ISSN 1472-2739 (on-line) 1472-2747 (printed)

Algebraic & Geometric Topology
Volume 2 (2002) 1119–1145
Published: 7 December 2002

Finite subset spaces of S^1

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Abstract Given a topological space X denote by $\exp_k X$ the space of non-empty subsets of X of size at most k, topologised as a quotient of X^k. This space may be regarded as a union over $1 \le l \le k$ of configuration spaces of l distinct unordered points in X. In the special case $X = S^1$ we show that: (1) $\exp_k S^1$ has the homotopy type of an odd dimensional sphere of dimension k or $k-1$; (2) the natural inclusion of $\exp_{2k-1} S^1 \simeq S^{2k-1}$ into $\exp_{2k} S^1 \simeq S^{2k-1}$ is multiplication by two on homology; (3) the complement $\exp_k S^1 \setminus \exp_{k-2} S^1$ of the codimension two strata in $\exp_k S^1$ has the homotopy type of a $(k-1,k)$–torus knot complement; and (4) the degree of an induced map $\exp_k f\colon \exp_k S^1 \to \exp_k S^1$ is $(\deg f)^{\lfloor (k+1)/2 \rfloor}$ for $f\colon S^1 \to S^1$. The first three results generalise known facts that $\exp_2 S^1$ is a Möbius strip with boundary $\exp_1 S^1$, and that $\exp_3 S^1$ is the three-sphere with $\exp_1 S^1$ inside it forming a trefoil knot.

AMS Classification 54B20; 55Q52, 57M25

Keywords Configuration spaces, finite subset spaces, symmetric product, circle

1 Introduction

1.1 Finite subset spaces

Given a topological space X let $\exp_k X$ denote the set of all nonempty finite subsets of X of cardinality at most k. There is a natural map

$$\underbrace{X \times \cdots \times X}_{k} \to \exp_k X$$
$$(x_1, \ldots, x_k) \mapsto \{x_1\} \cup \cdots \cup \{x_k\}$$

and we endow $\exp_k X$ with the quotient topology to obtain a topological space, the kth finite subset space of X. The first finite subset space $\exp_1 X$ is clearly X for all X, and $\exp_2 X$ co-incides with the second symmetric product $\mathrm{Sym}^2(X)$, but for $k \ge 3$ we have a proper quotient of $\mathrm{Sym}^k(X)$ since, for example, the points (a,a,b) and (a,b,b) in X^3 both map to $\{a,b\}$ in $\exp_3 X$. These extra identifications mean that $\exp_k X$ is in general highly singular, but give rise to natural inclusions

$$\begin{aligned} \exp_l X &\hookrightarrow \exp_k X \\ \{x_1,\dots,x_j\} &\mapsto \{x_1,\dots,x_j\} \end{aligned}$$

for $j \le l \le k$, maps that require a choice of basepoint for $\mathrm{Sym}^k(X)$. The space $\exp_k X$ may thus be regarded as a union over $1 \le l \le k$ of configuration spaces of l distinct unordered points in X. Moreover $\exp_k X$ is compact whenever X is, in which case it gives a compactification of the corresponding configuration space. Such spaces and their compactifications have been of considerable interest recently in algebraic topology. See, for example, Fulton and MacPherson [4], Levitt [10], Yoshida [16], and Ulyanov [15].

Given a map $f\colon X \to Y$ we obtain a map $\exp_k f\colon \exp_k X \to \exp_k Y$ in the obvious way, by sending $S \subseteq X$ to $f(S) \subseteq Y$. This construction turns $\exp_k$ into a functor. Moreover, if $\{h_t\}$ is a homotopy between f and g then $\{\exp_k h_t\}$ is a homotopy between $\exp_k f$ and $\exp_k g$, so that $\exp_k$ is in fact a functor on the level of homotopy classes of maps and spaces.

The space $\exp_k X$ was introduced by Borsuk and Ulam [2] in 1931 as the symmetric product, and has been re-introduced more recently by Handel [6] in a paper that establishes many general properties for Hausdorff X and some interesting homotopy properties when additionally X is path-connected. Various different notations have been used for $\exp_k X$, including $X(k)$, $X^{(k)}$, $F_k(X)$ and $Sub(X,k)$; our notation follows that used by Mostovoy [13] and reflects the idea that we are truncating the (suitably interpreted) series

$$\exp X = \emptyset \cup X \cup \frac{X^2}{2!} \cup \frac{X^3}{3!} \cup \cdots$$

at the $X^k/k! = X^k/S_k$ term. The name, however, is our own. There does not seem to be a satisfactory name in use among geometric topologists — indeed, recent authors Mostovoy and Handel do not use any name at all — and while symmetric product has remained in use among authors such as Illanes [9] and Macías [11] writing from the perspective of general topology we prefer to use this for X^k/S_k. We therefore propose the descriptive name kth finite subset space used here.

In what follows we will be concerned exclusively with the case $X = S^1$. The results are not only pretty, but also of topological interest due to their connection with configuration spaces and their compactifications.

1.2 Known and new results on $\exp_k S^1$

A sequence of pictures, outlined in section 2.2, shows that $\exp_2 S^1$ is a Möbius strip with boundary $\exp_1 S^1$. Note in particular that both $\exp_1 S^1$ and $\exp_2 S^1$ have the homotopy type of S^1 and that the inclusion map induces multiplication by two on H_1. The homeomorphism type of $\exp_3 S^1$ is also known and was calculated by Bott, correcting Borsuk's 1949 paper [1]:

Theorem 1 (Bott [3]) *The space $\exp_3 S^1$ is homeomorphic to the 3-sphere S^3.*

Bott proves this using a cut-and-paste argument, first showing that $\exp_3 S^1$ may be obtained from a single 3-simplex by gluing faces in pairs, then using this to find $\pi_1(\exp_3 S^1) = \{1\}$. He then divides the simplex into a number of pieces which he re-assembles to form solid tori, which give $\exp_3 S^1$ when glued along their boundary. This exhibits $\exp_3 S^1$ as a simply connected lens space, hence S^3. An explicit homeomorphism is not given, and indeed it is non-obvious, as the following theorem of E. Shchepin illustrates.

Theorem 2 (Shchepin, unpublished) *The inclusion $\exp_1 S^1 \hookrightarrow \exp_3 S^1$ is a trefoil knot.*

As further illustration, the two-cells in the above simplicial decomposition of S^3 form a Möbius strip and a dunce cap. The first is of course $\exp_2 S^1$ bounding $\exp_1 S^1$, and the second consists of those subsets containing $1 \in S^1$.

Shchepin's proof of Theorem 2 is apparently based on a direct calculation of the fundamental group of $\exp_3 S^1 \setminus \exp_1 S^1$. We will give two independent simultaneous proofs of Theorems 1 and 2, one via cut and paste topology, and the second via the classification of Seifert fibred spaces. The natural action of S^1 on itself gives an action of S^1 on $\exp_k S^1$ for each k, and for $k = 3$ we obtain the following refinement of Bott's and Shchepin's results:

Theorem 3 *The space $\exp_3 S^1$ is a Seifert fibred 3–manifold, and as such is oriented fibre preserving diffeomorphic to S^3 with the $(2, -3)$ S^1 action*

$$\lambda(z_1, z_2) = (\lambda^2 z_1, \lambda^{-3} z_2), \tag{1}$$

where we regard S^1 and S^3 as sitting in $\mathbb{C}$ and $\mathbb{C}^2$ respectively and give $\exp_3 S^1$ the canonical orientation it inherits from S^1.

We mention also an elegant geometric construction due to Mostovoy [13] showing that both Theorems 1 and 2 can be deduced from known facts about lattices in the plane. Namely, the space $SL(2,\mathbb{R})/SL(2,\mathbb{Z})$ of plane lattices modulo scaling is diffeomorphic to a trefoil complement (the proof, due to D. Quillen, may be found in [12, page 84] and uses the Weierstrass $\wp$ function associated to the lattice), and this space may be compactified by adding degenerate lattices to obtain S^3. Together with Theorems 1 and 2 this shows $\exp_3 S^1 \setminus \exp_1 S^1 \cong SL(2,\mathbb{R})/SL(2,\mathbb{Z})$, and Mostovoy's construction fills in the the third side of this triangle, associating to each lattice a finite subset of S^1, thought of as $\mathbb{R}P^1$. Each degenerate lattice corresponds to a one element subset, each rectangular lattice a two element subset, and all other lattices correspond to three element subsets. Moreover, his map is equivariant with respect to the natural actions of S^1 on $\exp_3 S^1$ and $PSO(2) \subset PSL(2,\mathbb{R})$ on $SL(2,\mathbb{R})/SL(2,\mathbb{Z})$.

