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Manifolds with non-stable fundamental groups at infinity

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Abstract

The notion of an open collar is generalized to that of a pseudo-collar. Important properties and examples are discussed. The main result gives conditions which guarantee the existence of a pseudo-collar structure on the end of an open n–manifold ($n \geq 7$). This paper may be viewed as a generalization of Siebenmann's famous collaring theorem to open manifolds with non-stable fundamental group systems at infinity.

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1 Introduction

One of the best known and most frequently applied theorems in the study of non-compact manifolds is found in L C Siebenmann's 1965 PhD thesis. It gives necessary and sufficient conditions for the end of an open manifold to possess the simplest possible structure — that of an open collar.

Theorem 1 (From [23]) *A one ended open n–manifold M^n ($n \geq 6$) contains an open collar neighborhood of infinity if and only if each of the following is satisfied:*

(1) *M^n is inward tame at infinity,*

(2) *$\pi_1(\varepsilon(M^n))$ is stable, and*

(3) *$\sigma_\infty(M^n) \in \tilde{K}_0(\mathbb{Z}[\pi_1(\varepsilon(M^n))])$ is trivial.*

A *neighborhood of infinity* is a subset $U \subset M^n$ with the property that $\overline{M^n - U}$ is compact. We say that U is an *open collar* if it is a manifold with compact boundary and $U \approx \partial U \times [0, \infty)$. Other terminology and notation used in this theorem will be discussed later.

Remark 1 A 3–dimensional version of Theorem 1 may be found in [16], while a 5–dimensional version (with some restrictions) may be found in [15]. In [19], it is shown that Theorem 1 fails in dimension 4.

One of the beauties of Theorem 1 is the simple structure it places on the ends of certain manifolds. At the same time, this simplicity greatly limits the class of manifolds to which the theorem applies. Indeed, many interesting and important non-compact manifolds are "too complicated at infinity" to be collarable. Frequently the condition these manifolds violate is π_1–stability. In this paper we present a program to generalize Theorem 1 so that it applies to manifolds with non-stable fundamental groups at infinity. Of course, a manifold with non-stable fundamental group at infinity cannot be collarable, so we must be satisfied with a less rigid structure on its end. The structure we have chosen to pursue will be called a *pseudo-collar*.

We say that a manifold U^n with compact boundary is a *homotopy collar* provided the inclusion $\partial U^n \hookrightarrow U^n$ is a homotopy equivalence. As it turns out, a homotopy collar may possess very little additional structure, hence, we define the following more rigid notion. A *pseudo-collar* is a homotopy collar that contains arbitrarily small homotopy collar neighborhoods of infinity.

With the above definition established, the goal of this paper can be described as a study of pseudo-collarability in high dimensional manifolds. For the sake of the experts, we state our principal result now. A more thorough development and motivation of this theorem can be found in Section 4.

Main Existence Theorem *A one ended open n–manifold M^n ($n \geq 7$) is pseudo-collarable provided each of the following is satisfied:*

(1) *M^n is inward tame at infinity,*

(2) *$\pi_1(\varepsilon(M^n))$ is perfectly semistable,*

(3) *$\sigma_\infty(M^n) = 0 \in \tilde{K}_0(\pi_1(\varepsilon(M^n)))$, and*

(4) *$\pi_2(\varepsilon(M^n))$ is semistable.*

As the reader can see, Condition 1 of Theorem 1 is unchanged in our more general setting. Condition 3 has been reformulated so that it applies to situations where the fundamental group at infinity is not stable — but it also is essentially unchanged. Both of these conditions are discussed in Section 3. The weakening of Condition 2 is the main task in this paper. Much of our work is done with no restrictions on the fundamental group at infinity; however, it eventually becomes necessary to focus on manifolds with *perfectly semistable* π_1–systems at infinity. These systems are semistable (also called *Mittag–Leffler*) and have bonding maps with perfect kernels. Condition 4 is different from the others — it has no analog in Theorem 1, and we are not sure whether it is necessary. It does, however, play a crucial role in our proof.

Semistability conditions are well-established in studies of non-compact 3–manifolds (see [18] or [3]) and also in studies of ends of groups (see [20]), so it seems fitting that they play a role in the study of high-dimensional manifolds. Precise definitions of these conditions may be found in Section 2.

In Section 3 we review some basics in the study of non-compact manifolds, then in Section 4 we explore the topology of pseudo-collars. Some examples are discussed and basic geometric and algebraic properties are derived. These provide the necessary framework and motivation for our Main Existence Theorem. Most of the remainder of the paper (Sections 5–8) is geared towards proving this theorem. The strategy is much the same as that used in [23]; however, since the hypothesis of π_1–stability is thoroughly ingrained in Siebenmann's work, nearly all steps require some revision. Sometimes these revisions are significant, while at other times the original arguments already suffice. For completeness, portions of [23] have been repeated. The reader who makes it to the end of this paper will reprove Siebenmann's theorem in the process. (See Remark 8.) In the final section of this paper we discuss some open questions.

We conclude this introduction by defending our choice of "pseudo-collarability" as the appropriate generalization of collarability.

At first glance, one might expect "homotopy collar" to be a good enough generalization of "collar". Unfortunately, homotopy collars carry very little useful structure beyond what is given by their definition. For example, every contractible open manifold (no matter how badly behaved at infinity) contains a homotopy collar neighborhood of infinity — just consider the complement of a small open ball in the manifold. Hence, some additional structure is desired. Propositions 2 and 3 and Theorem 2 show that pseudo-collars do indeed carry a great deal of additional structure.

A second reason for defining pseudo-collars as we have is to mimic a key property possessed by genuine collars. In particular, a collar structure on the end of a manifold guarantees the existence of arbitrarily small collar neighborhoods of that end. Although this observation is trivial, it is extremely important in applications. It seems that any useful generalization of "collar" should have an analogous property.

A third factor which focused our attention on pseudo-collarability was work by Chapman and Siebenmann on Z–compactifications of Hilbert cube manifolds. Although they advertise their main result as an infinite dimensional version of Theorem 1, it is really much more general. In particular, it applies to Hilbert cube manifolds with non-collarable ends. Their program can be broken into two parts. First they determine necessary and sufficient conditions for a one ended Hilbert cube manifold X to contain arbitrarily small neighborhoods U of infinity for which $Bdry(U) \hookrightarrow U$ is a homotopy equivalence. (In our language, they determine when X is pseudo-collarable.) Next they combine the structure supplied by the pseudo-collar with some powerful results from Hilbert cube manifold theory to determine whether a Z–compactification is possible. It is natural to ask if their program can be carried out in finite dimensions. In this paper we focus on the first part of that program. We intend to address the issue of Z–compactifiability for finite dimensional manifolds and its relationship to pseudo-collarability in a later paper.

