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# Generalized Orbifold Euler Characteristic of Symmetric Products and Equivariant Morava K-Theory

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**Abstract** We introduce the notion of generalized orbifold Euler characteristic associated to an arbitrary group, and study its properties. We then calculate generating functions of higher order ($p$-primary) orbifold Euler characteristic of symmetric products of a $G$-manifold $M$. As a corollary, we obtain a formula for the number of conjugacy classes of $d$-tuples of mutually commuting elements (of order powers of $p$) in the wreath product $G \wr \mathfrak{S}_n$ in terms of corresponding numbers of $G$. As a topological application, we present generating functions of Euler characteristic of equivariant Morava K-theories of symmetric products of a $G$-manifold $M$.

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## 1 Introduction and summary of results

Let $G$ be a finite group and let $M$ be a smooth $G$-manifold. We study generalized orbifold Euler characteristics of $(M;G)$. These are integer-valued invariants associated to any group $K$. (See (1-3) below.) The simplest of such invariants (corresponding to the trivial group $K = \{e\}$) is the usual Euler characteristic $\chi(M/G)$ of the orbit space. It is well known that $\chi(M/G)$ can be calculated as the average over $g \in G$ of Euler characteristic of corresponding fixed point submanifolds:

(1-1)
$$\chi(M/G) = \frac{1}{|G|} \sum_{g \in G} \chi(M^{\langle g \rangle}),$$

where $\langle g \rangle \subset G$ is the subgroup generated by $g \in G$. See for example, [Sh, p.127].

In 1980s, string physicists proposed a notion of *orbifold Euler characteristic* of $(M;G)$ defined by

$$\chi_{\mathrm{orb}}(M;G) = \frac{1}{|G|}\sum_{gh=hg} \chi(M^{\langle g,h\rangle}); \tag{1-2}$$

where the summation is over all commuting pairs of elements in $G$ [DHVW]. The orbifold Euler characteristic is always an integer, since (1-1) implies

$$\chi_{\mathrm{orb}}(M;G) = \sum_{[g]} \chi(M^{\langle g\rangle}/C_G(g)) \in \mathbb{Z};$$

where the summation is over all the conjugacy classes of $G$, and $C_G(g)$ is the centralizer of $g$ in $G$. This formula is of the form $\chi(M/G)$ + (correction terms).

## Generalized orbifold Euler characteristic

Let $K$ be any group. The generalized orbifold Euler characteristic of $(M;G)$ associated to $K$ is an integer

$$\chi_K(M;G) \stackrel{\mathrm{def}}{=} \sum_{[\phi]\in \mathrm{Hom}(K,G)/G} \chi\big(M^{\langle\phi\rangle}/C_G(\phi)\big) = \frac{1}{|G|}\sum_{\phi\in\mathrm{Hom}(K\times\mathbb{Z},G)} \chi(M^{\langle\phi\rangle}). \tag{1-3}$$

The first summation is over $G$-conjugacy classes of homomorphisms, and the second equality is a consequence of (1-1). Thus, either expression can be taken as the definition of $\chi_K(M;G)$. Here, $C_G(\phi)$ is the centralizer in $G$ of the image $\langle\phi\rangle$ of $\phi$. Note that when $K$ is the trivial group $\{e\}$ or $\mathbb{Z}$, our $\chi_K(M;G)$ specializes to (1-1) or (1-2). In section 2, we describe its various properties including multiplicativity and the following formula for a product $K\times L$ of two groups:

$$\chi_{K\times L}(M;G) = \sum_{[\phi]\in\mathrm{Hom}(K,G)/G} \chi_L\big(M^{\langle\phi\rangle}; C_G(\phi)\big). \tag{1-4}$$

This formula, which is easy to prove, is crucial for inductive steps in the proofs of our main results, Theorems A and B below.

In this paper, we are mostly concerned with the cases $K=\mathbb{Z}^d$ and $K=\mathbb{Z}_p^d$, where $\mathbb{Z}_p$ denotes the ring of $p$-adic integers. We use the following notations:

$$\chi_{\mathbb{Z}^d}(M;G) = \chi^{(d)}(M;G); \qquad \chi_{\mathbb{Z}_p^d}(M;G) = \chi_p^{(d)}(M;G). \tag{1-5}$$

We call these $d$-th order ($p$-primary) orbifold Euler characteristics.

Our definition (1-3) is partly motivated by consideration of a mapping space $\mathrm{Map}(\Sigma; M/G)$, where a manifold $\Sigma$ has fundamental group $K$. When $\Sigma$ is the genus $g$ orientable surface $\Sigma_g$ with $\pi_1(\Sigma_g)=\Gamma_g$, we call the corresponding quantity $\chi_{\Gamma_g}(M;G)$ genus $g$ orbifold Euler characteristic of $(M;G)$.

## Higher order orbifold Euler characteristics for symmetric products

It turns out that $\chi^{(d)}(M;G)$ admits a geometric interpretation in terms of the mapping space $\mathrm{Map}(T^d, M/G)$, where $T^d$ is the $d$-dimensional torus. See section 2 for more details. And as such, it is very well behaved. We demonstrate this point by calculating $\chi^{(d)}$ for symmetric products.

When $M$ is a $G$-manifold, the $n$-fold Cartesian product $M^n$ admits an action of a wreath product $G \wr \mathfrak{S}_n$. The orbit space $M^n/(G \wr \mathfrak{S}_n) = SP^n(M/G)$ is the $n$-th symmetric product of $M/G$.

**Theorem A** *For any $d \geq 0$ and for any $G$-manifold $M$,*

$$\sum_{n\geq 0} q^n \chi^{(d)}(M^n; G \wr \mathfrak{S}_n) = \Big[\prod_{r\geq 1}(1-q^r)^{j_r(\mathbb{Z}^d)}\Big]^{(-1)\chi^{(d)}(M;G)}, \tag{1-6}$$

*where $j_r(\mathbb{Z}^d) = \sum_{r_1 r_2 \cdots r_d = r} r_2 r_3^2 \cdots r_d^{d-1}$ is the number of index $r$ subgroups in $\mathbb{Z}^d$.*

For the case of $d = 1$ ($\chi^{(1)} = \chi_{\mathrm{orb}}$), the above formula was calculated by Wang [W]. By letting $M$ to be a point, and using the notation $G_n = G \wr \mathfrak{S}_n$, we obtain

**Corollary 1-1** *For any $d \geq 0$, we have*

$$\sum_{n\geq 0} q^n \big|\mathrm{Hom}\big(\mathbb{Z}^d, G_n\big)/G_n\big| = \Big[\prod_{r\geq 1}(1-q^r)^{j_r(\mathbb{Z}^d)}\Big]^{(-1)|\mathrm{Hom}(\mathbb{Z}^d,G)/G|}. \tag{1-7}$$

Special cases of our results are known. Macdonald [M1] calculated Euler characteristic of symmetric products of any topological space $X$. His formula reads

$$\sum_{n\geq 0} q^n \chi\big(SP^n(X)\big) = \frac{1}{(1-q)^{\chi(X)}}. \tag{1-8}$$

Hirzebruch-Höfer [HH] calculated the orbifold Euler characteristic (1-2), which is our $\chi^{(1)}(M;G)$, of symmetric products. Their formula is

$$\sum_{n\geq 0} q^n \chi_{\mathrm{orb}}(M^n; \mathfrak{S}_n) = \Big[\prod_{r\geq 1}(1-q^r)\Big]^{(-1)\chi(M)}. \tag{1-9}$$

After completing this research, the author became aware of the paper [BF] in which the formulae (1-6) and (1-7) with trivial $G$ were calculated. (That the exponent of their formula can be identified with $j_r(\mathbb{Z}^d)$ was pointed out by Allan Edmonds in Math. Review.) We remark that the formula (1-7) with trivial $G$ is straightforward once we observe that $|\mathrm{Hom}(\mathbb{Z}^d;\mathfrak{S}_n)/\mathfrak{S}_n|$ is the number of isomorphism classes of $\mathbb{Z}^d$-sets of order $n$, and $j_r(\mathbb{Z}^d)$ is the number of isomorphism classes of transitive (irreducible) $\mathbb{Z}^d$-sets of order $r$. The second fact is because the isotropy subgroup of transitive $\mathbb{Z}^d$-sets of order $r$ is a sublattice of index $r$ in $\mathbb{Z}^d$. See also an exercise and its solution in [St, p.76, p.113]. On the other hand, when $G$ is nontrivial, a geometric interpretation of elements in $\mathrm{Hom}(\mathbb{Z}^d;G_n)/G_n$ is more involved. Our method of proving (1-6) is a systematic use of the formula (1-4) for generalized orbifold Euler characteristic and the knowledge of centralizers of elements of the wreath product $G_n$ described in detail in section 3. Our method can also be applied to more general context including $p$-primary orbifold Euler characteristic $\chi_p^{(d)}(M;G)$.

The integer $j_r(\mathbb{Z}^d)$ has very interesting number theoretic properties. It is easy to prove that the Dirichlet series whose coefficients are $j_r(\mathbb{Z}^d)$ can be expressed as a product of Riemann zeta functions $\zeta(s) = \sum_{n\ge 1} 1/n^s$ with $s \in \mathbb{C}$:

(1-10)
$$\sum_{n\ge 1} \frac{j_n(\mathbb{Z}^d)}{n^s} = \zeta(s)\,\zeta(s-1)\cdots\zeta(s-d+1).$$

For the history of this result, see [So].

## Euler characteristic of equivariant Morava K-theory of symmetric products

Let $K(d)^*(X)$ be the $d$-th Morava K-theory of $X$ for $d \ge 0$. Since $K(d)^*$ is a graded field, we can count the dimension of $K(d)^*(X)$ over $K(d)^*$, if it is finite. We are interested in computing Euler characteristic of equivariant $d$-th Morava $K$-theory of a $G$-manifold $M$:

(1-11)
$$\tilde\chi_{K(d)_G}(M) = \dim K(d)^{\mathrm{even}}(EG\times_G M) - \dim K(d)^{\mathrm{odd}}(EG\times_G M).$$

In [HKR], they calculate this number in terms of Möbius functions [HKR, Theorem B]. It is a simple observation to identify (1-11) as the $d$-th order $p$-primary orbifold Euler characteristic $\chi_p^{(d)}(M;G)$ (see a paragraph before Proposition 5-1). Our second main result is as follows.

