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Intrinsic knotting and linking of complete graphs

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Abstract We show that for every $m \in \mathbb{N}$, there exists an $n \in \mathbb{N}$ such that every embedding of the complete graph K_n in $\mathbb{R}^3$ contains a link of two components whose linking number is at least m. Furthermore, there exists an $r \in \mathbb{N}$ such that every embedding of K_r in $\mathbb{R}^3$ contains a knot Q with $|a_2(Q)| \geq m$, where $a_2(Q)$ denotes the second coefficient of the Conway polynomial of Q.

AMS Classification 57M25; 05C10

Keywords Embedded graphs, intrinsic knotting, intrinsic linking

1 Introduction

The study of intrinsic knotting and linking began with the work of Conway and Gordon [CG], who showed that every embedding in $\mathbb{R}^3$ of the complete graph on six vertices, K_6, contains a non-trivial link of two components, and every embedding of K_7 in $\mathbb{R}^3$ contains a non-trivial knot. Since the existence of such a link or knot is intrinsic to the graph, and does not depend on the particular embedding of the graph in $\mathbb{R}^3$, we say that K_6 is *intrinsically linked* and K_7 is *intrinsically knotted*. We state Conway and Gordon's Theorem more precisely with the following notation. Let $L_1 \cup L_2$ be an oriented link, and let $lk(L_1, L_2)$ denote the linking number of L_1 and L_2; let Q be a knot, and let $a_2(Q)$ denote the second coefficient of the Conway polynomial of Q. Conway and Gordon proved that every embedding of K_6 contains a link $L_1 \cup L_2$ which has $lk(L_1, L_2) \equiv 1 \pmod 2$, and every embedding of K_7 contains a knot Q which has $a_2(Q) \equiv 1 \pmod 2$. Furthermore, they illustrated an embedding of K_6 such that the only non-trivial link $L_1 \cup L_2$ contained in K_6 is the Hopf link (which has $|lk(L_1, L_2)| = 1$); and they illustrated an embedding of K_7 such that the only non-trivial knot Q contained in K_7 is the trefoil knot (which has $|a_2(Q)| = 1$). In this sense K_6 exhibits the simplest type of intrinsic linking and K_7 exhibits the simplest type of intrinsic knotting.

In this paper, we will show that for larger values of n, the complete graph K_n can exhibit a more complex type of intrinsic linking or knotting. In particular, there exists an n such that every embedding of K_n contains a non-trivial 2-component link which is not the Hopf link, and there exists an n such that every embedding of K_n contains a non-trivial knot which is not the trefoil knot. In [FNP] we considered links of more than two components, and showed that every embedding of K_{10} in $\mathbb{R}^3$ contains a 3-component link $L_0 \cup L_1 \cup L_2$ such that both $lk(L_0, L_1) \equiv 1 \pmod 2$ and $lk(L_0, L_2) \equiv 1 \pmod 2$; furthermore, $n = 10$ is the smallest number such that K_n has this property. Here we shall generalize Conway and Gordon's Theorem by considering the linking number of 2-component links in $\mathbb{Z}$ rather than in $\mathbb{Z}_2$, and by considering the second coefficient of the Conway polynomial in $\mathbb{Z}$ rather than in $\mathbb{Z}_2$.

We begin by proving that every embedding of K_{10} contains a non-trivial link of two components other than the Hopf link. In particular we prove the following.

Theorem 1 *Every embedding of K_{10} in $\mathbb{R}^3$ contains a 2-component link $L = L_1 \cup J_1$ such that for some orientation of L we have $lk(L_1, J_1) \geq 2$.*

Theorem 2 will show that the complexity of the intrinsic linking of K_n (as measured by the linking number) can be made as large as we wish by making n sufficiently large.

