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Grafting Seiberg-Witten monopoles

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Abstract We demonstrate that the operation of taking disjoint unions of J-holomorphic curves (and thus obtaining new J-holomorphic curves) has a Seiberg-Witten counterpart. The main theorem asserts that, given two solutions (A_i, ψ_i), $i = 0, 1$ of the Seiberg-Witten equations for the $Spin^c$-structures $W^+_{E_i} = E_i \oplus (E_i \otimes K^{-1})$ (with certain restrictions), there is a solution (A, ψ) of the Seiberg-Witten equations for the $Spin^c$-structure W_E with $E = E_0 \otimes E_1$, obtained by "grafting" the two solutions (A_i, ψ_i).

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1 Introduction

In his series of groundbreaking works [4], [5], [6], Taubes showed that the Seiberg-Witten invariants and the Gromov-Witten invariants (as defined in [7]) for a symplectic 4-manifold (X, ω) are the same. His results opened the door to a whole new world of interactions between the two theories that had previously only been speculations. The most spectacular outcomes of this interplay were new results that in one theory were obvious but when translated into the other theory, became highly nontrivial. An example of such a phenomenon is the simple formula relating the Seiberg-Witten invariant of a $Spin^c$-structure W to the Seiberg-Witten invariant of its dual $Spin^c$-structure W^*, i.e. the one with $c_1(W^*) = -c_1(W)$. The formula reads:

$$SW_X(W^*) = \pm SW_X(W)$$

When translated into the Gromov-Witten language, this duality becomes

$$Gr_X(E) = \pm Gr_X(K - E) \qquad (1)$$

Here K is the canonical class of (X, ω) and $E \in H^2(X; \mathbb{Z})$ is related to W as $c_1(W^+) = 2E - K$. This is a highly nonobvious result about J-holomorphic curves, even in the simplest case when $E = 0$. In that case we obtain that

$Gr_X(K) = \pm Gr_X(0) = \pm 1$, the latter equation simply being the definition of $Gr_X(0)$. This gives an existence result of a J-holomorphic representative for the class K, a result unknown prior to Taubes' theorem. The formula (1) has recently been proved by S. Donaldson and I. Smith [1] without any reference to Seiberg-Witten theory (but under slightly stronger restrictions on (X, ω) than in Taubes' theorem).

In the author's opinion, proving a result about Gromov-Witten theory which had only been known through its relation with Seiberg-Witten theory, without relying on the latter, has a number of benefits. One is to understand Gromov-Witten theory from within better. But also to possibly generalize the theorem to a broader class of manifolds. Recall that Taubes' theorem equates the two invariants only on symplectic 4-manifolds. Both Seiberg-Witten and Gromov-Witten theory are defined over larger sets of manifolds, namely all smooth 4-manifolds and all symplectic manifolds (of any dimension) respectively. On the other hand, even within the category of symplectic 4-manifolds, one can hope for more nonvanishing theorems i.e. theorems of the type $Gr_X(E) \neq 0$ for classes $E \neq 0, K$. The techniques used by Donaldson and Smith are promising in that direction.

The aim of this paper is to prove a result in the same vein but going the opposite direction. Namely, on the Gromov-Witten side, given two classes $E_i \in H^2(X;\mathbb{Z})$, $i = 0, 1$ with $E_0 \cdot E_1 = 0$ and J-holomorphic curves Σ_i with $[\Sigma_i] = P.D.(E_i)$, one can define a new J-holomorphic curve $\Sigma = \Sigma_0 \sqcup \Sigma_1$. By the assumption $E_0 \cdot E_1 = 0$, the two curves Σ_i are either disjoint or share toroidal components (see [2]). In the former case, Σ is simply the disjoint union of Σ_0 and Σ_1 and in the latter case one needs to replace the tori shared by Σ_0 and Σ_1 with their appropriate multiple covers. This induces a map on moduli spaces:

$$\mathcal{M}_X^{Gr}(E_0) \times \mathcal{M}_X^{Gr}(E_1) \xrightarrow{\sqcup} \mathcal{M}_X^{Gr}(E_0 + E_1) \tag{2}$$

This article describes the Seiberg-Witten counterpart of (2). That is, given two complex line bundles E_0 and E_1 (with certain restrictions, see the assumption 3.1 below for a precise statement) and two solutions (A_i, Φ_i) of the Seiberg-Witten equations for the $Spin^c$-structures $W_{E_i} = E_i \oplus (E_i \otimes K^{-1})$, $i = 0, 1$ and with Taubes' large r perturbation, we show how to produce a solution $(A, \Phi) = (A_0, \Phi_0) \otimes (A_1, \Phi_1)$ for the $Spin^c$-structure W_E with $E = E_0 \otimes E_1$. This operation induces the following commutative diagram:

$$
\begin{array}{ccc}
M_X^{SW}(E_0)\times M_X^{SW}(E_1) & \longrightarrow & M_X^{SW}(E_0\otimes E_1)\\
\Big\downarrow & & \Big\downarrow\\
M_X^{Gr}(E_0)\times M_X^{Gr}(E_1) & \xrightarrow{\ t\ } & M_X^{Gr}(E_0+E_1)
\end{array}
\tag{3}
$$

Here the map $\Phi : M_X^{SW}(E) \to M_X^{Gr}(E)$ is the map described in [4] that associates to each solution of the Seiberg-Witten equations an embedded J-holomorphic curve. The solution (A, ψ) is obtained by "grafting" the two solutions (A_i, ψ_i). The key observation here is that for the large r version of Taubes' perturbation, a solution (B, ψ) of the Seiberg-Witten equations for the $Spin^c$-structure W_E is "concentrated" near the zero set of $\frac{\alpha}{\sqrt{r}}$, the E component of ψ. That is, the restriction of (B, ψ) to the complement of a regular neighborhood of $\alpha^{-1}(0)$ converges pointwise (under certain bundle identifications) to the unique solution $(A_0, \sqrt{r}\,u_0)$ for the anticanonical $Spin^c$-structure $W_0 = \underline{\mathbb{C}} \oplus K^{-1}$. This is used to define a first approximation of ψ by declaring it to be equal to ψ_i in a regular neighborhood V_i of $\alpha_i^{-1}(0)$ and equal to $\sqrt{r}\,u_0$ on the complement of $V_0 \cup V_1$. Bump functions are used to produce a smooth spinor. The first approximation of A is simply the product connection $A_0 \otimes A_1$. The contraction mapping principle is then evoked to deform this approximate solution to an honest solution of the Seiberg-Witten equations. The author has learned the techniques employed in this article from the inspiring work of Taubes on gauge theory of symplectic 4-manifolds, most notably from [5].

The article is organized as follows. In section 2 we review the needed Seiberg-Witten theory on symplectic 4-manifolds. Section 3 explains how to define an "almost" monopole (A', ψ') from a pair of monopoles (A_i, ψ_i), $i = 0, 1$. It also analyzes the asymptotic (as $r \to \infty$) regularity theory for the linearized operators $L_{(A_i, \psi_i)}$ and deduces a corresponding result for $L_{(A', \psi')}$. The latter is used in combination with the contraction mapping principle to obtain an "honest" monopole (A, ψ). Section 4 compares the present method of grafting monopoles to the one used in exploring Seiberg-Witten theory on manifolds X which are obtained as a fiber sum: $X = X_1 \#_\Sigma X_2$. Section 5 proves a converse to theorem 3.11. It explains which monopoles in the $Spin^c$-structure W_E can be obtained as products of monopoles (A_i, ψ_i) in the $Spin^c$-structures W_{E_i}, $i = 0, 1$ with $E_0 \otimes E_1 = E$ and with the property that $\Phi(A_i, \psi_i)$ does not contain multiply covered tori.

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2 The Seiberg-Witten equations on symplectic manifolds

2.1 Introduction

Let (X, ω) be a symplectic, smooth, compact 4-manifold with symplectic form ω. Denote by $\mathcal{J}$ the set of all almost complex structures J on TX that are compatible with ω, i.e. the ones for which

$$g(v, w) = g_J(v, w) = \omega(v, Jw) \qquad \forall\, v, w \in TX$$

defines a Riemannian metric on X. Given a $J \in \mathcal{J}$, the associated metric g_J will always be assumed throughout to be the metric of choice.

On any almost complex 4-manifold there is a anticanonical $Spin^c$-structure $W_0 = W_0^+ \oplus W_0^-$ determined by the almost complex structure as:

$$\begin{aligned} W_0^+ &= \Lambda^{0,0} \oplus \Lambda^{0,2} = \underline{\mathbb{C}} \oplus K^{-1} \\ W_0^- &= \Lambda^{0,1} \\ v.\alpha &= \sqrt{2}\,\left(v^{0,1} \wedge \alpha - \iota_v \alpha\right) \qquad v \in T_xX, \ \alpha \in W_{0,x}, \ x \in X \end{aligned}$$

In the above, $v^{0,1} \in \Lambda^{0,1}$ denotes the (0,1) projection of $v^* \in T_x^*X$, the dual of $v \in T_xX$. All other $Spin^c$-structures can be obtained from W_0 by tensoring it with a complex line bundle E and extending Clifford multiplication trivially over the E factor, i.e.

$$\begin{aligned} W_E &= E \otimes W_0 \\ v.(\varphi \otimes \alpha) &= \varphi \otimes (v.\alpha) \qquad \varphi \in E_x, \ v \in T_xX, \ \alpha \in W_{0,x}, \ x \in X \end{aligned}$$

The symplectic form ω induces a splitting of $\Lambda^{2,+}$ as

$$\Lambda^{2,+} = \mathbb{R}\,\omega \oplus \Lambda^{0,2} \tag{4}$$

which will be used below to write the curvature component of the Seiberg-Witten equations as two equations, one for each of the summands on the right-hand side of (4).

Given a $Spin^c$-structure W_E on X, the Seiberg-Witten equations are a coupled, elliptic system of equations for a pair (A, ψ) of a connection A on E and a positive spinor $\psi \in \Gamma(W_E^+) = \Gamma(E \oplus (E \otimes K^{-1}))$. The connection A on E together with a fixed connection A_0 on K^{-1} (which will be made specific in a bit), induces a $Spin^c$-connection on W_E which we will denote by ∇^A and which in turn gives rise to the Dirac operator $D_A : \Gamma(W_E) \to \Gamma(W_E)$. It proves convenient to write the spinor ψ in the form

$$\psi = \sqrt{r}(\alpha, \beta) \qquad \alpha \in \Gamma(E), \quad \beta \in \Gamma(E \otimes K^{-1})$$

where $r \geq 1$ is a parameter whose significance will become clear later. With this understood, the Seiberg-Witten equations read:

$$\begin{aligned} D_A(\psi) &= 0 \\ F_A^+ &= q(\psi, \psi) + \mu \end{aligned} \tag{5}$$

Here $\mu \in i\Omega^{2,+}$ is a fixed imaginary, self-dual two form on X and $q : \Gamma(W_E^+) \otimes \Gamma(W_E^+) \to i\Omega^{2,+}$ is the bilinear quadratic map given explicitly by

$$q(\psi, \psi) = \frac{ir}{8}(|\alpha|^2 - |\beta|^2)\omega + \frac{ir}{4}(\bar{\alpha}\beta + \alpha\bar{\beta}) \tag{6}$$

2.2 The anticanonical $Spin^c$-structure

It is another result of Taubes' [8] that the Seiberg-Witten invariant of the anticanonical $Spin^c$-structure on a symplectic manifold is equal to ± 1. Furthermore, the equations have exactly one solution $(A_0, \sqrt{r}\, u_0)$, $u_0 \in \Gamma(\underline{\mathbb{C}})$, for the choice of

$$\mu = F_{A_0}^+ - \frac{ir}{8}\omega \tag{7}$$

in (5) and for $r \geq 1$. The purpose of this section is to describe the solution $(A_0, \sqrt{r}\, u_0)$ and its linearized operator.