A final reason for the choices we have made lies with some key examples and current research trends in topology. For instance, the exotic universal covering spaces produced by M Davis in [10] are all pseudo-collarable but not collarable. Variations on those examples were produced in [11] with the aid of $CAT(0)$

geometry — they are also pseudo-collarable. Moreover, many of the basic conditions necessary for pseudo-collarability are satisfied by all $CAT(0)$ manifolds and also by universal covers of all aspherical manifolds with word hyperbolic or $CAT(0)$ fundamental groups. Thus, the collection of examples to which our techniques might be applied appears quite rich.

We wish to thank Steve Ferry for directing us to [12] and for sharing a copy of [13] which contains a clear and concise exposition of Siebenmann's thesis.

2 Inverse sequences and group theory

Throughout this section all arrows denote homomorphisms, while arrows of the type $\twoheadrightarrow$ or $\twoheadleftarrow$ denote surjections. The symbol $\cong$ denotes isomorphisms.

Let

$$G_0 \xleftarrow{\lambda_1} G_1 \xleftarrow{\lambda_2} G_2 \xleftarrow{\lambda_3} \cdots$$

be an inverse sequence of groups and homomorphisms. A *subsequence* of $\{G_i, \lambda_i\}$ is an inverse sequence of the form

$$G_{i_0} \xleftarrow{\lambda_{i_0+1}\circ\cdots\circ\lambda_{i_1}} G_{i_1} \xleftarrow{\lambda_{i_1+1}\circ\cdots\circ\lambda_{i_2}} G_{i_2} \xleftarrow{\lambda_{i_2+1}\circ\cdots\circ\lambda_{i_3}} \cdots.$$

In the future we will denote a composition $\lambda_i \circ \cdots \circ \lambda_j$ $(i \le j)$ by $\lambda_{i,j}$.

We say that sequences $\{G_i, \lambda_i\}$ and $\{H_i, \mu_i\}$ are *pro-equivalent* if, after passing to subsequences, there exists a commuting diagram:

$$\begin{tikzcd}
G_{i_0} & & G_{i_1} \arrow[ll, "\lambda_{i_0+1,i_1}"'] \arrow[dl] & & G_{i_2} \arrow[ll, "\lambda_{i_1+1,i_1}"'] \arrow[dl] & & \cdots \arrow[ll, "\lambda_{i_2+1,i_3}"'] \arrow[dl] \\
& H_{j_0} \arrow[ul] & & H_{j_1} \arrow[ll, "\mu_{i_0+1,i_1}"] \arrow[ul] & & H_{j_2} \arrow[ll, "\mu_{i_1+1,i_1}"] \arrow[ul] &
\end{tikzcd}$$

Clearly an inverse sequence is pro-equivalent to any of its subsequences. To avoid tedious notation, we often do not distinguish $\{G_i, \lambda_i\}$ from its subsequences. Instead we simply assume that $\{G_i, \lambda_i\}$ has the desired properties of a preferred subsequence — often prefaced by the words "after passing to a subsequence and relabelling".

The *inverse limit* of a sequence $\{G_i, \lambda_i\}$ is a subgroup of $\prod G_i$ defined by

$$\varprojlim \{G_i, \lambda_i\} = \left\{ (g_0, g_1, g_2, \cdots) \in \prod_{i=0}^{\infty} G_i \;\middle|\; \lambda_i(g_i) = g_{i-1} \right\}.$$

Notice that for each i, there is a *projection homomorphism* $p_i\colon \varprojlim\{G_i;\lambda_i\} \to G_i$. It is a standard fact that pro-equivalent inverse sequences have isomorphic inverse limits.

An inverse sequence $\{G_i;\lambda_i\}$ is *stable* if it is pro-equivalent to a constant sequence $\{H; id\}$. It is easy to see that $\{G_i;\lambda_i\}$ is stable if and only if, after passing to a subsequence and relabelling, there is a commutative diagram of the form:

$$\begin{tikzcd}
G_0 & & G_1 \arrow[ll, "\lambda_1"'] \arrow[dl] & & G_2 \arrow[ll, "\lambda_2"'] \arrow[dl] & & G_3 \arrow[ll, "\lambda_3"'] \arrow[dl] & & \cdots \arrow[ll, "\lambda_4"'] \\
& im(\lambda_1) \arrow[ul] & & im(\lambda_2) \arrow[ll, "\cong"'] \arrow[ul] & & im(\lambda_3) \arrow[ll, "\cong"'] \arrow[ul] & & \cdots \arrow[ll, "\cong"']
\end{tikzcd}$$

In this case $H = \varprojlim\{G_i;\lambda_i\} = im(\lambda_i)$ and each projection homomorphism takes $\varprojlim\{G_i;\lambda_i\}$ isomorphically onto the corresponding $im(\lambda_i)$.

The sequence $\{G_i;\lambda_i\}$ is *semistable* (or *Mittag–Leffler*) if it is pro-equivalent to an inverse sequence $\{H_i;\mu_i\}$ for which each μ_i is surjective. Equivalently, $\{G_i;\lambda_i\}$ is semistable if, after passing to a subsequence and relabelling, there is a commutative diagram of the form:

$$\begin{tikzcd}
G_0 & & G_1 \arrow[ll, "\lambda_1"'] \arrow[dl] & & G_2 \arrow[ll, "\lambda_2"'] \arrow[dl] & & G_3 \arrow[ll, "\lambda_3"'] \arrow[dl] & & \cdots \arrow[ll, "\lambda_4"'] \\
& im(\lambda_1) \arrow[ul] & & im(\lambda_2) \arrow[ll, two heads] \arrow[ul] & & im(\lambda_3) \arrow[ll, two heads] \arrow[ul] & & \cdots \arrow[ll, two heads]
\end{tikzcd}$$

We now describe a subclass of semistable inverse sequences which are of particular interest to us. Recall that a *commutator* element of a group H is an element of the form $xyx^{-1}y^{-1}$ where $x, y \in H$; and the *commutator subgroup* of H, denoted $[H, H]$, is the subgroup generated by all of its commutators. We say that H is *perfect* if $[H, H] = H$. An inverse sequence of groups is *perfectly semistable* if it is pro-equivalent to an inverse sequence

$$G_0 \overset{\lambda_1}{\twoheadleftarrow} G_1 \overset{\lambda_2}{\twoheadleftarrow} G_2 \overset{\lambda_3}{\twoheadleftarrow} \cdots$$

of finitely presentable groups and surjections where each $\ker(\lambda_i)$ is perfect. The following shows that inverse sequences of this type behave well under passage to subsequences.

