $I_0 = \{p_D(L) - L \sqcup O \mid L \in S_F(H_m)\}$. It is enough to show $S_F(H_m - D) \cap I_0 = 0$.

Let be the map from $S_F(H_m) \to S_F(H_m)$ given by $(L) = p_D(L) - L \sqcup O$ for $L \in S_F(H_m)$. $Image(\ ) = I_0$.

It also follows from Przytycki's basis that the map induced by inclusion $S(H_m - D) \to S(H_m)$ is injective. Let $B_0$ be the image of a free basis for the module $S(H_m - D)$ in $S(H_m)$. $B_0$ also a basis for injective image of $S_F(H_m - D)$ in $S_F(H_m)$. Let $B_n$ be the subspace of $S_F(H_m)$ generated by framed oriented links in $H_m$ which intersect the disk $D$ $2n$ times. Then we have a chain of vector spaces:

$$B_0 \quad B_1 \quad B_2 \qquad B_n$$

$B_0$ is a basis for $B_0$. The vector space $B_n/B_{n-1}$ is generated by elements of the form $_t$ $_s$ in a neighborhood of $D$, where $| \, | = | \, |$ is $n$. Let $B_n$ be a basis $B_n/B_{n-1}$, constructed by taking a maximal linearly independent subset of the above generating set. By the proof of Lemma 2.2, each element of $B_n$, where $n > 0$, is an eigenvector for with nonzero eigenvalue. $B = \cup_{n \ 0} B_n$ is a basis for $S_F(H_m)$. Let $B' = B - B_0$. Note $(B_0) = 0$. So $I_0 = Image( \ ) = (B')$.

It follows that induces a one to one map: $B_n/B_{n-1} \to B_n/B_{n-1}$. Thus $(< B' >) \cap S_F(H_m - D) = 0$. The result follows. □

**Theorem 5** *The embedding $H_m - D \to (H_m)_\gamma$ induces an isomorphism*

$$S(H_m - D) = S((H_m)_\gamma).$$

**Proof** From the above, we have the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & S(H_m - D) \cap I_0 & \longrightarrow & S(H_m - D) & \longrightarrow & S((H_m)_\gamma) & \longrightarrow & 0 \\
 & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \\
0 & \longrightarrow & I_0 & \longrightarrow & S(H_m) & \longrightarrow & S((H_m)_\gamma) & \longrightarrow & 0
\end{array}$$

□

$H_{m_1} \# H_{m_2}$ is equal to $H_{m_1+m_2}$ with a 2-handle added along the boundary of the meridian disc $D$ separating $H_{m_1}$ from $H_{m_2}$. Let $\gamma = \partial D$. Therefore we can consider $H_1 \# H_2 = (H_m)_\gamma$. As $H_{m_1+m_2} - D = H_{m_1} \sqcup H_{m_2}$, the above theorem says:

**Corollary 5** *Let $B_1$ and $B_2$ denote the 3-balls we remove from $H_{m_1}$ and $H_{m_2}$ while forming $H_{m_1} \# H_{m_2}$. The embedding $(H_{m_1} - B_1) \sqcup (H_{m_2} - B_2) \to H_{m_1} \# H_{m_2}$ induces*

$$S(H_{m_1}) \quad S(H_{m_2}) = S(H_{m_1} \# H_{m_2})$$

# 4 The general case for absolute skein modules

A connected oriented 3-manifold with nonempty boundary may be obtained from the handlebody $H$ by adding some 2-handles. If $M$ is closed, we will also need one 3-handle. As removing 3-balls from the interior of a 3-manifold does not change its Homflypt skein module, we may reduce Theorem 1 to the case that $M_1$ and $M_2$ are connected 3-manifolds with boundary.