The pair $(A_0, \sqrt{r}\, u_0)$ is characterized (up to gauge) by the condition

$$\langle \nabla^0 u_0, u_0 \rangle = 0 \tag{8}$$

(where ∇^0 is the $Spin^c$-connection induced by A_0) and can be obtained as follows: let u_0 be any section of $\underline{\mathbb{C}} \oplus K^{-1}$ with $|u_0| = 1$ and whose projection onto the second summand is zero. Likewise, let A be any connection on K^{-1} and let ∇^A be its induced $Spin^c$-connection on $W_0^+ = \underline{\mathbb{C}} \oplus K^{-1}$. Set $a = \langle u_0, \nabla^A u_0 \rangle$. This defines an imaginary valued 1-form as can easily be seen:

$$a + \bar{a} = \langle \nabla^A u_0, u_0 \rangle + \langle u_0, \nabla^A u_0 \rangle = d|u_0|^2 = 0$$

Define the connection A_0 on K^{-1} by $A_0 = A - a$ which induces the $Spin^c$-connection $\nabla^0 = \nabla^A - a$ on W_0^+. This connection clearly satisfies (8). With the choice of μ as in (7), the Seiberg-Witten equations (5) take the form:

$$\begin{aligned} D_A \psi &= 0 \\ F_A^+ &= \frac{ir}{8}(|\alpha|^2 - 1 - |\beta|^2)\omega + F_{A_0}^+ + \frac{ir}{4}(\bar{\alpha}\beta + \alpha\bar{\beta}) \end{aligned} \tag{9}$$

Since the β-component of u_0 is zero and since $|\alpha| = |u_0| = 1$, the pair (A_0, u_0) clearly solves the second equation of (9). The fact that is also solves the first equation relies on the property $d^*\omega = 0$ of ω as well as (8). Taubes [8] showed that there are, up to gauge, no other solutions to (9) and, as we shall presently see, that the solution (A_0, u_0) is a smooth solution in the sense that the linearization of (9) at (A_0, u_0) has trivial cokernel. These two facts together show that $SW_X(W_0) = \pm 1$.

Define $S : L^{1,2}(i\Lambda^1 \oplus W_0^+) \to L^2(i\Lambda^0 \oplus i\Lambda^{2,+} \oplus W_0^-)$ to be the linearized Seiberg-Witten operator for the solution (A_0, u_0). Thus, for $(b, (\alpha_0, \beta_2)) \in L^{1,2}(i\Lambda^1 \oplus (\underline{\mathbb{C}} \oplus K^{-1}))$ we have:

$$S(b,(\alpha_0,\beta_2)) = \begin{cases} d^*b + i\frac{\sqrt{r}}{\sqrt{2}}\,\mathrm{Im}(\bar{u}_0\alpha_0); \\ d^+b - \sqrt{r}\,q(\psi, u_0) - \sqrt{r}\,q(u_0,\psi); \\ D_{A_0}(\alpha_0,\beta_2) + \frac{\sqrt{r}}{2}\, b.u_0 \end{cases} \tag{10}$$

Let $S^* : L^2(i\Lambda^0 \oplus i\Lambda^{2,+} \oplus W_0^-) \to L^{1,2}(i\Lambda^1 \oplus W_0^+)$ be the formal adjoint of S. The following proposition and corollary are proved in [5], section 4.

Proposition 2.1 *Let S and S^* be as above. Then the operator SS^* on $L^2(i\Lambda^0 \oplus i\Lambda^{2,+} \oplus W_0^-)$ is given by*

$$SS^* = \frac{1}{4}\nabla^{0,*}\nabla^0 + R_0 + \sqrt{r}R_1 + \frac{r}{8} \tag{11}$$

where $\nabla^{0,*}$ is the adjoint of ∇^0 and where R_i, $i = 0, 1$ are certain r-independent endomorphism on $L^2(i(\Lambda^0 \oplus \Lambda^{2,+}) \oplus W_0^-)$.

The proof is a straightforward calculation, terms of the form $D_{A_0}D_{A_0}^*$ are simplified using the Weitzenböck formula for the Dirac operator. An important consequence of (11) is the following:

Corollary 2.2 *With S and S^* as above, the smallest eigenvalue λ_1 of SS^* is bounded from below by $r/16$. In particular, S is invertible and S^{-1} satisfies the bounds*

$$\|S^{-1}y\|_2 \le \frac{4}{\sqrt{r}}\|y\|_2 \quad \text{and} \quad \|S^{-1}y\|_{1,2} \le C\|y\|_2 \tag{12}$$

where C is r-independent.

2.3 The general case and bounds on (a,ψ)

Consider now a $Spin^c$-structure $W_E = E \otimes W_0$ on X. The connection A_0 on K^{-1} and a choice of a connection B_0 on E together induce a connection $B_0^{\otimes 2} \otimes A_0$ on $E^{\otimes 2} \otimes K^{-1} = c_1(W_E^+)$ by the product rule:

$$B_0^{\otimes 2} \otimes A_0(\varphi_1 \otimes \varphi_2 \otimes \eta) = B_0(\varphi_1) \otimes \varphi_2 \otimes \eta + \varphi_1 \otimes B_0(\varphi_2) \otimes \eta + \varphi_1 \otimes \varphi_2 \otimes A_0(\eta)$$

The space of connections on $E^{\otimes 2} \otimes K^{-1}$ is an affine space with associated vector space $i\Omega^1_X$. With the choice of a "base" connection $B_0^{\otimes 2} \otimes A_0$ in place, we will from now on regard solutions to the Seiberg-Witten equations as pairs $(a,\psi) \in i\Omega^1_X \oplus \Gamma(W_E^+)$ rather than $(A,\psi) \in \mathrm{Conn}(E^{\otimes 2} \otimes K^{-1}) \oplus \Gamma(W_E^+)$, the relation between the two being:

$$A = B_0^{\otimes 2} \otimes A_0 + a$$

We will agree to use henceforth the choice of η in (5) to be:

$$\eta = -\frac{ir}{8}\omega + F^+_{A_0} \tag{13}$$

For $\psi \in \Gamma(E \otimes (\mathbb{C} \oplus K^{-1}))$ we will write $\psi = \sqrt{r}(\alpha u_0, \beta)$ with $\alpha \in \Gamma(E)$ and $\beta \in \Gamma(E \otimes K^{-1})$ and u_0 as in the previous section.

With these conventions understood and with the use of (4), the Seiberg-Witten equations (5) become:

$$\begin{aligned} D_a\psi &= 0 \\ F_a^{1,1} &= \frac{ir}{8}(|\alpha|^2 - |\beta|^2 - 1)\,\omega \\ F_a^{0,2} &= \frac{ir}{4}\bar{\alpha}\beta \end{aligned} \tag{14}$$

Here $F_a^{i,j}$ is the orthogonal projection of $2F^+_{B_0} + d^+a$ onto $\Lambda^{i,j}$. The linearized operator $L_{(a,\psi)} : L^{1,2}(i\Lambda^1 \oplus W_E^+) \to L^2(i\Lambda^0 \oplus i\Lambda^{2,+} \oplus W_E^-)$ of the Seiberg-Witten equations for a solution (a,ψ) of (14) is:

$$L_{(a,\psi)}(b,(\zeta_0,\zeta_2)) = \begin{cases} d^*b + i\frac{\sqrt{r}}{\sqrt{2}}\,\mathrm{Im}(\langle\psi,\zeta_0\rangle + \langle\psi,\zeta_2\rangle); \\ d^+b - \sqrt{r}\,q(\zeta_0+\zeta_2,\psi) - \sqrt{r}\,q(\psi,\zeta_0+\zeta_2); \\ D_a(\zeta_0+\zeta_2) + \frac{\sqrt{r}}{2}\,b\cdot\psi. \end{cases} \tag{15}$$

It is another result of Taubes' that the operator $L_{(a,\psi)}$ has Fredholm index zero on a symplectic manifold with $b_2^+ \geq 2$, provided that E is a basic class. As we will use this fact repeatedly throughout the paper, we give a short proof of it here:

Theorem 2.3 (Taubes) *Let X be a symplectic manifold with $b_2^+ \geq 2$ and $E \in H^2(X;\mathbb{Z})$ a basic class, i.e. $SW(W_E) \neq 0$. Let (a,ψ) be a solution of* (14) *and $L_{(a,\psi)}$ be the operator defined by* (15)*. Then the Fredholm index of $L_{(a,\psi)}$ is equal to zero.*

Proof As E is assumed to be a Seiberg-Witten basic class, it has to also be a Gromov-Witten basic class. In particular, the dimension of the Gromov-Witten moduli space has to be non-negative:

$$\dim \mathcal{M}^{Gr}(E) = \frac{1}{2}(E^2 - K \cdot E) \geq 0$$

Let Σ be an embedded J-holomorphic curve in X with $[\Sigma] = P.D.(E)$. Then the adjunction formula for Σ states:

$$2g - 2 = E^2 + K \cdot E$$

Combining these last two relations we obtain two inequalities:

$$E^2 \geq g - 1 \qquad \text{and} \qquad K \cdot E \leq g - 1$$

Let $n \geq 0$ be the integer such that $E^2 = g - 1 + n$ and $K \cdot E = g - 1 - n$. Since E is a Gromov-Witten basic class, by duality, so is $K - E$. But then (by positivity of intersection of J-holomorphic curves) we must have:

$$0 \leq E \cdot (K - E) = E \cdot K - E^2 = g - 1 - n - (g - 1 + n) = -2n \leq 0$$

This forces $n = 0$ and so $E^2 = g - 1 = K \cdot E$. Using these in the index formula for $L_{(A,\psi)}$ immediately yields the desired result:

$$\text{Ind}\, L_{(a,\psi)} = \frac{1}{4}\left((2E - K)^2 - (3\sigma + 2e)\right) = \frac{1}{4}\left(K^2 - (3\sigma + 2e)\right) = 0 \qquad \square$$

We also use this section to remind the reader of several useful bounds that a solution (a,ψ) of the Seiberg-Witten equations satisfies. These bounds are provided courtesy of [4] and their proofs rely solely on properties of the Seiberg-Witten equations.

A solution (a, ψ) of (14) satisfies the following bounds:

$$
\begin{aligned}
|\alpha| &\leq 1 + \frac{C}{r} \\
|\beta|^2 &\leq \frac{C}{r}(1 - |\alpha|^2) + \frac{C'}{\sqrt{r^3}} \\
|\nabla_A \alpha|_x^2 &\leq C\sqrt{r}\exp\left(-\frac{\sqrt{r}}{C}\operatorname{dist}(x, \alpha^{-1}(0))\right), \quad x \in X \\
|1 - |\alpha(x)|^2| &\leq C\exp\left(-\frac{\sqrt{r}}{C}\operatorname{dist}(x, \alpha^{-1}(0))\right), \quad x \in X \\
|F_a| &\leq \frac{r}{4\sqrt{2}}(1 - |\alpha|^2) + C \\
|F_a(x)| &\leq C\, r\exp\left(-\frac{1}{C}\sqrt{r}\operatorname{dist}(x, \alpha^{-1}(0))\right), \quad x \in X
\end{aligned}
\tag{16}
$$

Remark 2.4 The constant C appearing above may change its value from line to line. It is important to point out that C only depends on the $Spin^c$-structure W_E and the Riemannian metric g but **not** on the particular choice of the parameter r. This will be the case for all the numerous constants (all labeled C) appearing subsequently and we will henceforth tacitly adopt this misuse of notation.