Lemma 1 *Suppose $f\colon A \to B$ and $g\colon B \to C$ are each surjective group homomorphisms with perfect kernels. Then $g \circ f\colon A \to C$ is surjective and has perfect kernel.*

Proof Surjectivity is obvious. To see that $\ker(g \circ f)$ is perfect, begin with $a \in A$ such that $(g \circ f)(a) = 1$. Then $f(a) \in \ker(g)$, so by hypothesis we may write

$$f(a) = \prod_{i=1}^{k} x_i y_i x_i^{-1} y_i^{-1} \text{ where } x_i, y_i \in \ker(g) \text{ for } i = 1, \cdots, k.$$

For each i, choose $u_i, v_i \in A$ such that $f(u_i) = x_i$ and $f(v_i) = y_i$. Note that each u_i and v_i lies in $\ker(g \circ f)$, and let

$$a' = \prod_{i=1}^{k} u_i v_i u_i^{-1} v_i^{-1}.$$

Then $f(a') = f(a)$, which implies that $a(a')^{-1} \in \ker(f)$; so by hypothesis we may write

$$\left(a a'^{-1}\right) = \prod_{j=1}^{l} r_j s_j r_j^{-1} s_j^{-1} \text{ where } r_j, s_j \in \ker(f) \text{ for } j = 1, \cdots, l.$$

Moreover, since $\ker(f) \subset \ker(g \circ f)$, each r_j, s_j lies in $\ker(g \circ f)$.

Finally, we write

$$a = \left(a a'^{-1}\right) a' = \left(\prod_{j=1}^{l} r_j s_j r_j^{-1} s_j^{-1}\right)\left(\prod_{i=1}^{k} u_i v_i u_i^{-1} v_i^{-1}\right),$$

which shows that $a \in [\ker(g \circ f), \ker(g \circ f)]$. □

Corollary 1 *If $\{G_i, \alpha_i\}$ is an inverse sequence of groups and surjections with perfect kernels, then so is any subsequence.*

We conclude this section with three more group theoretic lemmas which will be used later. The first is from [26].

Lemma 2 *Let A be a finitely generated group and $f: A \to B$ and $g: B \to A$ be group homomorphisms with $f \circ g = id_B$. Then* $\ker(f)$ *is the normal closure of a finite set of elements. Therefore, if A is finitely presentable, then so is B.*

Proof Let $\{a_i\}_{i=1}^{k}$ be a generating set for A and let $X = \left\{a_i \cdot (g \circ f)(a_i^{-1})\right\}_{i=1}^{k}$. We will show that $\ker(f)$ is the normal closure of X.

First note that $f\left(a_i \cdot (g \circ f)(a_i^{-1})\right) = f(a_i)\,(f \circ g \circ f)\left(a_i^{-1}\right) = f(a_i)\, f\left(a_i^{-1}\right) = 1$, for each i. Hence X (and therefore the normal closure of X), is contained in $\ker(f)$.

As preparation for obtaining the reverse inclusion, let $w = w_1w_2 \in \ker(f)$ and observe that

$$\begin{aligned} w_1w_2 &= w_1w_2\cdot(g\circ f)((w_1w_2)^{-1}) \\ &= w_1w_2\cdot(g\circ f)(w_2^{-1})\cdot(g\circ f)(w_1^{-1}) \\ &= (w_1[w_2\cdot(g\circ f)(w_2^{-1})]w_1^{-1})(w_1\cdot(g\circ f)(w_1^{-1})). \end{aligned}$$

With this identity as the main tool, induction on word length shows that $\ker(f) \subset \textit{normal closure}(X)$. □

The next lemma is from [23], where it is used for purposes similar to our own.

Lemma 3 *Let $f\colon A \twoheadrightarrow B$ be a group homomorphism, and suppose $A = \langle a \mid r\rangle$ and $B = \langle b \mid s\rangle$ are presentations with $|a|$ generators and $|s|$ relators, respectively. Then* $\ker(f)$ *is the normal closure of a set containing $|a| + |s|$ elements.*

Proof Let β be a set of words so that $f(a) = \beta(b)$ in B. Since f is surjective, there exists a set of words α so that $b = \alpha(f(a))$ in B. Then Tietze transformations give the following isomorphisms:

$$\begin{aligned} \langle b \mid s\rangle &= \langle a, b \mid a = \beta(b), s(b)\rangle \\ &= \langle a, b \mid a = \beta(b), s(b), r(a), b = \alpha(a)\rangle \\ &= \langle a, b \mid a = \beta(\alpha(a)), s(\alpha(a)), r(a), b = \alpha(a)\rangle \\ &= \langle a \mid a = \beta(\alpha(a)), s(\alpha(a)), r(a)\rangle \end{aligned}$$

Now f is specified by the last presentation via the correspondence $a \mapsto a$. Hence $\ker(f)$ is the normal closure of the $|a| + |s|$ elements of $\beta(\alpha(a))$ and $s(\alpha(a))$. □

The following lemma was extracted from the proof of Theorem 4 in [13].

Lemma 4 *Each semistable inverse sequence $\{G_i, \lambda_i\}$ of finitely presented groups is pro-equivalent to an inverse sequence $\{G_i', \lambda_i'\}$ of finitely presented groups with surjective bonding maps.*

Proof After passing to a subsequence and relabelling we have a diagram:

$$\begin{tikzcd} G_0 & & G_1 \arrow[ll, "\lambda_1"'] \arrow[dl] & & G_2 \arrow[ll, "\lambda_2"'] \arrow[dl] & & G_3 \arrow[ll, "\lambda_3"'] \arrow[dl] & & \arrow[ll, "\lambda_4"'] \\ & im(\lambda_1) \arrow[ul] & & im(\lambda_2) \arrow[ll, two heads] \arrow[ul] & & im(\lambda_3) \arrow[ll, two heads] \arrow[ul] & & \arrow[ll, two heads] & \end{tikzcd}$$

The $im(\lambda_i)$'s are clearly finitely generated but may not be finitely presented. We will use this diagram to produce a new sequence with the desired properties. For each $i \geq 1$, let $\left\{g_j^i\right\}_{j=1}^{n_i}$ be a generating set for G_i, and choose $\left\{h_j^i\right\}_{j=1}^{n_i} \subseteq im(\lambda_{i+1})$ so that $\lambda_i(g_j^i) = \lambda_i(h_j^i)$.

Note The superscripts are indices, not powers.

Let $H_i \lhd G_i$ be the normal closure of the set $S_i = \left\{g_j^i \left(h_j^i\right)^{-1}\right\}_{j=1}^{n_i}$, define G_i' to be G_i/H_i, and let $q_i\colon G_i \to G_i'$ be the quotient map. Since $S_{i+1} \subseteq \ker(\lambda_{i+1})$ we get induced homomorphisms $\lambda'_{i+1}\colon G'_{i+1} \to G_i$. Define $\mu_{i+1} = q_i \circ \lambda'_{i+1}\colon G'_{i+1} \to G_i'$ to obtain the commuting diagram:

$$\begin{tikzcd}
G_1 \arrow[d,"q_1"'] & G_2 \arrow[l,"\lambda_2"'] \arrow[d,"q_2"] \arrow[dl,"\lambda'_2"'] & G_3 \arrow[l,"\lambda_3"'] \arrow[d,"q_3"] \arrow[dl,"\lambda'_3"'] & {} \arrow[l,"\lambda_4"'] \\
G'_1 & G'_2 \arrow[l, two heads,"\mu_2"'] & G'_3 \arrow[l, two heads,"\mu_3"'] & {} \arrow[l, two heads,"\mu_4"']
\end{tikzcd}$$

Since each G_i' is generated by $\left\{q_i(g_j^i)\right\}_{j=1}^{n_i}$ and each $q_i(g_j^i)$ has a preimage $h_j^i \in G_{i+1}$ under the map $q_i \circ \lambda_{i+1}$, it follows (from the commutativity of the diagram) that each μ_{i+1} is surjective. Lastly, each G_i' has a finite presentation which may be obtained from a finite presentation for G_i by adding relators corresponding to the elements of S_i. $\square$

3 Ends of manifolds: definitions and background information

In this section we review some standard notions involved in the study of non-compact manifolds and complexes. Since the terminology and notation used in this area are by no means standardized, the reader should be careful when consulting other sources. The remarks at the end of the section addresses a portion of this issue.