The first three of the following new results generalise the theorems and observations above. Proofs appear in subsequent sections. Since writing this paper I have learnt that Theorems 4, 5 and the observation on the map $\exp_{k-1} S^1 \to \exp_k S^1 : \Lambda \mapsto \Lambda \cup \{1\}$ that follows their proofs have been proved independently by David Handel [5, unpublished work], using essentially the same argument.

Theorem 4 *The space $\exp_k S^1$ has the homotopy type of an odd dimensional sphere, of dimension k or $k-1$ according to whether k is odd or even.*

Since $\exp_{2k-1} S^1 \simeq S^{2k-1} \simeq \exp_{2k} S^1$ we may ask how $\exp_{2k-1} S^1$ sits inside $\exp_{2k} S^1$. The following result falls out of the proof of Theorem 4 and shows that the situation is analogous to $\exp_1 S^1$ inside $\exp_2 S^1$:

Theorem 5 *The inclusion $\exp_{2k-1} S^1 \hookrightarrow \exp_{2k} S^1$ induces multiplication by two on H_{2k-1}.*

As our last generalisation, $\exp_{k-2} S^1$ inside $\exp_k S^1$ is in some sense a homotopy S^{l-2} inside a homotopy S^l, so it is natural to ask if this is embedded in some interesting way too. Analogously to Theorem 2 we have:

Theorem 6 *The complement of $\exp_{k-2} S^1$ in $\exp_k S^1$ has the homotopy type of a $(k-1,k)$–torus knot complement.*

Finally, a map $f\colon S^1 \to S^1$ induces a map $\exp_k f$ of homotopy spheres, and we calculate its degree in terms of k and the degree of f.

Theorem 7 *If $f\colon S^1 \to S^1$ then*

$$\deg \exp_k f = (\deg f)^{\lfloor \frac{k+1}{2} \rfloor}.$$

Remark It is perhaps worth noting that although $\exp_k S^1$ is a manifold for $k = 1, 2, 3$, it is not a manifold for any $k \geq 4$. Subsets of size k do have k–ball neighbourhoods in $\exp_k S^1$ and this transition may be understood in terms of neighbourhoods of $k-1$ element subsets as follows. Given such a subset Λ of S^1 there are $k-1$ choices for which point to consider "doubled" and split in two to obtain a k element subset. Each such choice leads to a k–dimensional halfspace containing Λ and we obtain a neighbourhood of Λ by gluing these together along their boundaries. The transition thus occurs when $k-1$ increases above two, and we see also that when $k = 2$ points in $\exp_1 S^1$ have halfspace neighbourhoods and thus form the boundary of a 2–manifold.

Later we shall see this more explicitly when we show that $\exp_k S^1$ may be obtained from a single k–simplex by identifying faces. Under the identifications the 0th and kth faces become one face and the remaining $k-1$ faces a second, corresponding to $\exp_{k-1} S^1$.

1.3 Notation and terminology

The proofs of our main results make use of arguments involving simplices and simplicial decompositions and we take a moment to fix language. For the most part we follow Hatcher [8, section 2.1], and will use "simplicial decomposition" in the sense of his Δ–complexes. In particular, we will not require that the simplices in our decompositions be determined by their vertices.

Given $k+1$ points $u_0, \ldots, u_k$ in an affine space let

$$[u_0, \ldots, u_k] = \left\{ \textstyle\sum_i t_i u_i \;\middle|\; \sum_i t_i = 1 \text{ and } t_i \geq 0 \text{ for all } i \right\},$$

the set of convex combinations of the u_i. We will typically only write this when the points are affinely independent, in which case $[u_0, \ldots, u_k]$ is a k–simplex with (ordered) vertices $u_0, \ldots, u_k$. For $\sigma \in S_{k+1}$ we regard $[u_{\sigma(0)}, \ldots, u_{\sigma(k)}]$ as the same simplex with orientation $(-1)^{\text{sign}\,\sigma}$ times that of $[u_0, \ldots, u_k]$.

The canonical map between two simplices $[u_0, \ldots, u_k]$ and $[v_0, \ldots, v_k]$ is the unique map given by sending u_i to v_i for each i and extending affinely. A hat ^ over a vertex means it is to be omitted, in other words

$$[u_0, \ldots, \hat{u}_i, \ldots, u_k] = [u_0, \ldots, u_{i-1}, u_{i+1}, \ldots, u_k],$$

and unless indicated otherwise the interior of a simplex $[u_0, \ldots, u_k]$ will always mean the open simplex

$$\operatorname{int}[u_0, \ldots, u_k] = \left\{ \sum_i t_i u_i \;\middle|\; \sum_i t_i = 1 \text{ and } t_i > 0 \text{ for all } i \right\}$$

regardless of whether this is an open subset of the ambient space.

2 Finite subset spaces of S^1 have the homotopy type of odd dimensional spheres

2.1 Introduction

To prove that $\exp_k S^1$ has the homotopy type of a sphere we will find a cell structure for it and use this to show it has the correct fundamental group and homology. Application of a standard argument that a simply connected homology sphere is a homotopy sphere then yields the result. Before doing so however, let us look at $\exp_2 S^1$ and $\exp_3 S^1$ in some detail, which will illustrate the situation in higher dimensions.

2.2 The homeomorphism type of $\exp_2 S^1$

To see that $\exp_2 S^1$ is a Möbius strip we will use the usual picture of $S^1 \times S^1$ as a square with opposite sides identified. In forming $\exp_2 S^1$ the point (x, y) is identified with (y, x), so the square is folded along the diagonal shown dotted in figure 1(a), resulting in the triangle with two edges identified shown in figure 1(b). This possible unfamiliar picture may be recognised as a Möbius

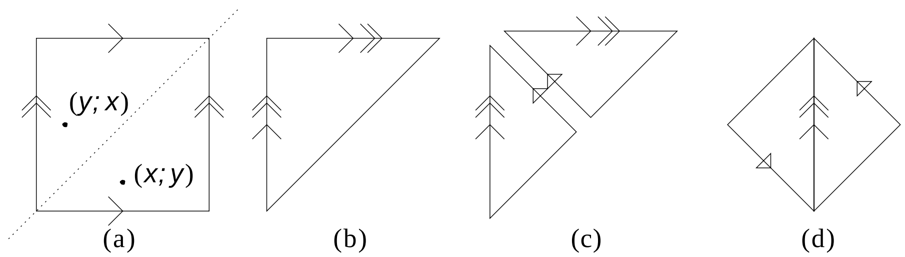


Figure 1: The space $\exp_2 S^1$ is a Möbius strip. Fold along the diagonal in (a) to identify (x, y) and (y, x); the result is shown in (b). Cut (c) and re-glue (d) to recognise this as a Möbius strip.

strip either by cutting and re-gluing as in figures 1(c) and (d), or by gluing just the ends of the hypotenuse together to get a punctured projective plane.

A Möbius strip with the edge corresponding to the glued sides of the triangle shown dotted appears in figure 2. The diagonal edge forms the boundary circle. Note that the diagonal comes from the diagonal embedding of S^1 in $S^1 \times S^1$, which maps to $\exp_1 S^1 \subset \exp_2 S^1$.

We mention also a construction pointed out by Chuck Livingston. There is a natural map from $\exp_2 S^1$ to $\mathbb{R}P^1$ sending each pair of points on S^1 to the line through the origin bisecting the arc between them. The fibre above each point in $\mathbb{R}P^1$ is an interval of length π and the bundle is easily seen to be non-orientable.

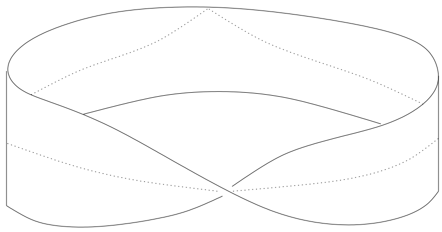
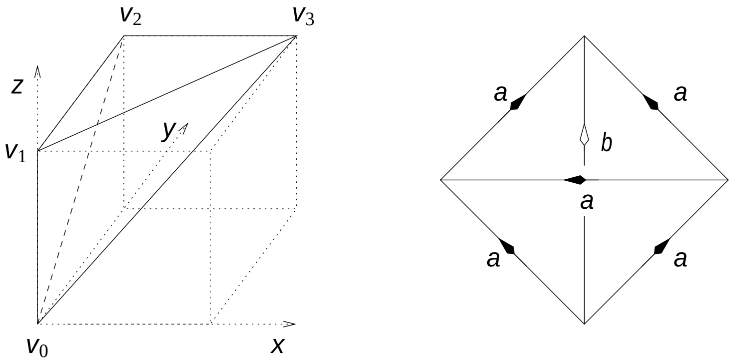


Figure 2: A Möbius strip, in which the glued edge-pair of the triangle of figure 1(b) corresponding to the set $\{\theta \in \exp_2 S^1 | 1 \in \theta\}$ is shown dotted.

