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The fundamental group of a Galois cover of $\mathbb{CP}^1 \times T$

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Abstract Let T be the complex projective torus, and X the surface $\mathbb{CP}^1 \times T$. Let X_{Gal} be its Galois cover with respect to a generic projection to $\mathbb{CP}^2$. In this paper we compute the fundamental group of X_{Gal}, using the degeneration and regeneration techniques, the Moishezon-Teicher braid monodromy algorithm and group calculations. We show that $\pi_1(X_{\mathrm{Gal}}) = \mathbb{Z}^{10}$.

AMS Classification 14Q10, 14J99; 14J80, 32Q55

Keywords Galois cover, fundamental group, generic projection, Moishezon-Teicher braid monodromy algorithm, Sieberg-Witten invariants

1 Overview

Let T be a complex torus in $\mathbb{CP}^2$. We compute the fundamental group of the Galois cover with respect to a generic map of the surface $X = \mathbb{CP}^1 \times T$ to $\mathbb{CP}^2$. We embed X into a projective space using the Segre map $\mathbb{CP}^1 \times \mathbb{CP}^2 \to \mathbb{CP}^5$ defined by $(s_0, s_1) \times (t_0, t_1, t_2) \mapsto (s_0t_0, s_1t_0, s_0t_1, s_1t_1, s_0t_2, s_1t_2)$. Then, a generic projection $f\colon X \to \mathbb{CP}^2$ is obtained by projecting X from a general plane in $\mathbb{CP}^5 - X$ to $\mathbb{CP}^2$. The Galois cover can now be defined as the closure of the n-fold fibered product $X_{\mathrm{Gal}} = \overline{X \times_f \cdots \times_f X - \Delta}$ where n is the degree of the map f, and Δ is the generalized diagonal. The closure is necessary because the branched fibers are excluded when Δ is omitted.

The fundamental group $\pi_1(X_{\mathrm{Gal}})$ is related to the fundamental group of the complement of the branch curve. The latter is an important invariant of X, which can be used to classify algebraic surfaces of a general type, up to deformations. Such an invariant is finer than the famous Sieberg-Witten invariants and thus can serve as a tool to distinguish diffeomorphic surfaces which are not deformation of each other (see [4]), [12] and [13]) a problem which is referred to as the Diff-Def problem. The algorithms and problems that arise in the computation of these two types of groups are related, and one hopes to be able to compute such groups for various types of surfaces.

Since the induced map $X_{\mathrm{Gal}} \to \mathbb{CP}^2$ has the same branch curve S as $f : X \to \mathbb{CP}^2$, the fundamental group $\pi_1(X_{\mathrm{Gal}})$ is related to $\pi_1(\mathbb{CP}^2 - S)$. In fact it is a normal subgroup of the quotient of $\pi_1(\mathbb{CP}^2 - S)$ by the normal subgroup generated by the squares of the standard generators. In this paper we employ braid monodromy techniques, the van Kampen theorem and various computational methods of groups to compute a presentation for the quotient $\tilde{\pi}_1$ from which $\pi_1(X_{\mathrm{Gal}})$ can be derived. Our main result is that $\pi_1(X_{\mathrm{Gal}}) = \mathbb{Z}^{10}$ (Theorem 9.3).

The fundamental group of the Galois cover of the surface $X = \mathbb{CP}^1 \times T$ is a step in computing the same group for $T \times T$ [2], and will later appear in local computations of fundamental groups of the Galois cover of $K3$-surfaces.

It turns out that a property of the Galois covers that were treated before (see [6], [7] or [9]) is lacking in the Galois cover of $X = \mathbb{CP}^1 \times T$. In all the cases computed so far, X had the property that the fundamental group of the graph defined on the planes of the degenerated surface X_0 by connecting every two intersecting planes, is generated by the cycles around the intersection points. Our surface, together with a parallel work on $T \times T$ [2], are the first cases for which this assumption does not hold. The significance of this 'redundancy' property of S_0 will be explained in Section 6 (and in more details in [2]).

The paper is organized as follows. In Section 2 we describe the degeneration of the surface $X = \mathbb{CP}^1 \times T$ and the degenerated branch curve. In Sections 3 and 4 we study the regeneration of this curve and its braid monodromy factorization. We also get a presentation for $\pi_1(\mathbb{C}^2 - S, u_0)$, the fundamental group of the complement of the regenerated branch curve in $\mathbb{C}^2$, see Theorem 4.3. In Section 5 we present the homomorphism $: \tilde{\pi}_1(\mathbb{C}^2 - S, u_0) \to S_6$, whose kernel is $\pi_1(X^{\mathrm{A}}_{\mathrm{Gal}})$. In Section 6 we study a natural Coxeter quotient of $\pi_1(\mathbb{C}^2 - S, u_0)$, and give its structure. In Sections 7 and 8 we use the Reidmeister-Schreir method to give a new presentation for $\pi_1(X^{\mathrm{A}}_{\mathrm{Gal}})$, see Theorem 8.10. In Section 9 we introduce the projective relation, and prove the main result about the structure of $\pi_1(X_{\mathrm{Gal}})$.

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2 Degeneration of $\mathbb{CP}^1 \times T$

In the computation of braid monodromies it is often useful to replace the surface with a degenerated object, made of copies of $\mathbb{CP}^2$. It is easy to see that T degenerates to a triangle of complex projective lines (see [1, Subsection 1.6.3]), so X degenerates to a union of three quadrics Q_1, Q_2, and Q_3, $Q_i = \mathbb{CP}^1 \times \mathbb{CP}^1$, which we denote by X_1, see Figure 1.

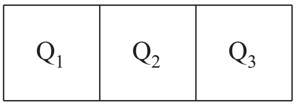


Figure 1: The space X_1

Each square in Figure 1 represents a quadric surface. Since T degenerates to a triangle, Q_1 and Q_3 intersect, so the left and right edges of X_1 are identified. Therefore, we can view X_1 as a triangular prism.

Each quadric in X_1 can be further degenerated to a union of two planes. In Figure 2 this is represented by a diagonal line which divides each square into two triangles, each one isomorphic to $\mathbb{CP}^2$.

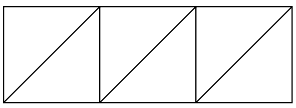


Figure 2: The simplicial complex X_0

We shall refer to this diagram as the simplicial complex of X_0. A common edge between two triangles represents the intersection line of the two corresponding planes. The union of the intersection lines is the ramification locus in X_0 of $f_0\colon X_0 \to \mathbb{CP}^2$, denoted by R_0. Let $S_0 = f_0(R_0)$ be the degenerated branch curve. It is a line arrangement, composed of all the intersection lines.

A vertex in the simplicial complex represents an intersection point of three planes. The vertices represent singular points of R_0. Each of these vertices is called a 3-point (reflecting the number of planes which meet there).

The vertices may be given any convenient enumeration. We have chosen left to right, bottom to top enumeration, see Figure 3. The extreme vertices are pairwise identi ed, as well as the left and right edges.

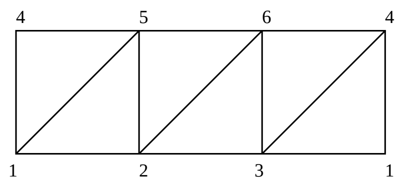


Figure 3

We create an enumeration of the edges based upon the enumeration of the vertices using reverse lexicographic ordering: if L_1 and L_2 are two lines with end points $_1; \ _1$ and $_2; \ _2$ respectively ($_1 < \ _1; \ _2 < \ _2$), then $L_1 < L_2$ i $_1 < \ _2$, or $_1 = \ _2$ and $_1 < \ _2$. The resulting enumeration is shown in Figure 4. This enumeration dictates the order of the regeneration of the lines to curves, see the next section. The horizontal lines at the top and bottom do not represent intersections of planes and hence are not numbered.

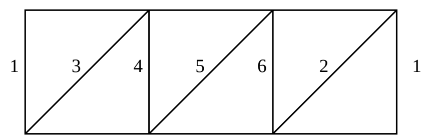


Figure 4

We enumerate the triangles $fP_ig_{i=1}^6$ also according to the enumeration of vertices in reverse lexicographic order. If P_i and P_j have vertices $_1; \ _2; \ _3$ and $_1; \ _2; \ _3$ respectively, with $_1 < \ _2 < \ _3$ and $_1 < \ _2 < \ _3$, then $P_i < P_j$ i $_3 < \ _3$, or $_3 = \ _3$ and $_2 < \ _2$, or $_3 = \ _3$, $_2 = \ _2$ and $_1 < \ _1$. The enumeration is shown in Figure 5.

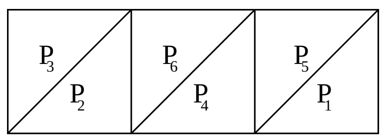


Figure 5

3 Regeneration of the Branch Curve

3.1 The Braid Monodromy of S_0

Starting from the branch curve S_0, we reverse the steps in the degeneration of X to regenerate the braid monodromy of S. Figure 6 shows the three steps to recover the original object $X_3 = X$.

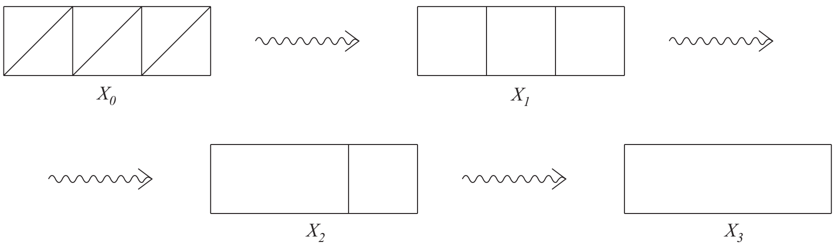


Figure 6: The regeneration process

Recall that X comes with an embedding to $\mathbb{CP}^5$. At each step of the regeneration, the generic projection $\mathbb{CP}^5 \to \mathbb{CP}^2$ restricts to a generic map $f_i\colon X_i \to \mathbb{CP}^2$. Let $R_i \subset X_i$ be the ramification locus of f_i and $S_i \subset \mathbb{CP}^2$ the corresponding branch curve.

We have enumerated the six planes $P_1, \dots, P_6$ which comprise X_0, the six intersection lines $\hat{L}_1, \dots, \hat{L}_6$, and their six intersection points $\hat{V}_1, \dots, \hat{V}_6$. Let L_i and V_j denote the projections of $\hat{L}_i$ and $\hat{V}_j$ to $\mathbb{CP}^2$ by the map f_0. Clearly $R_0 = \bigcup_{i=1}^{6} \hat{L}_i$ and $S_0 = \bigcup_{i=1}^{6} L_i$. Let C be the line arrangement consisting of all lines through pairs of the V_js. The degenerated branch curve S_0 is a sub-arrangement of C. Since C is a dual to generic arrangement, Moishezon's results in [5] (and later on Theorem IX.2.1 in [8]) gives us a braid monodromy factorization for C: $\Delta^2_C = \prod_{j=1}^{6} C_j \Delta_j^2$ where Δ_j^2 is the monodromy around V_j and the C_j consist of products of the monodromies around the other intersections points of C. This factorization can be restricted to S_0 by removing from the braids all strands which correspond to lines of C that do not appear in S_0, and deleting all factors which correspond to intersections in C that do not appear in S_0. Thus we get a braid monodromy factorization: $\Delta^2_{S_0} = \prod_{j=1}^{6} \tilde{C}_j \tilde{\Delta}_j^2$. The $\tilde{C}_j$ and $\tilde{\Delta}_j^2$ and their regenerations are formulated more precisely in the following subsections.

3.2 $\tilde{\ }^2_j$ and its Regeneration

Consider an affine piece of S_0 $\mathbb{CP}^2$ and take a generic projection $: \mathbb{C}^2 \to \mathbb{C}$. Let N be the set of the projections of the singularities and branch points with respect to . Choose $u \in \mathbb{C} - N$ and let $\mathbb{C}_u = {}^{-1}(u)$ be a generic fiber.

A path from a point j to a point k below the real line is denoted by $\underline{z}_{jk}$, and the corresponding halftwist by $\underline{Z}_{jk}$.