The symbol $\approx$ will denote homeomorphisms; $\simeq$ will denote homotopic maps or homotopy equivalent spaces. A manifold M^n is *open* if it is non-compact and has no boundary. We say that M^n is *one ended* if complements of compacta in M^n contain exactly one unbounded component. For convenience, we restrict our attention to one ended manifolds. In addition, we will work in the PL category. Equivalent results in the smooth and topological categories may be obtained in the usual ways. Results may be generalized to spaces with finitely many ends by considering one end at a time.

A set $U \subset M^n$ is a *neighborhood of infinity* if $\overline{M^n - U}$ is compact; U is a *clean neighborhood of infinity* if it is also a PL submanifold with bicollared boundary. It is easy to see that each neighborhood U of infinity contains a clean neighborhood V of infinity — just let $V = M^n - N$ where N is a regular neighborhood of a polyhedron containing $\overline{M^n - U}$. We may also arrange that V be connected by discarding all of its compact components. Thus we have:

Lemma 5 *Each one ended open manifold M^n contains a sequence $\{U_i\}_{i=0}^{\infty}$ of clean connected neighborhoods of infinity with $U_{i+1} \subset U_i$ for all $i \geq 0$, and $\bigcap_{i=0}^{\infty} U_i = \emptyset$.*

A sequence of the above type will be called *neat*. In the future, all neighborhoods of infinity are assumed to be clean and connected and sequences of these neighborhoods are neat.

We say that M^n is *inward tame* at infinity if, for arbitrarily small neighborhoods of infinity U, there exist homotopies $H\colon U \times [0,1] \to U$ such that $H_0 = id_U$ and $\overline{H_1(U)}$ is compact. Thus inward tameness means that neighborhoods of infinity can be pulled into compact subsets of themselves.

Recall that a CW complex X is *finitely dominated* if there exists a finite complex K and maps $u\colon X \to K$ and $d\colon K \to X$ such that $d \circ u \simeq id_X$. It is easy to see that X is finitely dominated if and only if it may be homotoped into a compact subset of itself. Hence, our manifold M^n is inward tame if and only if arbitrarily small neighborhoods of infinity are finitely dominated. This characterization of "inward tameness" will be useful to us later.

Next we study the *fundamental group system at the end* of M^n. Begin with a neat sequence $\{U_i\}_{i=0}^{\infty}$ of neighborhoods of infinity and basepoints $p_i \in U_i$. For each $i \geq 1$, choose a path $\alpha_i \subset U_{i-1}$ connecting p_i to p_{i-1}. Then, for each $i \geq 1$, let $\lambda_i\colon \pi_1(U_i, p_i) \to \pi_1(U_{i-1}, p_{i-1})$ be the homomorphism induced by inclusion followed by the change of basepoint isomorphism determined by α_i. Suppressing basepoints, this gives us an inverse sequence:

$$\pi_1(U_0) \xleftarrow{\lambda_1} \pi_1(U_1) \xleftarrow{\lambda_2} \pi_1(U_2) \xleftarrow{\lambda_3} \pi_1(U_3) \xleftarrow{\lambda_3} \cdots$$

Provided this sequence is semistable, one can show that its pro-equivalence class does not depend on any of the choices made above. We then denote the pro-equivalence class of this sequence by $\pi_1(\varepsilon(M^n))$. (In the absence of semistability, the choices become part of the data.) We will denote the inverse limit of the above sequence by $\check{\pi}_1(\infty)$.

Note For our purposes, $\check{\pi}_1(\infty)$ will only be used as a (rather trivial) convenience when $\pi_1(\varepsilon(M^n))$ is stable. Otherwise, we work with the inverse sequence.

Before moving to a new topic, notice that the same procedure may be used to define $\pi_k(\varepsilon(M^n))$ for $k > 1$.

We now understand the first two conditions in Theorem 1, and begin to look at the third. If Λ is a ring, we say that two finitely generated projective Λ–modules P and Q are *stably equivalent* if there exist finitely generated free Λ–modules F_1 and F_2 such that $P \oplus F_1 \cong Q \oplus F_2$. The stable equivalence classes of finitely generated projective modules form a group $\tilde{K}_0(\Lambda)$ under direct sum. Then P represents the trivial element of $\tilde{K}_0(\Lambda)$ if and only if it is *stably free*, ie, there exists a finitely generated free Λ–module F such that $P \oplus F$ is free. In [26], Wall shows that each finitely dominated X determines a well-defined element $\sigma(X) \in \tilde{K}_0(\mathbb{Z}[\pi_1 X])$ which vanishes if and only if X has the homotopy type of a finite complex. When an open one ended manifold M^n ($n \geq 6$) satisfies Conditions 1 and 2 of Theorem 1, Siebenmann isolated a single obstruction (which we have denoted $\sigma_\infty(M^n)$) in $\tilde{K}_0(\mathbb{Z}[\check{\pi}_1(\infty)])$ to finding an open collar neighborhood of infinity. In addition he observed that, up to sign, his obstruction is just the Wall obstruction of an appropriately chosen neighborhood of infinity. One upshot of this observation (requiring use of Siebenmann's Sum Theorem for the Finiteness Obstruction — see [23] or [13]) is that $\sigma_\infty(M^n)$ vanishes if and only if all clean neighborhoods of infinity in M^n have finite homotopy types.

When $\pi_1(\varepsilon(M^n))$ is not stable, the definition of $\sigma_\infty(M^n)$ becomes somewhat more complicated. Instead of measuring the obstruction in a single neighborhood of infinity, it will lie in the group $\tilde{K}_0(\pi_1(\varepsilon(M^n))) \equiv \varprojlim \tilde{K}_0(\mathbb{Z}[\pi_1 U_i])$, where $\{U_i\}$ is a neat sequence of neighborhoods of infinity. Then $\sigma_\infty(M^n)$ may be identified with the element $(-1)^n(\sigma(U_0), \sigma(U_1), \sigma(U_2), \cdots)$, with $\sigma(U_i)$ being the Wall finiteness obstruction for U_i. Again, this obstruction vanishes if and only if all clean neighborhoods of infinity in M^n have finite homotopy types. When $\pi_1(\varepsilon(M^n))$ is stable, this definition of $\sigma_\infty(M^n)$ reduces to the one discussed above. When $n \geq 6$ and $\pi_1(\varepsilon(M^n))$ is semistable, we will see $\sigma_\infty(M^n)$ arise naturally — without reference to the Wall finiteness obstruction — as an obstruction to pseudo-collarability (see Section 8). For a more general treatment of this obstruction — which, among other things, shows that $\tilde{K}_0(\pi_1(\varepsilon(M^n)))$ and $\sigma_\infty(M^n)$ are independent of the choice of $\{U_i\}$ — we refer the reader to [6].