3 The main part

3.1 Producing the approximate solution (a, ψ) from a pair (a_0, ψ_0), (a_1, ψ_1)

Let E_0 and E_1 be two complex line bundles over X. The aim of this section is to produce an approximate solution (a, ψ) of the Seiberg-Witten equations for the $Spin^c$-structure $W_{E_0 \otimes E_1}$ from two solutions (a_0, ψ_0) and (a_1, ψ_1) for the $Spin^c$-structures W_{E_0} and W_{E_1} respectively. Implicit to our discussion are the choices of two "base" connections B_0 and B_1 on E_0 and E_1 and the product connection $B_0 \otimes B_1$ they determine on $E_0 \otimes E_1$. As before, we will write $\psi_i = \sqrt{r}(\alpha_i u_0, \beta_i)$, $i = 0, 1$, and $\psi = \sqrt{r}(\alpha u_0, \beta)$. We define (a, ψ) as:

$$
\begin{aligned}
a &= a_0 + a_1 \\
\alpha &= \alpha_0 \alpha_1 \\
\beta &= \alpha_0 \beta_1 + \alpha_1 \beta_0
\end{aligned}
\tag{17}
$$

The first task at hand is to check how close (a, ψ) comes to solving the Seiberg-Witten equations. We begin by calculating $D_a \psi$ locally at a point $x \in X$. Choose an orthonormal frame $\{e_i\}$ in a neighborhood of x and let $\{e^i\}$ be its dual frame.

$$
\begin{aligned}
D_a(\psi) &= \sqrt{r} D_a(\alpha_0 \otimes \alpha_1 \otimes u_0 + \alpha_0 \otimes \beta_1 + \alpha_1 \otimes \beta_0) \\
&= \sqrt{r} D_{a_0}(\alpha_0 \otimes u_0) \otimes \alpha_1 + \sqrt{r}\, \alpha_0 \otimes D_{a_1}(\alpha_1 \otimes u_0) + \\
&\quad + \sqrt{r} e^i \cdot \nabla^a_{e_i}(\alpha_0 \otimes \beta_1 + \alpha_1 \otimes \beta_0) \\
&= \sqrt{r} D_{a_0}(\alpha_0 \otimes u_0) \otimes \alpha_1 + \sqrt{r}\, \alpha_0 \otimes D_{a_1}(\alpha_1 \otimes u_0) + \\
&\quad + \sqrt{r}(\alpha_0 \otimes e^i \cdot (\nabla^{a_1}_{e_i} \beta_1) + \alpha_1 \otimes e^i \cdot (\nabla^{a_0}_{e_i} \beta_0) + \\
&\quad + (\nabla^{a_0}_{e_i} \alpha_0) \otimes e^i \cdot \beta_1 + (\nabla^{a_1}_{e_i} \alpha_1) \otimes e^i \cdot \beta_0) \\
&= \sqrt{r} D_{a_0}(\alpha_0 \otimes u_0) \otimes \alpha_1 + \sqrt{r}\, \alpha_0 \otimes D_{a_1}(\alpha_1 \otimes u_0) + \\
&\quad + \sqrt{r}(\alpha_0 \otimes D_{a_1} \beta_1 + \alpha_1 \otimes D_{a_0} \beta_0) + \\
&\quad + \sqrt{r}((\nabla^{a_0}_{e_i} \alpha_0) \otimes e^i \cdot \beta_1 + (\nabla^{a_1}_{e_i} \alpha_1) \otimes e^i \cdot \beta_0) \\
&= (D_{a_0} \psi_0) \otimes \alpha_1 + \alpha_0 \otimes (D_{a_1} \psi_1) + \\
&\quad + \sqrt{r}((\nabla^{a_0}_{e_i} \alpha_0) \otimes e^i \cdot \beta_1 + (\nabla^{a_1}_{e_i} \alpha_1) \otimes e^i \cdot \beta_0) \\
&= \sqrt{r}(\nabla^{a_0}_{e_i} \alpha_0) \otimes e^i \cdot \beta_1 + \sqrt{r}(\nabla^{a_1}_{e_i} \alpha_1) \otimes e^i \cdot \beta_0
\end{aligned} \tag{18}
$$

It is easy to see, using the bounds in (16), that the first term in (18) satisfies the following pointwise estimate :

$$
\begin{aligned}
& r|(\nabla^{a_0}_{e_i} \alpha_0) \otimes e^i \cdot \beta_1|^2_x \leq \\
& \qquad \leq C r \exp\left(-\frac{\sqrt{r}}{C}\mathrm{dist}(x, \alpha_0^{-1}(0))\right) \cdot \exp\left(-\frac{\sqrt{r}}{C}\mathrm{dist}(x, \alpha_1^{-1}(0))\right)
\end{aligned} \tag{19}
$$

The second term in (18) satisfies the same bound. In order for the right hand side of (19) to pointwise converge to zero, it is sufficient and necessary that there exist some $r_0 \geq 1$ such that for all $r \geq r_0$, the distance from $\alpha_0^{-1}(0)$ to $\alpha_1^{-1}(0)$ be bounded from below by some r-independent $M > 0$. This condition, under the map Φ from (3), is the Seiberg-Witten equivalent of the condition that $C_i = \Phi(A_i, \psi_i)$ be disjoint curves. Thus, from now onward we will make the following assumption.

Assumption 3.1 *As above, let $E_0, E_1 \in H^2(X; \mathbb{Z})$ be two line bundles over X. Let (ψ_i, a_i), $i = 0, 1$, be two solutions to the Seiberg-Witten equations (14) for the $Spin^c$-structures W_{E_i} with $\psi_i = \sqrt{r}(\alpha_i \otimes u_0, \beta_0)$ and $\alpha_i \in \Gamma(E_i)$. We henceforth make the assumption that there exists an $r_0 \geq 1$ and $M > 0$ such that for all $r \geq r_0$ the inequality*

$$
dist(\alpha_0^{-1}(0), \alpha_1^{-1}(0)) \geq M \tag{20}
$$

holds.

We now proceed by looking at the second equation in (14):

$$
\begin{aligned}
F_a^{1,1} - \frac{i}{8} r(|\alpha|^2 - 1 - |\beta|^2)\,\omega &= \\
&= F_{a_0}^{1,1} + F_{a_1}^{1,1} - \frac{i}{8} r(|\alpha_0|^2|\alpha_1|^2 - 1 - |\beta_0|^2|\alpha_1|^2 - |\beta_1|^2|\alpha_0|^2 - \\
&\qquad - 2\langle \beta_0\bar\alpha_1, \beta_1\bar\alpha_0\rangle)\,\omega \\
&= F_{a_0}^{1,1} + F_{a_1}^{1,1} - \frac{i}{8} r|\alpha_1|^2(|\alpha_0|^2 - 1 - |\beta_0|^2)\,\omega - \frac{i}{8} r|\alpha_0|^2(|\alpha_1|^2 - 1 - |\beta_1|^2)\,\omega \\
&\qquad + \frac{i}{8} r(|\alpha_0|^2 - 1)(|\alpha_1|^2 - 1)\,\omega + \frac{i}{4} r\langle \beta_0\bar\alpha_1, \beta_1\bar\alpha_0\rangle\,\omega \\
&= \frac{i}{8} r(1 - |\alpha_1|^2)(|\alpha_0|^2 - 1 - |\beta_0|^2)\,\omega - \frac{i}{8} r(1 - |\alpha_0|^2)(|\alpha_1|^2 - 1 - |\beta_1|^2)\,\omega + \\
&\quad + \frac{i}{8} r(|\alpha_0|^2 - 1)(|\alpha_1|^2 - 1)\,\omega + \frac{i}{4} r\langle \beta_0\bar\alpha_1, \beta_1\bar\alpha_0\rangle\,\omega
\end{aligned}
$$

From this last equation, and again using (16), one easily deduces that:

$$
\begin{aligned}
&\left|F_a^{1,1} - \frac{i}{8} r(|\alpha|^2 - 1 - |\beta|^2)\,\omega\right| \qquad (21)\\
&\quad \le Cr \exp\left(-\frac{\sqrt{r}}{C}\operatorname{dist}(x, \alpha_0^{-1}(0))\right) \cdot \exp\left(-\frac{\sqrt{r}}{C}\operatorname{dist}(x, \alpha_1^{-1}(0))\right) + \frac{C}{\sqrt{r}}
\end{aligned}
$$

Finally, we consider the third equation in (14):

$$
\begin{aligned}
F_a^{0,2} - \frac{i}{4} r\bar\alpha\beta &= F_{a_0}^{0,2} + F_{a_1}^{0,2} - \frac{i}{4} r\overline{\alpha_0\alpha_1}(\alpha_0\beta_1 + \alpha_1\beta_0) \\
&= \frac{i}{4} r\bar\alpha_0\beta_0 + \frac{i}{4} r\bar\alpha_1\beta_1 - \frac{i}{4} r|\alpha_0|^2\bar\alpha_1\beta_1 - \frac{i}{4} r|\alpha_1|^2\bar\alpha_0\beta_0 \\
&= \frac{i}{4} r(1 - |\alpha_1|^2)\bar\alpha_0\beta_0 + \frac{i}{4} r(1 - |\alpha_0|^2)\bar\alpha_1\beta_1
\end{aligned}
$$

Once again using the bounds (16), we find from this last equation:

$$
\begin{aligned}
&\left|F_a^{0,2} - \frac{i}{4} r\bar\alpha\beta\right| \qquad (22)\\
&\quad \le Cr \exp\left(-\frac{\sqrt{r}}{C}\operatorname{dist}(x, \alpha_0^{-1}(0))\right) \cdot \exp\left(-\frac{\sqrt{r}}{C}\operatorname{dist}(x, \alpha_1^{-1}(0))\right) + \frac{C}{\sqrt{r}}
\end{aligned}
$$

To summarize, we have proved the following result.

Proposition 3.2 *Let* (a, ψ) *be defined as in* (17) *and assume that there exists an* $r_0 \ge 1$ *and* $M > 0$ *such that for all* $r \ge r_0$*, the distance* $\operatorname{dist}(\alpha_0^{-1}(0), \alpha_1^{-1}(0))$

is bounded from below by M. Then for large enough r and any $x \in X$ the pointwise bound below holds:

$$|(D_a(\psi); F_a^{1,1} - \frac{i}{8} r (|\alpha|^2 - 1 - |\beta|^2)\, \omega ; F_a^{0,2} - \frac{i}{4} r \bar{\alpha}\beta)|_x \le \frac{C}{\sqrt{r}} \tag{23}$$

3.2 Inverting the linearized operators of (a_i, ψ_i)

This section serves as a digression. The main result of the section is theorem 3.6, an asymptotic (as $r \to \infty$) regularity statement for the linear operators $L_{(a_i, \psi_i)}$.

We start with two easy auxiliary lemmas:

Lemma 3.3 *Let $L : V \to W$ be a surjective Fredholm operator between Hilbert spaces. Then there exists a $\epsilon > 0$ such that for every linear operator $\ell : V \to W$ with $||\ell(x)||_W \le \epsilon ||x||_V$, the operator $L + \ell$ is still surjective.*

Proof Since L is Fredholm, we can orthogonally decompose V as $V = \mathrm{Ker}(L) \oplus \mathrm{Im}(L^*)$. Let L_1 be the restriction of L to $\mathrm{Im}(L^*)$. Then $L_1 : \mathrm{Im}(L^*) \to W$ is an isomorphism with bounded inverse L_1^{-1}.