Two lines $\hat{L}_j$ and $\hat{L}_k$ which meet in X_0 give rise to braids connecting $j = L_j \cap \mathbb{C}_u$ and $k = L_k \cap \mathbb{C}_u$, namely a fulltwist $\underline{Z}^2_{jk}$ of j and k. This is done in the following way: let $V_i = L_j \cap L_k$ be the intersection point, then $\tilde{\ }^2_i = \underline{Z}^2_{jk}$.

We shall analyze the regeneration of the local braid monodromy of S in a small neighborhood of each V_i. The case of non-intersecting lines (which give 'parasitic intersections') is discussed in the next subsection.

The degenerated branch curve S_0 has six singularities coming from the 3-points of R_0, shown in Figure 7.

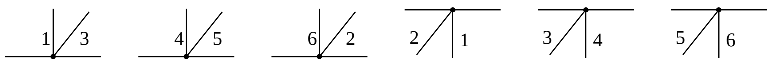


Figure 7: Enumeration around the 3-points

Each pair of lines intersecting at a 3-point regenerates in S_1 (the branch curve of X_1) as follows: the diagonal line becomes a conic, and the vertical line is tangent to it. In the next step of the regeneration the point of tangency becomes three cusps according to the third regeneration rule (which was quoted in [5] and proven in [10, p.337]). This is enough information to compute H_{V_i}, the local braid monodromy of S in a neighborhood of V_i, see these specific computations for this case in [1, Subsection 1.10.4].

In the regeneration, each point on the typical fiber is replaced by two close points $;\ ^{0}$. Denote by $= Z_{\ 0}$ the counterclockwise halftwist of and ^{0}.

The following table presents the global form of the local braid monodromies, as quoted in [5], and presents also the application of this global form to our case.

Table 3.1 *The local braid monodromies H_{V_i} are as follows. For every fixed i, let $<$ be the lines intersecting at V_i. Let $Z^3_{\ 0;} = (\underline{Z}^3_{\ 0})\ \underline{Z}^3_{\ 0}\ (\underline{Z}^3_{\ 0})^{-1}$ and $Z^3_{0;\ 0} = (\underline{Z}^3_{\ 0})\ \underline{Z}^3_{\ 0}\ (\underline{Z}^3_{\ 0})^{-1}$.*

For $i = 1;2;4$ we have $H_{V_i} = Z^3_{\ 0;\ 0}\ Z_{\ 0(\ )}$, where $Z_{\ 0(\ )}$ is the halftwist corresponding to the path shown in Figure 8. For $i = 3;5;6$ we have $H_{V_i} = Z^3_{\ 0;\ 0}\ Z_{\ 0(\ )}$, where $Z_{\ 0(\ )}$ is the halftwist corresponding to the path shown in Figure 9.

In paricular we have

$$\begin{aligned}
H_{V_1} &= Z^3_{11';3'}\ Z_{33'(1)};\\
H_{V_2} &= Z^3_{44';5'}\ Z_{55'(4)};\\
H_{V_3} &= Z^3_{2';66'}\ Z_{22'(6)};\\
H_{V_4} &= Z^3_{11';2'}\ Z_{22'(1)};\\
H_{V_5} &= Z^3_{3';44'}\ Z_{33'(4)};\\
H_{V_6} &= Z^3_{5';66'}\ Z_{55'(6)}:
\end{aligned}$$

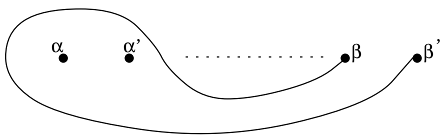


Figure 8: $Z_{\ 0(\ )}$ for $i = 1;2;4$

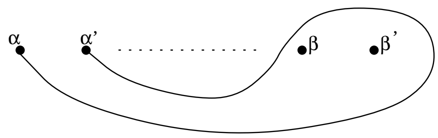


Figure 9: $Z_{\ 0(\ )}$ for $i = 3;5;6$

The table given in Figure 10 presents the six monodromy factorization, one for every point $V_1;\ldots;V_6$. For each point, the rst path represents three factors obtained from cusps, and the other represents the fourth factor, obtained from the branch point (as shown in Figures 8 and 9). The relations obtained from these braids are given in Theorem 4.3.

3.3 C_j and its Regeneration

There are lines which do not meet in X_0 but whose images meet in $\mathbb{C}^2$. Such an intersection is called a parasitic intersection. Each pair of disjoint lines $\hat{L}_i$ and $\hat{L}_j$ give rise to a certain fulltwist, see [8, Theorem IX.2.1]. This is denoted as Z^2_{ij}, corresponding to a path z_{ij}, running from i over the points up to j_0, then under j_0 up to j, where j_0 is the least numbered line which shares the upper vertex of L_j.

As discussed in [7] and [1] the degree of the regenerated branch curve S is twice the degree of S_0. Consequently each line $L_i \quad S_0$ divides locally into two branches of S and each $i = L_i \setminus \mathbb{C}_u$ divides into two points, i and i'.

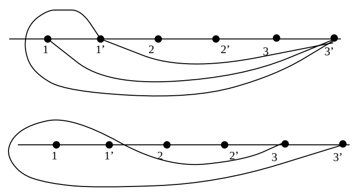
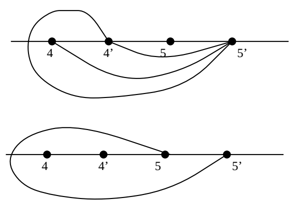
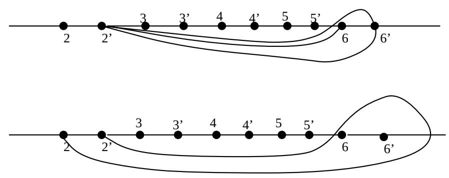
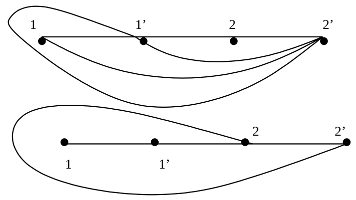
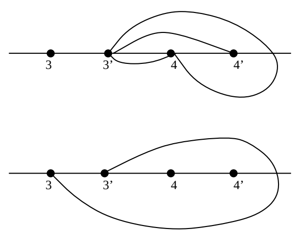
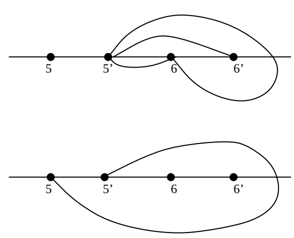
pt.	the braid	exp.	the path representing the braid
V_1	${}_1^{i}\underline{Z}_{1'3'}\,{}_1^{-i}$ $i = 0, 1, 2$ $Z_{33'(1)}$	3 1	
V_2	${}_4^{i}\underline{Z}_{4'5'}\,{}_4^{-i}$ $i = 0, 1, 2$ $Z_{55'(4)}$	3 1	
V_3	${}_6^{i}\underline{Z}_{2'6'}\,{}_6^{-i}$ $i = 0, 1, 2$ $Z_{22'(6)}$	3 1	
V_4	${}_1^{i}\underline{Z}_{1'2'}\,{}_1^{-i}$ $i = 0, 1, 2$ $Z_{22'(1)}$	3 1	
V_5	${}_4^{i}\underline{Z}_{3'4}\,{}_4^{-i}$ $i = 0, 1, 2$ $Z_{33'(4)}$	3 1	
V_6	${}_6^{i}\underline{Z}_{5'6}\,{}_6^{-i}$ $i = 0, 1, 2$ $Z_{55'(6)}$	3 1	

Figure 10: Monodromy factorizations

According to the second regeneration rule (quoted in [5] and proved in [10, p. 337]) the fulltwist Z_{ij}^2 becomes $Z_{ii',jj'}^2$, which compounds four nodes of S, namely Z_{ij}^2, $Z_{ij'}^2$, $Z_{i'j}^2$ and $Z_{i'j'}^2$ as shown in Figure 11. These are the factors in the regenerations C_j' of the C_j. In Table 3.2 we construct the paths which correspond to these braids in our case.

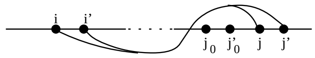
factor	corresponding paths	singularity types	degrees
$Z_{ii',jj'}$		four nodes	2,2,2,2

Figure 11

Pick a base point $u_0 \in \mathbb{C}_u$ in the generic fiber $\mathbb{C}_u = \pi^{-1}(u)$. The fundamental group $\pi_1(\mathbb{C}_u - S, u_0)$ is freely generated by $\{\Gamma_j, \Gamma_{j'}\}_{j=1}^6$, where Γ_j and $\Gamma_{j'}$ are loops in $\mathbb{C}_u$ around j and j' respectively. Let us explain how to create such generators. Define the generators Γ_j and Γ_j' in two steps. First select a path γ_j from u_0 to a point u_j close to j and j'. Next choose small counterclockwise circles δ_j and δ_j' starting from u_j and circling j and j' respectively. Use these three paths to build the generators $\Gamma_j = \gamma_j \delta_j \gamma_j^{-1}$ and $\Gamma_{j'} = \gamma_j \delta_j' \gamma_j^{-1}$.

By the van Kampen Lemma [15], there is a surjection from $\pi_1(\mathbb{C}_u - S, u_0)$ onto $\pi_1 = \pi_1(\mathbb{C}^2 - S, u_0)$. The images of $\{\Gamma_j, \Gamma_{j'}\}_{j=1}^6$ generate π_1. By abuse of notation we denote them also by $\{\Gamma_j, \Gamma_{j'}\}_{j=1}^6$. By the van Kampen Theorem [15], each braid in the braid monodromy factorization of S induces a relation on π_1 through its natural action on $\mathbb{C}_u - \{j, j'\}_{j=1}^6$ [5]. A presentation for

$$\tilde{\pi}_1 = \frac{\pi_1(\mathbb{C}^2 - S, u_0)}{\left\langle \Gamma_j^2, \Gamma_{j'}^2 \right\rangle} \tag{1}$$

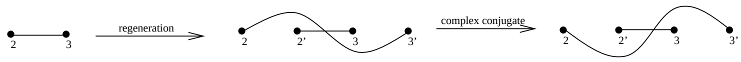
is thus immediately obtained from a presentation of π_1.

The algorithm used to compute a relation from a braid is explained in [7, Section 0.7], see also [1, Section 1.11].

Moishezon claimed in [5] that the braid monodromy factorization is invariant under complex conjugation of $\mathbb{C}_u$. Later it was proven in Lemma 19 of [10]. Therefore we can include the complex conjugate paths and relations in the table. For simplicity of notation we will use the following shorthand: $\Gamma_{ii'}$ will stand for either Γ_i or $\Gamma_{i'}$; $\Gamma_{ii'}$ will stand for either $\Gamma_{i'}\Gamma_i\Gamma_{i'}$ or $\Gamma_{i'}$; $\Gamma_{j\hat{j}'}$ will stand for either Γ_j or $\Gamma_j\Gamma_{j'}\Gamma_j$.

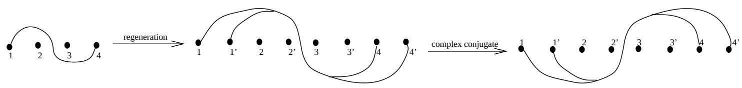
Table 3.2 *We present the relations induced by the van Kampen Theorem from the paths, one for every pair of non-intersecting lines $\hat{L}_i, \hat{L}_j$.*

$\mathbf{Z}^2_{22',33'}$:



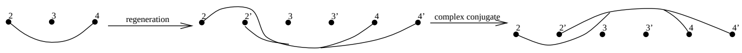
The relations: $[\Gamma_{22'}, \Gamma_{33'}] = 1$ *(from the path itself), and* $[\Gamma_{22'}, \Gamma_{3\hat{3}}] = 1$ *(from the complex conjugate).*

$\mathbf{Z}^2_{11',44'}$:



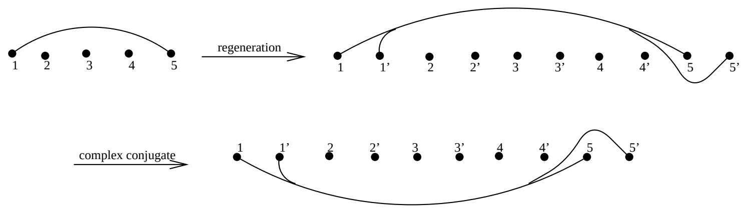
Relations: $[\Gamma_{2'}\Gamma_2\Gamma_{11'}\Gamma_2\Gamma_{2'}, \Gamma_{44'}] = 1$ *and* $[\Gamma_{11'}, \Gamma_3\Gamma_{3'}\Gamma_{4\hat{4}}\Gamma_{3'}\Gamma_3] = 1$.