In this case, each $M_i$ is obtained from the handlebody $H_{m_i}$ by adding some 2-handles. Let $m = m_1 + m_2$. Let $N$ be the manifold obtained by adding both sets of 2-handles to the boundary connected sum of $H_{m_1}$ and $H_{m_2}$ which we identify with $H_m$. Let $D$ be the disc in $H_m$ separating $H_{m_1}$ from $H_{m_2}$. Let $\gamma = \partial D$, so $H_{m_1} \# H_{m_2} = (H_m)_\gamma$. Here and below $P_\alpha$ denotes the result of adding a 2-handle to a 3-manifold $P$ along a curve $\alpha$ in $\partial N$. We can consider $M_1 \# M_2$ as obtained from $(H_m)_\gamma$ by adding those 2-handles. Thus $N - D = M_1 \sqcup M_2$, and $M_1 \# M_2 = N_\gamma$.

**Theorem 6** *The embedding $N - D \to N_\gamma$ induces an isomorphism*

$$S(N - D) = S(N_\gamma).$$

**Proof** We proceed by induction on $n$, the number of the 2-handles to be added to $(H_m)_\gamma$ to obtain $N_\gamma$. If $n = 0$, we are done by Theorem 5. If $n \geq 1$, let $N'$ be the 3-manifold obtained from $(H_m)_\gamma$ by adding $(n-1)$ of those 2-handles added to $(H_m)_\gamma$. Suppose the result is true for $N'$, i.e.

$$S(N' - D) = S(N'_\gamma).$$

Suppose that the $n$th 2-handle is added along a curve $\gamma'$ in the boundary of $(H_m)_\gamma$, where $\gamma'$ is disjoint from $\gamma$ and the curves where the other $(n-1)$ 2-handles are attached. Let $A'$ and $B'$ be two points on $\gamma'$. By the proof of the Epimorphism Lemma 2.2,

$$S(N' - D; A', B') \twoheadrightarrow S(N'_\gamma; A', B').$$

Using [4, section 2], we have the following commutative diagram with exact rows.

$$\begin{array}{ccccccc}
S(N' - D; A', B') & \longrightarrow & S(N' - D) & \longrightarrow & S((N' - D)_{\gamma'}) & \longrightarrow & 0 \\
\downarrow \scriptstyle{\text{onto}} & & \downarrow \scriptstyle{=} & & \downarrow & & \\
S(N'_\gamma; A', B') & \longrightarrow & S(N'_\gamma) & \longrightarrow & S((N'_\gamma)_{\gamma'}) & \longrightarrow & 0
\end{array}$$

The vertical map on the right is an isomorphism by the five-lemma. $N$ is obtained from $N'$ by adding the $n$th 2-handle along $\gamma'$. Thus $(N' - D)_{\gamma'} = N - D$ and $(N'_\gamma)_{\gamma'} = N_\gamma$. □

**Corollary 6** *Let $B_1$ and $B_2$ denote the 3-balls we remove from $M_1$ and $M_2$ while forming $M_1 \# M_2$: The embedding $(M_1 - B_1)\ t\ (M_2 - B_2)\ !\ M_1 \# M_2$ induces an isomorphism*

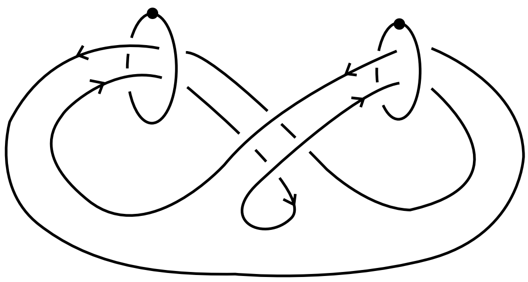
$$S(M_1) \quad S(M_2) = S(M_1 \# M_2):$$

**Proof** Since $S(M - D) = S(M_1) \quad S(M_2)$. □

The above corollary holds whether or not $M_1$ or $M_2$ have boundary.