If the lemma were not true then we could find for all integers $n \ge 1$ an operator $\ell_n : V \to W$ with $||\ell_n x||_W \le 1/n \, ||x||_V$ and with $\mathrm{Coker}(L + \ell_n) \ne \{0\}$. Let $0 \ne y_n \in \mathrm{Coker}(L + \ell_n)$ with $||y_n||_W = 1$ and $x_n = L_1^{-1}(y_n)$. Notice that the sequence $\{x_n\}_n$ is bounded by $||L_1^{-1}||$. Since $y_n \in \mathrm{Coker}(L + \ell_n)$, y_n is orthogonal to $\mathrm{Im}(L + \ell_n)$. In particular,

$$\langle (L + \ell_n) x_n, y_n \rangle = 0$$

This immediately leads to a contradiction for large enough n since $\langle L x_n, y_n \rangle = 1$ and $|\langle \ell_n x_n, y_n \rangle| \le ||L_1^{-1}||/n$. □

Lemma 3.4 *Let V and W be two finite rank vector bundles over X and $L_r : L^{1,2}(V) \to L^2(W)$ a smooth one-parameter family (indexed by $r \ge 1$) of elliptic, first order, differential operators of index zero. Assume further that there exists a $\epsilon > 0$ and $r_0 \ge 1$ such that for any zeroth order linear operator $\ell : L^{1,2}(V) \to L^2(W)$ with $||\ell(x)||_2 < \epsilon ||x||_{1,2}$, the operator $L_r + \ell$ is onto. Then there exists a $r_1 \ge r_0$ and a $M > 0$ such that for all $r \ge r_1$ the inverses of the operators L_r are uniformly bounded by M, i.e. $||L_r^{-1} y||_{1,2} \le M ||y||_2$.*

Proof Notice that a universal upper bound on L_r^{-1} is equivalent to a universal lower bound on L_r. Suppose the lemma were not true: then there would be a sequence $r_n \to \infty$ and $x_n \in L^{1,2}(V)$ with $||x_n||_{1,2} = 1$ and $||L_{r_n} x_n||_2 < 1/n$. Choose n large enough so that $1/n < \epsilon$ and define the operator $\varphi : L^{1,2}(V) \to L^2(W)$ by $\varphi(x) = -\langle x_n, x\rangle_{1,2} \, L_{r_n}(x_n)$. For this φ the assumption of the lemma is met, namely

$$||\varphi(x)||_2 \leq \frac{1}{n}||x||_{1,2} < \epsilon ||x||_{1,2}$$

Thus the operator $L_{r_n} + \varphi$ should be onto and injective (since the index of $L_r + \varphi$ is zero). But x_n is clearly a nonzero kernel element. This is a contradiction. □

Recall that the set $\mathcal{J}$ of almost-complex structures compatible with the symplectic form ω, contains a Baire subset $\mathcal{J}_0$ of generic almost-complex structures in the sense of Gromov-Witten theory (see [7]). Also, as in the introduction, let

$$\Phi : \mathcal{M}_X^{SW}(W_E) \to \mathcal{M}_X^{Gr}(E) \tag{24}$$

be the map introduced in [4] which associates an embedded J-holomorphic curve to a Seiberg-Witten monopole.

Proposition 3.5 *Let J be chosen from $\mathcal{J}_0$ and let (a, ψ) be a solution of the Seiberg-Witten equations* (14) *such that $\Phi(a, \psi)$ doesn't contain any multiply covered components. Then there exists a $\epsilon > 0$ and an $r_0 \geq 1$ such that for all linear operators $\varphi : L^{1,2}(i\Lambda^1 \oplus E \otimes W_0^+) \to L^2(i\Lambda^0 \oplus i\Lambda^{2,+} \oplus E \otimes W_0^-)$ with norm $||\varphi(x)||_2 < \epsilon ||x||_{1,2}$, the operator $L_{(a,\psi)} + \varphi$ is surjective.*

Before proceeding to the proof, notice that proposition 3.5 and lemma 3.4 immediately imply the following theorem, the main result of this section:

Theorem 3.6 *Choose $J \in \mathcal{J}_0$ and let (a, ψ) be a solution of the Seiberg-Witten equations for the $Spin^c$-structure W_E with parameter r. Assume that $\Phi(a, \psi)$ contains no multiply covered components. Then there exists a $r-$independent $M > 0$ and $r_0 \geq 1$ such that for all $r \geq r_0$:*

$$||L^{-1}_{(a,\psi)} x||_{1,2} \leq M ||x||_2 \tag{25}$$

Proof of proposition 3.5 The proof is a bit technical and relies on the even more technical account from [5] on the connection between the deformation theory of the Seiberg-Witten equations on one hand and the Gromov-Witten equation on the other. The idea is however very simple: for large $r \gg 1$, a certain perturbation of the operator L (with the size of the perturbation getting smaller with larger r) has no cokernel if a certain perturbation of the linearization of the generalized del-bar operator has no cokernel. The latter is ensured by the choice of a generic almost complex structure J from the Baire set J_0 of almost complex structures compatible with ω.

Before proceeding, the (interested) reader is advised to familiarize him/her-self with the notation from [5], in particular, sections 4 and 6 as the remainder of the proof heavily relies on it. For convenience we restate here the parts of lemma 4.11 and a slightly modi ed version of lemma 6.7 from [5] relevant to our situation.

Lemma 4.11 *The equation* $Lq + \; q = g$ *is solvable if and only if, for each* k

$$c_k w^k + y_0^k(w) + \;_k(w) = x(g^k) + y_1^k(g)$$

Notice that the assignment of $\;_k$ to is linear i.e. for two operators and $'$, we have $(+ \;')_k = \;_k + \;'_k$.

Lemma 6.7$'$ *The equation* $(L_{\Psi_r(y)} + \ell)p = g$ *has an* $L^{1,2}$ *solution* p *if and only if there exists* $u = (u^1, \ldots, u^k) \in \;_k L^{1,2}(N^{(k)})$ *for which*

$$_y u^k + \;_0^k(u) + \ell_k(u) = \;_1^{-1} x(g^k) + \;_1^k(g)$$

holds for each k.

The proof of lemma 6.7$'$ is almost identical to that of the original lemma 6.7 in [5]. The only di erence is in *Step 2* where Taubes shows that one can write the equation $L_{\Psi_r(y)}p = g$ in the form $Lp + \; p = g$ with L as in lemma 4.11 above and with an appropriate (bounded) correction term (see (6.30) in [5] for a precise de nition). The di erence here is that in our case one can write $(L_{\Psi_r(y)} + \ell)p = g$ as $Lp + \;'p = g$ (with L again as in lemma 4.11 of [5]) but with $\;' = \; + \ell$. Since ℓ is assumed bounded, lemma 4.11 applies to $\;'$ in the exact same way as it applied to the original and the proof of lemma 6.7 in [5] transfers verbatim to our case. Note also that the operators $\;_i^k$ occurring in lemmas 6.7 and 6.7$'$ are identical so in particular they continue to satisfy the bounds asserted by lemma 6.7 of [5].

According to lemma 3.3 there exists a $\;' > 0$ such that $\;_y + \ell'$ is still surjective if $\|\ell'\| < \;'$. Choose r large enough so that $\|\;_0^k\| < \;'/2k$. On the other hand, since $\ell_k(v) = \;(\;_{25,k}\,\ell(\sum_{k'}\;_{100,k'}\underline{v}^{k'}))$ we nd that $\|\ell_k\| \;\; C\|\ell\|$. Thus choosing $\; = \;'/2C$ ensures that $L_{\Psi_r(y)} + \ell$ is surjective provided that $\|\ell\| < \;$. This nishes the proof of proposition 3.5. □

3.3 The linearized operator at (a, ψ)

In order to use the contraction mapping principle to deform the approximate solution (a, ψ) to an honest solution of the Seiberg-Witten equations, we need to know that $L = L_{(a,\psi)}$ admits an inverse whose norm is bounded independently of r. We start by exploring when the equation

$$L\xi = g \tag{26}$$

has a solution ξ for a given g. Here:

$$\xi \in L^{1,2}(i\Lambda^1 \oplus (E_0 \otimes E_1 \otimes W_0^+)) \quad \text{and} \quad g \in L^2(i\Lambda^0 \oplus i\Lambda^{2,+} \oplus (E_0 \otimes E_1 \otimes W_0^-))$$

The idea is to restrict equation (26) first to a neighborhood of $\alpha_0^{-1}(0)$. Over such a neighborhood the bundle E_1 is trivial and, under an isomorphism trivializing E_1, the equation (26) becomes a zero-th order perturbation of the equation $L_0\xi_0 = g_0$ (with ξ_0 and g_0 being appropriately defined in terms of ξ and g). This allows one to take advantage of the results of theorem 3.6 about the inverse of $L_0 = L_{(a_0,\psi_0)}$. Then one restricts (26) to a neighborhood of $\alpha_1^{-1}(0)$ where the bundle E_0 trivializes and once again uses theorem 3.6, this time for the inverse of $L_1 = L_{(a_1,\psi_1)}$. Finally, one restricts to the complement of a neighborhood of $\alpha_0^{-1}(0) \cup \alpha_1^{-1}(0)$ where both E_0 and E_1 become trivial and L becomes close to S - the linearized operator of the unique solution $(A_0, \sqrt{r}\, u_0)$ for the anticanonical $Spin^c$-structure W_0.

To begin this process, choose regular neighborhoods V_i of $\alpha_i^{-1}(0)$, $i = 0, 1$ subject to the condition:

$$\mathrm{dist}(V_0, V_1) \geq M \qquad \text{for some } M > 0$$

The existence of such neighborhoods V_i follows from our main assumption (20). A priori, as one chooses larger values of r, it seems that the sets V_i may need to be chosen anew as well. However, it was shown in [4], section 5c, that in fact this is not necessary. An initial "smart" choice of V_i for large enough r ensures that for $r' > r$, the zero sets $\alpha_i^{-1}(0)$ continue to lie inside of V_i. Choose an open set U such that $X = V_0 \cup V_1 \cup U$ and such that

$$U \cap (\alpha_0^{-1}(0) \cup \alpha_1^{-1}(0)) = \emptyset$$

Arrange the choices of V_i and U further so that ∂V_i is an embedded 3-manifold of X and so that $U \cap V_i$ contains a collar $\partial V_i \times I$. Here I is some segment $[0, d]$ and ∂V_i corresponds to $\partial V_i \times \{d\}$. For the sake of simplicity of notation, we shall make the assumption that for large values of r, the sets $\alpha_i^{-1}(0)$, $i = 0, 1$, are connected. The case of disconnected zero sets of the α_i's is treated much in the same way except for that in the following, one would have to choose a bump function $\ _{;i}$ (see below) for each connected component. This complicates notation to a certain degree but doesn't lead to new phenomena.

Fix once and for all a bump function $\ : [0;\infty) \to [0;1]$ which is 1 on [0,1] and 0 on $[2;\infty)$. For $0 < \ < d/1000$ define $\ _{;i} : X \to [0;1]$ by:

$$\ _{;i}(x) = \begin{cases} 1 & x \in V_i \setminus (\partial V_i \ I) \\ \ (t/\ ) & x = (y;t) \in \partial V_i \ I \\ 0 & x \notin V_i \end{cases} \tag{27}$$

Set $V_0^0 = V_0 \cup U$ and $V_1^0 = V_1 \cup U$. Define the isomorphisms $\ _0 : \mathbb{C} \ V_0^0 \to E_1|_{V_0^0}$ and $\ _1 : \mathbb{C} \ V_1^0 \to E_0|_{V_1^0}$ as $\ _0(\ ;x) = \ _1(x)\ $ and $\ _1(\ ;x) = \ _0(x)\ $. Also, for $i = 0, 1$ define the operators

$$M_i : L^{1;2}(i^1 \ (E_i \ W_0^+);V_i^0) \to L^2(i^0 \ i^{2;+} \ (E_i \ W_0^-);V_i^0)$$

and

$$T : L^{1;2}(i^1 \ W_0^+;U) \to L^2(i^0 \ i^{2;+} \ W_0^-;U)$$

by demanding the diagrams

$$\begin{array}{ccc} L^{1;2}(i^1 \ (E_0 \ E_1 \ W_0^+);V_i^0) & \xrightarrow{\ \ _i\ } & L^{1;2}(i^1 \ W_i^+;V_i^0) \\ \Big\downarrow L & & \Big\downarrow M_i \\ L^2(i^0 \ i^{2;+} \ (E_0 \ E_1 \ W_0^-);V_i^0) & \xrightarrow{\ \ _i\ } & L^2(i^0 \ i^{2;+} \ W_i^-;V_i^0) \end{array}$$

and

$$\begin{array}{ccc} L^{1;2}(i^1 \ (E_0 \ E_1 \ W_0^+);U) & \xrightarrow{\ \ _0 \ _1\ } & L^{1;2}(i^1 \ W_0^+;U) \\ \Big\downarrow L & & \Big\downarrow T \\ L^2(i^0 \ i^{2;+} \ (E_0 \ E_1 \ W_0^-);U) & \xrightarrow{\ \ _0 \ _1\ } & L^2(i^0 \ i^{2;+} \ W_0^-;U) \end{array}$$

to be commutative diagrams (with $W_i = E_i \ W_0$).