Remark 2 Our use of the phrase “inward tame” is not standard. In [6] the same notion is simply called “tame”, while in [23], “tame” means “inward tame and π_1–stable”. Quinn and others (see, for example, [17]) have given “tame” a different and inequivalent meaning which involves pushing neighborhoods of infinity toward the end of the space, while referring to our brand of tameness as “reverse tameness”. We hope that by referring to our version of tameness as “inward tame” and Quinn’s version as “outward tame” we can avoid some confusion.

Remark 3 One should be careful not to interpret the symbol $\sigma_\infty(M^n)$ as the Wall finiteness obstruction $\sigma(M^n)$ of the manifold M^n. Indeed, M^n can have finite homotopy type even when its neighborhoods of infinity do not. (The Whitehead contractible 3–manifold is one well-known example.) This situation can arise even when $\pi_1(\varepsilon(M^n))$ is stable.

4 Pseudo-collars and the Main Theorem

Recall that a manifold U^n with compact boundary is an *open collar* if $U^n \approx \partial U^n \times [0,\infty)$; it is a *homotopy collar* if the inclusion $\partial U^n \hookrightarrow U^n$ is a homotopy equivalence. If U^n is a homotopy collar which contains arbitrarily small homotopy collar neighborhoods of infinity, then we call U^n a *pseudo-collar*. We say that an open n–manifold M^n is *collarable* if it contains an open collar neighborhood of infinity, and that M^n is *pseudo-collarable* if it contains a pseudo-collar neighborhood of infinity. The following easy example is useful to keep in mind.

Example 1 Let M^n be a contractible n–manifold and $B^n \subset M^n$ a standardly embedded n–ball. Then $U = M^n - B^n$ is a homotopy collar; however, in general M^n need not be pseudo-collarable (see Example 2).

Remark 4 A standard duality argument guarantees that any connected homotopy collar (hence any connected pseudo-collar) is one ended. See, for example, [24].

When discussing collars, some complementary notions are useful. A compact codimension 0 submanifold C of an open manifold M^n is called a *core* if $C \hookrightarrow M^n$ is a homotopy equivalence; it is called a *geometric core* if $M^n - C$ is a homotopy collar; and it is called an *absolute core* if $M^n - C$ is an open collar. The following is immediate.

Proposition 1 *Let M^n be a one ended open n–manifold. Then:*

(1) *M^n is collarable if and only if M^n contains an absolute core (hence, arbitrarily large absolute cores), and*

(2) *M^n is pseudo-collarable if and only if M^n contains arbitrarily large geometric cores.*

Example 2 The Whitehead contractible 3–manifold M^3 is not pseudo-collarable. Indeed, if M^3 were pseudo-collarable, it would contain arbitrarily large geometric cores each of which — by standard 3–manifold topology — would be a 3–ball. But then M^3 would be a monotone union of open 3–balls, and hence, homeomorphic to $\mathbb{R}^3$ by [4] or by an application of the Combinatorial Annulus Theorem (Corollary 3.19 of [22]).

On the positive side we have:

Example 3 Although they are not collarable, the exotic universal covering spaces produced by Davis in [10] are pseudo-collarable. If M^n is one of these covering spaces with compact contractible manifold C^n as a "fundamental chamber", then M^n contains arbitrarily large geometric cores homeomorphic to finite sums

$$C^n \#_\partial C^n \#_\partial \cdots \#_\partial C^n$$

(with increasing numbers of summands). Here $\#_\partial$ denotes a "boundary connected sum", ie, the union of two n–manifolds with boundary along boundary $(n-1)$–disks. However, M^n is not collarable since $\pi_1(\varepsilon(M^n))$ is not stable — in fact $\pi_1(\varepsilon(M^n))$ may be represented by the sequence

$$G \leftarrow G * G \leftarrow (G * G) * G \leftarrow (G * G * G) * G \leftarrow \cdots$$

where $G = \pi_1(\partial C^n)$ and each homomorphism is projection onto the first term. It is interesting to note that this sequence is perfectly semistable.

A compact cobordism $\left(W^n, M^{n-1}, N^{n-1}\right)$ is a *one-sided h-cobordism* if one (but not necessarily both) of the inclusions $M^{n-1} \hookrightarrow W^n$ or $N^{n-1} \hookrightarrow W^n$ is a homotopy equivalence. The following property of one-sided h-cobordisms is a well-known consequence of duality (see, for example, Lemma 2.5 of [8]).

Lemma 6 *Let $\left(W^n, M^{n-1}, N^{n-1}\right)$ be a compact connected one-sided h-cobordism with $M^{n-1} \overset{\simeq}{\hookrightarrow} W^n$. Then the inclusion induced homomorphism $\pi_1\left(N^{n-1}\right) \to \pi_1(W^n)$ is surjective and has perfect kernel.*

Non–trivial one-sided h-cobordisms are plentiful. In fact, if we are given a closed $(n-1)$–manifold N^{n-1} ($n \geq 6$), a finitely presented group G, and a homomorphism $\phi\colon \pi_1\left(N^{n-1}\right) \twoheadrightarrow G$ with perfect kernel, then the "Quillen plus construction" (see [21] or Sections 11.1 and 11.2 of [15]) produces a one-sided h-cobordism $\left(W^n, M^{n-1}, N^{n-1}\right)$ with $\pi_1(W^n) = G$ and

$$\ker\left(\pi_1\left(N^{n-1}\right) \to \pi_1(W^n)\right) = \ker(\phi).$$

The role played by one-sided h-cobordisms in the study of pseudo-collars is clearly illustrated by the following easy proposition.

Proposition 2 *Let $\{(W_i, M_i, N_i)\}_{i=1}^{\infty}$ be a collection of one-sided h-cobordisms with $M_i \overset{\simeq}{\hookrightarrow} W_i$, and suppose that for each $i \geq 1$ there is a homeomorphism $h_i\colon N_i \to M_{i+1}$. Then the adjunction space*

$$U = W_1 \cup_{h_1} W_2 \cup_{h_2} W_3 \cup_{h_3} \cdots$$

is a pseudo-collar. Conversely, every pseudo-collar may be expressed as a countable union of one-sided h-cobordisms in this manner.