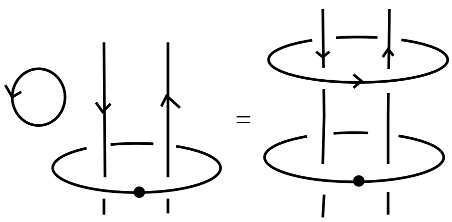
## 5 An example in $S^1 \quad S^2 \# S^1 \quad S^2$

In [4], we showed that $S(S^1 \quad S^2)$ is a free $R[x; x^{-1}]$-module generated by the empty link. It follows that $S(S^1 \quad S^2 \# S^1 \quad S^2)$ is also a free module generated by the empty link. Let $K$ be a knot in $S^1 \quad S^2 \# S^1 \quad S^2$ pictured by the following diagram:

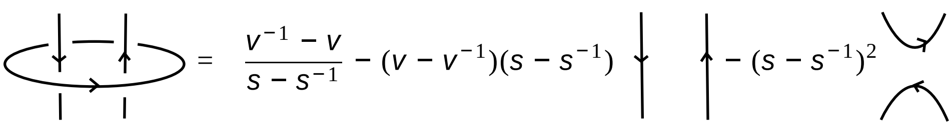


Here the two circles with a dot are a framed link description of $S^1 \quad S^2 \# S^1 \quad S^2$. Note this same knot was studied with respect to the Kau man Bracket skein modules in [3].

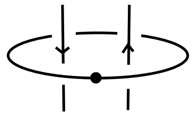
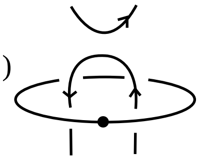
In $S(S^2 \quad S^1 \# D^3; 4pts)$, isotopy yields,



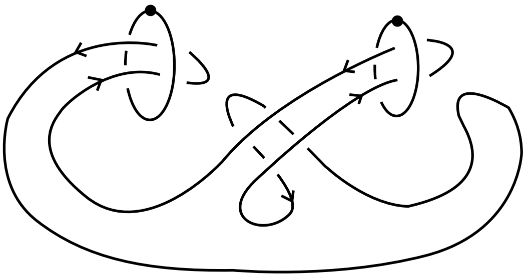

Using the Homflypt skein relations in $S(D^2 \quad I; 4pts)$,



Therefore, in $S(S^2 \quad S^1 \# D^3; 4pts)$, we have:

$(v - v^{-1})$  $-(s - s^{-1})$ 

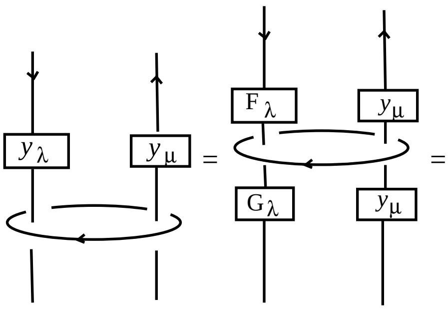
Thus

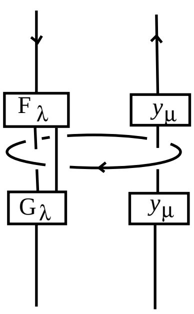
$(v - v^{-1})^2 K = (s - s^{-1})^2$ 

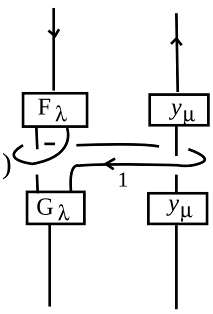
$= (s - s^{-1})^2 \frac{v^{-1}-v}{s-s^{-1}}$ : i.e. $K = \dfrac{s - s^{-1}}{v^{-1} - v}$ in $S(S^1 \quad S^2 \# S^1 \quad S^2)$.

