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# Diffeomorphisms, symplectic forms and Kodaira fibrations

Claude LeBrun

*Department of Mathematics, SUNY at Stony Brook*
*Stony Brook, NY 11794-3651, USA*

Email: claude@math.sunysb.edu

**Abstract**

As was recently pointed out by McMullen and Taubes [7], there are 4–manifolds for which the diffeomorphism group does not act transitively on the deformation classes of orientation-compatible symplectic structures. This note points out some other 4–manifolds with this property which arise as the orientation-reversed versions of certain complex surfaces constructed by Kodaira [3]. While this construction is arguably simpler than that of McMullen and Taubes, its simplicity comes at a price: the examples exhibited herein all have large fundamental groups.

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Let $M$ be a smooth, compact oriented 4–manifold. If $M$ admits an orientation-compatible symplectic form, meaning a closed 2–form $\omega$ such that $\omega \wedge \omega$ is an orientation-compatible volume form, one might well ask whether the space of such forms is connected. In fact, it is not difficult to construct examples where the answer is negative. A more subtle question, however, is whether the group of orientation-preserving diffeomorphisms $M \to M$ acts transitively on the set of connected components of the orientation-compatible symplectic structures of $M$. As was recently pointed out by McMullen and Taubes [7], there are 4–manifolds $M$ for which this subtler question also has a negative answer. The purpose of the present note is to point out that many examples of this interesting phenomenon arise from certain complex surfaces with Kodaira fibrations.

A *Kodaira fibration* is by definition a holomorphic submersion $f\colon M \to B$ from a compact complex surface to a compact complex curve, with base $B$ and fiber $F_z = f^{-1}(z)$ both of genus $\geq 2$. (In $C^\infty$ terms, $f$ is thus a locally trivial fiber bundle, but nearby fibers of $f$ may well be non-isomorphic as complex curves.) One says that $M$ is a *Kodaira-fibered surface* if it admits such a fibration $f$. Now any Kodaira-fibered surface $M$ is algebraic, since $K_M \otimes f^* K_B^{\otimes \ell}$ is obviously positive for sufficiently large $\ell$. On the other hand, recall that a holomorphic map from a curve of lower genus to a curve of higher genus must be constant.[1] If $f\colon M \to B$ is a Kodaira fibration, it follows that $M$ cannot contain any rational or elliptic curves, since composing $f$ with the inclusion would result in a constant map, and the curve would therefore be contained in a fiber of $f$; contradiction. The Kodaira–Enriques classification [2] therefore tells us that $M$ is a minimal surface of general type. In particular, the only non-trivial Seiberg–Witten invariants of the underlying oriented 4–manifold $M$ are [8] those associated with the canonical and anti-canonical classes of $M$. Any orientation-preserving self-diffeomorphism of $M$ must therefore preserve $\{\pm c_1(M)\}$.

We have just seen that $M$ is of Kähler type, so let $\psi$ denote some Kähler form on $M$, and observe that $\psi$ is then of course a symplectic form compatible with the usual 'complex' orientation of $M$. Let $\varphi$ be any area form on $B$, compatible with *its* complex orientation, and, for sufficiently small $\varepsilon > 0$, consider the closed 2–form

$$\omega = \varepsilon\psi - f^*\varphi.$$

[1] Indeed, by Poincaré duality, a continuous map $h\colon X \to Y$ of non-zero degree between compact oriented manifolds of the same dimension must induce inclusions $h^*\colon H^j(Y,\mathbb{R}) \hookrightarrow H^j(X,\mathbb{R})$ for all $j$. Such a map $h$ therefore cannot exist whenever $b_j(X) < b_j(Y)$ for some $j$.

Then
$$\frac{\omega \wedge \omega}{\varepsilon} = -2(f^*\varphi) \wedge \psi + \varepsilon\psi \wedge \psi = (\varepsilon - \langle f^*\varphi, \psi\rangle)\, \psi \wedge \psi,$$
where the inner product is taken with respect to the Kähler metric corresponding to $\psi$. Now $\langle f^*\varphi, \psi\rangle$ is a positive function, and, because $M$ is compact, therefore has a positive minimum. Thus, for a sufficiently small $\varepsilon > 0$, $\omega \wedge \omega$ is a volume form compatible with the *non-standard* orientation of $M$; or, in other words, $\omega$ is a symplectic form for the reverse-oriented 4–manifold $\overline{M}$. For related constructions of symplectic structures on fiber-bundles, cf [6].