We now start our search for a solution $\ $ of (26) in the form:

$$\ = \ _0(\ _{100\ ;0}\ _0) + \ _1(\ _{100\ ;1}\ _1) + \ _0\ _1((1 - \ _{4\ ;0})(1 - \ _{4\ ;1})\ ) \tag{28}$$

Here $\ _i \in L^{1;2}(i^1 \ (E_i \ W_0^+))$ and $\ \in L^{1;2}(i^1 \ W_0^+)$. Given a $g \in L^2(i^0 \ i^{2;+} \ (E_0 \ E_1 \ W_0^-))$, define $g_i \in L^2(i^0 \ i^{2;+} \ (E_i \ W_0^-))$ and $\gamma \in L^2(i^0 \ i^{2;+} \ W_0^-)$ as

$$g_i = \ _i^{-1}(\ _{25\ ;i}g) \quad \text{and} \quad \gamma = (\ _0\ _1)^{-1}((1 - \ _{25\ ;0})(1 - \ _{25\ ;1})g) \tag{29}$$

It is easy to check that g, g_i and γ satisfy a relation similar to (28), namely:

$$g = \ _0(\ _{100\ ;0}g_0) + \ _1(\ _{100\ ;1}g_1) + \ _0\ _1((1 - \ _{4\ ;0})(1 - \ _{4\ ;1})\gamma) \tag{30}$$

Putting the form (28) of and the form (30) of g into equation (26), after a few simple manipulations, yields the equation

$$
\begin{aligned}
& {}_0(\ {}_{100\,;0}(M_0(\ {}_0) - \ {}_1P(d\ {}_{4\,;0};\ ) - g_0)) + \\
+\ & {}_1(\ {}_{100\,;1}(M_1(\ {}_1) - \ {}_0P(d\ {}_{4\,;1};\ ) - g_1)) + \\
+\ & {}_0\ {}_1((1 - \ {}_{4\,;0})(1 - \ {}_{4\,;1})\ T\ + \ {}_1^{-1}P(d\ {}_{100\,;0};\ {}_0) + \\
+\ & {}_0^{-1}P(d\ {}_{100\,;1};\ {}_1) - \gamma \\
=\ & 0
\end{aligned}
\qquad (31)
$$

In the above, P denotes the principal symbol of L. This last equation suggests a splitting into three equations (each corresponding to one line in (31)):

$$
\begin{aligned}
M_0(\ {}_0) - \ {}_1P(d\ {}_{4\,;0};\ ) &= g_0 \\
M_1(\ {}_1) - \ {}_0P(d\ {}_{4\,;1};\ ) &= g_1 \\
T\ + \ {}_1^{-1}P(d\ {}_{100\,;0};\ {}_0) + \ {}_0^{-1}P(d\ {}_{100\,;1};\ {}_1) &= \gamma
\end{aligned}
\qquad (32)
$$

Equation (31) (and hence also equation (26)) can be recovered from (32) by multiplying the three equations by ${}_0\ {}_{100\,;0}$, ${}_1\ {}_{100\,;1}$ and ${}_0\ {}_1\ ((1 - \ {}_{4\,;0})(1 - \ {}_{4\,;1})$ respectively and then adding them. Thus, given a g and with g_i and γ defined by (29), solutions ${}_i$ and of (32) lead to a solution of (26) via (28). However, the problem with (32) is that the operators M_i and T are not defined over all of X. We remedy this in the next step.

Define new operators:

$$
M_i^0 : L^{1;2}(i^{\ 1}\ (E_i\ \ W_0^+)) \ ! \ L^2(i^{\ 0}\ \ {}^{2;+}\ (E_i\ \ W_0^-))
$$

and

$$
T^0 : L^{1;2}(i^{\ 1}\ \ W_0^+) \ ! \ L^2(i^{\ 0}\ \ {}^{2;+}\ \ W_0^-)
$$

by

$$
\begin{aligned}
M_i^0 &= \ {}_{200\,;i}M_i + (1 - \ {}_{200\,;i})L_i \\
T^0 &= (1 - \ {}_{;0})(1 - \ {}_{;1})T + (\ {}_{;0} + \ {}_{;1})S
\end{aligned}
\qquad (33)
$$

Here $L_i = L_{(a_i;\ i)}$. Now replace the coupled equations (32) by the following system:

$$
\begin{aligned}
M_0^0(\ {}_0) - \ {}_1P(d\ {}_{4\,;0};\ ) &= g_0 \\
M_1^0(\ {}_1) - \ {}_0P(d\ {}_{4\,;1};\ ) &= g_1 \\
T^0\ + \ {}_1^{-1}P(d\ {}_{100\,;0};\ {}_0) + \ {}_0^{-1}P(d\ {}_{100\,;1};\ {}_1) &= \gamma
\end{aligned}
\qquad (34)
$$

The advantage of (34) over (32) is that the former is defined over all of X (notice that the support of $P(d\ {}_{100\,;0};\ {}_0)$ lies in the domain of ${}_1^{-1}$ and the support of $P(d\ _{100,1};\ _1)$ lies in the domain of $\ _0^{-1})$. On the other hand, solutions of (34) give rise to solutions of (26) in the same way as solutions of (32) did because

$$\ _{100,i}\ M_i' = \ _{100,i}\ M_i \qquad i = 0; 1$$
$$(1 - \ _{4,0})(1 - \ _{4,1})T' = (1 - \ _{4,0})(1 - \ _{4,1})T$$

Lemma 3.7 *For every $\ > 0$ there exists an $r\ \ 1$ such that for $r\ \ r\ $ the following hold:*

$$\|(M_i' - L_i)x_i\|_2 \quad \|x_i\|_2$$
$$\|(T' - S)y\|_2 \quad \|y\|_2$$

Here $x_i \in L^{1,2}(i^{\ 1}\ E_i\ \ W_0^+)$ and $y \in L^{1,2}(i^{\ 1}\ W_0^+)$.

Proof The above Sobolev inequalities are proved by first calculating pointwise bounds for $|(M_i' - L_i)x_i|_p$ and $|(T' - S)y|_p$, $p \in X$. Notice firstly that $|(M_i' - L_i)x_i|_p = 0$ if $p \notin V_i$ and $|(T' - S)y|_p = 0$ if $p \notin U$. For $p \in V_i$ and for $q \in U$, a straightforward but somewhat tedious calculation shows that:

$$|(M_i' - L_i)x_i|_p \quad C\ \frac{p}{r}|1 - |\ _i|^2| + \frac{p}{r}|\ _i||\ _i| + |r^{a_i}\ _i|\ |x_i|_p$$
$$|(T' - S)y|_q \quad C\Big(\frac{p}{r}|1 - |\ _0|^2| + \frac{p}{r}|1 - |\ _1|^2| + \frac{p}{r}|\ _0| + \frac{p}{r}|\ _1| + |r^{a_0}\ _0| + |r^{a_1}\ _1|\Big)|y|_q$$

Squaring and then integrating both sides over X together with a reference to (16) gives the desired Sobolev inequalities. □

The lemma suggests that the system (34) can be replaced by the system:

$$
\begin{aligned}
L_0(\ _0') - \ _1 P(d\ _{4,0};\ ') &= g_0\\
L_1(\ _1') - \ _0 P(d\ _{4,1};\ ') &= g_1 \qquad (35)\\
S\ ' + \ _1^{-1}P(d\ _{100,0};\ _0') + \ _0^{-1}P(d\ _{100,1};\ _1') &= y
\end{aligned}
$$

Lemmas 3.7 and 3.3 say that for $r\ \ 0$, (34) has a solution $(\ _0;\ _1;\ )$ if (35) has a solution $(\ _0';\ _1';\ ')$. It is this latter set of equations that we now proceed to solve.

Since S is onto, we can solve the third equation in (35), regarding $\ _0'$ and $\ _1'$ as parameters. Thus:

$$\ ' = \ '(\ _0';\ _1') = S^{-1}(y - \ _1^{-1}P(d\ _{100,0};\ _0') - \ _0^{-1}P(d\ _{100,1};\ _1')) \qquad (36)$$

Recall that the inverse of S satisfies the bound (12):

$$\|S^{-1}y\|_2 \quad \frac{4}{\sqrt{r}}\|y\|_2 \quad \text{for} \quad y \in L^2(i^{\ 0}\ i^{\ 2,+}\ W_0^-)$$

We will solve the first two equations in (35) simultaneously by first rewriting them in the form:

$$\begin{aligned} \psi_0' &= L_0^{-1}(g_0 + \beta_1 P(d_{4\,;0};\eta'(\psi_0';\psi_1'))) \\ \psi_1' &= L_1^{-1}(g_1 + \beta_0 P(d_{4\,;1};\eta'(\psi_0';\psi_1'))) \end{aligned} \tag{37}$$

To solve (37) is the same as to find a fixed point of the map $Y : L^2(i\,\Lambda^1 \otimes W^+_{E_0}) \oplus L^2(i\,\Lambda^1 \otimes W^+_{E_1}) \to L^2(i\,\Lambda^1 \otimes W^+_{E_0}) \oplus L^2(i\,\Lambda^1 \otimes W^+_{E_1})$ given by

$$\begin{aligned} Y(\psi_0';\psi_1') &= \\ &= (L_0^{-1}(g_0 + \beta_1 P(d_{4\,;0};\eta')); L_1^{-1}(g_1 + \beta_0 P(d_{4\,;1};\eta'))) \end{aligned} \tag{38}$$

with η' given by (36). The existence and uniqueness of such a fixed point will be guaranteed by the fixed point theorem for Banach spaces if we can show that Y is a contraction mapping. To see this, let $x, y \in L^2(i\,\Lambda^1 \otimes W^+_{E_0}) \oplus L^2(i\,\Lambda^1 \otimes W^+_{E_1})$ be two arbitrary sections. Using the first bound of (12) and the result of theorem 3.6 to bound the norms of L_i^{-1}, one finds:

$$\begin{aligned} &\|Y(x) - Y(y)\|_2^2 = \\ &= \|L_0^{-1}(g_0 + \beta_1 P(d_{4\,;0};\eta(x))) - L_0^{-1}(g_0 + \beta_1 P(d_{4\,;0};\eta(y)))\|_2^2 \\ &+ \|L_1^{-1}(g_1 + \beta_0 P(d_{4\,;1};\eta(x))) - L_1^{-1}(g_1 + \beta_0 P(d_{4\,;1};\eta(y)))\|_2^2 \\ &\quad\le C_0\|\eta(x) - \eta(y)\|_2^2 + C_1\|\eta(x) - \eta(y)\|_2^2 \\ &\quad\le C\|S^{-1}(\beta_1^{-1} P(d_{100\,;0};y) - \beta_1^{-1} P(d_{100\,;0};x) + \\ &\quad + \beta_0^{-1} P(d_{100\,;1};y) - \beta_0^{-1} P(d_{100\,;1};x))\|_2^2 \end{aligned} \tag{39}$$

$$\le \frac{C}{r}\|x - y\|_2^2 \tag{40}$$

Choosing $r > 2C$, where C is the constant in the last line of (40), makes Y a contraction mapping. Thus we finally arrive at an L^2 solution $(\psi_0';\psi_1')$. It is in fact an $L^{1;2}$ solution because of (37). This, together with equation (36) provides a solution $(\psi_0';\psi_1';\eta')$ of (35). As explained above, this gives rise to a solution $(\psi_0;\psi_1;\eta)$ of (34) and thus provides a solution $\psi \in L^{1;2}\,\Gamma(i\,\Lambda^1 \otimes W_0^+)$ of (26). In particular, we have proved half of the following theorem.