Theorem 2 *Let $p \in \mathbb{N}$ be given, and let $n = p(15p - 9)$. Then every embedding of K_n in $\mathbb{R}^3$ contains a 2-component link $L = L_p \cup J_p$ such that for some orientation of L we have $lk(L_p, J_p) \geq p$.*

It is natural to ask whether the complexity of the intrinsic knotting of K_n can also be made as large as we wish by choosing n sufficiently large. While the linking number seems like the natural measure of the complexity of a 2-component link, there are various ways to measure the complexity of a knot. The second coefficient of the Conway polynomial $a_2(Q)$ of an oriented knot Q is a convenient invariant to use because it relates knotting and linking. In particular, Kauffman [Ka] has shown that it satisfies the following equation

$$a_2(K_+) = a_2(K_-) + lk(L_1, L_2) \tag{1}$$

where K_+ and K_- are identical oriented knots outside of the crossing illustrated in Figure 1, and the oriented link $L_1 \cup L_2$ is obtained by smoothing this crossing as illustrated.

Due to the utility of equation (1), the second coefficient of the Conway polynomial has been used to prove various theorems about knots contained in spatial graphs (see for example [CG], [Fo], [TY], [Sh]). Using $a_2(Q)$ as a measure of knot complexity we prove the following theorem.

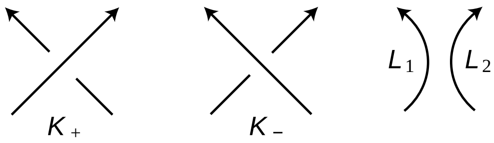

Figure 1: The skein moves

Theorem 3 *Let $m \in \mathbb{N}$ be given, let p be an integer such that $p \geq 4\sqrt{m}$, and let $n = p(15p - 9)$. Then every embedding of K_{2n} in $\mathbb{R}^3$ contains a knot Q with $|a_2(Q)| \geq m$.*

Observe that by picking p to be the smallest integer such that $p \geq 4\sqrt{m}$, we have $p \leq 4\sqrt{m} + 1$ and hence $2n \leq 720m - 60$. In particular, this means that $2n$ only grows linearly with m.

The question of finding the minimum number of vertices necessary to guarantee a certain type of intrinsic linking or knotting remains open. Since K_5 does not have enough vertices to contain a link, $n = 6$ is the smallest number of vertices necessary such that every embedding of K_n contains a link with $L_1 \cup L_2$ with $|lk(L_1, L_2)| \geq 1$. Theorem 1 shows every embedding of K_{10} in $\mathbb{R}^3$ contains a link with $L_1 \cup L_2$ with $|lk(L_1, L_2)| \geq 2$. However, it is not known whether 10 is the smallest such n. In general, it is an open question to find a function $\varphi : \mathbb{N} \to \mathbb{N}$ such that for every $p \in \mathbb{N}$, $\varphi(p)$ is the minimum number of vertices necessary such that every embedding of $K_{\varphi(p)}$ contains a 2-component link $L_1 \cup L_2$ with $|lk(L_1, L_2)| \geq p$. With respect to intrinsic knotting, the embedding of K_6 given by Conway and Gordon contains no non-trivial knot, thus $n = 7$ is the minimum number of vertices necessary such that every embedding of K_n contains a knot Q with $|a_2(Q)| \geq 1$. It follows from Theorem 3 that every embedding of K_{972} in $\mathbb{R}^3$ contains a non-trivial knot other than the trefoil knot. However it is not known what the smallest n is such that every embedding of K_n in $\mathbb{R}^3$ contains a knot other than the trefoil knot. Furthermore, it is an open question to find a function $\psi : \mathbb{N} \to \mathbb{N}$ such that for every $m \in \mathbb{N}$, $\psi(m)$ is the minimum number of vertices necessary such that every embedding of $K_{\psi(m)}$ contains a knot Q with $|a_2(Q)| \geq m$.

2 Intrinsic Linking

The following lemma allows us to go from a 3-component link (with a sufficient number of vertices) to a 2-component link whose linking number is at least the sum of the linking numbers of two pairs of components of the 3-component link.