2.3 The homeomorphism types of $\exp_3 S^1$ and $\exp_3 S^1 \setminus \exp_1 S^1$

Now consider $\exp_3 S^1$. Begin again with $[0,1]^3 = I^3$ with opposite faces identified. Each $\xi \in \exp_3 S^1$ has at least one representative $(x,y,z) \in I^3$ with $0 \le x \le y \le z \le 1$, so we may restrict our attention to the simplex with vertices $v_0 = (0,0,0)$, $v_1 = (0,0,1)$, $v_2 = (0,1,1)$ and $v_3 = (1,1,1)$ shown in figure 3(a). Now $(0,x,y) \sim (x,y,1)$ in $\exp_3 S^1$, so the face $[v_0,v_1,v_2]$ is glued to the face $[v_1,v_2,v_3]$; next $(x,x,y) \sim (x,y,y)$ in $\exp_3 S^1$, so the face $[v_0,v_1,v_3]$ is glued to $[v_0,v_2,v_3]$. This accounts for all the identifications of the simplex arising from $\exp_3 S^1$, and the result, taking account of edge identifications, is shown in figure 3(b), viewed from a different angle. There is just one vertex (the set $\{1\}$), two edges and two 2–simplices, one forming a dunce cap and the second a Möbius strip. The dunce cap comes from the two faces $x = 0$ and $z = 1$ and corresponds to the set $\{\xi \in \exp_3 S^1 | 1 \in \xi\}$; the Möbius strip comes from the faces $x = y$ and $y = z$ and is of course $\exp_2 S^1 \subset \exp_3 S^1$.



(a) (b)

Figure 3: Simplicial decomposition of $\exp_3 S^1$. $\exp_3 S^1$ may be formed from the simplex $[v_0, v_1, v_2, v_3]$ in (a) by identifying the face $[v_0, v_1, v_2]$ with $[v_1, v_2, v_3]$, and the face $[v_0, v_1, v_3]$ with $[v_0, v_2, v_3]$. This corresponds to the edge and associated face gluings of (b).

From above we have $\chi(\exp_3 S^1) = 1-2+2-1 = 0$; a well known but still magical fact (see [14, p. 122]) then guarantees that the space formed by gluing the faces of the 3–simplex in this fashion is a 3–manifold. Calculating $\pi_1(\exp_3 S^1)$ using figure 3(b) we obtain the presentation $\langle a, b \mid a, a^2b^{-1}\rangle$, so $\exp_3 S^1$ is in fact a simply connected 3–manifold — and hence almost certainly S^3, especially given its simple construction. Bott [3] completes the proof by showing it has a genus one Heegaard splitting and appealing to the classification of lens spaces. A more informative approach is the proof of Theorem 3 given below. In appendix A (beginning page 1140) we will also show directly that $\exp_3 S^1$ is a 3–sphere (and that $\exp_1 S^1$ inside it is a trefoil) using the genus two Heegaard splitting obtained from the triangulation above.

Before proving Theorem 3 a word is necessary on orientation. Given an orientation of S^1 and a set $\Lambda = \{\alpha, \beta, \gamma\}$ of three distinct points in S^1 we may canonically orient

$$T_\Lambda \exp_3 S^1 = T_\alpha S^1 \oplus T_\beta S^1 \oplus T_\gamma S^1$$

by positively orienting each of the summands and requiring that the cyclic ordering of α, β, and γ agree with the orientation of S^1. This extends to an orientation of $\exp_3 S^1$ and we regard this as the canonical orientation of $\exp_3 S^1$. Note that the standard orientation of the simplex $0 \leq x \leq y \leq z \leq 1$ of figure 3 co-incides with the canonical orientation of $\exp_3 S^1$.

To orient S^3 we regard it as the boundary of the 4–ball in $\mathbb{C}^2$ with its canonical orientation and use the "outward first" convention for induced orientations on boundaries.

Proof of Theorem 3 We have seen that $\exp_3 S^1$ is a closed simply connected 3–manifold and we observe that it is Seifert fibred by the natural action of S^1. There are precisely two exceptional fibres, the orbits of $\{1, -1\}$ and $\{1, e^{\pi i/3}, e^{2\pi i/3}\}$, and these have multiplicities 2 and 3 respectively. Since S^3 with the $(2, -3)$ action of equation (1) on page 1121 shares these properties our aim will be to show that they are enough to completely determine the unoriented fibre type of $\exp_3 S^1$.

To this end let M be a closed simply connected Seifert fibred space with precisely two exceptional fibres, of multiplicities two and three. Simple connectivity of M implies the orbit surface is simply connected also, and therefore S^2. Moreover, the fibres may be consistently oriented. Removing fibred solid torus neighbourhoods of each of the exceptional fibres thus leaves an oriented circle bundle over a twice punctured sphere, which we may write as $S^1 \times I \times S^1$, the base corresponding to the first two factors.

M is completely determined by specifying slopes $\beta_0/\alpha_0, \beta_1/\alpha_1 \in \mathbb{Q}$ along which to glue back in meridional discs to $S^1 \times \{0\} \times S^1$ and $S^1 \times \{1\} \times S^1$. To get the correct multiplicities we must have $\{\alpha_0, \alpha_1\} = \{2, 3\}$ so without loss of generality let $\alpha_0 = 2$, $\alpha_1 = 3$. The classification of orientable Seifert fibred spaces (see for example Hatcher's 3–manifold notes [7, p. 25]) tells us that the slopes are only determined mod 1 subject to their sum being fixed (this ambiguity comes from the choice of trivialisation of the circle bundle) so we may further assume $\beta_0 = 1$ and write $\beta_1 = \beta$. We calculate π_1 of the resulting manifold.

The fundamental group of $S^1 \times I \times S^1$ is free abelian, generated by b and f, where b and f are positively oriented generators of π_1 of the base and fibre respectively. Gluing a disc in to $S^1 \times \{0\} \times S^1$ along a line of slope $1/2$ kills $2b + f$ while gluing a disc in to $S^1 \times \{1\} \times S^1$ along a line of slope $\beta/3$ kills $-3b + \beta f$, the minus sign coming from the fact that $S^1 \times \{1\} \times S^1$ has orientation $[-b, f]$ (recall that we are using "outward first" to orient boundaries). Thus simple connectivity of M implies

$$\left| \det \begin{pmatrix} 2 & -3 \\ 1 & \beta \end{pmatrix} \right| = |2\beta + 3| = 1.$$

Setting the determinant equal to 1 and -1 in turn gives $\beta = -1$, $\beta = -2$. Thus there are exactly two possibilities for the oriented fibre type of M; in Hatcher's notation they are $M(\pm g, b; \beta_0/\alpha_0, \beta_1/\alpha_1) = M(0, 0; 1/2, -1/3)$ and

$M(0,0;1/2,-2/3)$, where ϵg specifies the genus and orientability of the orbit surface and b the number of boundary components.

Reversing the orientation of $M(\epsilon g, b; \alpha_1/\beta_1, \ldots, \alpha_k/\beta_k)$ simply changes the signs of all the attaching slopes. Thus

$$-M(0,0;1/2,-1/3) = M(0,0;-1/2,1/3) = M(0,0;1/2,-2/3)$$

and the unoriented fibre type of M is completely determined as claimed. It follows that $\exp_3 S^1$ and S^3 with the $(2,-3)$ action are fibre preserving diffeomorphic and all that remains is to determine whether orientation is preserved or reversed.

To determine orientations we look at the return map on a disc D transverse to the exceptional fibre of multiplicity three at a point p. The fibre F is oriented by the S^1 action and we orient D such that $T_pF \oplus T_pD$ is positive. In the case of $\exp_3 S^1$ we use the point $p = (1/3, 2/3, 1) \in I^3$ and a small disc containing it in the plane $z = 1$. The vector $(1,1,1)$ forms a positive basis for T_pF so T_pD has orientation $[e_x, e_y]$, and the action of the return map is the $1/3$ anti-clockwise twist given by the canonical map from $[v_1, v_2, v_3]$ to $[v_3, v_1, v_2]$. For S^3 we take $p = (0,1)$ and consider the $(2,3)$ action. The tangent space T_pS^3 has orientation $[(0,i),(1,0),(i,0)]$, in which $(0,i)$ forms a positive basis for T_pF, so for a suitable choice of disc D its tangent space at p has positive basis $\{(1,0),(i,0)\}$. The first return is when $\theta = \omega := e^{2\pi i/3}$ and we see that the derivative of the first return map is multiplication by ω^2 on T_pD. This is a clockwise rotation through $2\pi/3$, so to match orientations with $\exp_3 S^1$ we must reverse the orientation of the orbit through $(0,1)$, giving the $(2,-3)$ S^1 action as claimed. □