## 6 Proof of Lemma 2.1

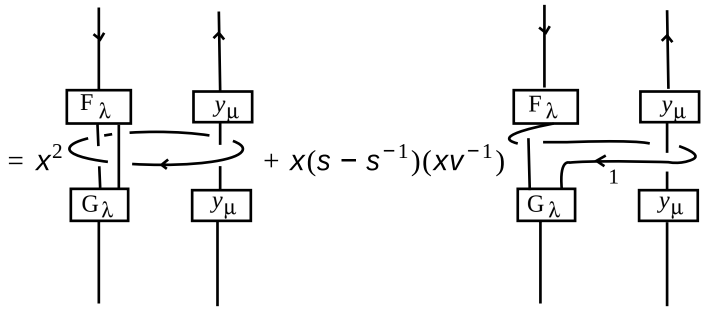
Note $y \quad = F \quad G \quad$. We start with the following:



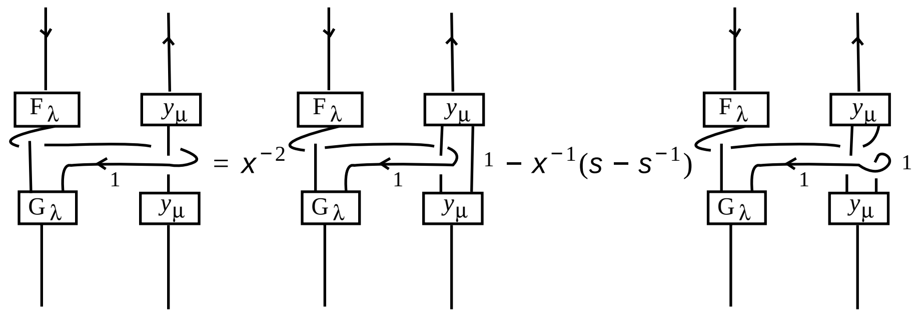

$x^2$ 

$+ x(s - s^{-1})$ 

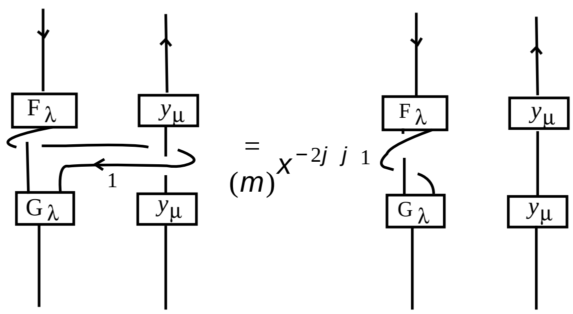
(I*)




We pulled out the string corresponding to the last cell in the last row of : Therefore in the above diagram, a 1 by the side of the string indicates the string related to the last cell in the last row of . Applying the Homflypt skein relation to the last diagram:

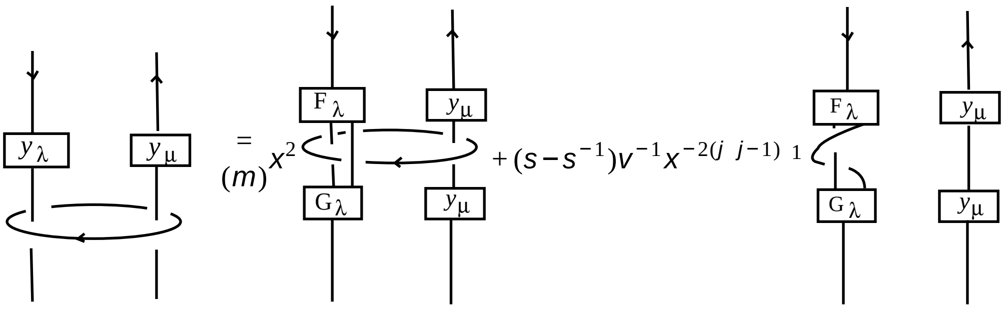



We pulled out the string corresponding to the last cell in the last row of : Continuing to pull out strings which correspond to cells of ; working to the left through columns and upward through the rows of ; we obtain:

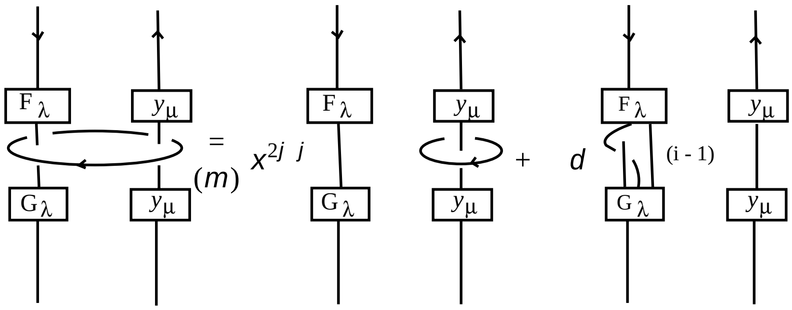



where the string corresponding to the last cell in the last row of encircles the remaining $j \; j - 1$ strings as shown.