Lemma 1 *Let $L \cup Z \cup W$ be a 3-component link contained in some embedding of K_n in $\mathbb{R}^3$. Suppose that $lk(L,Z) = p_1 > 0$ and $lk(L,W) = p_2 > 0$ for some orientation of $L \cup Z \cup W$. Suppose that Z and W each contain at least q vertices and $q > p_1 + p_2$. Then K_n contains a simple closed curve J with at least $2q$ vertices which is disjoint from L such that, for some orientation of $L \cup J$, we have $lk(L,J) \geq p_1 + p_2$.*

Proof On Z we select q vertices and label them consecutively by $v_1, \ldots, v_q$ in such a way that Z is oriented in the direction of increasing order of the v_j. On W we select q vertices and label them consecutively by $w_1, \ldots, w_q$ in such a way that W is oriented in the direction of decreasing order of the w_j. For each $j = 1, \ldots, q$, let A_j denote the simple closed curve $\overline{v_j w_j w_{j+1} v_{j+1} v_j}$, where the subscripts are taken mod q. We orient each A_j so that going from v_j to w_j along the edge $\overline{v_j w_j}$ is the positive direction.

Now in the homology group $H_1(\mathbb{R}^3 - L; \mathbb{Z})$ we have the equation

$$[Z] + [W] + [A_1] + \cdots + [A_q] = 0.$$

Thus

$$p_1 + p_2 = [Z] + [W] = -[A_1] - \cdots - [A_q].$$

Since $[A_j]$ is an integer for each j and $q > p_1 + p_2$, there is some j such that $[A_j] \geq 0$. Without loss of generality $[A_q] \geq 0$. Hence in $H_1(\mathbb{R}^3 - L; \mathbb{Z})$ we have the inequality

$$-[A_1] - \cdots - [A_{q-1}] \geq p_1 + p_2.$$

Now let J denote the simple closed curve obtained from $A_1 \cup \cdots \cup A_{q-1}$ by omitting the edges $\overline{v_j w_j}$ for $j = 2, \ldots, q-1$. We orient J so that going from w_1 to v_1 along the edge $\overline{w_1 v_1}$ is the positive direction. Then in $H_1(\mathbb{R}^3 - L; \mathbb{Z})$ we have

$$[J] = -[A_1] - \cdots - [A_{q-1}].$$

Hence $lk(L,J) \geq p_1 + p_2$ and J has at least $2q$ vertices. □

We shall prove Theorem 1 by using Lemma 1 together with [FNP].

Theorem 1 *Every embedding of K_{10} in $\mathbb{R}^3$ contains a 2-component link $L \cup J$ such that, for some orientation, we have $lk(L,J) \geq 2$.*

Proof Let K_{10} be embedded in $\mathbb{R}^3$. It follows from [FNP] that K_{10} contains a 3-component link $L \cup Z \cup W$ such that $lk(L,Z) \equiv 1$ (mod 2) and $lk(L,W) \equiv 1$ (mod 2). We orient the link $L \cup Z \cup W$ so that $lk(L,Z) = p_1 \geq 1$ and

$lk(L,W) = p_2 \geq 1$. If either $p_i \geq 3$ we are done. Otherwise, $p_1 = p_2 = 1$. Clearly Z and W each have at least 3 vertices. Now we apply Lemma 1 to obtain a simple closed curve J such that $lk(L,J) \geq 2$. □

Lemma 2 will allow us to go from a pair of disjoint 2-component links (each with a sufficient number of vertices) to a 3-component link, while bounding the linking numbers of the 3-component link below. The proof of Lemma 2 is similar in flavor to the proof of Lemma 1, though the details differ.

Lemma 2 *Let $X_1 \cup Y_1 \cup X_2 \cup Y_2$ be a 4-component link contained in some embedding of K_n in $\mathbb{R}^3$. Suppose that for some orientation of $X_1 \cup Y_1 \cup X_2 \cup Y_2$ we have $lk(X_1,Y_1) \geq 1$ and $lk(X_2,Y_2) = p \geq 1$. Also suppose that X_1, Y_1, X_2, Y_2 each contain at least q vertices and $q > p$. Then K_n contains disjoint simple closed curves L, Z and W, each with at least q vertices, such that $lk(L,Z) \geq 1$ and $lk(L,W) \geq p$ for some orientation of $L \cup Z \cup W$.*

Proof If $lk(X_2,Y_1)$ is non-zero, then let $L = X_2$, $Z = Y_1$, and $W = Y_2$. Now if we orient $L \cup Z \cup W$ appropriately we get a link with $lk(L,Z) \geq 1$ and $lk(L,W) \geq p$. If $lk(Y_2,X_1)$ is non-zero, let $L = Y_2$, $Z = X_1$, and $W = X_2$. Then $L \cup Z \cup W$ is the desired link. So from now on, we shall assume that both $lk(Y_1,X_2) = 0$ and $lk(X_1,Y_2) = 0$.