$\mathbf{Z}^2_{22',44'}$:



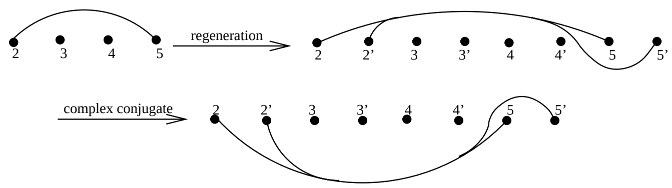
$[\Gamma_{22'}, \Gamma_{44'}] = 1$ *and* $[\Gamma_{22'}, \Gamma_3\Gamma_{3'}\Gamma_{4\hat{4}}\Gamma_{3'}\Gamma_3] = 1$.

$\mathbf{Z}^2_{11',55'}$:



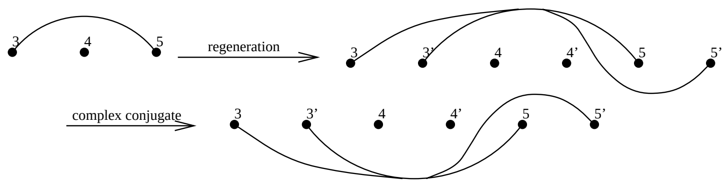
$[\Gamma_{4'}\Gamma_4\Gamma_{3'}\Gamma_3\Gamma_{2'}\Gamma_2\Gamma_{11'}\Gamma_2\Gamma_{2'}\Gamma_3\Gamma_{3'}\Gamma_4\Gamma_{4'}, \Gamma_{55'}] = 1$ *and* $[\Gamma_{11'}, \Gamma_{5\hat{5}}] = 1$.

$\mathbf{Z}^2_{22',55'}$:



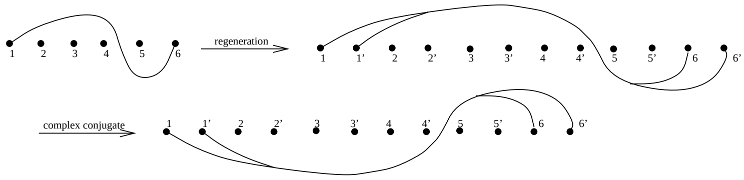
$[\Gamma_{4'}\Gamma_4\Gamma_{3'}\Gamma_3\Gamma_{22'}\Gamma_3\Gamma_{3'}\Gamma_4\Gamma_{4'}, \Gamma_{55'}] = 1$ *and* $[\Gamma_{22'}, \Gamma_{5\hat{5}}] = 1$.

$\mathbf{Z}^2_{33',55'}$:



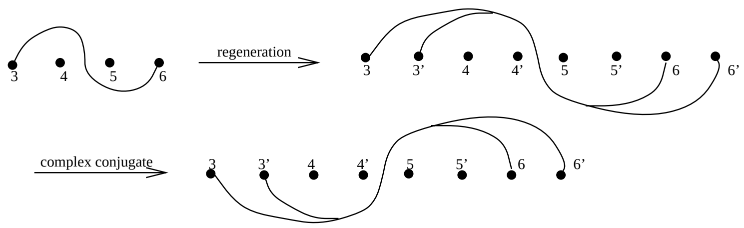
$[\Gamma_{4'}\Gamma_4\Gamma_{33'}\Gamma_4\Gamma_{4'}, \Gamma_{55'}] = 1$ *and* $[\Gamma_{33'}, \Gamma_{5\hat{5}}] = 1$.

$\mathbf{Z}^2_{11',66'}$:



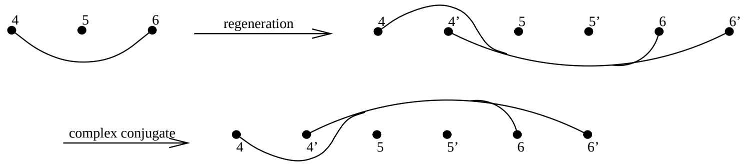
Relations: $[\Gamma_{4'}\Gamma_4\Gamma_{3'}\Gamma_3\Gamma_{2'}\Gamma_2\Gamma_{11'}\Gamma_2\Gamma_{2'}\Gamma_3\Gamma_{3'}\Gamma_4\Gamma_{4'}, \Gamma_{66'}] = 1$ *and* $[\Gamma_{11'}, \Gamma_5\Gamma_{5'}\Gamma_{6\hat{6}}\Gamma_{5'}\Gamma_5] = 1$.

$\mathbf{Z}^2_{33',66'}$:



$[\Gamma_{4'}\Gamma_4\Gamma_{33'}\Gamma_4\Gamma_{4'}, \Gamma_{66'}] = 1$ *and* $[\Gamma_{33'}, \Gamma_5\Gamma_{5'}\Gamma_{\hat{66}}\Gamma_{5'}\Gamma_5] = 1$.

$\mathbf{Z}^2_{44',66'}$:



$[\Gamma_{44'}, \Gamma_{66'}] = 1$ *and* $[\Gamma_{44'}, \Gamma_5\Gamma_{5'}\Gamma_{\hat{66}}\Gamma_{5'}\Gamma_5] = 1$.

3.4 Checking Degrees

Having computed the C_i' (Subsection 3.3) and the H_{V_i} (Figure 10), we obtain a regenerated braid monodromy factorization $\Delta^2_S = \prod_{i=1}^{6} C_i' H_{V_i}$. To verify that no factors are missing we compare degrees. First, since S is a curve of degree 12 (double the 6 lines in S_0), the braid Δ^2_S has degree $12 \cdot 11 = 132$. The six monodromies H_{V_i} each consist of three cusps and one branch point for a combined degree of $6 \cdot (3 \cdot 3 + 1) = 60$. The C_i' consist of four nodes for each parasitic intersection. The nine parasitic intersections (Table 3.2) give a combined degree of $9 \cdot (4 \cdot 2) = 72$. So together $\prod C_i' H_{V_i}$ has also a degree of $60 + 72 = 132$, which proves that no factor was left out.

4 Invariance Theorems and π_1

4.1 The Invariance Theorem

Invariance properties are results concerning the behavior of a braid monodromy factorization under conjugation by certain elements of the braid group. A factorization $g = g_1 \cdots g_k$ is said to be invariant under h if $g = g_1 \cdots g_k$ is Hurwitz equivalent to $(g_1)_h \cdots (g_k)_h$. Geometrically this means that if a braid monodromy factorization of Δ^2_C coming from a curve S is invariant under h, then the conjugate factorization is also a valid braid monodromy factorization for S.

The following rules [10, Section 3] give invariance properties of commonly occurring subsets of braid monodromy factorizations. Factors of the third type do not appear in our factorization.

(a) $Z^2_{ii',jj'}$ is invariant under $Z^p_{ii'}$ and $Z^p_{jj'}$, $\forall p \in \mathbb{Z}$.

(b) $Z^3_{i,jj'}$ is invariant under $Z^p_{jj'}$, $\forall p \in \mathbb{Z}$.

(c) Z^1_{ij} is invariant under $Z^p_{ii'} Z^p_{jj'}$, $\forall p \in \mathbb{Z}$.

Remark 4.1 *The elements* $Z_{ii'}$ *and* $Z_{jj'}$ *commute for all* $1 \le i, j \le 6$ *since the path from* i *to* i' *does not intersect the path from* j *to* j'.

Theorem 4.2 (Invariance Theorem) *The braid monodromy factorization* $\Delta^2_{12} = \prod_{i=1}^{6} C'_i H_{V_i}$ *is invariant under* $\prod_{j=1}^{6} Z^{m_j}_{jj'}$, *for all* $m_j \in \mathbb{Z}$.

Proof It is sufficient to show that the C'_i and the H_{V_i} are invariant individually. Corollary 14 of [10] proves that each H_{V_i} is invariant under $Z_{jj'}$, $1 \le j \le 6$. Since the $Z_{jj'}$ all commute the invariance extends to arbitrary products $\prod_{j=1}^{6} Z^{m_j}_{jj'}$. The C'_i are composed of quadruples of factors $Z_{kk',\ell\ell'}$, one from each parasitic intersection. Lemma 16 of that paper shows that each $Z_{kk',\ell\ell'}$ is invariant under $Z_{jj'}$, $1 \le j \le 6$. As before the invariance extends to products $\prod_{j=1}^{6} Z^{m_j}_{jj'}$. So the factorization $\Delta^2_{12} = \prod_{i=1}^{6} C'_i H_{V_i}$ is invariant under conjugation by these elements. □

We use $\Gamma_{(j)}$ to denote any element of the set $\{(\Gamma_j)_{Z^m_{jj'}}\}_{m \in \mathbb{Z}}$. These elements are odd length alternating products of Γ_j and $\Gamma_{j'}$. Thus $\Gamma_{(j)}$ represents any element of $\{(\Gamma_j \Gamma_{j'})^p \Gamma_j\}_{p \in \mathbb{Z}}$. The original generators Γ_j and $\Gamma_{j'}$ are easily seen to be members of this set for $p = 0, -1$.

As an immediate consequence of the Invariance Theorem, any relation satisfied by Γ_j is satisfied by any element of $\Gamma_{(j)}$. This infinitely expands our collection of known relations in $\tilde{\pi}_1$, however all of the new relations are consequences of our original finite set of relations. Δ^2_C is also invariant under complex conjugation [10, Lemma 19], so we can use the complex conjugates H_{V_i} and C'_i to derive additional relations. Once again these relations are already implied by the existing relations. On the other hand, many of the complex conjugate braids in Table 3.2 have simpler paths than their counterparts so they are a useful tool.

The paths corresponding to the H_{V_i} are already quite simple (see Figure 10) so nothing is gained there by using complex conjugates.

4.2 A presentation for $\tilde{\pi}_1$

Let S be the regenerated branch curve and let $\pi_1(\mathbb{C}^2 - S, u_0)$ be the fundamental group of its complement in $\mathbb{C}^2$. We know that this group is generated by the elements $\{\Gamma_j, \Gamma_{j'}\}_{j=1}^6$. Recall (Equation (1)) that $\tilde{\pi}_1 = \pi_1(\mathbb{C}^2 - S, u_0)/\left\langle \Gamma_j^2, \Gamma_{j'}^2 \right\rangle$.

We have listed the braids C_i (Table 3.2) and H_{V_i} (Figure 10). These are the only braids in the factoization of Δ^2_C, as explained in Subsection 3.4.

To the path of each braid there correspond two elements of $\pi_1(\mathbb{C}^2 - S, u_0)$, as explained in Subsection 3.3. From these, the van Kampen Theorem [15] produce the defining relations of $\pi_1(\mathbb{C}^2 - S, u_0)$.

Theorem 4.3 *The group $\tilde{\pi}_1$ is generated by $\{\Gamma_j, \Gamma_{j'}\}_{j=1}^6$ with the following relations:*

$$\Gamma_j^2 = 1 \qquad j = 1, \dots, 6 \tag{2}$$
$$\Gamma_{j'}^2 = 1 \qquad j = 1, \dots, 6 \tag{3}$$
$$\Gamma_{3'} = \Gamma_1\Gamma_{1'}\Gamma_3\Gamma_{1'}\Gamma_1 \tag{4}$$
$$\Gamma_{5'} = \Gamma_4\Gamma_{4'}\Gamma_5\Gamma_{4'}\Gamma_4 \tag{5}$$
$$\Gamma_2 = \Gamma_{6'}\Gamma_6\Gamma_{2'}\Gamma_6\Gamma_{6'} \tag{6}$$
$$\Gamma_{2'} = \Gamma_1\Gamma_{1'}\Gamma_2\Gamma_{1'}\Gamma_1 \tag{7}$$
$$\Gamma_3 = \Gamma_{4'}\Gamma_4\Gamma_{3'}\Gamma_4\Gamma_{4'} \tag{8}$$
$$\Gamma_5 = \Gamma_{6'}\Gamma_6\Gamma_{5'}\Gamma_6\Gamma_{6'} \tag{9}$$
$$[\Gamma_{ii'}, \Gamma_{jj'}] = 1 \qquad \text{if the lines } i, j \text{ are disjoint in } X_0 \tag{10}$$
$$\Gamma_{(i)}\Gamma_{(j)}\Gamma_{(i)} = \Gamma_{(j)}\Gamma_{(i)}\Gamma_{(j)} \qquad \text{if the lines } i, j \text{ intersect.} \tag{11}$$

The enumeration of the lines is given in Figure 4. $\Gamma_{ii'}$ represents either Γ_i or $\Gamma_{i'}$, and $\Gamma_{(i)}$ stands for any odd length word in the infinite dihedral group $\langle \Gamma_i, \Gamma_{i'} \rangle$.