2.4 The homotopy type of $\exp_k S^1$

Finally we turn our attention to the general case. Proceeding analogously to the two and three dimensional cases we may reduce to the k–simplex $0 \le x_1 \le \cdots \le x_k \le 1$. Working somewhat more formally than above let

$$v_i = (\underbrace{0, \ldots, 0}_{k-i}, \underbrace{1, \ldots, 1}_{i}) \in \mathbb{R}^k$$

for $i = 0, \ldots, k$, and let σ_k be the map of the simplex $[v_0, \ldots, v_k]$ to $\exp_k S^1$, σ_{k-1} that of the simplex $[v_1, \ldots, v_k]$. Being a little sloppy with notation we claim:

Lemma 1 *$\exp_k S^1$ has a simplicial decomposition with one 0-simplex, two i–simplices for each $1 \le i \le k-1$, and one k–simplex, namely, σ_i for $0 \le i \le k$ and τ_i for $1 \le i \le k-1$. The boundary map $\partial_i\colon \mathbb{Z}\sigma_i \oplus \mathbb{Z}\tau_i \to \mathbb{Z}\sigma_{i-1} \oplus \mathbb{Z}\tau_{i-1}$ has matrix*

$$D_i = \left(\frac{1+(-1)^i}{2}\begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$

for $2 \le i \le k-1$, is the zero map for $i = 1$ and has matrix $D_k|_{\mathbb{Z}\sigma_k}$ for $\partial_k\colon \mathbb{Z}\sigma_k \to \mathbb{Z}\sigma_{k-1} \oplus \mathbb{Z}\tau_{k-1}$, $k \ge 2$.

The lemma enables us to calculate the homology of $\exp_k S^1$, obtaining the following.

Corollary 1 *The space $\exp_k S^1$ has the homology of an odd dimensional sphere, of dimension k if k is odd and dimension $k-1$ if k is even. The inclusion map $\exp_{2k-1} S^1 \hookrightarrow \exp_{2k} S^1$ induces multiplication by two on H_{2k-1}.*

Proof of Lemma 1 and Corollary 1 The lemma is proved by induction. Taking $k = 1$ as our base case (although the cases $k = 2$ and 3 are largely established above) this is just the usual cell decomposition of S^1 as $I/\{0,1\}$, so consider $k \ge 2$. σ_k maps $[v_0,\ldots,v_k]$ onto $\exp_k S^1$, taking the interior of the simplex homeomorphically onto its image, so we need only sort out the face gluings. Faces of the form $[v_0,\ldots,\hat{v}_i,\ldots,v_k]$ where $1 \le i \le k-1$ correspond to $0 \le x_1 \le \cdots \le x_{k-i} = x_{k-i+1} \le \cdots \le x_k \le 1$, giving subsets of S^1 of size $k-1$ or less; therefore σ_k restricted to such a simplex factors through σ_{k-1}. More precisely, we should note that the map of simplices factoring this restriction preserves orientation. The simplex $[v_0,\ldots,v_{k-1}]$ is $0 = x_1 \le \cdots \le x_k \le 1$ which is identified with $0 \le x_1 \le \cdots \le x_k = 1$, so $\sigma_k|_{[v_0,\ldots,v_{k-1}]}$ factors through τ_{k-1} via the canonical map $[v_0,\ldots,v_{k-1}] \to [v_1,\ldots,v_k]$. Thus

$$\begin{aligned}
\partial\sigma_k &= \sum_{i=0}^{k} (-1)^i \sigma_k|_{[v_0,\ldots,\hat{v}_i,\ldots,v_k]} \\
&= \tau_{k-1} + \sum_{i=1}^{k-1} (-1)^i \sigma_{k-1} + (-1)^k \tau_{k-1} \\
&= \frac{1+(-1)^k}{2}(-\sigma_{k-1} + 2\tau_{k-1}),
\end{aligned}$$

giving the first column of D_k.

Turning our attention now to τ_{k-1}, observe that it maps $[v_1,\ldots,v_k]$ onto $\{A \in \exp_k S^1 \mid 1 \in A\}$, taking the interior of this simplex homeomorphically onto its image. A face of $[v_1, \dots, v_k]$ corresponds to replacing an inequality with an equality in $0 \le x_1 \le \dots \le x_{k-1} \le x_k = 1$, and in each case maps onto the $k-1$ or fewer element subsets containing 1. Thus σ_{k-1} factors through σ_{k-2} (via an orientation preserving map of simplices) when restricted to each face, and

$$\begin{aligned}
\partial \sigma_{k-1} &= \sum_{i=1}^{k} (-1)^{i-1} \sigma_{k-1}|_{[v_1,\dots,\hat{v}_i,\dots,v_k]} \\
&= \sum_{i=1}^{k} (-1)^{i-1} \sigma_{k-2} \\
&= \frac{1+(-1)^{k-1}}{2} \sigma_{k-2}, \qquad (2)
\end{aligned}$$

giving the second column of D_{k-1}, the first coming from the inductive hypothesis. This establishes the lemma.

It is now a simple matter to calculate the homology of $\exp_k S^1$. The matrix D_i is zero if i is odd and has determinant -1 if i is even, so the chain maps are alternately zeroes and isomorphisms in the middle dimensions. Thus $H_i(\exp_k S^1) = \{0\}$ for $1 \le i \le k-2$. Clearly $H_0(\exp_k S^1) = \mathbb{Z}$, so it remains to determine only H_{k-1} and H_k. When k is odd the top end of the chain complex is

$$0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \oplus \mathbb{Z} \longrightarrow 0,$$

so H_{k-1} is zero and $H_k(\exp_k S^1) = \mathbb{Z}$, generated by $[\sigma_k]$. When k is even we have instead

$$\begin{gathered}
0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow 0, \\
1 \longmapsto (-1, 2)
\end{gathered}$$

making H_k zero and imposing the relation $\sigma_{k-1} = 2\sigma_{k-1}$ on $\ker \partial_{k-1} = C_{k-1}$. Thus for k even we have $H_{k-1}(\exp_k S^1) = \mathbb{Z}$, generated by $[\sigma_{k-1}]$. This proves the first statement of the corollary, and for the second simply observe that the generator $[\sigma_{2k-1}]$ of $H_{2k-1}(\exp_{2k-1} S^1)$ is twice $[\sigma_{2k-1}]$, the generator of $H_{2k-1}(\exp_{2k} S^1)$. □

Theorem 4 for $k \ge 3$ now follows from Corollary 1 and the simple connectivity of $\exp_3 S^1$ by an application of a standard argument.

Corollary 2 (Theorem 4) $\exp_k S^1$ *has the homotopy type of an odd dimensional sphere of dimension* $d_k = 2\lceil k/2 \rceil - 1$.

Proof We have already seen this for $k = 1, 2$ so we may assume $k \geq 3$. Then the 2–skeleton of $\exp_k S^1$ co-incides with the 2–skeleton of $\exp_3 S^1$, which is simply connected, so $\pi_1(\exp_k S^1) = \{1\}$ too. By the Hurewicz theorem $\pi_{d_k}(\exp_k S^1) = H_{d_k}(\exp_k S^1) = \mathbb{Z}$, so let $\phi : S^{d_k} \to \exp_k S^1$ be a generator for $\pi_{d_k}(\exp_k S^1)$. ϕ induces an isomorphism on π_{d_k} and so on H_{d_k} also; since H_0 and H_{d_k} are the only non-vanishing homology groups of both S^{d_k} and $\exp_k S^1$ ϕ is an isomorphism on H_n for all n. By the simply connected version of Whitehead's theorem that only requires isomorphisms on homology, ϕ is a homotopy equivalence. □

We close this section with some remarks on two related spaces. Let

$$\begin{aligned}\exp_k(S^1, 1) &= \left\{\xi \in \exp_k S^1 \mid 1 \in \xi\right\} \\ &= \left\{\xi \cup \{1\} \mid \xi \in \exp_{k-1} S^1\right\};\end{aligned}$$

the subsets of S^1 of size k or less that contain 1. This subspace has a cell structure with one cell σ_i in each dimension less than or equal to $k-1$, and by (2) the boundary maps are alternately zero and isomorphisms. Thus the reduced homology of $\exp_k(S^1, 1)$ is given by

$$\tilde{H}_i(\exp_k(S^1, 1)) = \begin{cases} \mathbb{Z} & \text{if } i = k-1 \text{ is odd} \\ 0 & \text{otherwise,} \end{cases}$$

and moreover $\exp_k(S^1, 1)$ is simply connected for $k \geq 3$ since its 2-skeleton is a dunce cap. It follows (by the Whitehead theorem for k odd, and the argument of Corollary 2 for k even) that $\exp_k(S^1, 1)$ is contractible if k is odd, and homotopy equivalent to S^{k-1} if k is even. The natural map

$$\exp_{k-1} S^1 \to \exp_k(S^1, 1) : \xi \mapsto \xi \cup \{1\}$$

takes $[\sigma_{k-1}]$ to $[\sigma_{k-1}]$ and so is a homotopy equivalence for k even. Note however that this map is not a homeomorphism except when $k = 2$ since otherwise $\{\zeta\}$ and $\{\zeta, 1\}$ have the same image.