In this way Equation (I*) becomes:

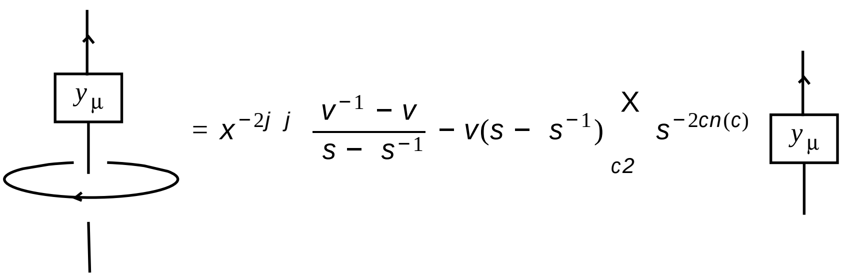



We continue in this way, pulling the encircling component successively through the vertical strings corresponding to cells of  , working to the left through columns and upward through the rows of  : We obtain:

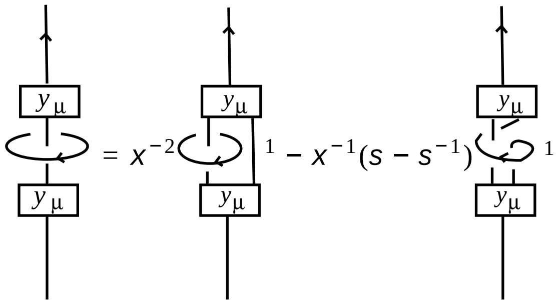



where $d$ denotes $v^{-1}(s - s^{-1}) \sum_{i=1}^{j\ j} x^{-2(j\ j-i)}$: In the last diagram, the $i-1$ vertical strings are related to the last $i-1$ cells of  by the index order, and the $i$th string encircles the remaining $j\ j - i$ strings. Lemma 2.1 follows from the following lemma and Lemma 6.2 (a) below.

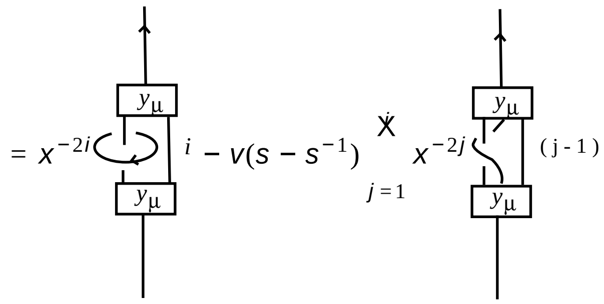
**Lemma 6.1** *Let  be a Young diagram of size $n$,*

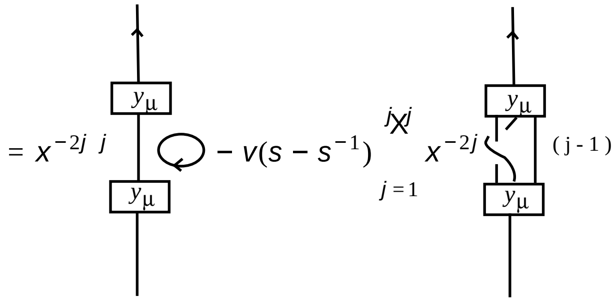


**Proof** We consider




Here we start with the string corresponding to the last cell in the last row of , we pull the encircling component successively through the vertical strings, working to the left through columns and upward through the rows. Repeating the above process, for $i$ 2:

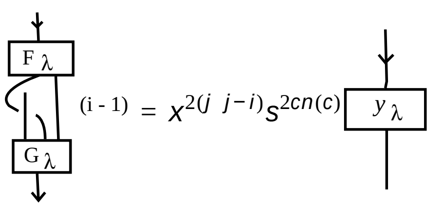


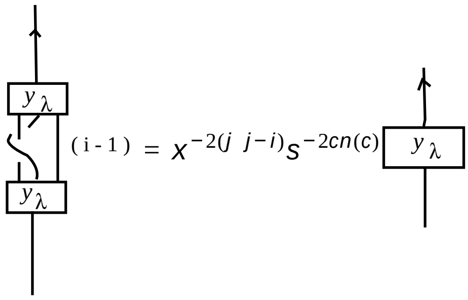

The result follows from Lemma 6.2 (b) below. □

**Lemma 6.2** *Let be a Young diagram and* $(h; l)$ *be the index of the cell after which* $(i - 1)$ *cells of follow.*

*(a)*



*(b)*

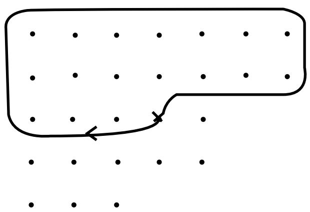



**Remark** The techniques used in this proof are similar to the proof of the framing factor in section 5 of [1] by H. Morton and A. Aiston.

**Proof** (a) We will borrow the notation of H. Morton and A. Aiston and use a schematic dot diagram to represent the element in the Hecke category $H_\square$ , which is between $F$ and $G$ as shown on the left-hand side.

Recall that $y$ = $F$ $G$ . Now in the diagram of the left-hand side of (a), introduce a schematic picture $T$ as follows:

$T$ =

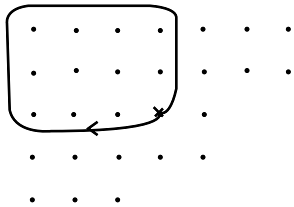


This indicates that the last $i-1$ strings were pulled out, the $i$th string marked by starts and nishes at $(h;l)$. The arrow on the $i$th string shows the string orientation when we look at it from above. The $i$th string encircles the remaining $j\ j-i$ strings in the clockwise direction. Here all strings shown by single dots are going vertical. The left-hand side of (a) can be expressed as $F\ TG$ . We will be working on $F\ TG$ . Using the Homflypt skein relations and the inseparability in Lemma 16 of [1], we have,

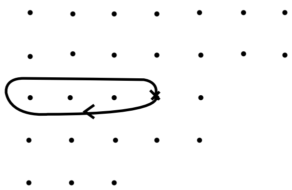
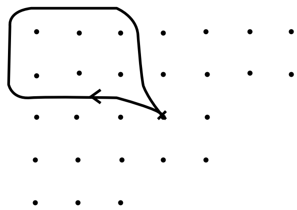
$$F\ TG\ = x^{2(j\ j-(i-1)-hl)}F\ SG$$

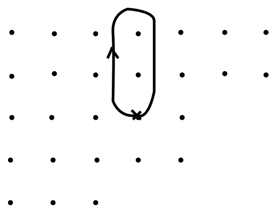
Where $S$ is given by:

$S$ =



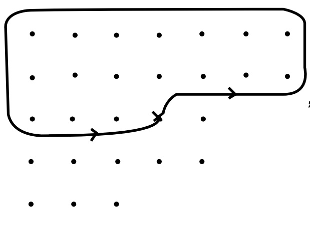
Since $S = S_1 T_1 S_2$, where:

$S_1 =$  $T_1 =$ 

$S_2 =$ 

First we have $F\ S_1 = (xs)^{2(l-1)}F$ by the property $_i f_m = xsf_m$ ; secondly, $S_2 G = (-xs^{-1})^{2(h-1)}G$ by the property $g_{m\ i} = -xs^{-1}g_m$, [1, Lemma 8]) . It follows that $F\ SG = x^{2(h+l-2)}s^{2(l-h)}F\ T_1 G$ . By a similar argument as in the proof of Theorem 17 in [1], $F\ T_1 G = x^{2(l-1)(h-1)}F\ G$ . Thus $F\ TG = x^{2(j\ j-i)}s^{2(l-h)}F\ G = x^{2(j\ j-i)}s^{2cn(c)}y$ , where c is the cell indexed by $(h;l)$.