**Theorem B** *Let $d \ge 0$ and let $M$ be a $G$-manifold. The Euler characteristic of equivariant Morava $K$-theory is equal to the $d$-th order $p$-primary orbifold Euler characteristic of $(M;G)$:*

(1-12)
$$\tilde\chi_{K(d)_G}(M) = \chi_p^{(d)}(M;G).$$

*The generating function of Euler characteristic of equivariant $d$-th Morava $K$-theory of symmetric products is given by*

$$\sum_{n\geq 0} q^n \chi\big(K(d)_{G_n}(M^n)\big) = \Big[\prod_{r\geq 0}(1-q^{p^r})^{j_{p^r}(\mathbb{Z}_p^d)}\Big]^{(-1)\chi(K(d)_G(M))}. \tag{1-13}$$

*Here, $G_n = G \wr \mathfrak{S}_n$, and $j_\ell(\mathbb{Z}_p^d)$ is the number of index $\ell$ subgroups in $\mathbb{Z}_p^d$ given by*

$$j_{p^r}(\mathbb{Z}_p^d) = \sum_{r_1 r_2 \cdots r_d = p^r} r_2 r_3^2 \cdots r_d^{d-1}; \textit{ and } j_\ell(\mathbb{Z}_p^d) = 0 \textit{ if } \ell \textit{ is not a power of } p. \tag{1-14}$$

Let $M$ be a point. The resulting formula is both topological and combinatorial:

**Corollary 1-2** *For any $d \geq 0$, we have*

$$\begin{aligned}\sum_{n\geq 0} q^n \chi\big(K(d)^*(BG_n)\big) &= \Big[\prod_{r\geq 0}(1-q^{p^r})^{j_{p^r}(\mathbb{Z}_p^d)}\Big]^{(-1)\chi(K(d)^*BG)} \\ \sum_{n\geq 0} q^n\, |\mathrm{Hom}(\mathbb{Z}_p^d, G_n)/G_n| &= \Big[\prod_{r\geq 0}(1-q^{p^r})^{j_{p^r}(\mathbb{Z}_p^d)}\Big]^{(-1)|\mathrm{Hom}(\mathbb{Z}_p^d,G)/G|}.\end{aligned} \tag{1-15}$$

When $G$ is a trivial group and hence $G_n = \mathfrak{S}_n$, the above formula is straightforward by an argument in terms of $\mathbb{Z}_p^d$-sets.

Again the integers $j_{p^r}(\mathbb{Z}_p^d)$ have number theoretic properties and it is well known that the corresponding Dirichlet series can be expressed as a product of $p$-local factors of Riemann zeta function. Namely, letting $\zeta_p(s) = (1-p^{-s})^{(-1)}$ denote the $p$-local factor in the Euler decomposition of $\zeta(s)$, we have

$$\sum_{r\geq 0} \frac{j_{p^r}(\mathbb{Z}_p^d)}{p^{rs}} = \zeta_p(s)\,\zeta_p(s-1)\cdots\zeta_p(s-d+1). \tag{1-16}$$

In particular, we have the following Euler decomposition of Dirichlet series:

$$\sum_{n\geq 1} \frac{j_n(\mathbb{Z}^d)}{n^s} = \prod_{p:\text{prime}} \Big(\sum_{r\geq 0} \frac{j_{p^r}(\mathbb{Z}_p^d)}{p^{rs}}\Big). \tag{1-17}$$

In [H], Hopkins shows that when $G$ is a trivial group, the exponents in (1-15) satisfy (1-16) by a method completely different from ours: by integrating a certain function over $\mathrm{GL}_n(\mathbb{Q}_p)$. And he identifies these exponents as Gaussian binomial coefficients [M2, p.26].

There is a physical reason why symmetric products of manifolds give rise to very interesting generating functions. In physics, the process of quantization of the state space of particles or strings moving on a manifold produces a Hilbert space of quantum states. The second quantization then corresponds to taking the total symmetric products of this Hilbert space, describing quantum states of many particles or strings. Reversing the order of these two procedures, we can first apply the second quantization to the manifold, by taking the total symmetric products of the manifold. This object quantizes well, and in [DMVV] (see also [D]) they calculate complex elliptic genera of second quantized Kähler manifolds and it is shown that they are genus 2 Siegel modular forms whose weight depend on holomorphic Euler characteristic of Kähler manifolds.

The organization of this paper is as follows. In section 2, we define generalized orbifold Euler characteristics associated to arbitrary groups, and describe their properties. We also express abelian orbifold Euler characteristics in terms of Möbius functions. We also describe a geometry behind our definition of generalized orbifold Euler characteristic. In section 3, we describe some properties of wreath products including the structure of centralizers. Material here is not new. However, detailed description on this topic seems to be rather hard to find in literature, so we worked out details and we decided to include it. This section is purely group theoretic and is independent from the rest of the paper. In sections 4 and 5, we compute higher order ($p$-primary) orbifold Euler characteristics of symmetric products in the form of generating functions. A relation to Euler characteristic of equivariant Morava $K$-theory is discussed in section 5.

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## 2 Generalized orbifold Euler characteristics

A generalization of physicists' orbifold Euler characteristic (1-2) was given in the introduction in (1-3). Here, group $K$ can be an arbitrary group. Properties enjoyed by $\chi^{(d)}(M;G)$ and $\chi_p^{(d)}(M;G)$ given in (1-5) become transparent in this generality. If the group $K$ is abelian, our generalized orbifold Euler characteristic is better behaved and it admits an expression in terms of Möbius functions defined on the family of abelian subgroups of $G$.

Later in this section, we will explain a geometric meaning of generalized orbifold Euler characteristics in terms of twisted mapping spaces.

## Generalized orbifold Euler characteristics

We prove basic properties of generalized orbifold Euler characteristic. Recall from section 1 that the generalized orbifold Euler characteristic associated to a group $K$ is given by

$$\chi_K(M;G) = \frac{1}{|G|}\sum_{\phi\in\mathrm{Hom}(K\times\mathbb{Z},G)} \chi(M^{\langle\phi\rangle}) = \sum_{[\phi]\in\mathrm{Hom}(K,G)/G} \chi\left(M^{\langle\phi\rangle}/C_G(\phi)\right) \tag{2-1}$$

See the last subsection of section 2 for a geometric motivation of this definition. Letting $M$ be a point, we obtain a useful formula

$$\chi_K(pt;G) = \frac{|\mathrm{Hom}(K\times\mathbb{Z},G)|}{|G|} = |\mathrm{Hom}(K,G)/G|. \tag{2-2}$$

Now we give a proof of (1-4).

**Proposition 2-1** *The orbifold Euler characteristic $\chi_K$ is multiplicative. Namely, for any $G_i$-manifolds $M_i$ for $i = 1, 2$, we have*

$$\chi_K(M_1\times M_2;G_1\times G_2) = \chi_K(M_1;G_1)\,\chi_K(M_2;G_2). \tag{2-3}$$

*Furthermore, for any two groups $K$ and $L$, we have*

$$\chi_{K\times L}(M;G) = \sum_{[\phi]\in\mathrm{Hom}(K,G)/G} \chi_L\left(M^{\langle\phi\rangle};C_G(\phi)\right). \tag{2-4}$$

**Proof** The first formula is straightforward from the definition of $\chi_K$ given in (2-1). For the second formula, first we observe that $\chi_L\left(M^{\langle\phi\rangle};C_G(\phi)\right)$ depends only on the conjugacy class of $\phi$, so the formula is well defined. Now

$$\begin{aligned}(\text{R.H.S}) &= \sum_{\phi\in\mathrm{Hom}(K,G)} \frac{1}{\#[\phi]}\frac{1}{|C_G(\phi)|}\sum_{\psi\in\mathrm{Hom}(L\times\mathbb{Z},C(\phi))}\chi\left(M^{\langle\phi,\psi\rangle}\right)\\ &= \frac{1}{|G|}\sum_{(\phi,\psi)\in\mathrm{Hom}(K\times L\times\mathbb{Z},G)}\chi\left(M^{\langle\phi,\psi\rangle}\right) = \chi_{K\times L}(M;G).\end{aligned}$$

Here, $\#[\phi]$ is the number of elements in the conjugacy class of $\phi$. Since $|C_G(\phi)|$ is the isotropy subgroup of the conjugation action of $G$ on the homomorphism set at $\phi$, we have $\#[\phi]\cdot|C_G(\phi)| = |G|$. This completes the proof. □

Next, we rewrite our orbifold Euler characteristic in terms of Möbius functions $\mu_H(X)$ defined for any subgroup $H$ and any $G$-CW complex $X$. These are defined by downward induction on $P \subseteq G$ by the formula

$$\sum_{P \subseteq H \subseteq G} \mu_H(X) = \chi(X^P). \tag{2-5}$$

It is known that any additive functions on $G$-CW complexes can be expressed as a linear combination of $\mu_H(\ )$'s with $\mathbb{Z}[1/|G|]$-coefficients [HKR, Proposition 4.6]. Our generalized orbifold Euler characteristic has the following expression in terms of Möbius functions.

**Lemma 2-2** *For any group $K$ and $G$-CW complex $M$, we have*

$$\chi_K(M;G) = \sum_{H \subseteq G} \frac{|\mathrm{Hom}(K \times \mathbb{Z}, H)|}{|G|}\mu_H(M) = \sum_{H \subseteq G} \frac{|H|}{|G|}\,\#\big(\mathrm{Hom}(K,H)/H\big)\,\mu_H(M).$$

**Proof** In the definition of $\chi_K(M;G)$ in (2-1), we replace $M^{\langle\phi\rangle}$ by (2-5). We have

$$\begin{aligned}
\chi_K(M;G) &= \frac{1}{|G|}\sum_{\phi} \chi(M^{\langle\phi\rangle}) = \frac{1}{|G|}\sum_{\phi}\sum_{\langle\phi\rangle \subseteq H} \mu_H(M) \\
&= \frac{1}{|G|}\sum_{H \subseteq G} \mu_H(M) \sum_{\phi: K \times \mathbb{Z} \to H} 1 = \frac{1}{|G|}\sum_{H \subseteq G} |\mathrm{Hom}(K \times \mathbb{Z}; H)|\,\mu_H(M).
\end{aligned}$$

Here in the second and third summation, $\phi$ runs over all homomorphisms in the set $\mathrm{Hom}(K \times \mathbb{Z}; G)$.

The second equality of the statement is due to (2-2). This proves the Lemma. □

## Abelian orbifold Euler characteristics and abelian Möbius functions

We recall some facts on complex oriented additive functions [HKR, sections 4.1, 4.2]. Let $\phi : \{G\text{-CW complexes}\} \to \mathbb{Z}$ be an integer-valued $G$-homotopy invariant function on $G$-CW complexes. Then the function $\phi$ is called *additive* if it satisfies

$$\phi(X \cup Y) + \phi(X \cap Y) = \phi(X) + \phi(Y); \qquad \phi(\emptyset) = 0$$

for any $G$-CW complexes $X; Y$. For a $G$-equivariant complex vector bundle $\xi$ on $X$, let $F(\xi)$ be the associated bundle of complete flags in $\xi$. An additive function $\phi$ is called *complex oriented* if $\phi$ satisfies $\phi\big(F(\xi)\big) = n!\,\phi(X)$ for any

$G$-equivariant complex $n$-dimensional bundle $\xi$ on $X$. It is known that any complex oriented additive function on $G$-CW complexes is completely determined by its value on the family of finite $G$-sets $\{G/A\}$, where $A$ runs over all abelian subgroups of $G$. In fact, the following formula holds [HKR, Proposition 4.10]:

$$\phi(\;) = \frac{1}{|G|}\sum_{\substack{A \subseteq G\\ A:\text{abelian}}} |A| \cdot \phi(G/A) \cdot \mu^{\mathbb{C}}_A(\;); \tag{2-6}$$

where $\mu_A(\;)$ is a complex oriented additive function defined by downward induction on an abelian subgroup $A$ by

$$\sum_{\substack{A \subseteq B \subseteq G\\ B:\text{abelian}}} \mu^{\mathbb{C}}_B(X) = \chi(X^A) \tag{2-7}$$

for any $G$-CW complex $X$.