Proof Relations (2)–(3) hold in $\tilde{\pi}_1$ by assumption. The other relations hold in $\pi_1(\mathbb{C}^2 - S, u_0)$. To see this, we now list the relations induced by the H_{V_i} (Table 3.1). Recall that each H_{V_i} is a product of regenerated braids induced from one branch point (the second path in each part of Figure 10), and three cusps (condensed in the first path of each part). Applying the van Kampen Theorem [15], we have two types of relations. The relations (4)–(9) are derived from the branch point braids, and the triple relations $\Gamma_i\Gamma_j\Gamma_i = \Gamma_j\Gamma_i\Gamma_j$ from the three other braids. Using the Invariance Theorem 4.2 to expand the pathes, we get (11) in its full generality. It remains to prove Equation (10). Note that the relations $[\Gamma_{ii'}, \Gamma_{jj'}] = 1$ and $[\Gamma_{(i)}, \Gamma_{(j)}] = 1$ imply each other.

We now consider the complex conjugates in Table 3.2. The relations $[\Gamma_{(2)}, \Gamma_{(3)}]$, $[\Gamma_{(2)}, \Gamma_{(4)}]$, $[\Gamma_{(1)}, \Gamma_{(5)}]$, $[\Gamma_{(2)}, \Gamma_{(5)}]$, $[\Gamma_{(3)}, \Gamma_{(5)}]$, and $[\Gamma_{(4)}, \Gamma_{(6)}]$ appear rather directly. We must derive the other commutators, namely $[\Gamma_{(1)}, \Gamma_{(4)}]$, $[\Gamma_{(1)}, \Gamma_{(6)}]$, and $[\Gamma_{(3)}, \Gamma_{(6)}]$.

By the second part of Table 3.2,

$$[\Gamma_{2'}\Gamma_2\Gamma_{(1)}\Gamma_2\Gamma_{2'}, \Gamma_{(4)}] = \Gamma_{2'}\Gamma_2\Gamma_{(1)}\Gamma_2\Gamma_{2'}\Gamma_{(4)}\Gamma_{2'}\Gamma_2\Gamma_{(1)}\Gamma_2\Gamma_{2'}\Gamma_{(4)},$$

but since $[\Gamma_{(2)}, \Gamma_{(4)}] = 1$ we get

$$\Gamma_{2'}\Gamma_2\Gamma_{(1)}\Gamma_{(4)}\Gamma_2\Gamma_{2'}\Gamma_{2'}\Gamma_2\Gamma_{(1)}\Gamma_{(4)}\Gamma_2\Gamma_{2'} = \Gamma_{2'}\Gamma_2\Gamma_{(1)}\Gamma_{(4)}\Gamma_{(1)}\Gamma_{(4)}\Gamma_2\Gamma_{2'}$$

from which $[\Gamma_{(1)}, \Gamma_{(4)}] = 1$ follows.

Now, by the seventh part of the table,

$$[\Gamma_{(1)}, \Gamma_5\Gamma_{5'}\Gamma_{(6)}\Gamma_{5'}\Gamma_5] = \Gamma_{(1)}\Gamma_5\Gamma_{5'}\Gamma_{(6)}\Gamma_{5'}\Gamma_5\Gamma_{(1)}\Gamma_5\Gamma_{5'}\Gamma_{(6)}\Gamma_{5'}\Gamma_5$$

but since $[\Gamma_{(1)}, \Gamma_{(5)}] = 1$ we get

$$\Gamma_5\Gamma_{5'}\Gamma_{(1)}\Gamma_{(6)}\Gamma_{5'}\Gamma_5\Gamma_5\Gamma_{5'}\Gamma_{(1)}\Gamma_{(6)}\Gamma_{5'}\Gamma_5 = \Gamma_5\Gamma_{5'}\Gamma_{(1)}\Gamma_{(6)}\Gamma_{(1)}\Gamma_{(6)}\Gamma_5\Gamma_{5'}$$

from which we get $[\Gamma_{(1)}, \Gamma_{(6)}] = 1$.

In the same way, since $\Gamma_{(3)}$ and $\Gamma_{(5)}$ commute we can get $[\Gamma_{(3)}, \Gamma_{(6)}] = 1$ from the relation $[\Gamma_{(3)}, \Gamma_5\Gamma_{5'}\Gamma_{(6)}\Gamma_{5'}\Gamma_5]$ (the eighth part of the table). This finishes the proof of (10). □

5 The homomorphism

$X^{\mathrm{A}} - f^{-1}(S)$ is a degree 6 covering space of $\mathbb{C}^2 - S$. Let

$$: {}_1(\mathbb{C}^2 - S, u_0) \to S_6$$

be the permutation monodromy of this cover. As before let $: \mathbb{C}^2 \to \mathbb{C}$ be a generic projection and choose $u \in \mathbb{C}$ such that S is unramified over u. For surfaces X close to the degenerated X_0 the points i and i' will be close to each other in $\mathbb{C}_u$. Finally choose a point $u_0 \in \mathbb{C}_u$.

We wish to determine what happens to the six preimages $f^{-1}(u_0)$ in X as they follow the lifts of Γ_i and $\Gamma_{i'}$. Again, for surfaces X close to the degenerate X_0 these six points inherit a unique enumeration based on which numbered plane of X_0 they are nearest. This enumeration remains valid all along ${}_i$ so we need only consider the monodromy around $_i$ and $'_i$. Take a small neighborhood $U_i \subset \mathbb{C}_u$ of i and i'. We can reduce the dimension of the question by restricting to $f^{-1}(U_i)$ which is a branched cover of U_i, branched over i and i'. It is clear that $f_0^{-1}(U_i)$ has a simple node over i where two planes containing $\hat{L}_i$ meet. When i divides, the node will factor into two simple branch points over i and i' involving the same sheets which met at the node. Thus we see that if P_k and P_ℓ intersect in $\hat{L}_i$ then $(\Gamma_i) = (\Gamma_{i'}) = (k\ \ell)$. Specifically, is defined by

Definition 5.1 *The map* $: {}_1(\mathbb{C}^2 - S; u_0) \to S_6$ *is given by*

$$
\begin{aligned}
(\Gamma_1) = (\Gamma_{1'}) &= (13);\\
(\Gamma_2) = (\Gamma_{2'}) &= (15);\\
(\Gamma_3) = (\Gamma_{3'}) &= (23);\\
(\Gamma_4) = (\Gamma_{4'}) &= (26);\\
(\Gamma_5) = (\Gamma_{5'}) &= (46);\\
(\Gamma_6) = (\Gamma_{6'}) &= (45).
\end{aligned}
$$

Figure 12 depicts the simplicial complex of X_0 with the planes and intersection lines numbered. From this we can determine the values of on the generators Γ_i and $\Gamma_{i'}$. Figure 13 below gives another graphical representation for , in which Γ_i connects the two vertices $;$ defined by $(\Gamma_i) = (\ \ )$.

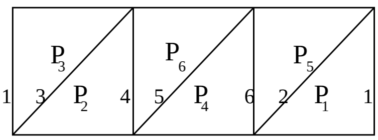


Figure 12

The reader may wish to check that is well defined (testing the relations given in Theorem 4.3), but this is of course guaranteed by the theory. From the definition is clearly surjective.

Since $(\Gamma_j^2) = 1$ and $(\Gamma_{j'}^2) = 1$, also defines a map $\tilde{e}_1 \to S_6$. We will use to denote this map as well. Let A be the kernel of $: \tilde{e}_1 \to S_6$. We have a short exact sequence sequence

$$1 \longrightarrow A \longrightarrow \tilde{e}_1 \longrightarrow S_6 \to 1. \tag{12}$$

Theorem 5.2 [7] *The fundamental group* ${}_1(X_{\mathrm{Gal}}^{\mathrm{A}})$ *is isomorphic to* A, *where* $X_{\mathrm{Gal}}^{\mathrm{A}}$ *is the affine part of* X_{Gal} *the Galois cover of* X *with respect to a generic projection onto* $\mathbb{CP}^2$.

6 A Coxeter subgroup of $\widetilde{\ }_1$

Our next step in identifying the group $\widetilde{\ }_1$ is to study a natural subgroup, which happens to be a Coxeter group.

Let $\ :\widetilde{\ }_1 \to C$ be the map defined by $\ (\Gamma_j) = \ (\Gamma_{j'}) = y_j$. The resulting group $C = \mathrm{Im}(\ )$ is formally defined by the generators $y_1, \ldots, y_6$ and the relations obtained by applying to the relations of $\widetilde{\ }_1$.

Since we have $\ (\Gamma_j) = \ (\Gamma_{j'})$, splits through : defining $\ : C \to S_6$ by $\ (y_j) = \ (\Gamma_j)$, we have that = .

Lemma 6.1 *In terms of the generators y_j, C has the following presentation*

$$\begin{aligned} C = \langle y_1, \ \ , y_6 \mid y_i^2 = 1, \\ & y_1y_3y_1 = y_3y_1y_3,\ y_3y_4y_3 = y_4y_3y_4,\ y_4y_5y_4 = y_5y_4y_5, \\ & y_5y_6y_5 = y_6y_5y_6,\ y_6y_2y_6 = y_2y_6y_2,\ y_2y_1y_2 = y_1y_2y_1, \\ & [y_1, y_4], [y_1, y_5], [y_1, y_6], [y_3, y_5], [y_3, y_6], [y_3, y_2], [y_4, y_6], [y_4, y_2], [y_5, y_2]\rangle. \end{aligned}$$

Proof We only need to apply on the relations of $\widetilde{\ }_1$ given in Theorem 4.3. Each of the relations in $\widetilde{\ }_1$ descends to a relation in C. The branch points all give relations of the form $\Gamma_{3'} = \Gamma_1\Gamma_{1'}\Gamma_3\Gamma_{1'}\Gamma_1$. When we equate $\Gamma_j = \Gamma_{j'}$ these relations vanish. If L_i, L_j intersect, then the relations $\Gamma_{(i)}\Gamma_{(j)}\Gamma_{(i)} = \Gamma_{(j)}\Gamma_{(i)}\Gamma_{(j)}$ descend to $y_iy_jy_i = y_jy_iy_j$. Similarly if the lines $\hat{L}_i, \hat{L}_j$ are disjoint, we get $[y_i, y_j]$. □

This presentation is easier to remember using Figure 13: y_i, y_j satisfy the triple relation if the corresponding lines intersect in a common vertex, and commute otherwise.

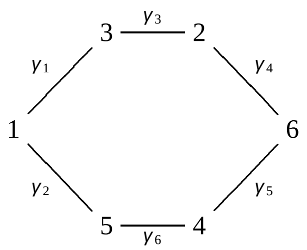


Figure 13

We will use Reidemeister-Schreier method to compute A. For this we need to split .

Lemma 6.2 *is split by the map* $s\colon C \to \widetilde{\ }_1$ *defined by* $\gamma_j \mapsto \Gamma_j$.