Lastly let

$$\exp S^1 = \bigcup_{k=1}^{\infty} \exp_k S^1 = \left\{\xi \subset S^1 \mid 0 < |\xi| < \infty\right\};$$

topologised as the direct limit of the $\exp_k S^1$, or equivalently, with the CW topology coming from the cell structure consisting of the σ_i, τ_i. The full finite subset space $\exp S^1$ has vanishing reduced homology in all dimensions and is therefore contractible.

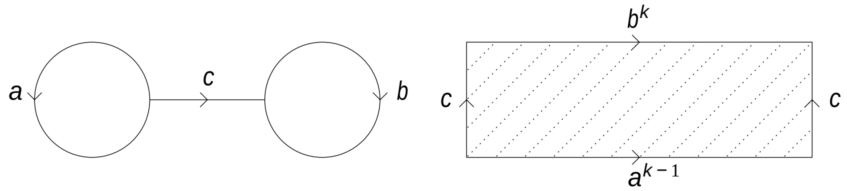


Figure 4: A cell structure consisting of three 1–cells a, b, c and one 2–cell, attached along $a^{k-1}cb^{-k}c^{-1}$.

3 Removing the codimension two strata gives a homotopy torus knot complement

3.1 A new model for $\exp_k S^1$

The strategy for proving Theorem 6 is to show that $\exp_k S^1 \setminus \exp_{k-2} S^1$ deformation retracts to a subspace having the cell structure shown in figure 4. In order to make calculations easier and the result more transparent we will adopt a slightly different picture of $\exp_k S^1$, using the action of S^1 to model it as a quotient of a $(k-1)$–simplex cross an interval. On the simplex level this action corresponds to the constant vector field equal to $(1, 1, \ldots, 1)$ everywhere and the orbits are unions of intervals of the form

$$[(0, x_2, \ldots, x_k), (y_1, \ldots, y_{k-1}, 1)], \tag{3}$$

where $y_i = x_i + 1 - x_k$ for each i. The effect of our new model will be to normalise the lengths of these intervals to one. Figure 5 depicts this in the case $k = 2$.

Let Δ be the simplex

$$[v_0, \ldots, v_{k-1}] = \{(0, a_1, \ldots, a_{k-1}) \mid 0 \leq a_1 \leq \cdots \leq a_{k-1} \leq 1\}$$

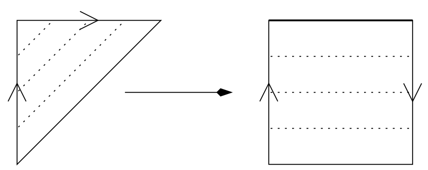


Figure 5: The new model in the case $k = 2$. We model $\exp_2 S^1$ as a quotient of a 1–simplex cross an interval by normalising the dotted lines in the triangle on the left (corresponding to subintervals of the S^1 action) to have length one, giving the square on the right. In doing this the top left vertex of the triangle is stretched to become the top edge of the square.

and consider the map from $\Delta \times I$ to our usual model

$$X = [v_0, \ldots, v_k] = \{(x_1, \ldots, x_k) \mid 0 \le x_1 \le \cdots \le x_k \le 1\}$$

given by

$$x_i = \begin{cases} (1 - a_{k-1})t & i = 1; \\ a_{i-1} + (1 - a_{k-1})t & 2 \le i \le k. \end{cases}$$

This has inverse

$$\begin{aligned} a_i &= x_{i+1} - x_1 \\ t &= \frac{x_1}{1 - x_k + x_1} \end{aligned}$$

well defined off the codimension two face $[v_1, \ldots, v_{k-1}]$, which has preimage the entire codimension one face $[v_1, \ldots, v_{k-1}] \times I$ due to the intervals (3) through these points being stretched from length zero to one. We will refer to this as the "fake face" of $\Delta \times I$ and denote the quotient map $\Delta \times I \to X \to \exp_k S^1$ by q.

In forming $\exp_k S^1$ from $\Delta \times I$ the $k-1$ faces $\{a_1 = 0\} \times I$ and $\{a_i = a_{i+1}\} \times I$, $1 \le i \le k-2$, are all identified according to the maps

$$[v_0, \ldots, \hat{v}_i, \ldots, v_{k-1}] \times I \to [v_0, \ldots, \hat{v}_j, \ldots, v_{k-1}] \times I \qquad (i, j \ne 0) \tag{4}$$

given by the product of the canonical map with the identity. The face $\{a_{k-1} = 1\} \times I$ is collapsed back down to $\{a_{k-1} = 1\}$ by projection on the first factor, and $\Delta \times \{0\}$ is glued to $\Delta \times \{1\}$ according to $(a, 1) \sim (\phi(a), 0)$, where

$$\phi_i(a) = \begin{cases} 1 - a_{k-1} & i = 1; \\ a_{i-1} + 1 - a_{k-1} & 2 \le i \le k. \end{cases}$$

ϕ is affine and permutes the vertices $v_0, \ldots, v_{k-1}$ cyclicly according to the permutation $i \mapsto i-1 \pmod{k-1}$ and so is the canonical map $[v_0, \ldots, v_{k-1}] \to [v_{k-1}, v_0, \ldots, v_{k-2}]$. In particular $\exp_k S^1 \setminus \exp_{k-1} S^1$, as the quotient of $(\operatorname{int} \Delta) \times I$, is the mapping torus of $\phi|_{\operatorname{int} \Delta}$ and has the homeomorphism type of an $\mathbb{R}^{k-1}$ bundle over S^1, with monodromy of order k. ϕ reverses orientation exactly when k is even so the bundle is trivial for k odd and nontrivial for k even.

Although we shall not explicitly do so, there is no loss in generality in regarding Δ as the more symmetrical standard $(k-1)$–simplex

$$\left\{ (t_0, t_1, \ldots, t_{k-1}) \in \mathbb{R}^k \;\middle|\; \textstyle\sum_i t_i = 1 \text{ and } t_i \ge 0 \text{ for all } i \right\},$$

and what follows may be read with this picture in mind.

3.2 The fundamental group of $\exp_k S^1 \setminus \exp_{k-2} S^1$

As a first application of our new model we calculate the fundamental group of $\exp_k S^1 \setminus \exp_{k-2} S^1$. This calculation is in some sense redundant, in that in the following section we will find a 2–complex to which it is homotopy equivalent. However, the proof given below that this space has the correct fundamental group strongly echoes the corresponding calculation for the torus knot complement. In doing so it carries the main insight as to why the two have the same homotopy type, while the proof of this fact, while geometric in nature, is somewhat technical. We therefore include both proofs to further understanding of the result.

We take as our base point the kth roots of unity. Let γ be the path from the kth roots of unity to the $(k-1)$th given by projecting to $\exp_k S^1$ the linear homotopy from $(0, 1/k, \ldots, (k-1)/k)$ to $(0, 0, 1/(k-1), \ldots, (k-2)/(k-1))$. Let α be the loop given by rotating the kth roots of unity anti-clockwise through $2\pi/k$, and β the loop based at $\{\zeta^k = 1\}$ given by taking γ to $\{\zeta^{k-1} = 1\}$, rotating the circle anti-clockwise through $2\pi/(k-1)$ then following γ back to the basepoint. Then:

Theorem 8 *$\pi_1(\exp_k S^1 \setminus \exp_{k-2} S^1)$ has presentation $\langle \alpha, \beta \mid \alpha^k = \beta^{k-1} \rangle$.*

Remark Another choice of second generator is the loop β' given by "teleporting" a point from $e^{2\pi i/k}$ to $e^{-2\pi i/k}$ as follows. Between time $t = 0$ and $1/2$ move one point from 1 to $e^{2\pi i/k}$, keeping the rest fixed. At time $t = 1/2$ the moving point merges with $e^{2\pi i/k}$ and we may regard it as being at $e^{-2\pi i/k}$ instead; between $t = 1/2$ and 1 move the extra point at $e^{-2\pi i/k}$ back to 1, keeping the rest fixed. Figure 6 illustrates that β' is homotopic to β, and we will use this below to show that the inclusion

$$\exp_k S^1 \setminus \exp_{k-2} S^1 \hookrightarrow \exp_k S^1 \setminus \exp_{k-3} S^1$$

is trivial on π_1.