For our generalized orbifold Euler characteristic, when the group $K$ is an abelian group $E$, then $\chi_E(\;;G)$ is a complex oriented additive function, since the image of any homomorphism $\phi: E \times \mathbb{Z} \to G$ is abelian. As such, $\chi_E(\;;G)$ satisfies a formula of the form (2-6). We will explicitly derive this formula in Proposition 2-3.

On the other hand, we can also show that $\chi_E(\;;G)$ can be written as a linear combination of complex oriented additive functions $\{\chi_E(\;;A)\}_A$ with $\mathbb{Z}[1/|G|]$-coefficients, where $A$ runs over all abelian subgroups of $G$. For this description, we need a function $\mu_A : \{\text{abelian subgroups of } G\} \to \mathbb{Z}$ defined by downward induction on an abelian subgroup $A$ by

$$\sum_{\substack{A \subseteq B \subseteq G\\ B:\text{abelian}}} \mu_A(B) = 1. \tag{2-8}$$

Note that when $G$ is abelian, this relation implies that $\mu_A(G) = 1$ and $\mu_A(A) = 0$ for any proper subgroup $A$ of $G$. Thus, (2-8) is of interest only when $G$ is non-abelian. We call $\mu^{\mathbb{C}}_A(\;)$ and $\mu_A(\;)$ abelian Möbius functions. We rewrite the generalized abelian orbifold Euler characteristic as follows. In (2-9) below, the first identity can be proved easily using (2-6), but here we give a different and amusing proof: we calculate a triple summation in three different orders.

**Proposition 2-3** *Let $E$ be an abelian group. Then the corresponding orbifold Euler characteristic $\chi_E(M;G)$ satisfies*

$$\chi_E(M;G) = \sum_{B \subseteq G} \frac{|B|}{|G|} \cdot |\mathrm{Hom}(E, B)| \cdot \mu^{\mathbb{C}}_B(M) = \sum_{A \subseteq G} \frac{|A|}{|G|} \cdot \mu_A(A) \cdot \chi_E(M;A). \tag{2-9}$$

*Here in the above summations, $A$ and $B$ run over all abelian subgroups of $G$.*

**Proof** We consider the following summation in three variables $\phi; A; B$:

$$\Phi(\ ) = \sum_{\phi;A;B} \mu_A(A)\,\chi^{\mathbb{C}}_B(M);$$

where $\phi : E \times \mathbb{Z} \longrightarrow G$ and $A; B$ are abelian subgroups satisfying $\langle\phi\rangle \subset A$ and $\langle\phi\rangle \subset B$. We compute this summation in three different ways:

$$(1) \sum_A \sum_\phi \sum_B ; \qquad (2) \sum_\phi \sum_A \sum_B ; \qquad (3) \sum_B \sum_\phi \sum_A .$$

For the case (1), the summation becomes

$$\Phi(\ ) = \sum_A \mu_A(A) \sum_{\phi:E\times\mathbb{Z}\longrightarrow A} \Big\{ \sum_{\langle\phi\rangle\subset B} \chi^{\mathbb{C}}_B(M) \Big\} = \sum_A \mu_A(A) \sum_{\phi:E\times\mathbb{Z}\longrightarrow A} \chi\big(M^{\langle\phi\rangle}\big)$$

$$= \sum_A \mu_A(A)\,|A|\,\chi_E(M;A).$$

Here, the summation over $A$ is over all abelian subgroups of $G$, and (2-7) was used for the second equality. This is allowed since $\langle\phi\rangle$ is abelian for any homomorphism $\phi : E \times \mathbb{Z} \longrightarrow A$. The third equality is the definition of $\chi_E(\ ;A)$ in (2-1).

For the case (2), we have

$$\Phi(\ ) = \sum_{\phi:E\times\mathbb{Z}\longrightarrow G} \Big\{ \sum_{\langle\phi\rangle\subset A} \mu_A(A) \Big\} \Big\{ \sum_{\langle\phi\rangle\subset B} \chi^{\mathbb{C}}_B(M) \Big\} = \sum_{\phi:E\times\mathbb{Z}\longrightarrow G} 1 \cdot \chi\big(M^{\langle\phi\rangle}\big) = |G|\,\chi_E(M;G).$$

Note that if $E$ is not abelian, then the image $\langle\phi\rangle$ of $\phi : E \times \mathbb{Z} \longrightarrow G$ can be non-abelian and the second equality above may not be valid. This is where we need to assume that $E$ is abelian.

For the third summation

$$\Phi(\ ) = \sum_B \chi^{\mathbb{C}}_B(M) \Big[ \sum_{\phi:E\times\mathbb{Z}\longrightarrow B} \Big\{ \sum_{\langle\phi\rangle\subset A} \mu_A(A) \Big\} \Big] = \sum_B \chi^{\mathbb{C}}_B(M) \cdot \mathrm{Hom}(E \times \mathbb{Z}; B)$$

$$= \sum_B |B| \cdot \mathrm{Hom}(E;B) \cdot \chi^{\mathbb{C}}_B(M).$$

Here the summation over $B$ is over all abelian subgroups of $G$. Since $B$ is abelian, $\mathrm{Hom}(E \times \mathbb{Z}; B)$ is a product of $\mathrm{Hom}(E;B)$ and $B$. This completes the proof. □

By letting $E$ be the trivial group, we get an interesting formula for $\chi(M/G)$.

**Corollary 2-4** *For any $G$-manifold $M$, we have*

$$\chi(M/G) = \frac{1}{|G|}\sum_{A:\text{abelian}} |A|\,\mu_A(A)\,\chi(M/A) = \frac{1}{|G|}\sum_{B:\text{abelian}} |B|\,\chi^{\mathbb{C}}_B(M).$$

It is interesting to compare this formula with (1-1). Note that the above formula does not imply that $\mu_A(A)\chi(M/A)$ is equal to $\chi^{\mathbb{C}}_A(M)$. The second equality holds only after summation over all abelian groups. A similar remark applies to (2-9).

## Higher order orbifold Euler characteristic

We specialize our previous results on generalized orbifold Euler characteristic to higher order orbifold Euler characteristic $\chi^{(d)}(M;G) = \chi_{\mathbb{Z}^d}(M;G)$. First, (2-1) specializes to

$$\chi^{(d)}(M;G) = \frac{1}{|G|}\sum_{\phi\in\mathrm{Hom}(\mathbb{Z}^{d+1},G)} \chi(M^{\langle\phi\rangle}) = \sum_{[\phi]\in\mathrm{Hom}(\mathbb{Z}^d,G)/G} \chi\big(M^{\langle\phi\rangle}/C_G(\phi)\big). \tag{2-10}$$

Notice that the second equality above is also a consequence of (2-2) with $K=\mathbb{Z}^d$ and $L=\{e\}$. If we apply (2-2) with $K=\mathbb{Z}$ and $L=\mathbb{Z}^{d-1}$, then we obtain the following inductive formula.

**Proposition 2-5** *For any $d\geq 1$, and for any $G$-manifold $M$, we have*

$$\chi^{(d)}(M;G) = \sum_{[g]} \chi^{(d-1)}\big(M^{\langle g\rangle};C_G(g)\big), \tag{2-11}$$

*where the summation is over all conjugacy classes $[g]\in\mathrm{Hom}(\mathbb{Z},G)/G$ of $G$.*

This is the formula which allows us to prove Theorem A inductively on $d\geq 0$. Lastly, formula (2-9) specializes in our case to

$$|G|\,\chi^{(d)}(M;G) = \sum_{A\leq G}|A|\,\mu_A(A)\,\chi^{(d)}(M;A) = \sum_{B\leq G}|B|^{d+1}\chi^{\mathbb{C}}_B(M). \tag{2-12}$$

Here the summations is over all abelian subgroups of $G$.

## Higher order $p$-primary orbifold Euler characteristic

Recall that the basic formula of this orbifold Euler characteristic $\chi^{(d)}_p(M;G) = \chi_{\mathbb{Z}_p^d}(M;G)$ is given by letting $K=\mathbb{Z}_p^d$ in (2-1):

$$\chi^{(d)}_p(M;G) = \frac{1}{|G|}\sum_{\phi\in\mathrm{Hom}(\mathbb{Z}_p^d\times\mathbb{Z},G)} \chi(M^{\langle\phi\rangle}) = \sum_{[\phi]\in\mathrm{Hom}(\mathbb{Z}_p^d,G)/G} \chi\big(M^{\langle\phi\rangle}/C_G(\phi)\big). \tag{2-13}$$

Now let $K=\mathbb{Z}_p$ and $L=\mathbb{Z}_p^{d-1}$ in the formula (2-4). We obtain the following inductive formula corresponding to (2-11) for the $p$-local case.

**Proposition 2-6** *For any $d \geq 1$ and for any $G$-manifold $M$, we have*

$$\chi_p^{(d)}(M;G) = \sum_{[\gamma]\in \mathrm{Hom}(\mathbb{Z}_p,G)/G} \chi_p^{(d-1)}\big(M^{\langle\gamma\rangle}; C_G(\gamma)\big). \tag{2-14}$$

*Here $[\gamma]$ runs over all $G$-conjugacy classes of elements of order powers of $p$.*

The following formula, which is a specialization of (2-9) in our setting, will be used later in section 5 to compare $\chi_p^{(d)}(M;G)$ with Euler characteristic of equivariant Morava K-theory.

**Proposition 2-7** *For any $d \geq 0$ and for any $G$-manifold $M$,*

$$\chi_p^{(d)}(M;G) = \sum_{A\subseteq G}\frac{|A|}{|G|}\mu_A(A)\,\chi_p^{(d)}(M;A) = \sum_{B\subseteq G}\frac{|B|}{|G|}|B_{(p)}|^d\,\chi_B^{\mathbb{C}}(M), \tag{2-15}$$

*where the summation is over all abelian subgroups of $G$.*

## Generalized orbifold Euler characteristic and twisted mapping space

We discuss a geometric origin of orbifold Euler characteristics. Physicists' orbifold Euler characteristic (1-2) originates in string theory. Higher order ($p$-primary) orbifold Euler characteristics $\chi^{(d)}(M;G)$ and $\chi_p^{(d)}(M;G)$ have similar geometric interpretations in terms of twisted mapping spaces. There is a very strong analogy between this geometric situation and methods used in orbifold conformal field theory. We can predict results in orbifold conformal field theory, for example a description of twisted sectors for the action of wreath products, simply by examining this geometric situation of twisted mapping spaces.