Theorem 3.8 *Let $(a;\psi)$ be constructed from $(a_i;\psi_i)$ as in (17). Suppose that the $(a_i;\psi_i)$ meet assumption 3.1, that $(a_i;\psi_i)$ contains no multiply covered tori and that J has been chosen from the Baire set J_0 of compatible almost complex structures. Then $L_{(a;\psi)} : L^{1;2}(i\,\Lambda^1 \oplus E_0 \oplus E_1 \oplus W_0^+) \to L^2(i\,\Lambda^0 \oplus i\,\Lambda^{2;+} \oplus E_0 \oplus E_1 \oplus W_0^-)$ is invertible with bounded inverse $\|L_{(a;\psi)}^{-1} y\|_{1;2} \le C\|y\|_2$ for all sufficiently large r. Here C is independent of r.*

Proof It remains to prove the inequality $\|L^{-1}_{(a,\ )} y\|_{1,2} \quad C\|y\|_2$. Each of the two lines of (37), together with the bound (25) on L_i^{-1}, yields:

$$\| \ ^{\prime}_i\|_{1,2} \quad C(\|g_i\|_2 + \| \ ^{\prime}(\ ^{\prime}_0, \ ^{\prime}_1)\|_2) \tag{41}$$

A bound for the second term on the right-hand side of (41) comes from (36) and the L^2 bound in (12):

$$\| \ ^{\prime}(\ ^{\prime}_0, \ ^{\prime}_1)\|_2 \quad \frac{C}{\sqrt{r}}(\|y\|_2 + \| \ ^{\prime}_0\|_2 + \| \ ^{\prime}_1\|_2) \tag{42}$$

Adding the two inequalities (41) for $i = 0, 1$ and using (42) gives:

$$(1 - \frac{C}{\sqrt{r}})(\| \ ^{\prime}_0\|_{1,2} + \| \ ^{\prime}_1\|_{1,2}) \quad C(\|g_0\|_2 + \|g_1\|_2 + \frac{1}{\sqrt{r}}\|y\|_2) \tag{43}$$

For large enough r, this last inequality gives a bound on the $L^{1,2}$ norm of $(\ ^{\prime}_0, \ ^{\prime}_1)$ in terms of an r-independent multiple of the L^2 norm of (g_0, g_1, y). With this established, the missing piece, namely the $L^{1,2}$ bound of $\ ^{\prime}$, comes from (36) and the $L^{1,2}$ bound in (12):

$$\| \ ^{\prime}\|_{1,2} \quad C(\|y\|_2 + \| \ ^{\prime}_0\|_2 + \| \ ^{\prime}_1\|_2) \quad C(\|y\|_2 + \|g_0\|_2 + \|g_1\|_2) \tag{44}$$

It remains to relate the now established bound on $(\ ^{\prime}_0, \ ^{\prime}_1, \ ^{\prime})$ to a bound for $(\ _0, \ _1, \ )$. To begin doing that, write the systems (35) and (34) schematically as:

$$F(\ ^{\prime}_0, \ ^{\prime}_1, \ ^{\prime}) = (g_0, g_1, y) \quad \text{and} \quad G(\ _0, \ _1, \ ) = (g_0, g_1, y)$$

Lemma 3.7 implies that for any $" > 0$ there exists a $r_" \quad 1$ such that for all $r \quad r_"$ the inequality $\|(F - G)x\|_2 \quad "\|x\|_2$ holds. The established surjectivity of F guarantees (by means of lemma 3.3) that G is also surjective. The proof of theorem 3.11 thus far, also shows that $\|F^{-1}\| \quad C$ where C is $r-$independent. Now the standard inequality

$$\|G^{-1}\| \quad \|F^{-1}\| + \|G^{-1} - F^{-1}\| \quad \|F^{-1}\| + \|F^{-1}\| \ \|G^{-1}\| \ \|G - F\|$$

implies the $r-$independent bound for $\|G^{-1}\|$:

$$\|G^{-1}\| \quad \frac{\|F^{-1}\|}{1 - \|F^{-1}\| \ \|G - F\|} \quad \frac{C}{1 - C"}$$

This last inequality provides $L^{1,2}$ bounds on $(\ _0, \ _1)$ and in terms of the L^2 norms of (g_0, g_1) and y which in turn imply an $r-$independent $L^{1,2}$ bound on $= L^{-1}g$ in terms of the L^2 norm of g through (28) and (29). This nishes the proof of theorem 3.8. □

3.4 Deforming (a,ψ) to an honest solution

The goal of this section is to show that the approximate solution (a,ψ) can be made into an honest solution of the Seiberg-Witten equations by a deformation whose size goes to zero as r goes to infinity.

To set the stage, let $SW : L^{1,2}(i\Lambda^1 \oplus W_E^+) \to L^2(i\Lambda^0 \oplus i\Lambda^{2,+} \oplus W_E^-)$ denote the Seiberg-Witten operator

$$SW(b,\phi) = (d^* b, F_b^+ - F_{A_0}^+ - q(\phi,\phi) + \frac{ir}{8}\omega, D_b\phi)$$

We will search for a zero of SW of the form $(a,\psi)+(a',\psi')$ with $(a',\psi') \in B(\varepsilon)$. Here $B(\varepsilon)$ is the closed ball in $L^{1,2}(i\Lambda^1 \oplus W_E^+)$ centered at zero and with radius $\varepsilon > 0$ which we will choose later but which should be thought of as being small. The equation $SW((a,\psi)+(a',\psi')) = 0$ can be written as:

$$0 = SW(a,\psi) + L_{(a,\psi)}(a',\psi') + Q(a',\psi') \tag{45}$$

Here $Q : L^{1,2}(i\Lambda^1 \oplus W_E^+) \to L^2(i\Lambda^0 \oplus i\Lambda^{2,+} \oplus W_E^-)$ is the quadratic map given by:

$$Q(b,\phi_0,\phi_2) = (b\cdot(\phi_0+\phi_2), \frac{i}{8}(|\phi_0|^2 - |\phi_2|^2)\omega + \frac{i}{4}(\bar\phi_0\phi_2 + \phi_0\bar\phi_2)) \tag{46}$$

Lemma 3.9 *For $x, y \in L^{1,2}(i\Lambda^1 \oplus W_E^+)$, the map Q satisfies the inequality:*

$$||Q(x) - Q(y)||_2 \leq C(||x||_{1,2} + ||y||_{1,2})||x - y||_{1,2} \tag{47}$$

Proof This is a standard inequality for quadratic maps and it can be explicitly checked using the definition of Q and the multiplication theorem for Sobolev spaces. We give the calculation for the first component of the right hand side of (46). Let $x = (b,\phi)$ and $y = (c,\varphi)$, then we have:

$$\begin{aligned} ||b\cdot\phi - c\cdot\varphi||_2 = ||b\cdot\phi - c\cdot\phi + c\cdot\phi - c\cdot\varphi||_2 &\leq ||(b-c)\cdot\phi||_2 + ||c\cdot(\phi-\varphi)||_2 \\ &\leq C||b-c||_{1,2}||\phi||_{1,2} + C||c||_{1,2}||\phi-\varphi||_{1,2} \\ &\leq C(||(b,\phi)-(c,\varphi)||_{1,2})(||(b,\phi)||_{1,2} + ||(c,\varphi)||_{1,2}) \end{aligned}$$

The other components are checked similarly. □

Solving equation (45) for $(a',\psi') \in L^{1,2}(i\Lambda^1 \oplus W_E^+)$ is equivalent to finding a fixed point of the map $Y : B(\varepsilon) \to B(\varepsilon)$ given by:

$$Y(b,\phi) = -L^{-1}_{(a,\psi)}(SW(a,\psi) + Q(b,\phi)) \tag{48}$$

In order for the image of Y to lie in $B(\delta)$ we need to choose r large enough and δ small enough. To make this precise, let $(b,\beta) \in B(\delta)$. Using the bounds in (23) we find that

$$\|SW(a,\psi)\|_2 \le \frac{C}{\sqrt{r}}$$

and so together with the results of theorem 3.8 and lemma 3.9 we get:

$$\|Y(b,\beta)\|_{1,2} \le \frac{C}{\sqrt{r}} + C\delta^2$$

Choosing $\delta < 1/2C$ and $r > 4C^2/\delta^2$ ensures that Y is well defined.

Lemma 3.10 *The map $Y : B(\delta) \to B(\delta)$ as defined by* (48) *is a contraction mapping for r large enough and δ small enough.*

Proof Let $x, y \in B(\delta)$, then using (47) we find:

$$\|Y(x) - Y(y)\|_{1,2} \le C\|Q(x) - Q(y)\|_2 \le C\|x + y\|_{1,2}\|x - y\|_{1,2} \tag{49}$$

Choosing $\delta < 1/2C$ makes $C\|x + y\|_{1,2} \le 2C\delta$ less than 1. □

We summarize in the following:

Theorem 3.11 *Let (a,ψ) be constructed from (a_i,ψ_i) as in* (17). *Suppose that the (a_i,ψ_i) meet assumption 3.1, that (a_i,ψ_i) contains no multiply covered tori and that J has been chosen from the Baire set J_0 of compatible almost complex structures.*

Then there exists a $\delta_0 > 0$ such that for any $0 < \delta \le \delta_0$ there exists an $r_\delta \ge 1$ such that for every $r \ge r_\delta$ there exists a unique solution $(a,\psi) + (a',\psi')$ of the Seiberg-Witten equations (with perturbation parameter r) with $(a',\psi') \in L^{1,2}(i\Lambda^1 \oplus W_E^+)$ satisfying the bound $\|(a',\psi')\|_{1,2} \le \delta$.

Remark 3.12 It is not known if theorem 3.11 holds under the relaxed hypothesis allowing (a_i,ψ_i) to contain multiply covered tori. The difficulty in dealing with this case stems from the fact that the operators $L_{(a_i,\psi_i)}$ may no longer have trivial cokernel.

4 Comparison with product formulas

Before proceeding further, we would like to take a moment to point out the similarities and differences between our construction of (A, Φ) from (A_i, Φ_i) on one hand and product formulas for the Seiberg-Witten invariants on manifolds that are fiber sums of simpler manifolds. We begin by briefly (and with few details) recalling the scenario of the latter.

Let X_i, $i = 0, 1$ be two compact smooth 4-manifolds and $\Sigma_i \hookrightarrow X_i$ embedded surfaces of the same genus and with $\Sigma_0 \cdot \Sigma_0 = -\Sigma_1 \cdot \Sigma_1$. In this setup one can construct the fiber sum

$$X = X_0 \#_{\Sigma_i} X_1$$

by cutting out tubular neighborhoods $N(\Sigma_i)$ in X_i and gluing the manifolds $X_i' = \overline{X_i \setminus N(\Sigma_i)}$ along their diffeomorphic boundaries.