Proof The result is an application of Van Kampen's theorem. Each of $\exp_k S^1 \setminus \exp_{k-1} S^1$ and $\exp_{k-1} S^1 \setminus \exp_{k-2} S^1$ has the homotopy type of a circle, with fundamental group generated (up to basepoint) by α and β respectively. We show that $\exp_{k-1} S^1 \setminus \exp_{k-2} S^1$ has a neighbourhood N in $\exp_k S^1 \setminus \exp_{k-2} S^1$ that deformation retracts to it and apply Van Kampen's theorem to the cover consisting of $\exp_k S^1 \setminus \exp_{k-1} S^1$ and N. The intersection N' of these two sets

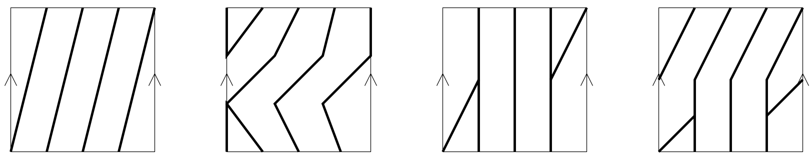


Figure 6: Movies of the generators of $_1(\exp_4 S^1 \setminus \exp_2 S^1)$. Each square represents a cylinder S^1 $[0;1]$; dark lines show the motion of points. The diagram for may be isotoped rel boundary (and respecting the fact that each slice S^1 ftg should meet the curves in either 3 or 4 points) to give the diagram on the right, showing ' .

will turn out to have the homotopy type of a circle also, and will lead to the relation $^k = {}^{k-1}$.

The preimage of $\exp_k S^1 \setminus \exp_{k-2} S^1$ in I is the product of less all faces of codimension two or more with the interval. Let b be the barycentre of , set

$$_i = [b; v_{i+1}; \ldots; v_{k-1}; v_0; \ldots; v_{i-1}];$$
$$_i = [v_{i+1}; \ldots; v_{k-1}; v_0; \ldots; v_{i-1}];$$

and note that is the union of the $_i$ and its boundary is the union of the $_i$. Further sends $_i; {}_i$ to $_{i-1}; {}_{i-1}$ respectively, where subscripts are taken mod $k-1$. Let

$$N = q \bigcup_{i=0}^{k-1} (\text{int } _i \cup \text{int } _i) \quad I$$

and observe that N is a neighbourhood of

$$\exp_{k-1} S^1 \setminus \exp_{k-2} S^1 = q \bigcup_{i=0}^{k-1} \text{int } _i \quad I$$

in $\exp_k S^1 \setminus \exp_{k-2} S^1$.

The half open simplex int $_i \cup$ int $_i$ deformation retracts to int $_i$ and moreover this may be done for each i simultaneously in a way compatible with the action of . Crossing this with I gives a deformation retraction of

$$\bigcup_{i=0}^{k-1} (\text{int } _i \cup \text{int } _i) \quad I \quad \text{to} \quad \bigcup_{i=0}^{k-1} \text{int } _i \quad I$$

that descends to a deformation retraction of N onto $\exp_{k-1} S^1 \setminus \exp_{k-2} S^1$. Thus $\exp_{k-1} S^1 \setminus \exp_{k-2} S^1$ does have a neighbourhood as desired and by the

Van Kampen theorem

$$_1(\exp_k S^1 \, n \exp_{k-2} S^1) = h \;\; i \quad _{1(N^0)} \; h \;\; i;$$

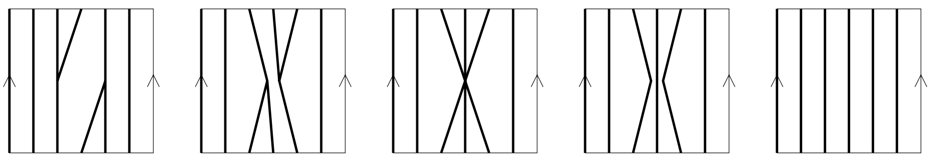
where

$$N^0 = N \setminus (\exp_k S^1 \, n \exp_{k-1} S^1) = q \bigcup_{i=0}^{k-1} \text{int} \quad _i \quad I \; :$$

Now N^0 is homeomorphic to $(\bigcup_i \text{int} \;\; _i \;\; I) = (\;\; _i \;\; f1g \quad _{i-1} \;\; f0g)$, where the gluing relations are given by . Thus $N^0 = \text{int} \;\; _{k-1} \;\; S^1$, where we have chosen $_{k-1}$ since $q(\;\; _{k-1} \;\; f0g)$ contains the path γ. If $_1(N^0) = h''i$ then clearly $'' = \;\; ^k$ in $_1(\exp_k S^1 \, n \exp_{k-1} S^1)$; it remains to determine the image of $''$ in $_1(\exp_{k-1} S^1 \, n \exp_{k-2} S^1)$.

Pushing $''$ onto $\exp_{k-1} S^1 \, n \exp_{k-2} S^1$ via the deformation retraction of N we see that upstairs in $\;\; I \;\; ''$ traverses the length of each int $_i \;\; I$ exactly once and positively. For $i = 1; \ldots; k-1$ we have $q(\text{int} \;\; _i \;\; I) = \exp_{k-1} S^1 \, n \exp_{k-2} S^1$ and we pick up a copy of the generator of $_1(\exp_{k-1} S^1 \, n \exp_{k-2} S^1)$ from each. However, recalling that $_0 \;\; I$ is the fake face mapping only to $\exp_{k-1}(S^1; 1)$ we see that the contribution from this face is just the constant loop. Thus $''$ maps to $k-1$ times the generator in $_1(\exp_{k-1} S^1 \, n \exp_{k-2} S^1)$ and we get the relation $\;\; ^k = \;\; ^{k-1}$ as required. □

Both and are null homotopic in $\exp_k S^1$ less the codimension three strata $\exp_{k-3} S^1$. To see this consider gure 7, which shows a movie of a homotopy in $\exp_k S^1 n \exp_{k-3} S^1$ from $= \;\; ^{-1}$ to the constant loop. The relation $\;\; ^k = \;\; ^{k-1}$ then gives $= \;\; = 1$.



(a) (b) (c) (d) (e)

Figure 7: The movie of a homotopy from to the constant loop in $\exp_k S^1$ less the codimension three strata. The gures show $k = 6$ but the method clearly generalises. Figure (a) shows , where we have cut the circle at -1 for clarity. Planar isotopy gives (b), then we merge the branch points (c) and separate them again (d) so that there are 6 distinct points throughout the path. Planar isotopy leads to the constant loop in (e).

3.3 The homotopy type of $\exp_k S^1 \setminus \exp_{k-2} S^1$

We now turn to the more delicate matter of showing that $\exp_k S^1 \setminus \exp_{k-2} S^1$ deformation retracts to a subspace having a cell structure as in figure 4. We will of course construct the deformation retraction upstairs in $\Delta \times I$ and some care will be required to ensure it descends to the quotient.

Let b_i be the barycentre of Δ_i. Since affine maps of simplices take barycentres to barycentres the identifications (4) in section 3.1 glue $\{b_i\} \times I$ to $\{b_j\} \times I$ for $i, j \neq 0$ and glues $[b, b_i] \times \{1\}$ to $[b, b_{i-1}] \times \{0\}$ for each i, while $\{b_0\} \times I$ is of course collapsed to $\{b_0\} \times \{0\}$. Letting B be the star graph $\bigcup_{i=0}^{k-1}[b, b_i]$ it is clear that $q(B \times I)$ may be given a cell structure as in figure 4. The fact that $\exp_k S^1 \setminus \exp_{k-2} S^1$ deformation retracts to this subspace is a consequence of the following technical lemma.

Lemma 2 *There is a deformation retraction ϕ from*

$$\operatorname{int} \Delta \cup \bigcup_{i=0}^{k-1} \operatorname{int} \Delta_i \quad \text{to} \quad B$$

such that

(a) $\phi_t(\operatorname{int} \Delta_i) \subset \operatorname{int} \Delta_i$ *for all t and $i = 0, \ldots, k-1$;*

(b) ϕ_t *commutes with the action of the symmetric group S_k on Δ, where $\sigma \in S_k$ acts by the canonical map $[v_0, \ldots, v_{k-1}] \to [v_{\sigma(0)}, \ldots, v_{\sigma(k-1)}]$.*

We will define ϕ using the barycentric subdivision of Δ, and first show:

Lemma 3 *Let $v_0, \ldots, v_n$ be affinely independent points. Then $[v_0, \ldots, v_n] \setminus [v_0, \ldots, v_{n-2}]$ deformation retracts to $[v_{n-1}, v_n]$ via a homotopy Ψ such that*

$$\Psi_t(\sigma \setminus [v_0, \ldots, v_{n-2}]) \subset \sigma \setminus [v_0, \ldots, v_{n-2}] \tag{5}$$

for all t and each face σ of $[v_0, \ldots, v_n]$.