To describe the geometry, first we consider the free loop space $L(M/G) = \mathrm{Map}(S^1, M/G)$ on the orbit space $M/G$. Our basic idea here is to study the orbit space $M/G$ by examining holonomies of loops passing through orbifold singularities of $M/G$. To be more precise, we consider lifting a loop $\bar{\gamma}: S^1 \longrightarrow M/G$ to a map $\gamma: \mathbb{R} \longrightarrow M$, where $S^1 = \mathbb{R}/\mathbb{Z}$. This lift may not close after moving 1 unit along $\mathbb{R}$ and the difference between $\gamma(t)$ and $\gamma(t+1)$ comes from the action of an element $g \in G$, the holonomy of $\bar{\gamma}$. When the loop $\bar{\gamma}$ does not pass through orbifold points of $M/G$, the conjugacy class of the holonomy is uniquely determined by $\bar{\gamma}$. However, when the loop $\bar{\gamma}$ passes through orbifold point, it can have lifts whose holonomies belong to different conjugacy classes. Furthermore, it can have a lift whose holonomy depend on the unit segment of $\mathbb{R}$ on which the holonomy is measured. To avoid this complication, we consider only $g$-periodic lifts. This is the notion of $g$-twisted free loop space $L_gM$ defined by

$$L_gM = \{\gamma: \mathbb{R} \longrightarrow M \mid \gamma(t+1) = g^{-1}\gamma(t),\ t \in \mathbb{R}\}. \tag{2-16}$$

Since any loop $\bar{\gamma} \in L(M/G)$ can be lifted to a $g$-periodic lift for some $g \in G$, we have a surjective map $\coprod_{g \in G} L_gM \longrightarrow L(M/G)$. On the space $\coprod_{g \in G} L_gM$, the group $G$ acts inducing a homeomorphism $h : L_gM \xrightarrow{\cong} L_{hgh^{-1}}M$ for any $h, g \in G$. Quotienting by this action, we get a surjective map

$$\text{(2-17)} \qquad \Big(\coprod_{g \in G} L_gM\Big)\Big/G = \coprod_{[g] \in G_*} L_gM/C_G(g) \xrightarrow{\text{onto}} L(M/G).$$

Here $G_*$ is the set of all conjugacy classes of $G$. This map is $1:1$ on the subset of $L(M/G)$ consisting of loops not passing through orbifold points. Thus, if the action of $G$ on $M$ is free, then the above map is a homeomorphism. When a loop passes through orbifold points, its inverse image is not unique, but finite, corresponding to finitely many possibilities of different conjugacy classes of lifts. Thus, in a sense, the above surjective map gives a mild resolution of orbifold singularities.

Since $L_gM/C(g)$ is again an orbifold space, we can apply the above procedure again on its free loop space. In fact, we can iterate this procedure. To describe this general case, for any $d \geq 1$ and for any homomorphism $\phi : \mathbb{Z}^d \longrightarrow G$, let $L_\phi M$ be the space of twisted $d$-dimensional tori defined by

$$\text{(2-18)} \qquad L_\phi M = \{\gamma : \mathbb{R}^d \longrightarrow M \mid \gamma(\mathrm{t} + \mathrm{m}) = \phi(\mathrm{m})^{-1}\gamma(\mathrm{t}),\ \mathrm{t} \in \mathbb{R}^d, \mathrm{m} \in \mathbb{Z}^d\}.$$

Here $\phi$ plays a role of holonomy of the map $\bar{\gamma} : T^d \longrightarrow M/G$, where $T^d = \mathbb{R}^d/\mathbb{Z}^d$. Observe that any $\gamma$ in $L_\phi M$ factors through the torus $T_\phi = \mathbb{R}^d/\mathrm{Ker}\,\phi$. As before, the action of any $h \in G$ induces a homeomorphism $h : L_\phi M \xrightarrow{\cong} L_{h\phi h^{-1}}M$. Let

$$\text{(2-19)} \qquad \mathbb{L}^d(M;G) = \Big(\coprod_{\phi \in \mathrm{Hom}(\mathbb{Z}^d,G)} L_\phi M\Big)\Big/G = \coprod_{[\phi] \in \mathrm{Hom}(\mathbb{Z}^d,G)/G} L_\phi M/C_G(\phi).$$

Here, $C_G(\phi) \subset G$ is the centralizer of $\phi$. We may call this space $d$-th order twisted torus space for $(M;G)$. Let $\mathbb{T} = \mathbb{R}^d/\bigcap_\phi \mathrm{Ker}\,\phi$, where $\phi$ runs over all homomorphisms $\mathrm{Hom}(\mathbb{Z}^d, G)$. Then $\mathbb{T}$ is a $d$-dimensional torus and it acts on $\mathbb{L}^d(M;G)$.

We have a canonical map $\mathbb{L}^d(M;G) \longrightarrow L^d(M/G)$ from the above space to $d$-th iterated free loop space on $M/G$. This map is no longer surjective nor injective in general. Of course when the action $G$ on $M$ is free, the above map is still a homeomorphism.

The space $\mathbb{L}^d(M;G)$ can be thought of as the space of pairs $(\bar{\gamma}, [\phi])$, where $\bar{\gamma} : T^d \longrightarrow M/G$ is a $d$-torus in $M/G$, and $[\phi]$ is the conjugacy class of the holonomy of a periodic lift of $\bar{\gamma}$ to a map $\gamma : \mathbb{R}^d \longrightarrow M$.

We want to calculate ordinary Euler characteristic of the space $\mathbb{L}^d(M;G)$. However, since this space is infinite dimensional, it may have nonzero Betti numbers in arbitrarily high degrees. We recall that for a finite dimensional manifold admitting a torus action, it is well known that Euler characteristic of the fixed point submanifold is the same as the Euler characteristic of the original manifold. In fact, a formal application of Atiyah-Singer-Segal Fixed Point Index Theorem predicts that the Euler characteristic of $\mathbb{L}^d(M;G)$ must be the same as the Euler characteristic of $\mathbb{T}$-fixed point subset. Thus, the Euler characteristic of $\chi\big(\mathbb{L}^d(M;G)\big)$ ought to be given by

$$\text{(2-20)} \qquad \chi\big(\mathbb{L}^d(M;G)^{\mathbb{T}}\big) = \sum_{[\phi]\in \mathrm{Hom}(\mathbb{Z}^d,G)/G} \chi\big(M^{\langle\phi\rangle}/C_G(\phi)\big) = \chi^{(d)}(M;G).$$

This is the geometric origin of our definition of higher order orbifold Euler characteristic $\chi^{(d)}(M;G)$.

We can give a similar geometric interpretation of the higher order $p$-primary orbifold Euler characteristic $\chi_p^{(d)}(M;G)$ as Euler characteristic of an infinite dimensional twisted mapping space with a torus action. The Euler characteristic of the fixed point subset under this torus action is precisely given by $\chi_p^{(d)}(M;G)$, as in (2-20). A similar consideration applies to generalized orbifold Euler characteristic $\chi_K(M;G)$ for a general group $K$: we replace the mapping space $\mathrm{Map}(\Sigma, M/G)$, where $\Sigma$ is a manifold with the fundamental group $\pi_1(\Sigma) = K$, by $G$-orbits of twisted mapping spaces parametrized by $\mathrm{Hom}(K,G)/G$, we then take the Euler characteristic of constant maps.

## 3 Centralizers of wreath products

This section reviews some facts on wreath products. In particular, we explicitly describe the structure of centralizers of elements in wreath products. This material may be well known to experts. For example, the order of centralizers in wreath product is discussed in Macdonald's book [M2, p.171]. However, since the precise details on this topic seem to be rather hard to locate in literature, our explicit and direct description will make this paper more self-contained and it may be useful for readers with different expertise. The structure of centralizers is described in Theorem 3-5. This section is independent from the rest of the paper.

Let $G$ be a finite group. The $n$-th symmetric group $\mathfrak{S}_n$ acts on the $n$-fold Cartesian product $G^n$ by $s(g_1,g_2,\dots,g_n) = (g_{s^{-1}(1)},g_{s^{-1}(2)},\dots,g_{s^{-1}(n)})$, where $s \in \mathfrak{S}_n$, $g_i \in G$. The semidirect product defined by this action is the wreath product $G \wr \mathfrak{S}_n = G^n \rtimes \mathfrak{S}_n$. We use the notation $G_n$ to denote this wreath product. Product and inverse are given by $(\mathbf{g}, s)(\mathbf{h}, t) = (\mathbf{g}\cdot s(\mathbf{h}), st)$ and $(\mathbf{g}, s)^{-1} = (s^{-1}(\mathbf{g}^{-1}), s^{-1})$.

### Conjugacy classes in wreath products

Let $s = \prod_i s_i$ be the cycle decomposition of $s$. If $s_i = (i_1, i_2, \dots, i_r)$ is a linear representation of $s_i$, then the product $g_{i_r}\cdots g_{i_2}g_{i_1}$ is called the cycle product of $(\mathbf{g}, s)$ corresponding to the above representation of the cycle $s_i$. For each cycle $s_i$, the conjugacy class of its cycle product is uniquely determined.

Corresponding to $s_i$, let $\mathbf{g}_i$ be an element of $G^n$ whose $a$-th component is given by $(\mathbf{g}_i)_a = (\mathbf{g})_a$ if $a \in \{i_1, \dots, i_r\}$, and $(\mathbf{g}_i)_a = 1$ otherwise. Then for each $i, j$, $(\mathbf{g}_i, s_i)$ and $(\mathbf{g}_j, s_j)$ commute and we have $(\mathbf{g}, s) = \prod_i (\mathbf{g}_i, s_i)$. We often write $\mathbf{g}_i = (g_{i_1}, g_{i_2}, \dots, g_{i_r})$ as if it is an element of $G^r$. The conjugacy class of the cycle product corresponding to $s_i$ is denoted by $[\mathbf{g}_i]$.

Let $G_*$ denote the set of all conjugacy classes of $G$, and we fix the representatives of conjugacy classes. For $[c] \in G_*$, let $m_r(c)$ be the number of $r$-cycles in the cycle decomposition $s = \prod_i s_i$ whose cycle products belong to $[c]$. This yields a partition-valued function $\rho : G_* \longrightarrow \mathcal{P}$, where $\mathcal{P}$ is the totality of partitions, defined by $\rho([c]) = (1^{m_1(c)}2^{m_2(c)}\cdots r^{m_r(c)}\cdots)$. Note that $\sum_{[c],r} r m_r(c) = n$. The function $\rho$ associated to $(\mathbf{g}, s) \in G_n$ is called the type of $(\mathbf{g}, s)$. It is well known that the conjugacy class of $(\mathbf{g}, s)$ in $G_n$ is determined by its type. This can be explicitly seen using the conjugation formulae in Proposition 3-1 below.