Proof From the definition of it is clear that s is the identity on C, so it remains to check that s is well defined. In Lemma 6.1 we gave a presentation of C. To prove that $\gamma_j \mapsto \Gamma_j$ is a splitting we must show that $\{\Gamma_j\}$ also satisfies the relations in $\widetilde{\ }_1$. s respects $\gamma_j^2 = 1$, since $\Gamma_j^2 = 1$ in $\widetilde{\ }_1$. Let i, j be indices such that L_i, L_j intersect, then $\gamma_i\gamma_j\gamma_i = \gamma_j\gamma_i\gamma_j$ is respected since in $\widetilde{\ }_1$ we have $\Gamma_{(i)}\Gamma_{(j)}\Gamma_{(i)} = \Gamma_{(j)}\Gamma_{(i)}\Gamma_{(j)}$ so specifically $\Gamma_i\Gamma_j\Gamma_i = \Gamma_j\Gamma_i\Gamma_j$. Finally if i, j are such that $\hat{L}_i \cap \hat{L}_j =$, $[\gamma_i, \gamma_j]$ is respected since $[\Gamma_{ii'}, \Gamma_{jj'}] = 1$ for disjoint i, j. □

Observe that Lemma 6.1 presents C as a Coxeter group on the generators $\gamma_1, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_2$, with a hexagon (the dual of that shown in Figure 13) as the Coxeter-Dynkin diagram of the group. The fundamental group of this defining graph is of course $\mathbb{Z}$. In previous works on the fundamental groups of Galois covers, the group C defined in a similar manner to what we define here, always happen to be equal to the symmetric group S_n (where n is the number of planes in the degeneration). Here, the map from C to S_6 is certainly not injective (C is known [3] to be the group $S_6 \ltimes \mathbb{Z}^5$, with an action of S_6 on $\mathbb{Z}^5$ by the nontrivial component of the standard representation). The connection of this fact to the fundamental group of S_0 is explained in more details in [2].

It will be useful for us to have a concrete isomorphism of C and $S_6 \ltimes \mathbb{Z}^5$.

Lemma 6.3 $C = S_6 \ltimes \mathbb{Z}^5$ *where* $\mathbb{Z}^5$ *is the nontrivial component of the standard representation.*

Proof First note that $\langle\gamma_2, \ldots, \gamma_6\rangle$ is the parabolic subgroup of C corresponding to the Dynkin diagram of type A_5, so that $C_0 = \langle\gamma_2, \ldots, \gamma_6\rangle = S_6$. We will therefore identify the subgroup C_0 with the symmetric group S_6 (using as the identifying map). Next, note that $(\gamma_1) = (13)$, so we set $x = (13)\gamma_1$, and consider the presentation of C on the new set of generators, namely $x, \gamma_2, \ldots, \gamma_6$. Substituting $\gamma_1 = (13)x$ in the presentation of Lemma 6.1, we obtain $C = \langle x, S_6\rangle$, with the relations

$$\begin{aligned} (13)x(23)(13)x &= (23)(13)x(23), \\ (13)x(15)(13)x &= (15)(13)x(15), \\ (26)x(26) &= x, \\ (46)x(46) &= x, \\ (45)x(45) &= x. \end{aligned}$$

Define $x \quad = \quad x \quad ^{-1}$, then the fact that x commute with $\langle(26),(46),(45)\rangle = S_{\{2,4,5,6\}}$ shows that x actually depends only on $^{-1}(1), \ ^{-1}(3)$. We can thus define $x_{k\ell} = x$ for some $\in \langle y_2, \dots, y_6\rangle$ such that $(k) = 1$ and $(\ell) = 3$ (so in particular $x_{13} = x$). With this definition one checks that $^{-1}x_{k\ell} = x_{(k),(\ell)}$. Adding this last relation as a definition of the $x_{k\ell}$, we obtain the following presentation: $C = \langle x_{k\ell}, S_6\rangle$, with the relations

$$
\begin{aligned}
(13)x_{13}(23)(13)x_{13} &= (23)(13)x_{13}(23),\\
(13)x_{13}(15)(13)x_{13} &= (15)(13)x_{13}(15),\\
{}^{-1}x_{k\ell} &= x_{(k),(\ell)}.
\end{aligned}
$$

Now, the first two relations translate to

$$
\begin{aligned}
x_{32}x_{13} &= x_{12},\\
x_{51}x_{13} &= x_{53},
\end{aligned}
$$

which after conjugating by an arbitrary give

$$
\begin{aligned}
x_{ij}x_{ki} &= x_{kj},\\
x_{ki}x_{ij} &= x_{kj},
\end{aligned}
$$

which shows that $\langle x_{k\ell}\rangle$ is generated by $x_{12}, x_{13}, \dots, x_{16}$ and is commutative (using the fact that the x_{1i} commute). Thus $\langle x_{k\ell}\rangle = \mathbb{Z}^5$, and $C = \mathbb{Z}^5 \rtimes S_6$ is the asserted group. □

The inclusion $S_6 \hookrightarrow C$ defined by sending the transpositions (15), (23), (26), (46) and (45) to $y_2, \dots, y_6$ respectively, splits the projection $: C \to S_6$. From now on we identify S_6 with the subgroup $\langle y_2, \dots, y_6\rangle$ of C, as well as the subgroup $\langle \Gamma_2, \dots, \Gamma_6\rangle$ of $\tilde{}_1$.

Corollary 6.4 *The sequence (12) is split (by the composition of the maps $S_6 \hookrightarrow C$ and $s\colon C \hookrightarrow \tilde{}_1$). We denote the splitting map by .*

7 The kernel of

We use the Reidemeister-Schreier method to find a presentation for the kernel A of the map $: \tilde{}_1 \to S_6$. Let $L = \mathrm{Ker}(\ : \tilde{}_1 \to C)$ and $K = \mathrm{Ker}(\ coC \to S_6)$, and consider the diagram of Figure 14, in which the rows are exact by definition of A and K, and the middle colomn by definition of L. The equality $L = L$ in the diagram follows from the nine lemma. Then, since $1 \to L \to \tilde{}_1 \to C \to 1$ splits (Lemma 6.2), we have that A is a semidirect product of $L = \mathrm{Ker}(\ : \tilde{}_1 \to C)$ and $K = \mathrm{Ker}(\ : C \to S_6)$, which is isomorphic to $\mathbb{Z}^5$ by Lemma 6.3.

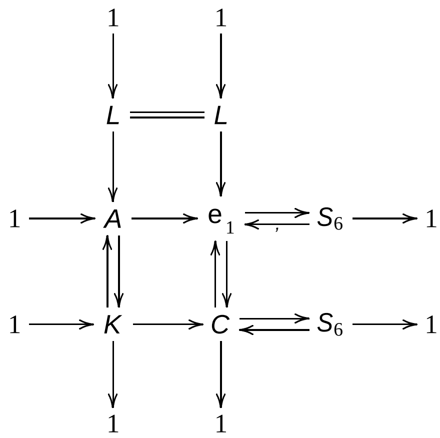


Figure 14

7.1 The Reidemeister-Schreier method

Let

$$1 \to K \to G \to H \to 1$$

be a short exact sequence, split by $: H \to G$. Assume that G is nitely generated, with generators $a_1; \quad ; a_n$. Then (g) is a representative for $g \in G$ in its class modulo H. It is easy to see that K is generated by the elements $ga_i(\quad (ga_i))^{-1}$, $1 \quad i \quad n$, $g \in \quad (H)$. Now, $ga_i(\quad (ga_i))^{-1} = ga_i(\quad (a_i))^{-1}(\quad (g))^{-1} = ga_i(\quad (a_i))^{-1}g^{-1}$, because is the identity on H. For $g \in \quad (H)$, denote the generators above by

$$\gamma(g; a_i) = ga_i(\quad (a_i))^{-1}g^{-1}.$$

The relations of G can be translated into expressions in these generators by the following process. If the word $! = a_{i_1} \quad a_{i_t}$ represents an element of K then $!$ can be rewritten as the product

$$(!) = \gamma(1; a_{i_1})\gamma(\quad (a_{i_1}); a_2) \quad \gamma(\quad (a_{i_1} \quad a_{i_{t-1}}); a_{i_t}).$$

Theorem 7.1 (Reidmeister-Schreier) *Let $\{R\}$ be a complete set of relations for G. Then $K = \text{Ker}(\ )$ is generated by the $\gamma(g; a_i)$ $(1 \quad i \quad n, g \in H)$, with the relations $\{ \ (trt^{-1})\}_{r \in R; t \in \ (H)}$.*

We will use this method to investigate $L = \ker$ and $A = \ker$.

7.2 Generators for L = Ker

For $c \in \mathrm{Im}(s) = \langle \Gamma_1, \ldots, \Gamma_6 \rangle$, we let

$$A_{c;j} = c\Gamma_j \Gamma_{j'} c^{-1}. \tag{13}$$

We start with the following:

Corollary 7.2 *The group* $L = \mathrm{Ker}(\ )$ *is generated by* $\{A_{c;j}\}_{1 \le j \le 6; c \in C}$.

Proof By Theorem 7.1, L is generated by elements of the form $c\Gamma_j (s\ (\Gamma_j))^{-1} c^{-1}$ and $c\Gamma_{j'} (s\ (\Gamma_{j'}^{-1})) c^{-1}$ for all $1 \le j \le 6$ and $c \in C$. We compute $s\ (\Gamma_j) = s(\gamma_j) = \Gamma_j$ and $s\ (\Gamma_{j'}) = s(\gamma_j) = \Gamma_j$, so the generators are $c\Gamma_j \Gamma_j^{-1} c^{-1} = I$ and $c\Gamma_{j'} \Gamma_j^{-1} c^{-1} = c\Gamma_{j'} \Gamma_j c^{-1}$ using $\Gamma_j^2 = I$. □

This set of generators is highly redundant as we shall later see, but for now we turn our attention to A = Ker .

7.3 Generators for A = Ker

By Theorem 7.1, A is generated by the elements $\ \Gamma_j(\varphi\ (\Gamma_j))^{-1}\ ^{-1}$ and $\ \Gamma_{j'}(\varphi\ (\Gamma_{j'}))^{-1}\ ^{-1}$, $1 \le j \le 6$, $\ \in S_6$. Again we compute $\varphi\ (\Gamma_j)$ and $\varphi\ (\Gamma_{j'})$. Recall that $\langle \Gamma_2, \ldots, \Gamma_6 \rangle = S_6$ is the image of φ (Corollary 6.4), so for $j \ne 1$ we get $\varphi\ (\Gamma_j) = \varphi\ (\Gamma_{j'}) = \Gamma_j$, and the generators are

$$A_{\ ;j} = \ \Gamma_j \Gamma_{j'}\ ^{-1}. \tag{14}$$

This agrees with our previous definition of $A_{c;j}$ for $c \in S_6$, see Equation (13).

The permutation $(1\,3)$ can be expressed in terms of the generators of S_6 corresponding to $\Gamma_2, \ldots, \Gamma_6$ as follows:

$$(1\,3) = (1\,5)(5\,4)(4\,6)(2\,6)(2\,3)(2\,6)(4\,6)(5\,4)(1\,5),$$

so for $j = 1$ we have that

$$\varphi\ (\Gamma_1) = \varphi((15)(54)(46)(62)(23)\ \ (15)) = \Gamma_2\Gamma_6\Gamma_5\Gamma_4\Gamma_3\Gamma_4\Gamma_5\Gamma_6\Gamma_2.$$

Likewise, $\varphi\ (\Gamma_1) = \varphi\ (\Gamma_{1'})$ since $\ (\Gamma_j) = \ (\Gamma_{j'})$. So we get generators

$$X = \ \Gamma_2\Gamma_6\Gamma_5\Gamma_4\Gamma_3\Gamma_4\Gamma_5\Gamma_6\Gamma_2\Gamma_1\ ^{-1} \tag{15}$$

$$B = \ \Gamma_2\Gamma_6\Gamma_5\Gamma_4\Gamma_3\Gamma_4\Gamma_5\Gamma_6\Gamma_2\Gamma_{1'}\ ^{-1}. \tag{16}$$

Since $X^{-1}B = \ \Gamma_1\Gamma_{1'}\ ^{-1} = A_{\ ;1}$, we have the following result:

Corollary 7.3 *The group* $A = \mathrm{Ker}(\ )$ *is generated by* $A_{\ ;j}$, X , *for* $\ 2\ S_6$ *and* $j = 1, \ldots, 6$.