Proof The proof is by induction on n, the case $n = 1$ being trivial. Define a deformation retraction ϕ to $[v_1, \ldots, v_n] \setminus [v_1, \ldots, v_{n-2}]$ by

$$\phi_t \sum_{i=0} \lambda_i v_i = (1-t) \sum_{i=0} \lambda_i v_i + t \sum_{i=1} \frac{\lambda_i}{1-\lambda_0} v_i ;$$

this is well defined since λ_0 is never equal to 1, and the co-efficient of at least one of v_{n-1}, v_n is nonzero for all t. Moreover if v is a convex combination of $v_{i_0}, \ldots, v_{i_l}$ then $\phi_t(v)$ is too so the face condition (5) is satisfied. Applying the induction hypothesis to $[v_1, \ldots, v_n] \setminus [v_1, \ldots, v_{n-2}]$ gives the result. □

Proof of Lemma 2 A typical simplex in the barycentric subdivision of has the form $[\ _0, \dots, \ _{k-1}]$ where each $\ _i$ is the barycentre of an i–dimensional face of containing $\ _0, \dots, \ _{i-1}$. In particular $\ _{k-1} = b$, $\ _0$ is some vertex v_i and $\ _{k-2}$ is b_j for some $j \neq i$. Deleting the codimension two faces of deletes precisely $[\ _0, \dots, \ _{k-3}]$ from $[\ _0, \dots, \ _{k-1}]$ and we define on this simplex using the deformation retraction Ψ given by lemma 3.

Suppose $[\ _0, \dots, \ _{k-1}]$ and $[\ ^0_0, \dots, \ ^0_{k-1}]$ share a common face $[\ _{i_0}, \dots, \ _{i_\ell}]$. Then necessarily $\ _{i_j} = \ ^0_{i_j}$ for $j = 1, \dots, \ell$ and this face is fixed pointwise by the canonical map $[\ _0, \dots, \ _{k-1}] \to [\ ^0_0, \dots, \ ^0_{k-1}]$; the face condition (5) in Lemma 3 then shows that is well defined. Condition (a) follows from the face condition (5) applied to each simplex of the form $[\ _0, \dots, \ _{k-2}]$, and the commutativity of with the action of the symmetric group is a consequence of the fact that any $\ \in S_k$ permutes the barycentres of the i–dimensional faces. □

Corollary 3 (implies Theorem 6) *The space* $\exp_k S^1 \setminus \exp_{k-2} S^1$ *deformation retracts to* $q(B \ I)$.

Proof Crossing with the identity gives a deformation retraction of

$$\operatorname{int} \ \bigcup_{i=0}^{k-1} \operatorname{int} \ _i \ \ I \quad \text{to} \quad B \ \ I$$

and we check that this is compatible with the gluings

(1) $\ : \ _i \ \{1\} \to \ _{i-1} \ \{0\}$,

(2) $[v_0, \dots, \hat{v}_i, \dots, v_{k-1}] \ I \to [v_0, \dots, \hat{v}_j, \dots, v_{k-1}] \ I$ for $i, j \neq 0$,

(3) $\ _0 \ I \to \ _0 \ \{0\}$.

Compatibility with (1) follows from commutativity of with S_k; compatibility with (2) uses commutativity with S_k together with condition (a) of Lemma 2; and compatibility with (3) comes from constructing the homotopy by crossing with the identity on I. □

4 The degree of an induced map

A map $f\colon S^1 \to S^1$ induces a map $\exp_k f\colon \exp_k S^1 \to \exp_k S^1$ of homotopy spheres, the degree of which depends only on k and the degree of f. We claim (Theorem 7) that

$$\deg \exp_k f = (\deg f)^{\lfloor \frac{k+1}{2} \rfloor}.$$

Proof of Theorem 7 We begin by reducing to the case where $k = 2\ell - 1$ is odd, using the commutative diagram

$$\begin{array}{ccc} \exp_{2\ell-1} S^1 & \xrightarrow{\exp_{2\ell-1} f} & \exp_{2\ell-1} S^1 \\ \Big\downarrow {\scriptstyle \cup\{1\}} & & \Big\downarrow {\scriptstyle \cup\{f(1)\}} \\ \exp_{2\ell} S^1 & \xrightarrow{\exp_{2\ell} f} & \exp_{2\ell} S^1 \end{array} .$$

The vertical arrows are degree one maps by the results of section 2.4, so we have

$$\deg \exp_{2\ell} f = \deg \exp_{2\ell-1} f$$

and it suffices to show that $\deg \exp_{2\ell-1} f = (\deg f)^{\ell}$. We do this by considering separately the cases $\deg f > 0$ and $\deg f = -1$; in both cases we assume that f is the map $\zeta \mapsto \zeta^d$ and count the preimages of a generic point with signs.

Suppose that $f(\zeta) = \zeta^d$ with d positive. A generic point Z of $\exp_k S^1$ will have d^k preimages under $\exp_k f$, corresponding to the d choices for the preimage of each element of Z. For concreteness let $\zeta_0, \ldots, \zeta_{k-1}$ be the kth roots of unity, cyclicly ordered so that

$$\zeta_r = e^{2\pi i r/k}.$$

Under f each ζ_r has d preimages $\xi_{r,s}$, $s = 0, \ldots, d-1$, which we again cyclicly order so that

$$\xi_{r,s} = e^{2\pi i(sk+r)/kd}.$$

Then a preimage of $\{\zeta \mid \zeta^k = 1\}$ is specified by a k–tuple $(s_0, \ldots, s_{k-1})$ of integers mod d, corresponding to the set

$$\{\xi_{0,s_0}, \ldots, \xi_{k-1,s_{k-1}}\},$$

and comes with a positive or negative sign according to whether this set is an even or odd permutation of cyclic order when ordered as written. Note that the sign relative to cyclic ordering makes sense since $k = 2\ell - 1$ is odd.

Our goal is to match the preimages up in cancelling pairs until those that are left are all positive. To this end consider the involution τ_{01} given by $(s_0, \ldots, s_{k-1}) \mapsto (s_1, s_0, s_2, \ldots, s_{k-1})$; we claim that if a preimage S is not fixed by τ_{01} then S and $\tau_{01}(S)$ have opposite signs. Indeed, if $s_0 \neq s_1$ and we move points from ξ_{0,s_0} to ξ_{0,s_1} and from ξ_{1,s_1} to ξ_{1,s_0} around the same arc of the circle then (since there can be no points between ξ_{0,s_j} and ξ_{1,s_j}) both must pass exactly the same points in between, and in addition one must pass the other (see figure 8). This involves an odd number of transpositions so S and $\tau_{01}(S)$ have opposite signs if $s_0 \neq s_1$.

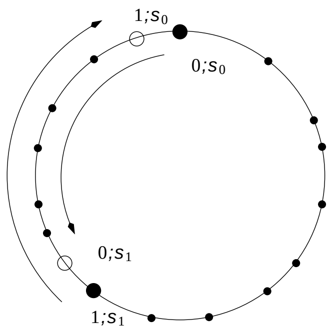


Figure 8: The preimages S and $\tau_{01}(S)$, if different, have opposite signs. We may move from one to the other by hopping a point at σ_{0,s_0} around the circle to σ_{0,s_1}, and then hopping a point at σ_{1,s_1} around the circle to σ_{1,s_0}. If we move both points around the same arc then both move past the same points in between, except that one must pass the other as well. This requires an odd number of transpositions.

Applying the same argument in turn to the involutions switching s_{2j} and s_{2j+1}, $1 \leq j \leq \ell - 2$, acting just on those preimages fixed by all previous involutions, we see that we may match up all preimages in cancelling pairs except those for which $s_{2j} = s_{2j+1}$, $0 \leq j \leq \ell - 2$. Since these can all be shuffled to cyclic order by moving points around in pairs they are all positive, and there are d^ℓ of them as there are d choices for each s_{2j}, $j = 0, \ldots, \ell - 1$. This gives $\deg \exp_k f = d^\ell$ as desired.

Now consider $f(\zeta) = \zeta^{-1}$. The sole preimage of the ordered set $\{\zeta_0, \ldots, \zeta_{k-1}\}$ is $\{\zeta_0, \zeta_{k-1}, \ldots, \zeta_1\}$ which may be put in cyclic order using $(k-1)/2 = \ell - 1$ transpositions. We get an additional factor of $(-1)^k$ from the product of the local degrees of f at each ζ_j, so

$$\deg \exp_k f = (-1)^{\ell-1} \cdot (-1)^{2\ell-1} = (-1)^{\ell-2} = (\deg f)^\ell$$

as required.