We choose q vertices on each of X_1 and X_2 and label the vertices of X_1 and X_2 as we did the vertices of Z and W in the proof of Lemma 1. We also define the oriented simple closed curves A_j as we did in the proof of Lemma 1. Now for both $i = 1$ and $i = 2$, in the first homology groups $H_1(\mathbb{R}^3 - Y_i;\mathbb{Z})$ we have the equation

$$[X_1] + [X_2] + [A_1] + \dots + [A_q] = 0.$$

Now by our assumptions that $lk(Y_1,X_2) = 0$ and $lk(X_1,Y_2) = 0$, we have $[X_2] = 0$ in $H_1(\mathbb{R}^3 - Y_1;\mathbb{Z})$ and we have $[X_1] = 0$ in $H_1(\mathbb{R}^3 - Y_2;\mathbb{Z})$. Thus in $H_1(\mathbb{R}^3 - Y_2;\mathbb{Z})$ we have

$$0 < p = [X_2] = -[A_1] - \dots - [A_q].$$

Since $q > p$, without loss of generality $[A_q] \geq 0$ in $H_1(\mathbb{R}^3 - Y_2;\mathbb{Z})$. Hence

$$-[A_1] - \dots - [A_{q-1}] \geq p$$

in $H_1(\mathbb{R}^3 - Y_2;\mathbb{Z})$.

In $H_1(\mathbb{R}^3 - Y_1;\mathbb{Z})$ we have

$$[X_1] = -[A_1] - \dots - [A_q].$$

First we suppose that $[A_q] = 0$ in $H_1(\mathbb{R}^3 - Y_1;\mathbb{Z})$. So $-[A_1] - \ldots - [A_{q-1}] = [X_1] \geq 1$ in $H_1(\mathbb{R}^3 - Y_1;\mathbb{Z})$. In this case, we let L denote the simple closed curve obtained from $A_1 \cup \ldots \cup A_{q-1}$ by omitting the edges $\overline{v_j w_j}$ for $j = 2, \ldots, q-1$. We orient L so that going from w_1 to v_1 along the edge $\overline{w_1 v_1}$ is the positive direction. Then L has at least $2q$ vertices and $lk(L, Y_1) = lk(X_1, Y_1) \geq 1$ and $|lk(L, Y_2)| \geq p$. We are done by letting $Z = Y_1$ and $W = Y_2$.

Now suppose that $[A_q] \neq 0$ in $H_1(\mathbb{R}^3 - Y_1;\mathbb{Z})$. Let L denote the simple closed curve obtained from $X_2 \cup A_q$ by omitting the edge $\overline{w_q w_1}$. We orient L so that going from v_1 to w_1 along the edge $\overline{v_1 w_1}$ is the positive direction. Then L has at least $q + 2$ vertices. Also since $lk(X_2, Y_1) = 0$ we have $|lk(L, Y_1)| = |lk(A_q, Y_1)| \geq 1$; Now by changing the orientation on Y_1, if necessary, $lk(L, Y_1) \geq 1$. Since $[A_q] \geq 0$ in $H_1(\mathbb{R}^3 - Y_2;\mathbb{Z})$ we have $lk(L, Y_2) \geq lk(X_2, Y_2) = p$. So we are done by letting $Z = Y_1$ and $W = Y_2$. □

Now we will use Lemmas 1 and 2, together with an inductive argument, to prove Theorem 2.

Theorem 2 *Let $p \in \mathbb{N}$ be given, and let $n = p(15p - 9)$. Then every embedding of K_n in $\mathbb{R}^3$ contains a 2-component link $L_p \cup J_p$ such that for some orientation of $L_p \cup J_p$, we have $lk(L_p, J_p) \geq p$.*

Proof Suppose that for every oriented link $L_p \cup J_p$ contained in K_n we have $lk(L_p, J_p) < p$.