Notice that we are now conjugating only by permutations $\ 2\ S_6$ instead of all elements $c\ 2\ C$ (as in Corollary 7.2) so this is a nite set of generators.

8 A better set of generators for A

We rst show that $A_{\ ;j}$ are not needed for $j = 2, \ldots, 6$

Theorem 8.1 A *is generated by* $\{A_{\ ;1}, X\ \}$.

Proof This follows immediately from the relations proven below. □

Table 8.2 *We have the following relations:*

$$A_{\ ;3} = A_{\ (23);1} A^{-1}_{\ ;1} \tag{17}$$
$$A_{\ ;3} = A_{\ (26)(23);4} \tag{18}$$
$$A_{\ ;5} = A_{\ (26)(46);4} \tag{19}$$
$$A_{\ ;5} = A_{\ (45)(46);6} \tag{20}$$
$$A_{\ ;2} = A_{\ (45)(15);6} \tag{21}$$
$$A_{\ ;2} = A_{\ (15);1} A^{-1}_{\ ;1}. \tag{22}$$

Proof We use the relations of Theorem 4.3. Let I denote the identity element of S_6, so that by de nition $A_{I;j} = \Gamma_j \Gamma_{j'}$. From (4) we have $1 = \Gamma_{3'}\Gamma_1\Gamma_{1'}\Gamma_3\Gamma_{1'}\Gamma_1 = (\Gamma_{3'}\Gamma_3)(\Gamma_3\Gamma_1\Gamma_{1'}\Gamma_3)(\Gamma_{1'}\Gamma_1) = A^{-1}_{I;3}A_{(23);1}A^{-1}_{I;1}$ so we get $A_{I;3} = A_{(23);1}A^{-1}_{I;1}$.

From (5) we have $1 = \Gamma_{5'}\Gamma_4\Gamma_{4'}\Gamma_5\Gamma_{4'}\Gamma_4 = (\Gamma_{5'}\Gamma_5)\Gamma_5\Gamma_4\Gamma_{4'}\Gamma_5\Gamma_{4'}\Gamma_4$
$= (\Gamma_{5'}\Gamma_5)\Gamma_5\Gamma_4\Gamma_5\Gamma_{4'}\Gamma_5\Gamma_4 = (\Gamma_{5'}\Gamma_5)\Gamma_4\Gamma_5\Gamma_4\Gamma_{4'}\Gamma_5\Gamma_4 = A^{-1}_{I;5}A_{(26)(46);4}$.

From (6) we have $1 = \Gamma_2\Gamma_{6'}\Gamma_6\Gamma_{2'}\Gamma_6\Gamma_{6'} = (\Gamma_2\Gamma_{6'}\Gamma_6\Gamma_2)(\Gamma_2\Gamma_{2'})(\Gamma_6\Gamma_{6'})$ so we get $A_{I;2} = \Gamma_2\Gamma_{2'} = \Gamma_2\Gamma_6\Gamma_{6'}\Gamma_2\Gamma_{6'}\Gamma_6 = \Gamma_2\Gamma_6\Gamma_2\Gamma_{6'}\Gamma_2\Gamma_6 = \Gamma_6\Gamma_2\Gamma_6\Gamma_{6'}\Gamma_2\Gamma_6 = A_{(45)(15);6}$.

From (7) we have $1 = \Gamma_{2'}\Gamma_1\Gamma_{1'}\Gamma_2\Gamma_{1'}\Gamma_1 = (\Gamma_{2'}\Gamma_2)(\Gamma_2\Gamma_1\Gamma_{1'}\Gamma_2)(\Gamma_{1'}\Gamma_1)$
$= A^{-1}_{I;2}A_{(15);1}A^{-1}_{I;1}$ so we get $A_{I;2} = A_{(15);1}A^{-1}_{I;1}$.

From (8) we have $1 = \Gamma_3\Gamma_{4'}\Gamma_4\Gamma_{3'}\Gamma_4\Gamma_{4'} = (\Gamma_3\Gamma_{4'}\Gamma_4\Gamma_3)(\Gamma_3\Gamma_{3'})(\Gamma_4\Gamma_{4'})$ so we get $\Gamma_3\Gamma_{3'} = \Gamma_3\Gamma_4\Gamma_{4'}\Gamma_3\Gamma_{4'}\Gamma_4 = \Gamma_3\Gamma_4\Gamma_3\Gamma_{4'}\Gamma_3\Gamma_4 = \Gamma_4\Gamma_3\Gamma_4\Gamma_{4'}\Gamma_3\Gamma_4 = A_{(26)(23);4}$.

Finally from (9) we have $1 = \Gamma_5\Gamma_{6'}\Gamma_6\Gamma_{5'}\Gamma_6\Gamma_{6'} = (\Gamma_5\Gamma_{6'}\Gamma_6\Gamma_5)(\Gamma_5\Gamma_{5'})(\Gamma_6\Gamma_{6'})$ so we get $\Gamma_5\Gamma_{5'} = \Gamma_5\Gamma_6\Gamma_{6'}\Gamma_5\Gamma_{6'}\Gamma_6 = \Gamma_5\Gamma_6\Gamma_5\Gamma_{6'}\Gamma_5\Gamma_6 = \Gamma_6\Gamma_5\Gamma_6\Gamma_{6'}\Gamma_5\Gamma_6 = A_{(45)(46),6}$. □

One may be tempted to use $(\Gamma_3\Gamma_1\Gamma_{1'}\Gamma_3)(\Gamma_{1'}\Gamma_1) = \Gamma_1\Gamma_3\Gamma_1\Gamma_{1'}\Gamma_3\Gamma_1$ in a similar manner to the other cases, to rewrite $A_{(23),1}A^{-1}_{,1}$ of Equation (17) as a single element of the form $A_{,1}$; however note that in the definition (14) we require $\in S_6 = \langle\Gamma_2, \ldots, \Gamma_6\rangle$, so we do not have the equality $\Gamma_1\Gamma_3\Gamma_1\Gamma_{1'}\Gamma_3\Gamma_1 = A_{(23)(13),1}$. The same remark applies for Equation (22).

Iterating the relations of Table 8.2, we obtain a new relation for $\langle A_{,1}\rangle$:

$$\begin{aligned}
A_{(23),1}A^{-1}_{,1} &= A_{,3}\\
&= A_{(26)(23),4}\\
&= A_{(26)(23)(46)(26),5}\\
&= A_{(26)(23)(46)(26)(45)(46),6}\\
&= A_{(26)(23)(46)(26)(45)(46)(15)(45),2}\\
&= A_{(36)(24)(56)(14),2}\\
&= A_{(36)(24)(56)(14)(15),1}A^{-1}_{(36)(24)(56)(14),1}
\end{aligned}$$

which may be rewritten as

$$A_{(23),1}A^{-1}_{,1} = A_{(142563),1}A^{-1}_{(142)(356),1} \tag{23}$$

Lemma 8.3 *For every $\in S_6$, X depends only on $^{-1}(1)$ and $^{-1}(3)$.*

Proof Viewing $S_6 \subset C$ as subgroups of e_1 (using the embedding $s\colon C \to e_1$), the elements X belong to $C = \langle\Gamma_1, \Gamma_2, \ldots, \Gamma_6\rangle$ by their definition (15).

Applying the isomorphism $C = S_6 \ltimes \mathbb{Z}^5$ of Lemma 6.3, we see that

$$\begin{aligned}
X &= \Gamma_2\Gamma_6\Gamma_5\Gamma_4\Gamma_3\Gamma_4\Gamma_5\Gamma_6\Gamma_2\Gamma_1{}^{-1}\\
&= s((15)(54)(46)(62)(23)(62)(46)(54)(15)(13)x_{13}{}^{-1})\\
&= s((13)(13)x_{13}{}^{-1})\\
&= s(x_{13}{}^{-1}) = s(x_{{}^{-1}(1)\,{}^{-1}(3)}).
\end{aligned}$$

□

A similar result holds for $\{A_{,1}\}$.

Lemma 8.4 *For every $\in S_6$, $A_{,1}$ depends only on $^{-1}(1)$ and $^{-1}(3)$.*

Proof If $\;_1^{-1}(1) = \;_2^{-1}(1)$ and $\;_1^{-1}(3) = \;_2^{-1}(3)$, then $\; = \;_1^{-1}\;_2$ stabilizes $1, 3$, so $\; \in S_{\{2,6,4,5\}} = \langle\Gamma_4, \Gamma_5, \Gamma_6\rangle$ which commute with both Γ_1 and $\Gamma_{1'}$. By definition (14), $A_{\;_2;1} = \;_2\Gamma_1\Gamma_{1'}\;_2^{-1} = \;_1\;\Gamma_1\Gamma_{1'}\;^{-1}\;_1^{-1} = \;_1\Gamma_1\Gamma_{1'}\;_1^{-1} = A_{\;_1;1}$. □

We can thus define

Definition 8.5 *For $k, \ell = 1, \ldots, 6$, $A_{k\ell}$ and $X_{k\ell}$ are defined by*

$$X_{k\ell} = \;\Gamma_2\Gamma_6\Gamma_5\Gamma_4\Gamma_3\Gamma_4\Gamma_5\Gamma_6\Gamma_2\Gamma_1\;^{-1} \quad (24)$$

$$A_{k\ell} = \;\Gamma_1\Gamma_{1'}\;^{-1} \quad (25)$$

where $\; \in S_6 = \langle\Gamma_2, \ldots, \Gamma_6\rangle$ is any permutation such that $\;(k) = 1$ and $\;(\ell) = 3$.

We need to know the action of S_6 on these generators:

Proposition 8.6 *For every $\; \in S_6 = \langle\Gamma_2, \ldots, \Gamma_6\rangle$ and $k, \ell = 1, \ldots, 6$, we have that*

$$\;^{-1}A_{k\ell}\; = A_{\;(k),\;(\ell)} \quad (26)$$

$$\;^{-1}X_{k\ell}\; = X_{\;(k),\;(\ell)} \quad (27)$$

Proof Let $\; \in S_6$ be such that $\;(k) = 1$ and $\;(\ell) = 3$. Since $A_{13} = \Gamma_1\Gamma_{1'}$ by Definition 8.5, we have $A_{k\ell} = \;A_{13}\;^{-1}$ and $\;^{-1}A_{k\ell}\; = \;^{-1}\;A_{13}\;^{-1}\; = A_{\;^{-1}\;(1),\;^{-1}\;(3)} = A_{\;(k)\;(\ell)}$. The same proof works for the $X_{k\ell}$. □

Note that $B_{k\ell} = X_{k\ell}A_{k\ell}$ can be defined in the same manner, and have the same S_6-action.

From Theorem 8.1 (with a little help from Lemma 8.3 and Lemma 8.4), we obtain

Corollary 8.7 *The group A is generated by $\{A_{k\ell}, X_{k\ell}\}_{1 \leq k,\ell \leq 6}$.*

Since $\;(X_{k\ell}) = x_{k\ell}$, we already proved

Corollary 8.8 *The elements $\{\;(X_{k\ell})\}$ generate $K = \mathrm{Ker}(\; : C \to S_6)$.*

In the new language, Equation (23) (for $= I$) can be written as $A_{12}A_{13}^{-1} = A_{36}A_{26}^{-1}$, so conjugating we get

$$A_{ij}A_{ik}^{-1} = A_{k\ell}A_{j\ell}^{-1} \tag{28}$$

for any four distinct indices i, j, k, ℓ. Using three consecutive applications of the relation (28) we can also allow $i = \ell$, and using just two applications we can get:

$$A_{ij}A_{ik}^{-1} = A_{\ell j}A_{\ell k}^{-1} \tag{29}$$

for any distinct indices i, j, k and $\ell \neq j, k$. In view of (28) and (29) and Table 8.2 we can write a translation table for the remaining generators $A_{;j}$ for $j \neq 1$.