To describe details of the structure of the wreath product $G_n$, we use the following notations. We express any element $(\mathbf{g}, s)$ as a product in two ways:

$$(\mathbf{g}, s) = \prod_i (\mathbf{g}_i, s_i) = \prod_{[c]\in G_*}\prod_{r\geq 1}\prod_{i=1}^{m_r(c)} (\mathbf{d}_{r,c;i}, \sigma_{r,c;i}). \tag{3-1}$$

In the second expression, the conjugacy class of the cycle product corresponding to $\mathbf{d}_{r,c;i}$ is $[\mathbf{d}_{r,c;i}] = [c]$.

Suppose the conjugacy class of the cycle product $[\mathbf{g}_i] = [g_{i_r}\cdots g_{i_1}]$ corresponding to $s_i$ is equal to $[c] \in G_*$. Choose and fix $p_i \in G$ such that $g_{i_r}\cdots g_{i_1} = p_i c p_i^{-1}$ for all $i$. Let

$$\mathbf{d}(\mathbf{g}_i) = (g_{i_1}, g_{i_2}g_{i_1}, \dots, g_{i_r}\cdots g_{i_1}), \quad \Delta_i(p_i) = (p_i, \dots, p_i) \in G^{r(i)} \subset G^n. \tag{3-2}$$

Here by $G^{r(i)} \subset G^n$, we mean that the components of the above elements at the position $a \notin \{i_1, i_2, \dots, i_r\}$ are $1 \in G$. This convention applies throughout this section. In the above, $\Delta_i$ is the diagonal map along the components of $s_i$. Let $\mathrm{c}_r = (c, 1, \dots, 1) \in G^r$ and $\mathrm{c}_{r,i} \in G^n$ is $\mathrm{c}_r$ along the positions which appear as components of $\zeta_{r,c,i}$ or $s_i$. The following proposition can be checked by direct calculation.

**Proposition 3-1** (1) *Let* $(\mathrm{g}; s) = \prod_i (\mathrm{g}_i; s_i)$. *For a given* $i$, *suppose* $[\mathrm{g}_i] = [c]$ *and* $|s_i| = r$. *Then* $\sigma_i = (\mathrm{g}_i)\,\Delta_i(p_i) \in G^r \overset{(i)}{\subset} G^n$ *has the property*

$$(\mathrm{g}_i; s_i) = (\sigma_i; 1)\big((c, 1, \dots, 1); s_i\big)(\sigma_i; 1)^{-1}. \tag{3-3}$$

*For* $i \neq j$, *elements* $(\sigma_i; 1)$ *and* $(\sigma_j; 1)$ *commute.*

(2) *Let* $(\sigma; 1) = \prod_i (\sigma_i; 1)$. *Then*

$$(\mathrm{g}; s) = (\sigma; 1)\prod_{[c]}\prod_{r \geq 1}\prod_{i=1}^{m_r(c)} (\mathrm{c}_{r,i}; \zeta_{r,c,i})\,(\sigma; 1)^{-1}. \tag{3-4}$$

>From this, it is clear that the conjugacy class of $(\mathrm{g}; s) \in G^n$ is determined by its type $\{m_r(c)\}_{r,[c]}$.

## Actions of wreath products

Let $M$ be a $G$-manifold. The wreath product $G_n$ acts on the $n$-fold Cartesian product $M^n$ by

$$\big((g_1, g_2, \dots, g_n); s\big)\cdot(x_1, x_2, \dots, x_n) = (g_1 x_{s^{-1}(1)}, g_2 x_{s^{-1}(2)}, \dots, g_n x_{s^{-1}(n)}).$$

The above conjugation formula shows that the fixed point subset of $M^n$ under the action of $(\mathrm{g}; s)$ is completely determined by its type.

**Proposition 3-2** *Suppose an element* $(\mathrm{g}; s) \in G_n$ *is of type* $\{m_r(c)\}$. *Then*

$$(M^n)^{\langle(\mathbf{g}; s)\rangle} \cong \prod_{[c]} \big(M^{\langle c\rangle}\big)^{\sum_r m_r(c)}. \tag{3-5}$$

**Proof** With respect to the decomposition (3-1) of $(\mathrm{g}; s)$, we have $(M^n)^{\langle(\mathbf{g}; s)\rangle} = \prod_i (M^{r_i})^{\langle(\mathbf{g}_i; s_i)\rangle}$, where $r_i = |s_i|$, and $M^{r_i} \subset M^n$ corresponds to components of $s_i$. If $[\mathrm{g}_i] = [c]$, then we have the following isomorphisms

$$M^{\langle c\rangle} \xrightarrow[\Delta_i]{\cong} (M^{r_i})^{\langle(\mathrm{c}_{r_i}; s_i)\rangle} \xrightarrow[(\sigma_i; 1)]{\cong} (M^{r_i})^{\langle(\mathbf{g}_i; s_i)\rangle}.$$

Now in terms of the other decomposition of $(\mathrm{g}; s)$ in (3-1), since $[\zeta_{r,c,i}] = [c]$, the above isomorphism implies

$$(M^n)^{\langle(\mathbf{g}; s)\rangle} = \prod_{[c]}\prod_{r \geq 1}\prod_{i=1}^{m_r(c)} (M^{r_i})^{\langle(\zeta_{r,c,i}; \zeta_{r,c,i})\rangle} = \prod_{[c]} (M^{\langle c\rangle})^{\sum_r m_r(c)}.$$

This completes the proof. □

## Centralizers in wreath products

Next, we describe the centralizer $C_{G_n}\big((g,s)\big)$ in the wreath product $G_n$. Let $(h,t) \in C_{G_n}\big((g,s)\big)$. Then $(h,t)(g,s)(h,t)^{-1} = (g,s)$. In terms of the cycle decomposition $(g,s) = \prod_i (g_i,s_i)$, we see that for each $i$ there exits a unique $j$ such that $(h,t)(g_i,s_i)(h,t)^{-1} = (g_j,s_j)$. Since the conjugation preserves the type, we must have $[g_i] = [g_j] = [c]$ for some $[c] \in G_*$, and $|s_i| = |s_j|$. Thus with respect to the second decomposition in (3-1), the conjugation by $(h,t)$ permutes $m_r(c)$ elements $\{(\alpha_{r,c,i}, \beta_{r,c,i})\}_i$ for each $[c] \in G_*$ and $r \geq 1$. Thus we have a homomorphism

$$p : C_{G_n}\big((g,s)\big) \longrightarrow \prod_{[c]}\prod_{r\geq 1} \mathfrak{S}_{m_r(c)}. \tag{3-6}$$

**Lemma 3-3** *The above homomorphism $p$ is split surjective.*

**Proof** First we construct a homomorphism $\varphi : \prod_{[c]}\prod_r \mathfrak{S}_{m_r(c)} \to \mathfrak{S}_n$ as follows. In the decomposition $(g,s) = \prod_i (g_i,s_i)$, we write each cycle $s_i$ starting with the smallest integer. If $\bar{t} \in \prod_{[c]}\prod_r \mathfrak{S}_{m_r(c)}$ sends the cycle $s_i = (i_1,i_2,\dots,i_r)$ to $s_j = (j_1,j_2,\dots,j_r)$, then let $\varphi(\bar{t}) = t \in \mathfrak{S}_n$ where $t(i_\ell) = j_\ell$, $1 \leq \ell \leq r$. It is clear that $\varphi$ defines a homomorphism, and any element in the image of $\varphi$ commutes with $s$, because $t$ permutes cycles preserving the smallest integers in cycles appearing in the cycle decomposition of $s$. Previously, we constructed an element $\xi \in G_n$ for each element $(g,s)$ in Proposition 3-1. Define a homomorphism $\psi : \prod_{[c]}\prod_r \mathfrak{S}_{m_r(c)} \to C_{G_n}\big((g,s)\big)$ by $\psi(\bar{t}) = (\xi,1)\big(1,\varphi(\bar{t})\big)(\xi,1)^{-1}$. Using (3-4), we can check that $\psi(\bar{t})$ commutes with $(g,s)$ for any $\bar{t}$. Since conjugation by $\psi(\bar{t})$ on $(g,s)$ induces the permutation $\bar{t}$ among $\{(g_i,s_i)\}_i$, we have $p\circ\psi$ = identity and $\psi$ is a splitting of $p$. This completes the proof. □

Next we examine the kernel of the homomorphism $p$ in (3-6). If $(h,t) \in \operatorname{Ker} p$, then $(h,t)(g_i,s_i)(h,t)^{-1} = (g_i,s_i)$ for all $i$. In particular, we have $ts_it^{-1} = s_i$ for all $i$, and consequently $t$ must be a product of powers of $s_i$'s. Thus, we may write $(h,t) = \prod_i \big((h_i,1)(g_i,s_i)^{k_i}\big)$ for some $h_i \in G^{|s_i|} \subset G^n$ and $0 \leq k_i < |s_i|$, where $(h_i,1)$ commutes with $(g_i,s_i)$ for any $i$. Let $G_r^{(i)} = G^r \rtimes \mathfrak{S}_r \subset G_n$ be a subgroup of $G_n$ isomorphic to $G_r$ corresponding to positions appearing in $s_i$. Recall that we defined $\xi_i$ in (3-3).

**Lemma 3-4** *For a given $i$, suppose the cycle product corresponding to $s_i$ is such that $[g_i] = [c]$. Then*

(3-7)
$$\begin{aligned} C_{G_r^{(i)}}\big((g_i;s_i)\big) &= \{(h_i;1)(g_i;s_i)^{k_i} \mid 0 \le k_i < |s_i|, [(h_i;1),(g_i;s_i)] = 1\} \\ &= \big(\alpha_i\,\Delta_i(C_G(c))\,\alpha_i^{-1};1\big)\cdot\langle(g_i;s_i)\rangle \\ &= C_G(c)\times\langle a_{r;c}\rangle, \quad \textit{where } (a_{r;c})^r = c \textit{ and } [a_{r;c}, C_G(c)] = 1. \end{aligned}$$

*Here* $a_{r;c} = \big(\mathbf{c}_r;(12\cdots r)\big)$.