Let $G_1, \ldots, G_p$ be p disjoint copies of K_{15p-9} which are contained in K_n. For each $i = 1, \ldots, p$, let H_i be a subgraph of G_i containing all $15p - 9$ vertices of G_i such that, as a topological space, H_i is homeomorphic to K_6, yet between every pair of vertices which have valence 5 in H_i there is a path in H_i containing $p - 1$ vertices which are each of valence 2 in H_i. Since K_6 contains 15 edges and $15(p - 1) + 6 = 15p - 9$, there is such a subgraph H_i in K_{15p-9}. Now by [CG], each H_i contains a link $X_i \cup Y_i$ such that with some orientation we have $lk(X_i, Y_i) \geq 1$ and X_i and Y_i each contain 3 vertices with valence 5 and $3(p - 1)$ vertices with valence 2 in H_i. Thus X_i and Y_i each contain a total of $3p$ vertices.

We will prove by induction that for every $m = 1, \ldots, p$, the $K_{m(15p-9)}$, which has all of its vertices in $\bigcup_{i=1}^{m} G_i$, contains a link $L_m \cup J_m$ such that, with some orientation, $lk(L_m, J_m) = p_m \geq m$ and L_m and J_m each have at least $3p$ vertices. We saw above that this is true for $m = 1$. Assume that it's true for m. Thus $lk(L_m, J_m) = p_m$, and by our initial assumption $p_m < p$.

Also G_{m+1} is disjoint from $K_{m(15p-9)}$ and contains a pair of simple closed curves X_{m+1} and Y_{m+1} each with $3p$ vertices such that $lk(X_{m+1}, Y_{m+1}) \geq 1$. Thus the disjoint simple closed curves L_m, J_m, X_{m+1}, and Y_{m+1} each contain at least $3p$ vertices and $3p > p_m$ since $p > p_m$. Thus by Lemma 2, the $K_{(m+1)(15p-9)}$, which has all of its vertices in $\bigcup_{i=1}^{m+1} G_i$, contains simple closed curves L_{m+1}, Z_{m+1} and W_{m+1} each with at least $3p$ vertices such that, for some orientation of $L_{m+1} \cup Z_{m+1} \cup W_{m+1}$, we have $lk(L_{m+1}, Z_{m+1}) = q_m \geq 1$ and $lk(L_{m+1}, W_{m+1}) = r_m \geq p_m \geq m$. By assumption $q_m < p$ and $r_m < p$, so $q_m + r_m < 3p$. Thus we can apply Lemma 1 to obtain a simple closed curve J_{m+1} which is contained in $K_{(m+1)(15p-9)}$ such that, with some orientation, $lk(L_{m+1}, J_{m+1}) \geq q_m + r_m \geq 1 + m$ and J_{m+1} has at least $6p$ vertices. Thus for all $m = 1, \dots, p$, the $K_{m(15p-9)}$, which has all of its vertices in $\bigcup_{i=1}^{m} G_i$, contains a link $L_m \cup J_m$ such that, with some orientation, $lk(L_m, J_m) \geq m$ and L_m and J_m each have at least $3p$ vertices. It follows that $K_{p(15p-9)}$ contains a link $L_p \cup J_p$ such that, with some orientation, $lk(L_p, J_p) \geq p$. However this contradicts our assumption that for every oriented link $L_p \cup J_p$ contained in K_n we have $lk(L_p, J_p) < p$. Hence we must, in fact, have a 2-component link $L_p \cup J_p$ in K_n such that for some orientation of the link we have $lk(L_p, J_p) \geq p$. □

We now observe as follows that for any given $p \in \mathbb{N}$, it is not possible to have an $n \in \mathbb{N}$, such that every embedding of K_n contains a 2-component link with linking number equal to precisely p. First recall that the linking number of any oriented 2-component link is equal to one half of the sum of $+1$ for every positive crossing and -1 for every negative crossing. We illustrate these two types of crossings in Figure 2.