Table 8.9 $A_{;j}$ *in terms of* $A_{k\ell}$

$A_{;1}$	$=$	$A_{13}{}^{-1}$					(30)
$A_{;2}$	$=$	$A_{x1}A_{x5}^{-1}{}^{-1}$	$=$	$A_{5x}A_{1x}^{-1}{}^{-1}$	*where* $x \neq 1, 5$		(31)
$A_{;3}$	$=$	$A_{x2}A_{x3}^{-1}{}^{-1}$	$=$	$A_{3x}A_{2x}^{-1}{}^{-1}$	*where* $x \neq 2, 3$		(32)
$A_{;4}$	$=$	$A_{x6}A_{x2}^{-1}{}^{-1}$	$=$	$A_{2x}A_{6x}^{-1}{}^{-1}$	*where* $x \neq 2, 6$		(33)
$A_{;5}$	$=$	$A_{x4}A_{x6}^{-1}{}^{-1}$	$=$	$A_{6x}A_{4x}^{-1}{}^{-1}$	*where* $x \neq 4, 6$		(34)
$A_{;6}$	$=$	$A_{x5}A_{x4}^{-1}{}^{-1}$	$=$	$A_{4x}A_{5x}^{-1}{}^{-1}$	*where* $x \neq 4, 5$		(35)

The indices 1 and 5 which appear in the formula for $A_{;2}$ arise because $(\Gamma_2) = (15)$. The conjugations by change the indices as in equation (26). Similar for $A_{;3}$ $A_{;6}$.

We have reduced the generating set for A to $\{X_{k\ell}, A_{k\ell}\}_{k\neq\ell}$ and we know that the subgroup $K = \langle X_{k\ell}\rangle = \mathbb{Z}^5$. Now we use the Reidmeister-Schreier rewriting process to translate all of the relations of $\tilde{}_1$. From now on, we denote $g = {}'(g)$ for every $g \in \tilde{}_1$. Using the notation of Subsection 7.1, $\gamma(;\Gamma_j) = I$ and $\gamma(;\Gamma_{j'}) = A_{;j}^{-1}$ for $j \neq 1$. For $j = 1$, $\gamma(;\Gamma_1) = X^{-1}$, and $\gamma(;\Gamma_{1'}) = B^{-1}$. We begin by translating some of the relations which involve Γ_1 but not $\Gamma_{1'}$. These will yield the relations among the $X_{k\ell}$ which we already know, but the exercise is useful nonetheless because other elements will satisfy identical sets of relations.

$\Gamma_1\Gamma_1 \mapsto \gamma(I;\Gamma_1)\gamma(\overline{\Gamma_1};\Gamma_1) = X_I^{-1}X_{(13)}^{-1} = X_{13}^{-1}X_{31}^{-1}$, so we deduce that $X_{31} = X_{13}^{-1}$ and conjugating we get

$$X_{\ell k} = X_{k\ell}^{-1}. \tag{36}$$

The relations $[\Gamma_1, \Gamma_4]$, $[\Gamma_1, \Gamma_5]$, and $[\Gamma_1, \Gamma_6]$ in $\tilde{}_1$ produce the same relations on $\langle X_{k\ell}\rangle$.

Now we translate the triple relations (11) for i, j adjacent. We start for example with $\Gamma_1\Gamma_2\Gamma_1\Gamma_2\Gamma_1\Gamma_2$. Later we will continue with $\Gamma_{1'}\Gamma_2\Gamma_{1'}\Gamma_2\Gamma_{1'}\Gamma_2$ and finish with $(\Gamma_{1'}\Gamma_1\Gamma_{1'})\Gamma_2(\Gamma_{1'}\Gamma_1\Gamma_{1'})\Gamma_2(\Gamma_{1'}\Gamma_1\Gamma_{1'})\Gamma_2$. For these two indices, there is no need to use any more relations from $\Gamma_{(1)}\Gamma_{(2)}\Gamma_{(1)} = \Gamma_{(2)}\Gamma_{(1)}\Gamma_{(2)}$, since the Invariance Theorem 4.2 showed that all of these relations were consequences of the three above.

The relation $\Gamma_1\Gamma_2\Gamma_1\Gamma_2\Gamma_1\Gamma_2$ translates through to the expression

$$\gamma(I;\Gamma_1)\gamma(\overline{\Gamma_1};\Gamma_2)\gamma(\overline{\Gamma_1\Gamma_2};\Gamma_1)\gamma(\overline{\Gamma_1\Gamma_2\Gamma_1};\Gamma_2)\gamma(\overline{\Gamma_1\Gamma_2\Gamma_1\Gamma_2};\Gamma_1)\gamma(\overline{\Gamma_1\Gamma_2\Gamma_1\Gamma_2\Gamma_1};\Gamma_2).$$

But since $\gamma(\ ;\Gamma_2) = I$ we get $\gamma(I;\Gamma_1)\gamma(\overline{\Gamma_1\Gamma_2};\Gamma_1)\gamma(\overline{\Gamma_1\Gamma_2\Gamma_1\Gamma_2};\Gamma_1)$, and since $\overline{\Gamma_1\Gamma_2} = (13)(15) = (135)$ we can further simplify to

$$X_I^{-1}X_{(135)}^{-1}X_{(531)}^{-1} = X_{13}^{-1}X_{51}^{-1}X_{35}^{-1}.$$

Thus $X_{35}X_{51}X_{13} = 1$ and including all conjugates

$$X_{k\ell}X_{\ell m}X_{mk} = 1. \tag{37}$$

Similarly the relation $\Gamma_1\Gamma_3\Gamma_1\Gamma_3\Gamma_1\Gamma_3$ translates through to the expression

$$\gamma(I;\Gamma_1)\gamma(\overline{\Gamma_1};\Gamma_3)\gamma(\overline{\Gamma_1\Gamma_3};\Gamma_1)\gamma(\overline{\Gamma_1\Gamma_3\Gamma_1};\Gamma_3)\gamma(\overline{\Gamma_1\Gamma_3\Gamma_1\Gamma_3};\Gamma_1)\gamma(\overline{\Gamma_1\Gamma_3\Gamma_1\Gamma_3\Gamma_1};\Gamma_3)$$

which equals $X_I^{-1}X_{(123)}^{-1}X_{(321)}^{-1} = X_{13}^{-1}X_{32}^{-1}X_{21}^{-1}$. Thus $X_{21}X_{32}X_{13} = 1$, and conjugating we obtain

$$X_{\ell m}X_{k\ell}X_{mk} = 1. \tag{38}$$

Together the relations (36)–(38) show that $\langle X_{\ell m}\rangle$ is generated by the five elements $X_{12},\ldots,X_{16}$ which will commute, so that $\langle X_{k\ell}\rangle = \mathbb{Z}^5$. These are precisely the relations we expected among the $X_{k\ell}$ and no more.

We continue with some of the relations of $\tilde{G}_1$ which involve $\Gamma_{1'}$ but not Γ_1. These yield identical relations among the $B_{k\ell}$.

$\Gamma_{1'}\Gamma_{1'} \mapsto \gamma(I;\Gamma_{1'})\gamma(\overline{\Gamma_{1'}};\Gamma_{1'}) = B_I^{-1}B_{(13)}^{-1} = B_{13}^{-1}B_{31}^{-1}$. Thus $B_{31} = B_{13}^{-1}$ and by all conjugations $B_{\ell k} = B_{k\ell}^{-1}$.

The relations $[\Gamma_{1'},\Gamma_4]$, $[\Gamma_{1'},\Gamma_5]$, and $[\Gamma_{1'},\Gamma_6]$ produce the same relations on $\langle B\ \rangle$.

The relation $\Gamma_{1'}\Gamma_2\Gamma_{1'}\Gamma_2\Gamma_{1'}\Gamma_2$ translates through to the expression

$$\gamma(I;\Gamma_{1'})\gamma(\overline{\Gamma_{1'}};\Gamma_2)\gamma(\overline{\Gamma_{1'}\Gamma_2};\Gamma_{1'})\gamma(\overline{\Gamma_{1'}\Gamma_2\Gamma_{1'}};\Gamma_2)\gamma(\overline{\Gamma_{1'}\Gamma_2\Gamma_{1'}\Gamma_2};\Gamma_{1'})\gamma(\overline{\Gamma_{1'}\Gamma_2\Gamma_{1'}\Gamma_2\Gamma_{1'}};\Gamma_2).$$

But since $\gamma(\ ;\Gamma_2) = I$ we get $\gamma(I;\Gamma_{1'})\gamma(\overline{\Gamma_{1'}\Gamma_2};\Gamma_{1'})\gamma(\overline{\Gamma_{1'}\Gamma_2\Gamma_{1'}\Gamma_2};\Gamma_{1'})$, and since $\overline{\Gamma_{1'}\Gamma_2} = (13)(15) = (135)$ we further simplify to $B_I^{-1}B_{(135)}^{-1}B_{(531)}^{-1} =$

$B_{13}^{-1}B_{51}^{-1}B_{35}^{-1}$. Thus $B_{35}B_{51}B_{13} = 1$ and including all conjugations we have $B_{k\ell}B_{\ell m}B_{mk} = 1$. Similarly the relation $\Gamma_{1'}\Gamma_3\Gamma_{1'}\Gamma_3\Gamma_{1'}\Gamma_3$ translates through to the expression $B_{13}^{-1}B_{32}^{-1}B_{21}^{-1}$. Thus $B_{21}B_{32}B_{13} = 1$ and including all conjugations $B_{\ell m}B_{k\ell}B_{mk} = 1$. By the arguments applied above for the $X_{k\ell}$, we also have that $\langle B_{k\ell}\rangle = \mathbb{Z}^5$ generated by B_{1k}.

We finish with the last necessary triple relations. Note that $\overline{\Gamma_{1'}\Gamma_1\Gamma_{1'}} = \overline{\Gamma}_1$ and $(\Gamma_{1'}\Gamma_1\Gamma_{1'}) = B_l^{-1}X_{(13)}^{-1}B_l^{-1} = B_{13}^{-1}X_{31}^{-1}B_{13}^{-1} = B_{13}^{-1}X_{13}B_{13}^{-1}$. So if we define $C_{k\ell}$ to be $B_{k\ell}X_{k\ell}^{-1}B_{k\ell} = X_{k\ell}A_{k\ell}^2$ then the additional relations are $C_{\ell k} = C_{k\ell}^{-1}$, $C_{k\ell}C_{\ell m}C_{mk} = 1$, and $C_{\ell m}C_{k\ell}C_{mk} = 1$. By the arguments above, the $\{C_{k\ell}\}$ generate another copy $\mathbb{Z}^5$ A. In fact for each exponent n the elements $X_{k\ell}A_{k\ell}^n = B_{k\ell}X_{k\ell}^{-1}$ $X_{k\ell}^{-1}B_{k\ell}$ or $A_{\ell k}^n = X_{k\ell}B_{k\ell}^{-1}$ $B_{k\ell}^{-1}X_{k\ell}$ generate a subgroup isomorphic to $\mathbb{Z}^5$.

The relations computed thus far turn out to be all of the relations in $\pi_1(X_{\mathrm{Gal}}^{\mathrm{A}})$. Once we show that the remaining relations translated from e_1 are consequences of the relations above we will have proven the following theorem:

Theorem 8.10 *The fundamental group* $\pi_1(X_{\mathrm{Gal}}^{\mathrm{A}})$ *is generated by elements* $\{X_{ij}, A_{ij}\}$ *with the relations*

$$X_{ji}A_{ji}^n = (X_{ij}A_{ij}^n)^{-1}, \tag{39}$$
$$(X_{ij}A_{ij}^n)(X_{jk}A_{jk}^n)(X_{ki}A_{ki}^n) = 1, \tag{40}$$
$$(X_{jk}A_{jk}^n)(X_{ij}A_{ij}^n)(X_{ki}A_{ki}^n) = 1, \tag{41}$$
$$A_{ij}A_{ik}^{-1} = A_{k\ell}A_{j\ell}^{-1} \tag{42}$$

for every $n \in \mathbb{Z}$.

Before we show that the remaining relations are redundant we prove that the relations above imply that some of the $A_{k\ell}$ commute. We shall frequently use the fact that $X_{k\ell}X_{\ell m} = X_{\ell m}X_{k\ell} = X_{km}$ which is a consequence of (36)–(38). $B_{k\ell}$ and $C_{k\ell}$ satisfy this as well.

Lemma 8.11 *In* $\pi_1(X_{\mathrm{Gal}}^{\mathrm{A}})$ *we have* $[A_{ij}, A_{ik}] = 1$ *and* $[A_{ji}, A_{ki}] = 1$ *for distinct* i, j, k.