**Proof** The first equality is obvious. For the second one, first observe that if $(\mathbf{h};1)$ with $\mathbf{h}\in G^r$ commutes with $\big(\mathbf{c}_r;(12\cdots r)\big)\in G_r$, then $\mathbf{h}$ must be of the form $(h,h,\dots,h)\in G^r$ with $h\in C_G(c)$. Conjugation by $\alpha_i$ gives the second description. For the third description, we simply observe that $\big(\mathbf{c}_r;(12\cdots r)\big)^r = \big(\Delta(h);1\big)\in G_r$. This completes the proof. □

Thus we have a split exact sequence

(3-8)
$$1 \longrightarrow \prod_{[c]}\prod_{r\ge 1}\prod_{i=1}^{m_r(c)} C_{G_r^{(i)}}\big((\xi_{r;c;i};\sigma_{r;c;i})\big) \longrightarrow C_{G_n}\big((g;s)\big) \xrightarrow{p} \prod_{[c]}\prod_{r\ge 1}\mathfrak{S}_{m_r(c)} \longrightarrow 1.$$

Since conjugation preserves the type of elements in $G_n$, the centralizer splits into $(r;[c])$-components
$$C_{G_n}\big((g;s)\big) = \prod_{[c]}\prod_{r\ge 1} C_{G_n}\big((g;s)\big)_{(r;[c])},$$
where $C_{G_n}\big((g;s)\big)_{(r;[c])}$ is the centralizer of the element $\prod_i(\xi_{r;c;i};\sigma_{r;c;i})$ in the subgroup $G_{rm_r(c)}$ corresponding to positions appearing in $\sigma_{r;c;i}$ for $1\le i\le m_r(c)$. The conjugation by $(\alpha_i;1)$ maps the following split exact sequence

(3-9)
$$1 \longrightarrow \prod_{i=1}^{m_r(c)} C_{G_r^{(i)}}\big((\xi_{r;c;i};\sigma_{r;c;i})\big) \longrightarrow C_{G_n}\big((g;s)\big)_{(r;[c])} \xrightarrow{p_{r;c}} \mathfrak{S}_{m_r(c)} \longrightarrow 1$$

isomorphically into the following split exact sequence
$$1 \longrightarrow \prod_{i=1}^{m_r(c)}\big(\Delta_i C_G(c)\times\langle a_{r;c}^{(i)}\rangle\big) \longrightarrow C_{G_{rm_r(c)}}\Big(\prod_{i=1}^{m_r(c)}(\mathbf{c}_{r;i};\sigma_{r;c;i})\Big) \longrightarrow \mathfrak{S}_{m_r(c)} \longrightarrow 1.$$

Here $a_{r;c}^{(i)}$ is $a_{r;c}$ along components of $\sigma_{r;c;i}$. Direct calculation shows that in the second exact sequence, $\mathfrak{S}_{m_r(c)}$ acts on the left side product by permuting factors. Hence the semidirect product structure in (3-9) is indeed isomorphic to a wreath product. Hence we obtain the following description of the centralizer of $(g;s)$ in $G_n$.

**Theorem 3-5** *Let $(\mathbf{g};\mathbf{s}) \in G_n$ have type $\{m_r(c)\}_{r,[c]}$. Then*

$$C_{G_n}\big((\mathbf{g};\mathbf{s})\big) \cong \prod_{[c]}\prod_{r\ge 1}\Big\{\big(C_G(c)\cdot\langle a_{r,c}\rangle\big)\wr\mathfrak{S}_{m_r(c)}\Big\}, \tag{3-10}$$

*where $(a_{r,c})^r = c \in C_G(c)$. Here, the isomorphism is induced by conjugation by $\gamma$ in Proposition* 3-1.

## 4 Higher order orbifold Euler characteristic of symmetric products

In this section, we prove Theorem A in the introduction. Explicitly writing out $j_r(\mathbb{Z}^d)$, the formula we prove is the following:

$$\sum_{n\ge 0} q^n\chi^{(d)}(M;G\wr\mathfrak{S}_n) = \prod_{r_1,r_2,\dots,r_d\ge 1}\Big[(1-q^{r_1r_2\cdots r_d})^{r_2r_3^2\cdots r_d^{d-1}}\Big]^{(-1)\chi^{(d)}(M;G)}. \tag{4-1}$$

When $d = 0$, this is Macdonald's formula applied to $M/G$:

$$\sum_{n\ge 0} q^n\chi\big(SP^n(M/G)\big) = \frac{1}{(1-q)^{\chi(M/G)}}. \tag{4-2}$$

When $d = 1$, the formula was proved by Wang [W]. We prove formula (4-1) by induction on $d \ge 0$, using Macdonald's formula as the start of induction. For the inductive step, we need the following Lemma.

**Lemma 4-1** *Let $G\cdot\langle a\rangle$ be a group generated by a finite group $G$ and an element $a$ such that $a$ commutes with any element of $G$ and $\langle a\rangle\cap G = \langle a^r\rangle$ for some integer $r \ge 1$. Suppose the element $a$ acts trivially on a $G$-manifold $M$. Then*

$$\chi^{(d)}(M;G\cdot\langle a\rangle) = r^d\chi^{(d)}(M;G). \tag{4-3}$$

**Proof** First note that $G\cdot\langle a\rangle = \coprod_{i=0}^{r-1} G\cdot a^i$ and $|G\cdot\langle a\rangle| = r\cdot|G|$. Observe that two elements of the form $ga^i$ and $ha^j$, where $g,h \in G$, commute if and only if $g,h$ commute, since $a$ is in the center of $G\cdot\langle a\rangle$. Now by definition,

$$\chi^{(d)}(M;G\cdot\langle a\rangle) = \frac{1}{r\cdot|G|}\sum_{\substack{(g_1,\dots,g_{d+1})\\ 0\le i_\ell<r}}\chi\big(M^{\langle g_1a^{i_1},g_2a^{i_2},\dots,g_{d+1}a^{i_{d+1}}\rangle}\big),$$

where $(g_1,\dots,g_{d+1})$ runs over all $(d+1)$-tuples of mutually commuting elements of $G$, and the index $\ell$ runs over $1 \le \ell \le d+1$. Since the element $a$

acts trivially on $M$, the fixed point subset above is the same as $M^{\langle g_1,\dots,g_{d+1}\rangle}$. Hence summing over $\ell$'s first, the above becomes

$$\chi^{(d)}(M;G\times\langle a\rangle)=\frac{r^{d+1}}{r\cdot|G|}\sum_{(g_1,\dots,g_{d+1})}\chi\big(M^{\langle g_1,\dots,g_{d+1}\rangle}\big)=r^d\cdot\chi^{(d)}(M;G).$$

This completes the proof. $\square$

**Proof of formula** (4-1) By induction on $d\ge 0$. When $d=0$, the formula is Macdonald's formula (4-2) and hence it is valid.

Assume the formula is valid for $\chi^{(d-1)}$ for $d\ge 1$. Let $G_*=\{[c]\}$ be the totality of conjugacy classes of $G$. By Proposition 2-5, we have

$$(*)\qquad \sum_{n\ge 0}q^n\chi^{(d)}(M;G_n)=\sum_{n\ge 0}q^n\sum_{[(\mathbf{g};\sigma)]}\chi^{(d-1)}\big((M^n)^{\langle(\mathbf{g};\sigma)\rangle};C_{G_n}((\mathbf{g};\sigma))\big).$$

Let $(\mathbf{g};\sigma)\in G_n$ have type $\{m_r(c)\}$. Then by Proposition 3-2 and Theorem 3-5, we have the following compatible isomorphisms:

$$(M^n)^{\langle(\mathbf{g};\sigma)\rangle}\cong\prod_{[c]}\prod_{r\ge 1}(M^{\langle c\rangle})^{m_r(c)},$$

$$C_{G_n}\big((\mathbf{g};\sigma)\big)\cong\prod_{[c]}\prod_{r\ge 1}\big((C_G(c)\times\langle a_{r,c}\rangle)\wr\mathfrak{S}_{m_r(c)}\big),$$

where $(a_{r,c})^r=c\in C_G(c)$ and $a_{r,c}$ acts trivially on $M^{\langle c\rangle}$. The above isomorphisms are compatible in the sense that the action of $C_{G_n}\big((\mathbf{g};\sigma)\big)$ on $(M^n)^{\langle(\mathbf{g};\sigma)\rangle}$ translates, via conjugation by $(\mathbf{h};1)$ in Proposition 3-1, to the action of the wreath product $(C_G(c)\times\langle a_{r,c}\rangle)\wr\mathfrak{S}_{m_r(c)}$ on $(M^{\langle c\rangle})^{m_r(c)}$ for any $[c]\in G_*$ and $r\ge 1$. Since the conjugacy classes of elements in $G_n$ are determined by their types, the summation over all conjugacy classes $[(\mathbf{g};\sigma)]$ corresponds to the summation over all $m_r(c)\ge 0$ for all $[c]\in G_*$ and $r\ge 1$ subject to $\sum_{[c],r}rm_r(c)=n$. By the multiplicativity of generalized orbifold Euler characteristic (2-3), the formula $(*)$ becomes

$$\begin{aligned}
&\sum_{n\ge 0}q^n\chi^{(d)}(M^n;G_n)\\
&\quad=\sum_{n\ge 0}q^n\sum_{\substack{m_r(c)\ge 0\\ \sum rm_r(c)=n}}\prod_{[c],r}\chi^{(d-1)}\big((M^{\langle c\rangle})^{m_r(c)};(C_G(c)\times\langle a_{r,c}\rangle)\wr\mathfrak{S}_{m_r(c)}\big)\\
&\quad=\sum_{m_r(c)\ge 0}\prod_{[c],r}(q^r)^{m_r(c)}\chi^{(d-1)}\big((M^{\langle c\rangle})^{m_r(c)};(C_G(c)\times\langle a_{r,c}\rangle)\wr\mathfrak{S}_{m_r(c)}\big)\\
&\quad=\prod_{[c],r}\sum_{m\ge 0}(q^r)^m\chi^{(d-1)}\big((M^{\langle c\rangle})^m;(C_G(c)\times\langle a_{r,c}\rangle)_m\big)
\end{aligned}$$

By inductive hypothesis, this is equal to

$$= \prod_{[c];\, r}\Bigl[\prod_{r_1,\dots,r_{d-1}\ge 1}\Bigl(1-(q^r)^{r_1\cdots r_{d-1}}\Bigr)^{r_2 r_3^2\cdots r_{d-1}^{d-2}}\Bigr]^{-\chi^{(d-1)}(M^{\langle c\rangle};\, C_G(c)\times\langle a_{r;c}\rangle)}$$

$$= \prod_{r;\, r_1,\dots,r_{d-1}}\Bigl[\Bigl(1-q^{r r_1\cdots r_{d-1}}\Bigr)^{r_2 r_3^2\cdots r_{d-1}^{d-2}}\Bigr]^{-\sum_{[c]}\chi^{(d-1)}(M^{\langle c\rangle};\, C_G(c)\times\langle a_{r;c}\rangle)}.$$

By Lemma 4-1, $\chi^{(d-1)}(M^{\langle c\rangle}; C_G(c)\times\langle a_{r;c}\rangle) = r^{d-1}\chi^{(d-1)}(M^{\langle c\rangle}; C_G(c))$. Hence summing over $[c]\in G$ and using Proposition 2-5, we see that the exponent is equal to $(-1)r^{d-1}\chi^{(d)}(M;G)$. Thus, renaming $r$ as $r_d$, the above is equal to the right hand side of (4-1). This completes the inductive step and the proof is complete. □