If follows that $\overline{M}$ carries a unique deformation class of almost-complex structures compatible with $\omega$. One such almost-complex structure can be constructed by considering the (non-holomorphic) orthogonal decomposition
$$TM = \ker(f_*) \oplus f^*(TB)$$
induced by the given Kähler metric, and then reversing the sign of the complex structure on the 'horizontal' bundle $f^*(TB)$. The first Chern class of the resulting almost-complex structure is thus given by
$$c_1(\overline{M}, \omega) = c_1(M) - 4(1 - \mathbf{g})F,$$
where $\mathbf{g}$ is the genus of $B$, and where $F$ now denotes the Poincaré dual of a fiber of $f$. For further discussion, cf [4, 5, 9].

Of course, the product $B \times F$ of two complex curves of genus $\geq 2$ is certainly Kodaira fibered, but such a product also admits orientation-reversing diffeomorphisms, and so, in particular, has signature $\tau = 0$. However, as was first observed by Kodaira [3], one can construct examples with $\tau > 0$ by taking *branched covers* of products; cf [1, 2].

**Example** Let $C$ be a compact complex curve of genus $k \geq 2$, and let $B_1$ be a curve of genus $\mathbf{g}_1 = 2k - 1$, obtained as an unbranched double cover of $C$. Let $\iota$: $B_1 \to B_1$ be the associated non-trivial deck transformation, which is a free holomorphic involution of $B_1$. Let $p$: $B_2 \to B_1$ be the unique unbranched cover of order $2^{4k-2}$ with $p_*[\pi_1(B_2)] = \ker[\pi_1(B_1) \to H_1(B_1, \mathbb{Z}_2)]$; thus $B_2$ is a complex curve of genus $\mathbf{g}_2 = 2^{4k-1}(k-1)+1$. Let $\Sigma \subset B_2 \times B_1$ be the union of the graphs of $p$ and $\iota \circ p$. Then the homology class of $\Sigma$ is divisible by 2. We may therefore construct a ramified double cover $M \to B_2 \times B_1$ branched over $\Sigma$. The projection $f_1$: $M \to B_1$ is then a Kodaira fibration, with fiber $F_1$ of genus $2^{4k-2}(4k-3)+1$. The projection $f_2$: $M \to B_2$ is also a Kodaira fibration, with fiber $F_2$ of genus $4k - 2$. The signature of this doubly Kodaira-fibered complex surface is $\tau(M) = 2^{4k}(k - 1)$.

We now axiomatize those properties of these examples which we will need.

**Definition** Let $M$ be a complex surface equipped with two Kodaira fibrations $f_j\colon M \to B_j$, $j = 1, 2$. Let $\mathbf{g}_j$ denote the genus of $B_j$, and suppose that the induced map

$$f_1 \times f_2\colon M \to B_1 \times B_2$$

has degree $r > 0$. We will then say that $(f_1, f_2)$ is a *Kodaira double-fibration* of $M$ if $\tau(M) \neq 0$ and

$$(\mathbf{g}_2 - 1) \not| \; r(\mathbf{g}_1 - 1).$$

In this case, $(M, f_1, f_2)$ will be called a *Kodaira doubly-fibered* surface.

Of course, the last hypothesis depends on the ordering of $(f_1, f_2)$, and is automatically satisfied, for fixed $r$, if $\mathbf{g}_2 \gg \mathbf{g}_1$. The latter may always be arranged by simply replacing $M$ and $B_2$ with suitable covering spaces.

Note that $r = 2$ in the explicit examples given above.