Figure 2: Positive and negative crossings

Let $n \in \mathbb{N}$ be fixed and let K_n be embedded in $\mathbb{R}^3$. We shall find a (possibly different) embedding of K_n in $\mathbb{R}^3$ which contains no 2-component link $L_1 \cup L_2$ with $lk(L_1, L_2) = p$. Let $q = max\{|lk(L_i, L_j)|\}$ taken over all disjoint pairs of simple closed curves L_i and L_j, contained in K_n. Suppose there is some link $L_1 \cup L_2$ in K_n such that for some orientation, $lk(L_1, L_2) = p$. Let e be an edge in L_1 and let f be an edge in L_2. We create a new embedding of K_n by adding $2p + 2q + 2$ half twists between the edges e and f without changing anything else about the embedding. We call this new embedding K'_n and this new pair of edges e' and f'. Now for any pair of disjoint oriented simple closed curves L_i and L_j in K_n, there is a corresponding pair of disjoint oriented simple closed curves L'_i and L'_j in K'_n. Furthermore, L'_i and L'_j contain e' and f' respectively if and only if L_i and L_j contain e and f respectively. Now suppose that one of L'_i and L'_j contains e' and the other contains f'. Then $lk(L'_i, L'_j) = lk(L_i, L_j) \pm (p + q + 1)$. Since $lk(L_i, L_j) \geq -q$ we know that $lk(L_i, L_j) + (p + q + 1) \geq p + 1 > 0$, so $|lk(L_i, L_j) + (p + q + 1)| \geq p + 1$. Since $lk(L_i, L_j) \leq q$ we know that $lk(L_i, L_j) - (p + q + 1) \leq -p - 1 < 0$, so $|lk(L_i, L_j) - (p + q + 1)| \geq p + 1$. Thus $|lk(L'_i, L'_j)| = |lk(L_i, L_j) \pm (p + q + 1)| \geq p + 1$.

On the other hand, for any pair L'_i and L'_j which do not contain e' and f', then $lk(L'_i, L'_j) = lk(L_i, L_j)$. It follows that K'_n contains fewer links which have linking number equal to p than K_n did. If we repeat this process enough times, we will eventually have an embedding of K_n which contains no links that have linking number equal to p.

3 Intrinsic Knotting

In order to consider intrinsic knotting of complete graphs, we shall make use of the pseudo-graph D_4 (illustrated in Figure 3) in a similar manner to that of [TY] and [Fo].

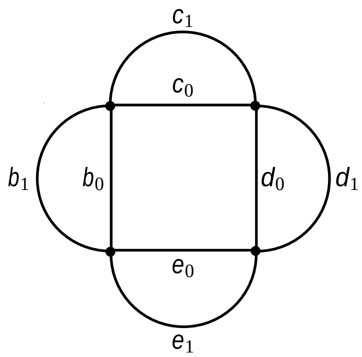

Figure 3: The pseudo-graph G_4

A *Hamiltonian cycle* in a graph or pseudo-graph G is a simple closed curve in G which contains every vertex of G. Each Hamiltonian cycle of D_4 has the form $b_i c_j d_k e_l$ where $i, j, k, l \in \{0, 1\}$. For each such cycle we define

$$\varepsilon(b_i c_j d_k e_l) = \begin{cases} +1 & \text{if } i + j + k + l \text{ is even} \\ -1 & \text{if } i + j + k + l \text{ is odd} \end{cases}$$

Let Γ denote the set of all Hamiltonian cycles in D_4. Recall that if γ is a simple closed curve in $\mathbb{R}^3$, then $a_2(\gamma)$ denotes the second coefficient of the Conway polynomial of γ. For any embedding of D_4 in $\mathbb{R}^3$ we define

$$= \Big|\sum_{\gamma \in \Gamma} \varepsilon(\gamma) a_2(\gamma)\Big|$$

Let B, C, D, and E denote the simple closed curves $b_0 \cup b_1$, $c_0 \cup c_1$, $d_0 \cup d_1$, and $e_0 \cup e_1$ respectively. It is shown in [TY] that, for a given embedding of D_4 in $\mathbb{R}^3$, we have the equation

$$= |lk(E;C)\, lk(B;D)| \qquad (2)$$

Equation (2) enables us to relate knotting and linking in an embedded graph. We use this equation to prove Theorem 3.