Now let $M=\mathrm{pt}$. Using (2-2) with $K=\mathbb{Z}^d$ and $G$ replaced by $G$ or $G_n$, we get

**Corollary 4-2** *For each $d\ge 0$ and for any finite group $G$, we have*

(4-4)
$$\sum_{n\ge 0} q^n\,\bigl|\mathrm{Hom}\bigl(\mathbb{Z}^d, G_n\bigr)/G_n\bigr| = \Bigl[\prod_{r_1,\dots,r_d\ge 1}\bigl(1-q^{r_1\cdots r_d}\bigr)^{r_2 r_3^2\cdots r_d^{d-1}}\Bigr]^{(-1)|\mathrm{Hom}(\mathbb{Z}^d,G)/G|}.$$

The above formula is the formula (1-7) in the introduction. Furthermore, letting $G$ be the trivial group, we get

$$\sum_{n\ge 0} q^n\,\bigl|\mathrm{Hom}\bigl(\mathbb{Z}^d, \mathfrak{S}_n\bigr)/\mathfrak{S}_n\bigr| = \Bigl[\prod_{r_1,\dots,r_d\ge 1}\bigl(1-q^{r_1\cdots r_d}\bigr)^{r_2 r_3^2\cdots r_d^{d-1}}\Bigr]^{(-1)}. \tag{4-5}$$

Here, as remarked in the introduction, we recognize $\bigl|\mathrm{Hom}\bigl(\mathbb{Z}^d,\mathfrak{S}_n\bigr)/\mathfrak{S}_n\bigr|$ as the number of isomorphism classes of $\mathbb{Z}^d$-sets of order $n$. Any finite $\mathbb{Z}^d$-set decomposes into a union of transitive $\mathbb{Z}^d$-sets, and any isomorphism class of transitive $\mathbb{Z}^d$-set of order $r$ corresponds to a unique subgroup of $\mathbb{Z}^d$ of index $r$, by taking the isotropy subgroup. Thus, letting $j_r(\mathbb{Z}^d)$ be the number of index $r$ subgroups of $\mathbb{Z}^d$, we have

$$\sum_{n\ge 0} q^n\,\bigl|\mathrm{Hom}\bigl(\mathbb{Z}^d, \mathfrak{S}_n\bigr)/\mathfrak{S}_n\bigr| = \Bigl[\prod_{r\ge 1}\bigl(1-q^r\bigr)^{j_r(\mathbb{Z}^d)}\Bigr]^{(-1)}. \tag{4-6}$$

By comparing (4-5) and (4-6), we get a formula for $j_r(\mathbb{Z}^d)$. However, we can easily directly calculate the number $j_r(\mathbb{Z}^d)$ as follows. This calculation is well known (for its history, see [So]) and gives an alternate proof of (4-5).

**Lemma 4-4** *For any $r \geq 1$, $d \geq 1$, we have*

(4-7) $$j_r(\mathbb{Z}^d) = \sum_{r_1 \cdots r_d = r} r_2 r_3^2 \cdots r_d^{d-1}; \quad and \quad j_r(\mathbb{Z}^d) = \sum_{m|r} m \cdot j_m(\mathbb{Z}^{d-1}).$$

**Proof** Let $e_1, e_2, \ldots, e_d$ be the standard basis of the lattice. It is easy to see that any sublattice of index $r$ has a unique basis $\{x_i\}_{i=1}^d$ of the form $x_i = r_i e_i + \sum_{i<j\leq d} a_{ij} e_j$ for $1 \leq i \leq d$, where $r_1 r_2 \cdots r_d = r$ and $0 \leq a_{ij} < r_j$ for $1 \leq i < j \leq d$. For any choice of $\{a_{ij}\}$ satisfying the condition, there exists a sublattice of rank $d$ of index $r = r_1 \cdots r_d$. Since given $r_1, \ldots, r_d \geq 1$, there are $r_2 r_3^2 \cdots r_d^{d-1}$ choices of $a_{ij}$'s, the total number of sublattices of index $r$ in $\mathbb{Z}^d$ is given by $\sum_{r_1 \cdots r_d = r} r_2 r_3^2 \cdots r_d^{d-1}$.

The second equality easily follows from the first. This completes the proof. □

The proof of the identity (1-10) mentioned in the introduction can be readily proved using (4-7) by induction on $d \geq 1$. We will discuss the corresponding $p$-local situation in the next section.

# 5 Euler characteristic of equivariant Morava K-theory of symmetric products

As before, let $G$ be a finite group and let $M$ be a $G$-manifold. Let $d \geq 0$ be an integer. The equivariant $d$-th Morava $K$-theory of $M$ is defined as $K(d)^*_G(M) = K(d)^*(EG \times_G M)$. Since $K(d)^* = \mathbb{F}_p[v_d, v_d^{-1}]$ with $|v_d| = -2(p^d-1)$ is a graded field, any $K(d)^*$-module is free and we can count the dimension over $K(d)^*$, if it is finite. Let $\chi(K(d)^*_G(M))$ be its Euler characteristic defined by

(5-1) $$\chi(K(d)^*_G(M)) = \dim K(d)^{\text{even}}(EG \times_G M) - \dim K(d)^{\text{odd}}(EG \times_G M).$$

When $d = 0$, we have $K(0)^*(\ \ ) = H^*(\ \ ;\mathbb{Q})$ and $\chi(K(0)^*_G(M)) = \chi(M/G)$, the ordinary Euler characteristic of the orbit space. Hopkins, Kuhn, and Ravenel compute this number (5-1) using a general theory of complex oriented additive functions [HKR, Theorem 4.12]. Their result is

(5-2) $$\chi(K(d)^*_G(M)) = \frac{1}{|G|} \sum_{\substack{A \leq G \\ A:\text{abelian}}} |A| \cdot |A_{(p)}|^d \cdot \mu^{\mathbb{C}}_A(M),$$

where the Möbius function $\mu^{\mathbb{C}}_A(M)$ is defined in (2-7), and the summation is over all abelian subgroups of $G$. Now comparing (5-2) with our calculation of higher order $p$-primary orbifold Euler characteristic of $(M;G)$ in Proposition 2-7, we realize that these two quantities are in fact equal for all $d \geq 0$.

**Proposition 5-1** *For any $G$-manifold $M$, and for any $d \geq 0$,*

$$\chi\big(K(d)^*_G(M)\big) = \chi_p^{(d)}(M;G) = \frac{1}{|G|}\sum_{\phi:\mathbb{Z}_p^d\times\mathbb{Z}\to G}\chi(M^{\langle\phi\rangle}). \tag{5-3}$$

Our objective in this section is to calculate the Euler characteristic of equivariant Morava K-theory of symmetric products $(M^n; G\wr\mathfrak{S}_n)$ for $n \geq 1$. By Proposition 5-1, this homotopy theoretic number can be calculated as the higher order $p$-primary orbifold Euler characteristic. We prove

**Theorem 5-2** *For any $d \geq 0$ and $G$-manifold $M$,*

$$\sum_{n\geq 0} q^n\chi_p^{(d)}(M^n; G\wr\mathfrak{S}_n) = \Big[\prod_{\ell_1,\dots,\ell_d\geq 0}\big(1-q^{p^{\ell_1}p^{\ell_2}\cdots p^{\ell_d}}\big)^{p^{\ell_2}p^{2\ell_3}\cdots p^{(d-1)\ell_d}}\Big]^{(-1)\chi_p^{(d)}(M;G)}. \tag{5-4}$$

The proof is very similar to the one for the formula (4-1). Note the similarities of exponents. Since the formula (5-4) is $p$-primary, there are differences at many parts of the proof, although the idea of the proof is the same. Thus, we believe that it is better to give a complete proof of the above formula (5-4) rather than explaining differences of proofs between formulae (4-1) and (5-4).

**Proof of Theorem 5-2** By induction on $d \geq 0$. When $d = 0$, by (2-13) we have $\chi_p^{(0)}(M^n; G\wr\mathfrak{S}_n) = \chi\big(SP^n(M/G)\big)$, and the formula (5-4) in this case asserts

$$\sum_{n\geq 0} q^n\chi\big(SP^n(M/G)\big) = \frac{1}{(1-q)^{\chi(M/G)}},$$

which is valid due to Macdonald's formula (1-8).

Assume that the formula (5-4) is valid for $\chi_p^{(d-1)}$ for $d \geq 1$. By Proposition 2-6,

$$\sum_{n\geq 0} q^n\chi_p^{(d)}(M^n; G\wr\mathfrak{S}_n) = \sum_{n\geq 0} q^n\sum_{[\gamma]}\chi_p^{(d-1)}\big((M^n)^{\langle\gamma\rangle}; C_{G_n}(\gamma)\big). \tag{$*$}$$

Here the second summation on the right hand side is over all $G_n$-conjugacy classes $[\gamma]$ of elements of $p$-power order in $G_n$, that is $[\gamma] \in \mathrm{Hom}(\mathbb{Z}_p, G_n)/G_n$. Let $\gamma = (g, s) \in G_n$. Since $\gamma$ has order a power of $p$, the second component $s \in \mathfrak{S}_n$ must have order a power of $p$. Thus, the type of $\gamma = (g, s)$ must be of the form $\{m_{p^r}(c)\}_{r;[c]}$. Here, $[c]$ runs over all $G$-conjugacy classes of elements of order powers of $p$. We indicate this by the notation $[c]_p$. So

$[c]_p \in \mathrm{Hom}(\mathbb{Z}_p, G)/G$. By Proposition 3-1, $\alpha$ is conjugate to an element of the form

$$\prod_{r\ge 0}\prod_{[c]_p}\prod_{i=1}^{m_{p^r}(c)}\Big(\overbrace{\underbrace{(c,1,\dots,1)}_{p^r};\,\zeta_{p^r,c,i}}^{a_{p^r,c,i}}\Big);\quad \text{where } (a_{p^r,c,i})^{p^r} = \Big(\underbrace{(c,c,\dots,c)}_{p^r};1\Big).$$

By Proposition 3-2 and Theorem 3-5, the fixed point subset of $M^n$ under the action of $\alpha$, and the centralizer of $\alpha$ in $G_n$ are each isomorphic to

$$(M^n)^{\langle\alpha\rangle} = \prod_{[c]_p}(M^{\langle c\rangle})^{\sum_r m_{p^r}(c)},$$

$$C_{G_n}(\alpha) = \prod_{[c]_p}\prod_{r\ge 0}\big(C_G(c)\langle a_{p^r,c}\rangle\big)\wr\mathfrak{S}_{m_{p^r}(c)}, \quad (a_{p^r,c})^{p^r} = c\in C_G(c).$$