Proof Starting with $C_{ki}C_{jk}C_{ij} = 1$ and use the definition of C_{ij} to rewrite it as $(B_{ki}X_{ik}B_{ki})(B_{jk}X_{kj}B_{jk})(B_{ij}X_{ji}B_{ij}) = B_{ki}X_{ik}B_{ji}X_{kj}B_{ik}X_{ji}B_{ij} =$ $(B_{ki}X_{ik})(B_{ji}X_{ij})(X_{ki}B_{ik})(X_{ji}B_{ij}) = A_{ik}^{-1}A_{ij}^{-1}A_{ik}A_{ij} = 1$. Thus the commutator $[A_{ij}, A_{ik}] = 1$. The relation (28) can be used to show that $[A_{ji}, A_{ki}] = 1$ as well. □

Now we treat the remainder of the relations in e_1. For $j \neq 1$ the relation $\Gamma_j \Gamma_j$ translates immediately to the null relation. Next consider $\Gamma_{j'}\Gamma_{j'}$.

$\Gamma_{3'}\Gamma_{3'} \mapsto A_{I,3}^{-1}A_{(23),3}^{-1}$. But taking the inverse and using Table 8.9 we get $A_{(23),3}A_{I,3} = (A_{x3}A_{x2}^{-1})(A_{x2}A_{x3}^{-1})$ which cancels completely. Identical computations treat all other values of j.

$\Gamma_1\Gamma_{4'}\Gamma_1\Gamma_{4'} \mapsto X_I^{-1}A_{(13),4}^{-1}X_{(13)(26)}^{-1}A_{(26),4}^{-1}$ and taking the inverse we get $A_{(26),4}X_{(13)(26)}A_{(13),4}X_I$. Using Table 8.9 we have $A_{12}A_{16}^{-1}X_{31}A_{36}A_{32}^{-1}X_{13} = (X_{21}B_{12})(B_{61}X_{16})X_{31}(X_{63}B_{36})(B_{23}X_{32})X_{13} = X_{21}B_{62}B_{26}X_{12} = 1$. Similar calculations show that $[\Gamma_1, \Gamma_{5'}] = 1$ and $[\Gamma_1, \Gamma_{6'}] = 1$ are also redundant.

$\Gamma_{1'}\Gamma_{4'}\Gamma_{1'}\Gamma_{4'} \mapsto B_I^{-1}A_{(13),4}^{-1}B_{(13)(26)}^{-1}A_{(26),4}^{-1}$ and taking the inverse we get $A_{(26),4}B_{(13)(26)}A_{(13),4}B_I$. Using Table 8.9 we have $A_{12}A_{16}^{-1}B_{31}A_{36}A_{32}^{-1}B_{13}$. Now, by Lemma 8.11 we can commute A_{12} and A_{16} as well as A_{36} and A_{32} to get $A_{16}^{-1}A_{12}B_{31}A_{32}^{-1}A_{36}B_{13} = (B_{61}X_{16})(X_{21}B_{12})B_{31}(B_{23}X_{32})(X_{63}B_{36})B_{13} = B_{61}X_{26}X_{62}B_{16} = 1$. Again, similar calculations work for $[\Gamma_{1'}, \Gamma_{5'}] = 1$ and $[\Gamma_{1'}, \Gamma_{6'}] = 1$.

For non-adjacent $i, j \neq 1$ the relation $[\Gamma_i, \Gamma_j]$ translates directly to the null relation. So next we treat $[\Gamma_{i'}, \Gamma_j]$.

$\Gamma_{2'}\Gamma_3\Gamma_{2'}\Gamma_3 \mapsto A_{I,2}^{-1}A_{(15)(23),2}^{-1}$ and inverting we have $A_{(15)(23),2}A_{I,3} = (A_{x5}A_{x1}^{-1})(A_{x1}A_{x5}^{-1}) = 1$. The same happens for every other non-adjacent pair $i, j \neq 1$.

$\Gamma_{2'}\Gamma_{3'}\Gamma_{2'}\Gamma_{3'} \mapsto A_{I,2}^{-1}A_{(15),3}^{-1}A_{(15)(23),2}^{-1}A_{(23),3}^{-1}$ and taking inverses again we get $A_{(23),3}A_{(15)(23),2}A_{(15),3}A_{I,2} = (A_{x3}A_{x2}^{-1})(A_{1y}A_{5y}^{-1})(A_{z2}A_{z3}^{-1})(A_{5w}A_{1w}^{-1})$. Substituting speci c values $x = 1$, $y = 2$, $z = 5$, and $w = 3$ we get $A_{13}A_{12}^{-1}A_{12}A_{52}^{-1}A_{52}A_{53}^{-1}A_{53}A_{13}^{-1} = 1$. Identical arguments work for every other non-adjacent $i, j \neq 1$.

All that remains are the triple relations for $i, j \neq 1$. As before we need only three such relations for each pair of indices. The relation $\Gamma_i\Gamma_j\Gamma_i\Gamma_j\Gamma_i\Gamma_j$ translates trivially, so we begin with $\Gamma_{i'}\Gamma_j\Gamma_{i'}\Gamma_j\Gamma_{i'}\Gamma_j$.

$\Gamma_{4'}\Gamma_3\Gamma_{4'}\Gamma_3\Gamma_{4'}\Gamma_3 \mapsto A_{I,4}^{-1}A_{(632),4}^{-1}A_{(236),4}^{-1}$ and taking the inverse we get $A_{(236),4}A_{(632),4}A_{I,4} = (A_{6x}A_{3x}^{-1})(A_{3x}A_{2x}^{-1})(A_{2x}A_{6x}^{-1}) = 1$.

Finally consider $\Gamma_4\Gamma_{4'}\Gamma_4\Gamma_3\Gamma_4\Gamma_{4'}\Gamma_4\Gamma_3\Gamma_4\Gamma_{4'}\Gamma_4\Gamma_3 \mapsto A_{(26),4}^{-1}A_{(36),4}^{-1}A_{(23),4}^{-1}$ and $A_{(23),4}A_{(36),4}A_{(26),4} = (A_{x6}A_{x3}^{-1})(A_{x3}A_{x2}^{-1})(A_{x2}A_{x6}^{-1}) = 1$. So all of the relations in e_1 are included in Theorem 8.10.

9 The Projective Relation

To complete the computation of $\pi_1(X_{\mathrm{Gal}})$ we need only to add the projective relation

$$\Gamma_1\Gamma_{1'}\Gamma_2\Gamma_{2'}\Gamma_3\Gamma_{3'}\Gamma_4\Gamma_{4'}\Gamma_5\Gamma_{5'}\Gamma_6\Gamma_{6'} = 1.$$

This relation translates in A as the product $P = A_{I,1}A_{I,2}A_{I,3}A_{I,4}A_{I,5}A_{I,6}$. We must translate the $A_{I,j}$ to the language of the $A_{k\ell}$, using Table 8.9:

P translates to $A_{13}(A_{21}A_{25}^{-1})(A_{31}A_{21}^{-1})(A_{21}A_{61}^{-1})(A_{61}A_{41}^{-1})(A_{41}A_{51}^{-1})$ which cancels to $A_{13}A_{21}A_{25}^{-1}A_{31}A_{51}^{-1}$. Using Equation (28), we get $A_{13}A_{21}A_{25}^{-1}A_{25}A_{23}^{-1}$. Thus the projective relation may be written as $A_{13}A_{21}A_{23}^{-1} = 1$ or equivalently $A_{23} = A_{13}A_{21}$. Conjugating, this becomes

$$A_{ij} = A_{kj}A_{ik}. \tag{43}$$

Substituting back into (28), writing $A_{ij} = A_{kj}A_{ik}$ and $A_{k\ell} = A_{j\ell}A_{kj}$, we obtain

$$A_{kj}A_{j\ell} = A_{j\ell}A_{kj}. \tag{44}$$

Lemma 9.1 *The subgroup $\langle A_{k\ell}\rangle$ of $\pi_1(X_{\mathrm{Gal}})$ is commutative of rank of at most* 5

Proof We will compute the centralizer of A_{ij} for fixed i,j. Let i,j,k,ℓ be four distinct indices. We already know from Lemma 8.11 that A_{ij} commutes with A_{ik} and $A_{\ell j}$. By equation (44) it also commutes with A_{ki} and $A_{j\ell}$. Now equation (43) allows us to write $A_{k\ell} = A_{i\ell}A_{\ell j}$, both of which commute with A_{ij}, so $\langle A_{k\ell}\rangle$ is commutative.

Now, since $A_{jk}A_{ij} = A_{ik}$ and $A_{ik}A_{ji} = A_{jk}$, we have $A_{jk}A_{ij}A_{ji} = A_{ik}A_{ji} = A_{jk}$, so that $A_{ji} = A_{ij}^{-1}$, the group is generated by the A_{1k} ($k = 2,\dots,6$), and the rank is at most 5. □

We see that $\pi_1(X_{\mathrm{Gal}}) = \langle A_{ij}, X_{ij}\rangle$ with the two subgroups $\langle A_{ij}\rangle, \langle X_{ij}\rangle$ isomorphic to $\mathbb{Z}^5$. The only question left is how these two subgroups interact.

Lemma 9.2 *In $\pi_1(X_{\mathrm{Gal}})$ the A_{ij} and $X_{k\ell}$ commute.*

Proof We need only consider the commutators of A_{13} and X_{ij} since all others are merely conjugates of these. First consider the commutator $[X_{13}, A_{13}]$. Since $X_{13} = (13)\Gamma_1$ (choose = 1 in (24) and note that as elements of S_6, we have $(13) = \Gamma_2\Gamma_6\Gamma_5\Gamma_4\Gamma_3\Gamma_4\Gamma_5\Gamma_6\Gamma_2$) and $A_{13} = \Gamma_1\Gamma_{1'}$ this becomes

$(13)\Gamma_1(\Gamma_1\Gamma_{1'})\Gamma_1(13)A_{13}^{-1} = (13)\Gamma_{1'}\Gamma_1(13)A_{13}^{-1} = A_{31}^{-1}A_{13}^{-1} = A_{13}A_{13}^{-1} = 1$. So X_{13} and A_{13} commute.

Next consider the commutator $X_{12}A_{13}X_{12}^{-1}A_{13}^{-1}$. By de nition we have that $X_{12} = (23)X_{13}(23) = (23)(13)\Gamma_1(23) = (321)\Gamma_1\Gamma_3$. Thus the commutator becomes $(321)\Gamma_1\Gamma_3(\Gamma_1\Gamma_{1'})\Gamma_3\Gamma_1(123)A_{13}^{-1}$. We use the triple relations $\Gamma_{(1)}\Gamma_{(3)}\Gamma_{(1)} = \Gamma_{(3)}\Gamma_{(1)}\Gamma_{(3)}$ to get $(321)\Gamma_3\Gamma_1\Gamma_3\Gamma_{1'}\Gamma_3\Gamma_1(123)A_{13}^{-1} =$ $(321)\Gamma_3\Gamma_1\Gamma_{1'}\Gamma_3\Gamma_{1'}\Gamma_1(123)A_{13}^{-1} = (321)\Gamma_3\Gamma_1\Gamma_{1'}\Gamma_3(123)(321)\Gamma_{1'}\Gamma_1(123)A_{13}^{-1}$ which is equal to $A_{(321)\Gamma_3;1}A_{(321);1}^{-1}A_{13}^{-1} = A_{(12);1}A_{(321);1}^{-1}A_{13}^{-1} = A_{23}A_{21}^{-1}A_{13}^{-1} = 1$, thus proving that X_{12} and A_{13} commute.

Conjugating by $(2j)$ we see that X_{1j} commutes with A_{13} and since $X_{ij} = X_{1i}^{-1}X_{1j}$ we see that every X_{ij} commutes with A_{13}. □

Theorem 9.3 *The fundamental group* $_1(X_{\mathrm{Gal}}) = \mathbb{Z}^{10}$.

Proof $_1(X_{\mathrm{Gal}})$ is generated by A_{1j} and X_{1j} which all commute. Hence the group they generate is $\mathbb{Z}^{10}$. □

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