If $\deg f = 0$ then clearly $\deg \exp_k f = 0$, so putting the cases $\deg f > 0$ and $\deg f = -1$ together using $\deg(g \circ h) = (\deg g) \cdot (\deg h)$ gives the result. □

A Cut and paste proof of Theorems 1 and 2

We give a mostly pictorial proof that $\exp_3 S^1$ is S^3 and that $\exp_1 S^1$ inside it is a left-handed trefoil knot.

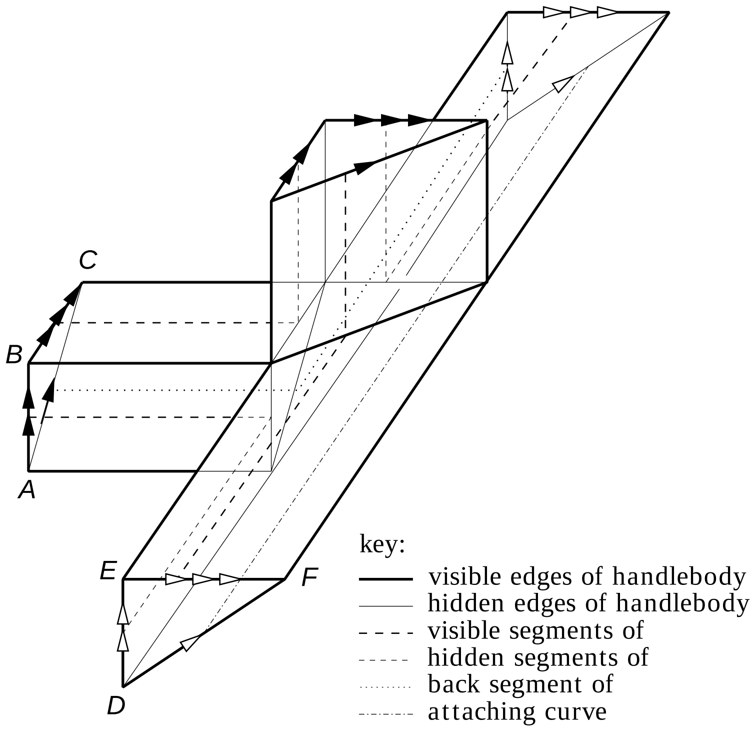


Figure 9: A regular neigbourhood of the dual 1{skeleton of the triangulation of $\exp_3 S^1$ in gure 3. Glue the triangular faces as indicated to get a genus two solid handlebody H forming half of a Heegaard splitting of $\exp_3 S^1$. The curves and bound discs in the second handlebody H^0.

We found in section 2.3 that $\exp_3 S^1$ is a 3{manifold with a triangulation consisting of just one 3{simplex, and in the standard way we obtain a Heegaard splitting by regarding it as the union H^0 [$_@$ H of regular neighbourhoods of the 1{ and dual 1{skeletons. A regular neighbourhood of the dual 1{skeleton of the 3{simplex of gure 3(a) is shown in gure 9 and H is the genus two solid handlebody given by gluing the four triangular faces at the end of each \arm" as indicated by the arrows. We keep track of H^0 by recording loops and linking each of the edges, the loops forming the attaching circles for the 2{handles of $\exp_3 S^1$. The loop linking the edge a of gure 3(b) appears in ve pieces in gure 9, four of which are shown dashed and the fth dotted, while the loop linking the edge b appears in just one piece, indicated by the dash-dot-dash-dot line.

Bending the arms and giving the top arm a one-third twist so the arrows match

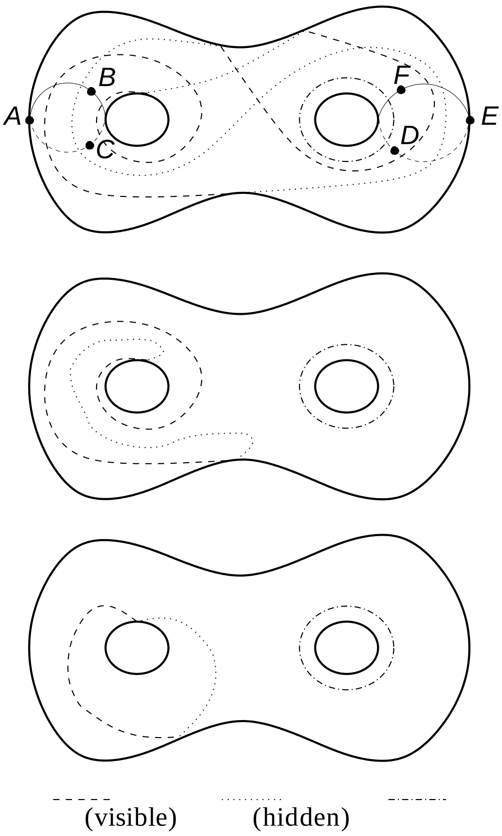


Figure 10: Heegaard diagrams of $\exp_3 S^1$. Top: The handlebody H obtained by gluing the triangular faces of gure 9. Middle: Attaching curve after sliding the loops going around the right 1{handle over the cancelling 2{handle. Bottom: After isotopy forms a (1; 1){curve around the left 1{handle, giving a Heegaard diagram of S^3.

we glue the triangular faces to obtain H, shown in the top diagram in gure 10. The curve is again indicated by a dash-dot-dash-dot pattern and we see immediately that the attached 2{handle cancels the right 1{handle. The curve linking a, shown dashed when it is on top and dotted when it is underneath, goes geometrically twice over the right 1{handle and we slide each loop over the cancelling 2{handle to get the middle diagram. Further isotopy leads to the bottom diagram in which appears as a (1; 1){curve around the left 1{handle. This may be recognised as a Heegaard diagram for S^3 but we nevertheless apply a Dehn twist to convert it to the standard diagram. Writing the meridian rst and giving ∂H the right-hand orientation induced by H the appropriate Dehn

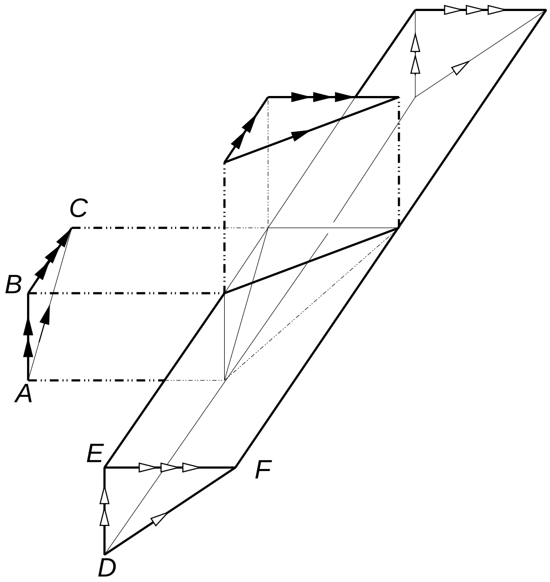


Figure 11: Generic orbits of the S^1 action on $\exp_3 S^1$ meet the 3–simplex in three lines parallel to the direction $(1,1,1)$. Perturbing them to lie on the Heegaard surface we may obtain the three arcs shown dash-dot-dotted.

twist acts by

$$T = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

on $H_1(\partial(H \cup (D^2 \times I)))$.

We now turn our attention to $\exp_1 S^1 \hookrightarrow \exp_3 S^1$, corresponding to the edge b of figure 3(b). We take a push-off of b into the interior of the simplex and perturb it to lie on the Heegaard surface. This is most easily done by recalling that $\exp_1 S^1$ forms a generic (here meaning trivial stabiliser) orbit of the S^1 action on $\exp_3 S^1$, and that generic orbits passing through the interior of the simplex break into three lines parallel to the vector $(1,1,1)$. Pushing them onto the Heegaard surface we obtain the arcs shown dash-dot-dotted in figure 11. The resulting curve on ∂H appears in figure 12, shown dashed when it is on top and dotted underneath, and as might be expected from figure 11 it forms a $(1,3)$–curve around the left 1–handle. The Dehn twist with matrix T that converts the bottom Heegaard diagram of figure 10 to the standard diagram takes this curve to a $(-2,3)$–curve, giving a left-handed trefoil as promised.

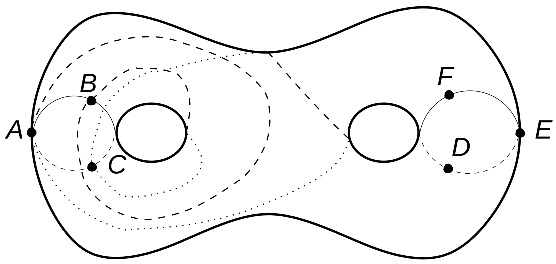


Figure 12: The perturbed orbit, shown dashed when on top and dotted when below, after gluing gure 11 up to form H. It traces a $(1;3)${curve on the left 1{handle and is transformed to a $(-2;3)${curve by the Dehn twist.

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Received: 22 October 2002