Proof For the forward implication, we begin by observing that U is a homotopy collar. First note that $\partial U = M_1 \hookrightarrow W_1 \cup_{h_1} \cdots \cup_{h_1} W_k$ is a homotopy equivalence for any finite k. A direct limit argument then shows that $\partial U \hookrightarrow U$ is a homotopy equivalence. Alternatively, we may observe that $\pi_*(U, \partial U) \equiv 0$ and apply the Whitehead theorem. To see that U is a pseudo-collar we apply the same argument to the subsets $U_i = W_{i+1} \cup_{h_{i+1}} W_{i+2} \cup_{h_{i+2}} W_{i+3} \cup_{h_{i+3}} \cdots$.

For the converse, assume that U is a pseudo-collar. Choose a homotopy collar $U_1 \subset U$ and let $W_1 = U - U_1$. Then $\partial U \overset{\simeq}{\hookrightarrow} W_1$, so $(W_1, \partial U, \partial U_1)$ is a one-sided h-cobordism. Next choose a homotopy collar $U_2 \subset U_1$ and let $W_2 = U_1 - U_2$. Repeating this procedure gives the desired result. See Figure 1. □

The next result provides a striking similarity between pseudo-collars and genuine open collars. It follows immediately from Proposition 2 and the main result of [9] which shows that one-sided h-cobordisms in dimensions ≥ 6 may be "laminated".

Proposition 3 *Let U^n be a pseudo-collar ($n \geq 6$). Then there exists a proper continuous surjection $p\colon U^n \to [0, \infty)$ with the following properties.*

(1) $p^{-1}(0) = \partial U^n$;

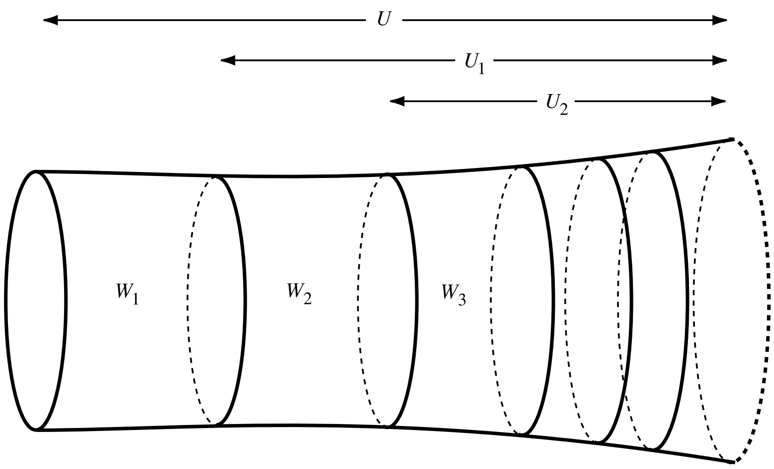

Figure 1

(2) *each* $p^{-1}(r)$ *is a closed* $(n-1)$*–manifold with the same* $\mathbb{Z}$*–homology as* ∂U^n*, and*

(3) $p^{-1}(r)$ *is nicely embedded, ie, has a product neighborhood in* U^n*, for* $r \neq 1, 2, 3, \cdots$.

Our next result provides the fundamental conditions necessary for pseudo-collarability.

Theorem 2 *Suppose a one ended open manifold* M^n *is pseudo-collarable. Then*

(1) M^n *is inward tame at infinity,*

(2) $\pi_1(\varepsilon(M^n))$ *is perfectly semistable, and*

(3) $\sigma_\infty(M^n) = 0 \in \tilde{K}_0(\pi_1(\varepsilon(M^n)))$.

Proof Properties 1 and 3 follow easily from the definition of pseudo-collar; while Property 2 is obtained from Proposition 2 and Lemma 6. □

One might hope that the above conditions are also sufficient for M^n ($n \geq 6$) to be pseudo-collarable. This would be an ideal generalization of Theorem 1; but, although we have not ruled it out, we are thus far unable to prove it. Our main result — which, for easy reference, we now restate — requires an additional hypothesis and one additional dimension.

Theorem 3 (Main Existence Theorem) *A one ended open n–manifold M^n ($n \geq 7$) is pseudo-collarable provided each of the following is satisfied:*

(1) *M^n is inward tame at infinity,*

(2) *$\pi_1(\varepsilon(M^n))$ is perfectly semistable,*

(3) *$\sigma_\infty(M^n) = 0 \in \widetilde{K}_0(\mathbb{Z}[\pi_1(\varepsilon(M^n))])$, and*

(4) *$\pi_2(\varepsilon(M^n))$ is semistable.*

Remark 5 Several interesting classes of manifolds are known to satisfy some or all of the conditions in the above theorems, thus making them ideal candidates for pseudo-collarability. We mention a few of them.

(a) We already know that the exotic universal coverings of [10] are pseudo-collarable, and therefore satisfy Conditions 1–3. It can also be shown that they satisfy Condition 4.

(b) Every piecewise flat $CAT(0)$ manifold satisfies Conditions 1–3. Some of the most interesting of these — the exotic universal covers produced by Davis and Januszkiewicz in [11] — also satisfy Condition 4 (and are therefore pseudo-collarable).

(c) A more general class of open n–manifolds which are of current interest are those admitting Z–compactifications (see [1], [2], [14] and [5] for discussions). These manifolds satisfy Conditions 1 and 3, and also have semistable fundamental groups at infinity (whether these are perfectly semistable is unknown).

Most of the remainder of this paper is devoted to proving the Main Existence Theorem.

5 Proof of the Main Existence Theorem: an outline

Let M^n be a 1–ended open manifold and U a connected clean neighborhood of infinity. According to [23], U is a *0–neighborhood of infinity* if ∂U is connected. Under the assumption that $\pi_1(\varepsilon(M^n))$ is stable, [23] then defines U to be a *1–neighborhood of infinity* provided it is a 0–neighborhood infinity and both $\pi_1(\infty) \to \pi_1(U)$ and $\pi_1(\partial U) \to \pi_1(U)$ are isomorphisms. For $k \geq 2$, U is a *k–neighborhood of infinity* if it is a 1–neighborhood of infinity and $\pi_i(U, \partial U) = 0$ for $i \leq k$.

We may now describe Siebenmann's proof of Theorem 1. Beginning with a neat sequence $\{U_i\}$ of neighborhoods of infinity, perform geometric alterations to obtain a neat sequence of 0–neighborhoods of infinity. This is easy — given a U_i with non-connected boundary, choose finitely many disjoint properly embedded arcs in U_i connecting the components of ∂U_i. Then "drill out" regular neighborhoods of these arcs to connect up the boundary components, thus obtaining a 0–neighborhood $U_i' \subset U_i$. After passing to a subsequence (if necessary) to maintain the "nestedness" condition, we have the desired sequence. Assuming then that $\{U_i\}$ is a neat sequence of 0–neighborhoods of infinity and that Conditions 1 and 2 of Theorem 1 are both satisfied, convert the U_i's into 1–neighborhoods of infinity. This stage of the proof is more complicated. We view it as the first of three major steps in obtaining Theorem 1. Some algebra (Lemmas 2 and 3) is required, neighborhoods of arcs are drilled out, and neighborhoods of disks are "traded" — sometimes removed and sometimes added. Ultimately one obtains a neat sequence of 1–neighborhoods of infinity. Next, in the middle step of the proof, the U_i's are inductively improved until they are $(n-3)$–neighborhoods. The key tools here are: general position, handle theory, and Lemma 9. The final step in Siebenmann's proof is to improve $(n-3)$–neighborhoods of infinity to $(n-2)$–neighborhoods — which turn out to be open collars. This step is very delicate. More algebra is required, the need for Condition 3 becomes clear, and π_1–stability plays a crucial role.