Under certain conditions one can calculate some of the Seiberg-Witten invariants of X in terms of the Seiberg-Witten invariants of the building blocks X_i (see e.g. [3]). One accomplishes this by showing that from solutions (B_i, Ψ_i), $i = 0, 1$ on X_i one can construct a solution (B, Ψ) on X (this isn't possible for any pair of solutions (B_i, Ψ_i) but the details are not relevant to the present discussion). This is done by inserting a "neck" of length $r \gg 1$ between the X_i' so as to identify X with

$$X = X_0' \cup ([0, r] \times Y) \cup X_1'$$

with $Y = \partial N(\Sigma_0) = \partial N(\Sigma_1)$. A partition of unity $\{\varphi_0, \varphi_1\}$ is chosen for each value of $r \gg 1$ subject to the conditions:

$$\begin{aligned} \varphi_i &= 1 \text{ on } X_i' \\ \varphi_i &= 0 \text{ outside of } X_i' \cup [0, r] \times Y \\ |\varphi_i'| &\leq \frac{C}{r} \text{ on } [0, r] \times Y \end{aligned}$$

An approximation Ψ' of Ψ is then defined to be $\Psi' = \varphi_0 \Psi_0 + \varphi_1 \Psi_1$ (similarly for B', a first approximation for B). The measure of the failure of (B', Ψ') to solve the Seiberg-Witten equations can be made as small as desired by making r large. The honest solution (B, Ψ) is then sought in the form $(B', \Psi') + (b, \psi)$ with (b, ψ) small. The correction term (b, ψ) is found as a fixed point of the map

$$(b, \psi) \mapsto Z(b, \psi) = -L^{-1}_{(B', \Psi')}(Q(b, \psi) + \text{err})$$

Here "err" is the size of $SW(B', \Psi')$ and L and Q are as in the previous section. Choosing r large enough and $\|(b, \psi)\|$ small enough makes Z a contraction mapping and so the familiar fixed point theorem for Banach spaces guarantees the existence of a unique fixed point.

In the case of fiber sums there are product formulas that allow one to calculate the Seiberg-Witten invariants of X in terms of the invariants of the manifolds X_i. The formulas typically have the form:

$$SW_X(W_E) = \sum_{E_0+E_1=E} SW_{X_0}(W_{E_0}) \cdot SW_{X_1}(W_{E_1}) \qquad (50)$$

Due to the similarity of our construction of grafting monopoles to the one used to construct (B,Ψ) from (B_i,Ψ_i), it is natural to ask if such or similar formulas exist for the present case, that is, can one calculate $SW_X(W_{E_0\otimes E_1})$ in terms of $SW_X(W_{E_0})$ and $SW_X(W_{E_1})$? The author doesn't know the answer. However, if they do exist, they can't be expected to be as simple as (50). The reason for this can be understood by trying to take the analogy between our setup and that for fiber sums further.

In the case of fiber sums, once one has established that the two solutions (B_i,Ψ_i) on X_i can be used to construct a solution (B,Ψ) on X, one needs to establish a converse of sorts. That is, one needs to show that every solution (B,Ψ) on X is of that form. It is at this point where the analogy between the two situations breaks down. It is conceivable in our setup, that there will be solutions for the $Spin^c$-structure $(E_0 \otimes E_1) \oplus W_0^+$ that can not be obtained as products of solutions for the $Spin^c$-structures $E_i \oplus W_0^+$. Worse even, there might be monopoles that can not be obtained as products of solutions for any $Spin^c$-structures $F_j \oplus W_0^+$ with the choice of F_j, $j=0,1$ such that $E = F_0 \otimes F_1$ and $F_j \neq 0$. Those are the monopoles where $\alpha^{-1}(0)$ is connected. Thus if a product formula for our situation exists, it must in addition to a term similar to the right hand side of (50) also contain terms which count these "undecomposable" solutions. But then again, they might not exist.

The next section describes which solutions of the Seiberg-Witten equations for the $Spin^c$-structure $(E_0 \otimes E_1) \oplus W_0^+$ are obtained as products of solutions for the $Spin^c$-structures $E_i \oplus W_0^+$, $E = E_0 \otimes E_1$.

5 The image of the multiplication map

This section describes a partial converse to theorem 3.11. Recall that

$$\Phi : M_X^{SW}(W_E) \to M_X^{Gr}(E)$$

is the map assigning a J-holomorphic curve to a Seiberg-Witten monopole.

Theorem 5.1 *Let $E = E_0 \otimes E_1$ and let (A, ψ) be a solution of the Seiberg-Witten equations in the $Spin^c$-structure W_E with perturbation term $\eta = F^+_{A_0} - ir\omega/8$ and with $\psi = \sqrt{r}(\alpha u_0, \beta)$. Assume further that J has been chosen from the Baire set $\mathcal{J}_0$ and that $\alpha(A, \psi)$ contains no multiply covered components. If there exists an r_0 such that for all $r \geq r_0$, $\alpha^{-1}(0)$ splits into a disjoint union $\alpha^{-1}(0) = \Sigma_0 \sqcup \Sigma_1$ with $[\Sigma_i] = P.D.(E_i)$ then (A, ψ) lies in the image of the multiplication map*

$$M^{SW}_X(E_0) \times M^{SW}_X(E_1) \to M^{SW}_X(E_0 \otimes E_1)$$

The proof of theorem 5.1 is divided into 3 sections. In section 5.1 we give the definition of (A^0_i, ψ^0_i) - first approximations of Seiberg-Witten monopoles (A_i, ψ_i) for the $Spin^c$-structure W_{E_i} which when multiplied give the monopole (A, ψ) from theorem 5.1. Section 5.2 shows that for large values of r, (A^0_i, ψ^0_i) come close to solving the Seiberg-Witten equations. In the final section 5.3 we show that $L_{(A^0_i, \psi^0_i)}$ is surjective with inverse bounded independently of r. The contraction mapping principle is then used to deform the approximate solutions (A^0_i, ψ^0_i) to honest solutions (A_i, ψ_i). Section 5.3 also explains why $(A_0, \psi_0) \cdot (A_1, \psi_1) = (A, \psi)$.

We tacitly carry the assumptions of the theorem until the end of the section.

5.1 Defining (A^0_i, ψ^0_i)

The basic idea behind the definition of (A^0_i, ψ^0_i) is again that of grafting existing solutions. For example, one would like ψ^0_0 to be defined as the restriction of ψ to a neighborhood of Σ_0 (under an appropriate bundle isomorphism trivializing E_1 over that neighborhood) and to be the restriction of $\sqrt{r}u_0$ outside that neighborhood. This is essentially how the construction goes even though a bit more care is required, especially in splitting the connection A into A^0_0 and A^0_1.

To begin with, choose regular neighborhoods V_0 and V_1 of Σ_0 and Σ_1. Once r is large enough, these choices don't need to be readjusted for larger values of r. Choose, as in section 3.3, an open set U such that:

$$X = V_0 \cup U \cup V_1$$
$$U \cap \Sigma_i = \emptyset$$

Also, just as in section 3.3, arrange the choices so that $U \cap V_i$ contains a collar $\partial V_i \times [0, d]$ (with ∂V_i corresponding to $\partial V_i \times \{d\}$) and choose $\epsilon > 0$ smaller than $d/1000$. Assume that the curves Σ_i are connected, the general case goes through with little difficulty but with a bit more complexity of notation.

Over $U \cup V_1$, choose a section $\gamma_0 \in \Gamma(E_0; U \cup V_1)$ with $|\gamma_0| = 1$. Choose a connection B_0 on E_0 with respect to which γ_0 is covariantly constant over $U \cup V_1$, i.e.

$$\nabla_{B_0}(\gamma_0(x)) = 0 \qquad \forall x \in U \cup V_1 \tag{51}$$

Notice that such a connection is automatically flat over $U \cup V_1$. Choose a connection B_1 on E_1 such that $B_0 \otimes B_1 = A$ over X. Now define $\tilde{\alpha}_1' \in \Gamma(E_1; U \cup V_1)$ and $\tilde{\beta}_1' \in \Gamma(E_1 \otimes K^{-1}; U \cup V_1)$ by:

$$\alpha = \gamma_0 \otimes \tilde{\alpha}_1' \tag{52}$$

$$\beta = \gamma_0 \otimes \tilde{\beta}_1' \tag{53}$$

Proceed similarly over V_0. However, since some of the data is now already defined, more caution is required. Choose a section $\gamma_1 \in \Gamma(E_1; V_0)$ with:

$$\begin{aligned} \gamma_1 &= \tilde{\alpha}_1' \quad \text{on } (U \cap V_0)\setminus(\partial V_0 \times [0,4 i)) \\ |\gamma_1| &= 1 \quad \text{on } (V_0 \setminus U) \cup (\partial V_0 \times [0,2 i)) \end{aligned} \tag{54}$$

We continue by defining $\tilde{\alpha}_0'$ and $\tilde{\beta}_0'$ over V_0 by:

$$\alpha = \tilde{\alpha}_0' \otimes \gamma_1 \tag{55}$$

$$\beta = \tilde{\beta}_0' \otimes \gamma_1 \tag{56}$$

Choose one forms a_0 and a_1 such that over V_0 the following two relations hold:

$$\nabla_{(B_1 + i a_1)} \gamma_1 = 0 \tag{57}$$

$$(B_0 + i a_0) \otimes (B_1 + i a_1) = A \tag{58}$$

With these preliminaries in place, we are now ready to define (A_i', ψ_i'):

$$\begin{array}{ll} \tilde{\alpha}_0 = \chi_{4,0}\,\tilde{\alpha}_0' + (1-\chi_{4,0})\gamma_0 & \tilde{\beta}_0 = \chi_{4,0}\,\tilde{\beta}_0' \\ \tilde{\alpha}_1 = (1-\chi_{4,0})\,\tilde{\alpha}_1' + \chi_{4,0}\gamma_1 & \tilde{\beta}_1 = (1-\chi_{4,0})\,\tilde{\beta}_1' \\ A_0' = B_0 + i\,\chi_{4,0}\,a_0 & A_1' = B_1 + i\,\chi_{4,0}\,a_1 \end{array} \tag{59}$$

Lemma 5.2 *The (A_i', ψ_i') defined above, satisfy the following properties:*

(a) $A_0' \otimes A_1' = A$ *on all of* X.

(b) $\tilde{\alpha}_0 = \gamma_0$ *on* $(U \cap V_0)\setminus(\partial V_0 \times [0,4 i))$.

(c) $F_{B_0} = 0$ *on* $U \cup V_1$ *and* $F_{B_1 + i a_1} = 0$ *on* V_0.

(d) *On* $(U \cap V_0)\setminus(\partial V_0 \times [0,4 i))$, $|a_i|$ *and* $|d a_i|$ *converge exponentially fast to zero as* $r \to \infty$.

Proof (a) This is trivially true everywhere except possibly on the support of $d\beta_{4\tau,0}$ which is contained in $U \cap V_0$. However, on $U \cap V_0$ we have $A = B_0 \otimes B_1$ and $A = (B_0 + ia_0) \otimes (B_1 + ia_1)$ and thus $a_0 + a_1 = 0$. In particular, $A_0' \otimes A_1' = B_0 \otimes B_1 + i\beta_{4\tau,0}(a_0 + a_1) = B_0 \otimes B_1 = A$.

(b) Notice that on $(U \cap V_0)\backslash(\partial V_0 \times [0, 4\tau))$, $\psi_1 = \tilde{\psi}_1'$. Thus, $\psi = \psi_0 \otimes \psi_1$ and $\psi = \tilde{\psi}_0' \otimes \psi_1$ imply that $\psi_0 = \tilde{\psi}_0'$. The claim now follows from the definition of ψ_0'.

(c) Follows from the fact that both connection annihilate nowhere vanishing sections on the said regions.