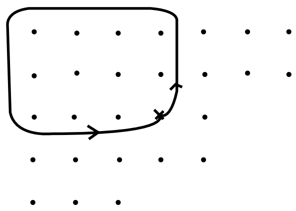
(b) We prove the result with all string orientations reversed. As string reversal de nes a skein module isomorphism, this su ces. As $y = F\ G$ , we can use the following schematic picture to denote the left-hand side of (b) as $F\ G\ T^{-1}F\ G$ , where

$T^{-1} =$  ;

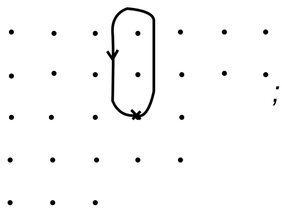
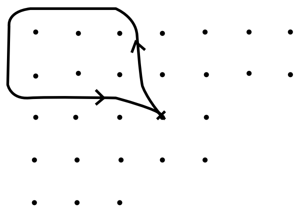
the $i$th string is indexed by $(h;l)$ and circles the remaining strings in the clockwise direction. Again, we have

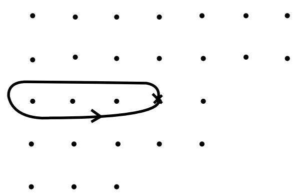
$$G\ T^{-1}F = x^{-2(j\ j-(i-1)-hl)}G\ S^{-1}F :$$

Where $S^{-1}$ is given by:

$S^{-1} =$ 

Since $S^{-1} = S_2^{-1}T_1^{-1}S_1^{-1}$, where:

$S_2^{-1} =$  ; $T_1^{-1} =$ 

$S_1^{-1} =$  :

We have $G\ S_2^{-1} = (-x^{-1}s)^{2(h-1)}G$ and $S_1^{-1}F = (x^{-1}s^{-1})^{2(l-1)}F$ by the properties $_i f_m = xsf_m$ and $g_m\ _i = -xs^{-1}g_m$.

We have

$$G\ S^{-1}F = x^{-2(h+l-2)}s^{2(h-l)}G\ T_1^{-1}F = x^{-2(h+l-2)}s^{2(h-l)}x^{-2(h-1)(l-1)}G\ F\ :$$

It follows that $G\ T^{-1}F = x^{-2(j\ j-i)}s^{-2cn(c)}G\ F$ . By the idempotent property, $F\ G\ T^{-1}F\ G = x^{-2(j\ j-i)}s^{-2cn(c)}F\ G$ . The result follows. □

## 7 Discussion of the proofs of Theorems 2 & 3

The proof of Theorem 2 is basically the same as the proof of Theorem 1. However as noted in the introduction we do not yet know that the relative Homflypt skein of a handlebody is free. So we must work over $F$:

For the proof of Theorem 3, we note that every relative link in $(M; X; Y)$ is isotopic to a link which intersects a tubular neighborhood of with $m$

straight strands going in one direction and $m+r$ straight strands going the other direction. We will write such elements as linear combinations of $_t$ $_s$, and $t$ and are standard tableaux of a Young diagram , and $s$ are standard tableaux of a Young diagram with $j\ j = m;$ and $j\ j = m+r$. As $x^{2r}-1-c$ $_;$ is invertible over $k_r;$ we have that $_t$ $_s$ $2\ L(j\ j+j\ j):$ We may repeat this argument until the class of our original relative link is represented by a linear combination of links each of which intersects less than $r$ times. This must be the empty linear combination.

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