Given a Kodaira doubly-fibered surface $(M, f_1, f_2)$, let $\overline{M}$ denote $M$ equipped with the non-standard orientation, and observe that we now have two different symplectic structures on $\overline{M}$ given by

$$\begin{aligned} \omega_1 &= \varepsilon\psi - f_1^*\varphi_1 \\ \omega_2 &= \varepsilon\psi - f_2^*\varphi_2 \end{aligned}$$

for any given area forms $\varphi_j$ on $B_j$ and any sufficiently small $\varepsilon > 0$.

**Theorem 1** *Let $(M, f_1, f_2)$ be any Kodaira doubly-fibered complex surface. Then for any self-diffeomorphism $\Phi\colon M \to M$, the symplectic structures $\omega_1$ and $\pm\Phi^*\omega_2$ are deformation inequivalent.*

That is, $\omega_1$, $-\omega_1$, $\Phi^*\omega_2$, and $-\Phi^*\omega_2$ are always in different path components of the closed, non-degenerate 2–forms on $\overline{M}$. (The fact that $\omega_1$ and $-\omega_1$ are deformation inequivalent is due to a general result of Taubes [10], and holds for any symplectic 4–manifold with $b^+ > 1$ and $c_1 \neq 0$.)

Theorem 1 is actually a corollary of the following result:

**Theorem 2** *Let $(M, f_1, f_2)$ be any Kodaira doubly-fibered complex surface. Then for any self-diffeomorphism $\Phi\colon M \to M$,*

$$\Phi^*[c_1(\overline{M}, \omega_2)] \neq \pm c_1(\overline{M}, \omega_1).$$

**Proof** Because $\tau(M) \neq 0$, any self-diffeomorphism of $M$ preserves orientation. Now $M$ is a minimal complex surface of general type, and hence, for the standard 'complex' orientation of $M$, the only Seiberg–Witten basic classes [8] are $\pm c_1(M)$. Thus any self-diffeomorphism $\Phi$ of $M$ satisfies

$$\Phi^*[c_1(M)] = \pm c_1(M).$$

Letting $F_j$ be the Poincaré dual of the fiber of $f_j$, and letting $\mathbf{g}_j$ denote the genus of $B_j$, we have

$$c_1(\overline{M}, \omega_j) = c_1(M) + 4(\mathbf{g}_j - 1)F_j$$

for $j = 1, 2$. The adjunction formula therefore tells us that

$$[c_1(\overline{M}, \omega_j)] \cdot [c_1(M)] = (2\chi + 3\tau)(M) - 2\chi(M) = 3\tau(M) \neq 0,$$

where the intersection form is computed with respect to the 'complex' orientation of $M$.

If we had a diffeomorphism $\Phi\colon M \to M$ with $\Phi^*[c_1(\overline{M}, \omega_2)] = \pm c_1(\overline{M}, \omega_1)$, this computation would tell us that that

$$\Phi^*[c_1(M)] = c_1(M) \implies \Phi^*[c_1(\overline{M}, \omega_2)] = c_1(\overline{M}, \omega_1)$$

and that

$$\Phi^*[c_1(M)] = -c_1(M) \implies \Phi^*[c_1(\overline{M}, \omega_2)] = -c_1(\overline{M}, \omega_1).$$

In either case, we would then have

$$4(\mathbf{g}_1 - 1)F_1 = c_1(\overline{M}, \omega_1) - c_1(M) = \pm\Phi^*[c_1(\overline{M}, \omega_2) - c_1(M)] = \pm 4(\mathbf{g}_2 - 1)\Phi^*(F_2).$$

On the other hand, $F_1 \cdot F_2 = r$, so intersecting the previous formula with $F_2$ yields

$$4(\mathbf{g}_1 - 1)r = 4(\mathbf{g}_1 - 1)F_1 \cdot F_2 = 4(\mathbf{g}_2 - 1)[\pm\Phi^*(F_2) \cdot F_2],$$

and hence

$$(\mathbf{g}_2 - 1) \mid r(\mathbf{g}_1 - 1),$$

in contradiction to our hypotheses. The assumption that $\Phi^*[c_1(\overline{M}, \omega_1)] = \pm c_1(\overline{M}, \omega_2)$ is therefore false, and the claim follows. □

Theorem 1 is now an immediate consequence, since the first Chern class of a symplectic structure is deformation-invariant.

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