(d) On $(U \cap V_0)\backslash(\partial V_0 \times [0, 4\tau))$ we have $\psi = \psi_0 \otimes \psi_1$ and $\nabla^A = \nabla^{B_0+ia_0} \otimes \nabla^{B_1+ia_1}$. Also, recall that $\nabla^{B_0}\psi_0 = 0$ and $\nabla^{B_1+ia_1}\psi_1 = 0$. Thus:

$$\nabla^A \psi = (\nabla^{B_0+ia_0} \otimes \nabla^{B_1+ia_1})(\psi_0 \otimes \psi_1) = i a_0 \psi_0 \otimes \psi_1$$

This equation yields:

$$|a_0| = \frac{|\nabla^A \psi|}{|\psi|} \tag{60}$$

The claim follows now for a_0 by evoking the bounds (16). The same result holds for a_1 by the proof of part (a) where it is shown that $a_0 + a_1 = 0$ on $U \cap V_0$. The statement for da_i follows from part (c), the equation $F_A = F_{B_0+ia_0} + F_{B_1+ia_1}$ and the bounds (16) for $|F_A|$. □

5.2 Pointwise bounds on $SW(A_i', \psi_i')$

Proposition 5.3 *Let (A_i', ψ_i') be defined as above, then there exists a constant C and an $r_0 \geq 1$ such that for all $r \geq r_0$ the inequality*

$$|SW(A_i', \psi_i')|_x \leq \frac{C}{\sqrt{r}}$$

holds for all $x \in X$.

Proof We calculate the size of the contribution of each of the three Seiberg-Witten equations separately. The only nontrivial part of the calculation is in the region of X which contains the support of $d\beta_{4\tau,0}$ i.e. in $\partial V_0 \times [4\tau, 8\tau]$. We will tacitly use the results of lemma 5.2 in the calculations below.

a) The Dirac equation

To begin with, we calculate the expression $D_A((\tilde{\ }_0 \quad u_0 + \tilde{\ }_0) \quad \gamma_1)$ in two di erent ways. On one hand we have:

$$D_A((\tilde{\ }_0 \quad u_0 + \tilde{\ }_0) \quad \gamma_1) = D_A(\quad + \quad {}_{4;0} \quad) = (1 - \quad {}_{4;0}) D_A \quad + d \quad {}_{4;0}.$$

On the other hand we get:

$$\begin{aligned} D_A((\tilde{\ }_0 \quad u_0 + \tilde{\ }_0) \quad \gamma_1) &= \\ &= \gamma_1 \quad D_{A_0^0}(\tilde{\ }_0 \quad u_0 + \tilde{\ }_0) + e^i.(\tilde{\ }_0 \quad u_0 + \tilde{\ }_0) \quad A_1^0(\gamma_1) \\ &= \gamma_1 \quad D_{A_0^0}(\tilde{\ }_0 \quad u_0 + \tilde{\ }_0) + i(\quad {}_{4;0} - 1) a_1 \gamma_1 \end{aligned} \tag{61}$$

Equating the results of the two calculations we obtain:

$$\begin{aligned} |\ |\ |D_{A_0^0}(\tilde{\ }_0 \quad u_0 + \tilde{\ }_0)| &= |\gamma_1 \quad D_{A_0^0}(\tilde{\ }_0 \quad u_0 + \tilde{\ }_0)| \\ &\quad C(|\ || a_1| + |D_A \quad | + |\ |) \quad \frac{C}{\sqrt{r}} \end{aligned}$$

Since over $\partial V_0 \quad [4;\infty]$, $|\ | \to 1$ exponentially fast as $r \to \infty$ we obtain that:

$$|D_{A_0^0}(\tilde{\ }_0 \quad u_0 + \tilde{\ }_0)| \quad \frac{C}{\sqrt{r}} \tag{62}$$

b) The $(1,1)$-component of the curvature equation

Again, we only calculate for $x \in \partial V_0 \quad [4;\infty]$:

$$\begin{aligned} F_{A_0^0}^{(1,1)} - F_{A_0}^{(1,1)} - \frac{ir}{8} \quad |\tilde{\ }_0|^2 - 1 - |\tilde{\ }_0|^2 \quad ! &= \quad {}_{4;0}(da_0)^{(1,1)} + \frac{ir}{8}|\tilde{\ }_0|^2 ! \\ &= \quad {}_{4;0}(da_0)^{(1,1)} + \frac{ir}{8|\ |^2}| \quad {}_{4;0}|^2 |\ |^2 ! \end{aligned}$$

Both terms in the last line converge in norm exponentially fast to zero on $\partial V_0 \quad [4;\infty]$ as $r \to \infty$.

c) The $(0,2)$-component of the curvature equation

Similar to the calculation for the $(1,1)$-component of the curvature equation on $\partial V_0 \quad [4;\infty]$, we have for the $(0,2)$-component of the same equation:

$$F_{A_0^0}^{(0,2)} - F_{A_0}^{(0,2)} - \frac{ir}{4}\overline{\tilde{\ }_0} \quad \tilde{\ }_0 = \quad {}_{4;0}(da_0)^{(0,2)} - \frac{ir}{4|\ |^2} \quad {}_{4;0} \overline{\ }$$

Once again, both terms on the right-hand side of the above equation converge in norm exponentially fast to zero as r converges to in nity. The proofs for the case of $(A_1^0, \ {}_1^0)$ are similar and are left to the reader. □

5.3 Surjectivity of $L_{(A'_i;\ '_i)}$ and deforming $(A'_i;\ '_i)$ to an exact solution

The strategy employed here is very similar to the one used in section 3.4 and we only spell out part of the details. We start by showing that $L_{(A'_0;\ '_0)}$ is surjective, the case $L_{(A'_1;\ '_1)}$ is identical.

We begin by asking ourselves when the equation

$$L_{(A'_0;\ '_0)}\ _0 = g_0 \tag{63}$$

has a solution $_0 \in L^{1;2}(i^1\ W^+_{E_0})$ for a given $g_0 \in L^2(i^0\ \ i^{2;+}\ W^-_{E_0})$. Define the analogues of the isomorphisms $_i$ from section 3.3 to be:

$$_0 : \mathbb{C}\ \ (U \cup V_1) \to \Gamma(E_0; U \cup V_1) \text{ given by } _0(\ ; x) = \ \ y_0(x) \text{ and}$$
$$_1 : \mathbb{C}\ \ V_0 \to \Gamma(E_1; V_0) \text{ given by } _1(\ ; x) = \ \ y_1(x)$$

Let $y \in L^2(i^0\ \ i^{2;+}\ W^-_0; U \cup V_1)$ be determined by the equation $_{25\ ;0}\, g_0 = \ _0(y)$ on $U \cup V_1$ and $\& \in L^2(i^0\ \ i^{2;+}\ W^-_E; V_0)$ be given by the equation $_1^{-1}(\&) = (1 - \ _{25\ ;0})\, g_0$ on V_0. Thus we can write g_0 as:

$$g_0 = \ _{100\ ;0}\ \ _0(y) + (1 - \ _{4\ ;0})\ \ _1^{-1}(\&) \tag{64}$$

This last form suggests that, in order to split equation (63) into two components involving $L_{(A;\ )}$ and S, one should search for $_0$ in the form

$$_0 = \ _{100\ ;0}\ \ _0(\ ) + (1 - \ _{4\ ;0})\ \ _1^{-1}(\ ) \tag{65}$$

with $\ \in L^{1;2}(i^1\ W^+_0; U \cup V_1)$ and $\ \in L^{1;2}(i^1\ W^+_E; V_0)$. Using relations (64) and (65) in (63) one obtains the analogue of equation (31):

$$\begin{aligned} &_{100\ ;0}\ \ _0(T(\ ) - \ _0^{-1}\ _1^{-1} P(d\ _{4\ ;0};\ \ ) - y) + \\ &\quad + (1 - \ _{4\ ;0})\ \ _1^{-1}(M(\ ) + \ _1\ \ _0 P(d\ _{100\ ;0};\ \ ) - \&) = 0 \end{aligned} \tag{66}$$

The operators T' and M' are defined over $U \cup V_1$ and V_0 respectively, through the relations:

$$\begin{aligned} L_{(A'_0;\ '_0)}\ \ _0 &= \ _0 T \\ L_{(A'_0;\ '_0)}\ \ _1^{-1} &= \ _1^{-1} M \end{aligned}$$

We use these operators, defined only over portions of X, to define the operators T' and M' defined on all of X by:

$$\begin{aligned} T' &= (1 - \ _{;0})T + \ _{;0} S \\ M' &= \ _{200\ ;0} M + (1 - \ _{200\ ;0}) L_{(A;\ )} \end{aligned}$$

Split equation (66) into the following two equations:

$$\begin{aligned} T^0(\ ) - \ {}_0^{-1}\ {}_1^{-1}P(d\ {}_{4\,;0};\ ) &= y \\ M^0(\ ) + \ {}_1\ {}_0P(d\ {}_{100\,;0};\ ) &= \& \end{aligned} \qquad (67)$$

It is easy to see that solutions to the system of equations (67) provide solutions to (66) by multiplying the two lines with ${}_{100\,;0}\ {}_0$ and $(1 - \ {}_{4\,;0}\ {}_1^{-1})$ respectively and adding them.

The following lemma is the analogue of lemma 3.6, its proof is identical to that of lemma 3.6 and will be skipped here.

Lemma 5.4 *For every > 0 there exists an r 1 such that for r r the following hold:*

$$\begin{aligned} ||(M^0 - L_{(A;\ )})x||_2 &\quad ||x||_2 \\ ||(T^0 - S)y||_2 &\quad ||y||_2 \end{aligned}$$

Here $x \in L^{1;2}(i\ ^1\ W_E^+)$ and $y \in L^{1;2}(i\ ^1\ W_0^+)$.

The lemma allows us to replace the system (67) by the system:

$$\begin{aligned} S(\ ) - \ {}_0^{-1}\ {}_1^{-1}P(d\ {}_{4\,;0};\ ) &= y \\ L_{(A;\ )}(\ ) + \ {}_1\ {}_0P(d\ {}_{100\,;0};\ ) &= \& \end{aligned} \qquad (68)$$

The process of solving (68) is now step by step the analogue of solving (35). In particular, we solve the first of the two equations in (68) for in terms of :

$$= (\ ) = S^{-1}(\ {}_0^{-1}\ {}_1^{-1}P(d\ {}_{4\,;0};\ ) + y)$$

Use this in the second equation of (68) and rewrite it as:

$$= L^{-1}_{(A;\ )}(\& - \ {}_1\ {}_0P(d\ {}_{100\,;0};\ (\ )))$$

To solve this last equation is the same as to find a fixed point of the map $Y : L^2(i\ ^1\ W_E^+) \to L^2(i\ ^1\ W_E^+)$ (the analogue of the map described by (38)) given by:

$$Y(\ ) = L^{-1}_{(A;\ )}(\& - \ {}_1\ {}_0P(d\ {}_{100\,;0};\ (\ )))$$

The proof of the existence of a unique fixed point of Y follows from a word by word analogue of the proof of theorem 3.8 together with the discussion preceding the theorem.

With the surjectivity of $L_{(A_i^0;\ {}_i^0)}$ proved, the process of deforming $(A_i^0;\ {}_i^0)$ to an honest solution $(A_i;\ {}_i)$ is accomplished by the same method as used in section 3.4 and will be skipped here.

To finish the proof theorem 5.1, we need to show that:

$$(A_0, \ _0) \quad (A_1, \ _1) = (A, \ )$$

This follows from the fact that as $r \to \infty$, the distance $\mathrm{dist}((A_i, \ _i), (A_i' \ _i'))$ converges to zero, together with the following relations which follow directly from the definitions:

$$\tilde{}_0 \quad \tilde{}_1 =$$
$$\tilde{}_0 \quad \tilde{}_1 + \tilde{}_1 \quad \tilde{}_0 =$$
$$A_0' \quad A_1' = A$$

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