The above isomorphisms are compatible with the action of the centralizer on the fixed point subset. The summation over all conjugacy classes $[\alpha]_p$ can be replaced by the summation over all the types $\{m_{p^r}(c)\}_{r,[c]_p}$. By multiplicativity of $\chi_p^{(d-1)}$, the right hand side of $(*)$ becomes

$$\begin{aligned}
(*) &= \sum_{n\ge 0} q^n \sum_{\substack{m_{p^r}(c)\ge 0\\ \sum p^r m_{p^r}(c)=n}}\ \prod_{[c]_p,r}\chi_p^{(d-1)}\Big((M^{\langle c\rangle})^{m_{p^r}(c)};\big(C_G(c)\langle a_{p^r,c}\rangle\big)\wr\mathfrak{S}_{m_{p^r}(c)}\Big)\\
&= \sum_{m_{p^r}(c)\ge 0}\ \prod_{[c]_p,r}(q^{p^r})^{m_{p^r}(c)}\chi_p^{(d-1)}\Big((M^{\langle c\rangle})^{m_{p^r}(c)};\big(C_G(c)\langle a_{p^r,c}\rangle\big)\wr\mathfrak{S}_{m_{p^r}(c)}\Big)\\
&= \prod_{[c]_p,r}\ \sum_{m\ge 0}(q^{p^r})^m\chi_p^{(d-1)}\Big((M^{\langle c\rangle})^m;\big(C_G(c)\langle a_{p^r,c}\rangle\big)\wr\mathfrak{S}_m\Big)
\end{aligned}$$

By inductive hypothesis, the summation inside is given by

$$\begin{aligned}
&= \prod_{[c]_p,r}\Big[\prod_{\ell_1,\dots,\ell_{d-1}\ge 0}\big(1-(q^{p^r})^{p^{\ell_1}\cdots p^{\ell_{d-1}}}\big)^{p^{\ell_2}p^{2\ell_3}\cdots p^{(d-2)\ell_{d-1}}}\Big]^{(-1)\chi_p^{(d-1)}(M^{\langle c\rangle};C_G(c)\langle a_{p^r,c}\rangle)}\\
&= \Big[\prod_{\ell_1,\dots,\ell_{d-1},r\ge 0}\big(1-q^{p^{\ell_1}\cdots p^{\ell_{d-1}}p^r}\big)^{p^{\ell_2}p^{2\ell_3}\cdots p^{(d-2)\ell_{d-1}}}\Big]^{(-1)\sum_{[c]_p}\chi_p^{(d-1)}(M^{\langle c\rangle};C_G(c)\langle a_{p^r,c}\rangle)}.
\end{aligned}$$

At this point, we need a sublemma which is completely analogous to Lemma 4-1.

**Sublemma** *Let $G\langle a\rangle$ be a group generated by a finite group $G$ and an element $a$ of order a power of $p$ such that $a$ commutes with any element in $G$ and $G\cap\langle a\rangle = \langle a^{p^r}\rangle\in G_{(p)}$. Suppose $\langle a\rangle$ acts trivially on $M$. Then*

$$\chi_p^{(d)}(M;G\langle a\rangle) = p^{rd}\chi_p^{(d)}(M;G). \tag{5-5}$$

The proof of this sublemma is analogous to Lemma 4-1. Using this sublemma and the fact that $(a_{p^r,c})^{p^r} = c \in C_G(c)$, we see that the exponent of the previous expression is equal to

$$\sum_{[c]_p} \chi_p^{(d-1)}(M^{\langle c\rangle}; C_G(c)/\langle a_{p^r,c}\rangle) = p^{r(d-1)} \sum_{[c]_p} \chi_p^{(d-1)}\big(M^{\langle c\rangle}; C_G(c)\big)$$
$$= p^{r(d-1)}\chi_p^{(d)}(M;G).$$

Thus, renaming $r$ as $\ell_d$, the expression $(*)$ finally becomes

$$(*) = \prod_{\ell_1,\dots,\ell_{d-1},\ell_d \geq 0} \Big[(1-q^{p^{\ell_1}\cdots p^{\ell_{d-1}}p^{\ell_d}})^{p^{\ell_2}p^{2\ell_3}\cdots p^{(d-2)\ell_{d-1}}p^{(d-1)\ell_d}}\Big]^{(-1)\chi_p^{(d)}(M;G)},$$

which is the right hand side of formula (5-4). This completes the proof. □

Now letting $M$ be a point and using (2-2) with $K = \mathbb{Z}_p^d$, we get

**Corollary 5-3** *Let $G_n = G \wr \mathfrak{S}_n$ for $n \geq 0$. For any $d \geq 0$, we have*

(5-6)
$$\sum_{n\geq 0} q^n\, |\mathrm{Hom}(\mathbb{Z}_p^d, G_n)/G_n| = \prod_{\ell_1,\dots,\ell_d\geq 0}\Big[(1-q^{p^{\ell_1}\cdots p^{\ell_d}})^{p^{\ell_2}p^{2\ell_3}\cdots p^{(d-1)\ell_d}}\Big]^{(-1)|\mathrm{Hom}(\mathbb{Z}_p^d,G)/G|}.$$

In particular, letting $G$ to be the trivial group, we get

**Corollary 5-4** *For any $d \geq 0$,*

$$\text{(5-7)}\qquad \sum_{n\geq 0} q^n\, |\mathrm{Hom}(\mathbb{Z}_p^d, \mathfrak{S}_n)/\mathfrak{S}_n| = \prod_{\ell_1,\ell_2,\dots,\ell_d\geq 0}\Big[(1-q^{p^{\ell_1}p^{\ell_2}\cdots p^{\ell_d}})^{p^{\ell_2}p^{2\ell_3}\cdots p^{(d-1)\ell_d}}\Big]^{(-1)}.$$

Observe that $|\mathrm{Hom}(\mathbb{Z}_p^d, \mathfrak{S}_n)/\mathfrak{S}_n|$ is the number of isomorphism classes of $\mathbb{Z}_p^d$-sets of order $n$. Any finite $\mathbb{Z}_p^d$-sets can be decomposed into transitive $\mathbb{Z}_p^d$-sets which must have order powers of $p$. For any $r \geq 0$, isomorphism classes of transitive $\mathbb{Z}_p^d$-sets of order $p^r$ are in 1 : 1 correspondence with index $p^r$ subgroup of $\mathbb{Z}_p^d$, by taking isotropy subgroups. Let $j_{p^r}(\mathbb{Z}_p^d)$ be the number of index $p^r$ subgroup of $\mathbb{Z}_p^d$. Note that $j_\ell(\mathbb{Z}_p^d)$ is zero unless $\ell$ is a power of $p$. This consideration of decomposing finite $\mathbb{Z}_p^d$-sets into transitive ones immediately gives the following formula.

$$\text{(5-8)}\qquad \sum_{n\geq 0} q^n\, |\mathrm{Hom}(\mathbb{Z}_p^d, \mathfrak{S}_n)/\mathfrak{S}_n| = \prod_{r\geq 0}\Big[(1-q^{p^r})^{j_{p^r}(\mathbb{Z}_p^d)}\Big]^{(-1)}.$$

There is an easy way to calculate the number $j_{p^r}(\mathbb{Z}_p^d)$.

**Lemma 5-5** *For any $r \ge 0$ and $d \ge 1$, we have*

$$(5\text{-}9)\qquad \begin{aligned} j_{p^r}(\mathbb{Z}_p^d) &= \sum_{\sum_i \ell_i = r} p^{\ell_2} p^{2\ell_3} \cdots p^{(d-1)\ell_d} = \sum_{n_1 \cdots n_d = p^r} n_2 n_3^2 \cdots n_d^{d-1}, \\ j_{p^r}(\mathbb{Z}_p^d) &= \sum_{0 \le \ell \le r} p^{\ell} j_{p^{\ell}}(\mathbb{Z}_p^{d-1}). \end{aligned}$$

**Proof** Any subgroup $H$ of $\mathbb{Z}_p^d$ of index $p^r$ for any $r \ge 0$ is a closed subgroup and hence it has a structure of a $\mathbb{Z}_p$-submodule, and as such it is a free module. Let the standard basis of $\mathbb{Z}_p^d$ be $e_1, e_2, \dots, e_d$. It is easy to see that $H$ has a unique $\mathbb{Z}_p$-module basis $\{x_i\}_{i=1}^n$ of the form $x_i = p^{\ell_i} e_i + \sum_{i<j\le d} a_{ij} e_j$ for some uniquely determined integers $\ell_1, \ell_2, \dots, \ell_d \ge 0$ with $\sum_i \ell_i = r$ and $0 \le a_{ij} < p^{\ell_j}$. Any choice of such integers gives rise to a subgroup of $\mathbb{Z}_p^d$ of index $p^r$. Thus, counting all possible choices of these integers subject to $\sum_i \ell_i = r$, we obtain the expression of $j_{p^r}(\mathbb{Z}_p^d)$ given in (5-9).

The second formula in (5-9) is straightforward from the first. This completes the proof. □

Note that formulae (5-8) and (5-9) give an alternate proof of (5-7).

Now, formulas (1-16) and (1-17) in the introduction easily follow from (5-9) by induction on $d \ge 1$. In terms of these numbers $j_{p^r}(\mathbb{Z}_p^d)$, the formula (5-4) can be rewritten as

$$(5\text{-}10)\qquad \sum_{n \ge 0} q^n \chi_p^{(d)}(M^n; G \wr \mathfrak{S}_n) = \Big[\prod_{r \ge 0} (1 - q^{p^r})^{j_{p^r}(\mathbb{Z}_p^d)}\Big]^{(-1)\chi_p^{(d)}(M;G)}.$$

Since the Euler characteristic of the equivariant Morava K-theory $K(d)_{G_n}(M^n)$ is equal to the $d$-th order $p$-primary orbifold Euler characteristic $\chi_p^{(d)}(M^n; G_n)$ by Proposition 5-1, we obtain our final result of this paper:

**Theorem 5-6** *For any $d \ge 0$ and for any $G$-manifold $M$,*

$$(5\text{-}11)\qquad \sum_{n \ge 0} q^n \chi\big(K(d)_{G_n}(M^n)\big) = \Big[\prod_{r \ge 0} (1 - q^{p^r})^{j_{p^r}(\mathbb{Z}_p^d)}\Big]^{(-1)\chi(K(d)_G(M))},$$

*where $G_n = G \wr \mathfrak{S}_n$ is a wreath product. When $M$ is a point, this formula gives*

$$(5\text{-}12)\qquad \sum_{n \ge 0} q^n \chi\big(K(d)^*(BG_n)\big) = \Big[\prod_{r \ge 0} (1 - q^{p^r})^{j_{p^r}(\mathbb{Z}_p^d)}\Big]^{(-1)\chi(K(d)^* BG)},$$

*where $BG$ and $BG_n$ are classifying spaces of $G$ and $G_n$.*

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