Theorem 3 *Let $m \in \mathbb{N}$ be given, let p denote an integer such that $p \geq 4\sqrt{m}$, and let $n = p(15p - 9)$. Then every embedding of K_{2n} in $\mathbb{R}^3$ contains a knot Q with $|a_2(Q)| \geq m$.*

Proof Let K_{2n} be embedded in $\mathbb{R}^3$. Let G_1 and G_2 denote two disjoint K_n's which are contained in K_{2n}. By Theorem 2, each G_i contains a link $X_i \cup Y_i$ such that, for some orientation of $X_i \cup Y_i$, we have $lk(X_i; Y_i) \geq p \geq 4\sqrt{m}$. For $i = 1$ and $i = 2$, let u_i and v_i be distinct vertices on X_i, and let z_i and w_i be distinct vertices on Y_i. Consider the embedded subgraph G of K_{2n} consisting of $X_1 \cup Y_1 \cup X_2 \cup Y_2$ together with the edges $\overline{v_1u_2}$, $\overline{w_1v_2}$, $\overline{z_1w_2}$, and $\overline{u_1z_2}$. Figure 4 illustrates the abstract graph G, with all of the other vertices omitted.

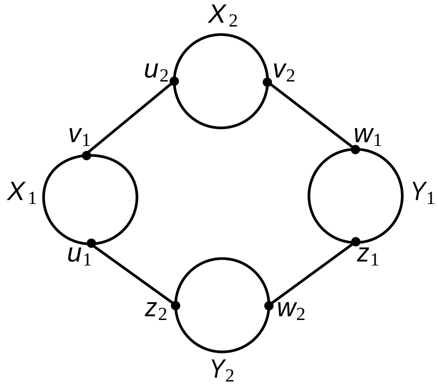

Figure 4: The graph G

From the embedded graph G, we obtain an embedding of the pseudo-graph D_4 by omitting all vertices of valence 2 and collapsing the edges $\overline{v_1u_2}$, $\overline{w_1v_2}$, $\overline{z_1w_2}$, and $\overline{u_1z_2}$. Now by equation (2) we have $\sum_{\gamma\in\Gamma}\varepsilon(\gamma)a_2(\gamma) = |lk(X_1,Y_1)lk(X_2,Y_2)| \geq 16m$. Thus $\sum_{\gamma\in\Gamma}|a_2(\gamma)| \geq |\sum_{\gamma\in\Gamma}\varepsilon(\gamma)a_2(\gamma)| \geq 16m$. However D_4 has precisely 16 Hamiltonian cycles. Thus there is some $\gamma_0 \in \Gamma$ such that $|a_2(\gamma_0)| \geq m$. Now observe that there is a simple closed Q in K_{2n} such that γ_0 is obtained from Q by collapsing precisely the edges $\overline{v_1u_2}$, $\overline{w_1v_2}$, $\overline{z_1w_2}$, and $\overline{u_1z_2}$. Thus Q has the same knot type as γ_0 and hence $|a_2(Q)| \geq m$. □

We can see as follows that for a given $m \in \mathbb{N}$ there is no $n \in \mathbb{N}$ such that every embedding of K_n contains a knot Q with $|a_2(Q)| = m$. First observe that if Q is any knot and R is a right handed trefoil knot, then it follows from equation (1) in the introduction that $a_2(Q\#R) = a_2(Q) + 1$. Let $m \in \mathbb{N}$ and $n \in \mathbb{N}$ be fixed, and let K_n be embedded in $\mathbb{R}^3$. Consider all simple closed curves Q_i which are contained in K_n, and let $q = \min\{a_2(Q_i), 0\} \leq 0$. We modify the embedding of K_n by adding $m - q \geq m$ right handed trefoil knots to every edge within a ball which does not intersect the graph elsewhere. We call this modified embedding K_n'. Now for each simple closed curve Q_i' in K_n' there is a corresponding simple closed curve Q_i in K_n. Since every simple closed curve in K_n' contains at least 3 edges, each Q_i' is a connected sum which has at least $3(m - q)$ more right-handed trefoil knots as summands than Q_i had. Thus for every simple closed curve Q_i' in K_n', we have $a_2(Q_i') \geq a_2(Q_i) + 3(m - q) \geq 3m - 2q \geq 3m$. Thus K_n' contains no knot Q' such that $|a_2(Q')| = m$.

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