To a large extent, the proof of our Main Existence Theorem is a careful reworking of [23]. In fact, the reader will find Siebenmann's proof properly embedded in ours. However, since the π_1–stability hypothesis so thoroughly permeates [23], a great deal of revision and generalization is necessary. First, we define a *generalized 1–neighborhood of infinity* to be a 0–neighborhood of infinity U with the property that $\pi_1(\partial U) \to \pi_1(U)$ is an isomorphism. Then for $k \geq 2$, a *generalized k–neighborhood of infinity* is a generalized 1–neighborhood of infinity with the property that $\pi_i(U, \partial U) = 0$ for $i \leq k$. The point here is that, when $\pi_1(\varepsilon(M^n))$ is not stable, there is no "preferred fundamental group" for our neighborhoods of infinity. Later we will see that there are sometimes "preferred sequences of fundamental groups". To avoid confusion, we will often refer to the k–neighborhoods of infinity defined earlier as *strong k–neighborhoods of infinity*. Of course, this only makes sense when $\pi_1(\varepsilon(M^n))$ is stable.

We break our proof into the same three major steps as above. In the first step (Section 6) we obtain neat sequences of generalized 1–neighborhoods of infinity. For this, only Condition 1 of the Main Existence Theorem is required; however, given additional assumptions about $\pi_1(\varepsilon(M^n))$ (eg, stability, semistability, or perfect semistability), we show how these may be incorporated. The middle step of the proof (Section 7) requires the least revision of [23]. Only Condition 1 is needed to obtain a neat sequence of generalized $(n-3)$–neighborhoods of infinity. As before, additional assumptions on the fundamental group at infinity can be incorporated into this step. The final step (Section 8) is the most difficult. In order to make any progress beyond generalized $(n-3)$–neighborhoods of infinity, it becomes necessary to assume that $\pi_1(\varepsilon(M^n))$ is semistable (a part of Condition 2). We show that a neat sequence of generalized $(n-2)$–neighborhoods, with π_1–semistability appropriately built in, determines a pseudo-collar structure; hence, obtaining generalized $(n-2)$–neighborhoods is our goal. In our attempt to mimic Siebenmann, we rediscover the $\tilde{K}_0$–obstruction much as it appeared in [23]. The difference is that, since $\pi_1(U_i)$ changes with i, so must the $\tilde{K}_0$–obstruction. Hence, our obstruction becomes a *sequence* of obstructions. When this obstruction dies, most of the algebraic and handle theoretic steps from [23] may be duplicated. Unfortunately, at the last instant — a final application of the Whitney Lemma — the lack of π_1–stability creates major problems. To complete the proof in the non-stable situation, we are forced to develop a new strategy. It is only here that we require Condition 4 and the "perfect" part of Condition 2.

6 Obtaining generalized 1–neighborhoods of infinity

In this section we show how to obtain a neat sequence $\{U_i\}$ of generalized 1–neighborhoods of infinity in a one ended open n–manifold when $n \geq 5$. This requires only that M^n be inward tame at infinity. (In fact, it would be enough to assume that clean neighborhoods of infinity have finitely presentable fundamental groups.) In addition we show that, when $\pi_1(\varepsilon(M^n))$ is pro-equivalent to certain preferred inverse sequences of surjections, we can make our sequence $\{\pi_1(U_i)\}$ isomorphic to corresponding subsequences This covers situations where $\pi_1(\varepsilon(M^n))$ is stable, semistable and perfectly semistable.

Lemma 7 *Let M^n ($n \geq 5$) be a one ended open n–manifold which is inward tame at infinity and let V be a 0–neighborhood of infinity. Then V contains a generalized 1–neighborhood U of infinity with the property that $\pi_1(U) \to \pi_1(V)$ is an isomorphism.*

Proof First we construct a 0–neighborhood $V' \subset V$ so that $\pi_1(\partial V') \to \pi_1(V')$ is surjective and $\pi_1(V') \to \pi_1(V)$ is an isomorphism.

Since V is finitely dominated, $\pi_1(V)$ is finitely generated, so we may choose a finite collection $\{\alpha_1, \alpha_2, \cdots, \alpha_k\}$ of disjoint properly embedded p.l. arcs in V

so that $\pi_1(\partial V \cup (\bigcup_{i=1}^{k} \Gamma_i)) \to \pi_1(V)$ is surjective. Choose a collection $\{N_i\}_{i=1}^{k}$ of disjoint regular neighborhoods of the Γ_i's in V and let

$$V' = \overline{V - \bigcup_{i=1}^{k} N_i}.$$

Clearly $\pi_1(\partial V')$ (and thus $\pi_1(V')$) surjects onto $\pi_1(V)$; moreover, since disks in V may be pushed off the N_i's, then $\pi_1(V') \to \pi_1(V)$ is also injective.

Next we modify V' to be a generalized 1–neighborhood. Since V' is finitely dominated, Lemma 2 implies that $\pi_1(V')$ is finitely presentable. Hence, by Lemma 3, $\ker(\pi_1(\partial V') \to \pi_1(V'))$ is the normal closure of a finite set of elements. Let $\{\alpha_1, \alpha_2, \cdots, \alpha_r\}$ be a collection of pairwise disjoint embedded loops in $\partial V'$ representing these elements, then choose $\{D_1, D_2, \cdots, D_r\}$ a pairwise disjoint collection of properly embedded 2–disks in V' with $\partial D_i = \alpha_i$ for each i. Let $\{P_1, P_2, \cdots, P_r\}$ be a pairwise disjoint collection of regular neighborhoods of the D_i's in V' and define

$$U = \overline{V' - \bigcup_{i=1}^{r} P_i}.$$

By VanKampen's theorem $\pi_1(\partial V' \cup (\bigcup_{i=1}^{r} P_i)) \to \pi_1(V')$ is an isomorphism, and by general position $\pi_1(\partial U) \to \pi_1(\partial V' \cup (\bigcup_{i=1}^{r} P_i))$ and $\pi_1(U) \to \pi_1(V')$ are isomorphisms. It follows that U is a generalized 1– neighborhood of infinity and $\pi_1(U) \to \pi_1(V)$ is an isomorphism. □

Combining the above lemma with the method described in the previous section for obtaining 0–neighborhoods of infinity gives:

Corollary 2 *Every one ended open n–manifold ($n \geq 5$) that is inward tame at infinity contains a neat sequence of generalized 1–neighborhoods of infinity.*

Lemma 8 *Let M^n ($n \geq 5$) be a one ended n–manifold that is inward tame at infinity and suppose the fundamental group system $\pi_1(\varepsilon(M^n))$ is pro-equivalent to an inverse sequence $G\colon\ G_1 \twoheadleftarrow G_2 \twoheadleftarrow G_3 \twoheadleftarrow \cdots$ of finitely presentable groups and surjections. Then there is a neat sequence $\{U_i\}_{i=1}^{\infty}$ of 1–neighborhoods of infinity so that the inverse sequence $\pi_1(U_1) \leftarrow \pi_1(U_2) \leftarrow \pi_1(U_3) \leftarrow \cdots$ is isomorphic to a subsequence of G.*

Proof By the hypothesis and Corollary 2, there exists a neat sequence $\{V_i\}$ of generalized 1–neighborhoods of infinity, a subsequence $G_{k_1} \twoheadleftarrow G_{k_2} \twoheadleftarrow G_{k_3} \twoheadleftarrow \cdots$ of G, and a commutative diagram:

$$\begin{tikzcd}
\pi_1(V_0) & & \pi_1(V_1) \arrow[ll, "\lambda_1"'] \arrow[dl, "f_1"] & & \pi_1(V_2) \arrow[ll, "\lambda_2"'] \arrow[dl, "f_2"] & & \pi_1(V_3) \arrow[ll, "\lambda_3"'] \arrow[dl, "f_3"] & \cdots \arrow[l, "\lambda_4"'] \\
& G_{k_1} \arrow[ul, "g_1"] & & G_{k_2} \arrow[ll, two heads] \arrow[ul, "g_2"] & & G_{k_3} \arrow[ll, two heads] \arrow[ul, "g_3"] & & \cdots \arrow[ll, two heads]
\end{tikzcd}$$

Each f_i is necessarily surjective, so by Lemmas 2 and 3 each $\ker(f_i)$ is the normal closure of a finite set of elements $F_i \subset \pi_1(V_i)$. For each $i \geq 1$, choose a finite collection $\left\{\alpha^i_j\right\}_{j=1}^{n_i}$ of pairwise disjoint embedded loops in ∂V_i representing the elements of F_i. By the commutativity of the diagram, each α^i_j contracts in V_{i-1}. For each α^i_j choose an embedded disk $D^i_j \subset V_{i-1}$ with $\partial D^i_j = \alpha^i_j$. Arrange that the D^i_j's are pairwise disjoint, and all intersections between D^i_j and $\bigcup \partial V_k$ are transverse.

In order to kill the kernels of the f_i's, we would like to add to each V_i regular neighborhoods of the D^i_j's. This would work if each D^i_j was contained in $V_{i-1} - V_i$; for then we would be attaching a finite collection of 2–handles to each V_i and each would kill the normal closure of its attaching 1–sphere α^i_j in $\pi_1(V_i)$, and no more. Since this ideal situation may not be present, we must first perform some alterations on the V_i's.

Claim *There exists a nested cofinal sequence $\{V'_i\}$ of 0–neighborhoods of infinity which satisfy the following properties for all $i \geq 1$:*

(i) $V'_i \subset V_i$,

(ii) $\pi_1(V'_i) \to \pi_1(V_i)$ *is an isomorphism,*

(iii) $\bigcup_{j=1}^{n_i} \alpha^i_j \subset \partial V'_i$, *and*

(iv) *each* α^i_j *bounds a* 2*–disk in* $V'_{i-1} - V'_i$.

Roughly speaking, a V'_q will be constructed by removing regular neighborhoods of the D^q_j's from V_q; but in order arrange condition (iii) and to maintain "nestedness", some extra care must be taken.

We already have that $\partial D^i_j = \alpha^i_j \subset \partial V_i$ and D^i_j intersects finitely many ∂V_l ($l \geq i$) transversely. In addition, we would like the outermost component of $D^i_j - \partial V_i$ to lie in $V_{i-1} - V_i$. If this is not already the case, it can easily be arranged by pushing a small annular neighborhood of ∂D^i_j into $V_{i-1} - V_i$ while leaving $\partial D^i_j = \alpha^i_j$ fixed. Now choose a pairwise disjoint collection $\left\{L^i_j\right\}$ of regular neighborhoods of the collection $\left\{D^i_j\right\}$; then for each D^i_j, choose a smaller regular neighborhood $N^i_j \subset L^i_j$. Between each N^i_j and L^i_j there exists a sequence $2N^i_j, 3N^i_j, 4N^i_j, \cdots$ of regular neighborhoods of D^i_j such that

$$N^i_j \subset \operatorname{int} 2N^i_j \subset 2N^i_j \subset \operatorname{int} 3N^i_j \subset 3N^i_j \subset \cdots \subset L^i_j.$$

For each $q \geq 1$, let

$$V'_q = \overline{V_q - \left(\bigcup_{i=1}^{q} \bigcup_{j=1}^{n_i} qN^i_j \right)}.$$

Conditions (i), (iii) and (iv) are obvious, and since each V'_i was obtained from V_i by removing regular neighborhoods of 2–complexes, condition (ii) follows from general position.

Now, along each γ^i_j it is possible to attach an ambiently embedded 2–handle $h^i_j \subset V'_{i-1} - V'_i$ to V'_i. For each $q \geq 1$, let

$$V''_q = V'_q \cup \left(\bigcup_{j=1}^{n_q} h^q_j \right).$$

The naturally induced homomorphisms $f''_q : \pi_1(V''_q) \to G_{k_q}$ are now isomorphisms.

Lastly, we must apply Lemma 7 to each V''_q to create a sequence $\{U_q\}$ of generalized 1–neighborhoods of infinity with the same fundamental groups. To regain nestedness, we may then have to pass to a subsequence of $\{U_q\}$ (and to the corresponding subsequence of $\{G_{k_q}\}$) to complete the proof. □

The main consequences of this section are summarized by the following:

Theorem 4 (Generalized 1–Neighborhoods Theorem) *Let M^n $(n \geq 5)$ be a one ended, open n–manifold which is inward tame at infinity. Then:*

1. *M^n contains a neat sequence $\{U_i\}$ of generalized 1–neighborhoods of infinity,*
2. *if $\pi_1(\varepsilon(M^n))$ is stable, we may arrange that the U_i's are strong 1–neighborhoods of infinity,*
3. *if $\pi_1(\varepsilon(M^n))$ is semistable, we may arrange that each $\pi_1(U_i) \leftarrow \pi_1(U_{i+1})$ is surjective, and*
4. *if $\pi_1(\varepsilon(M^n))$ is perfectly semistable, we may arrange that each $\pi_1(U_i) \leftarrow \pi_1(U_{i+1})$ is surjective and has perfect kernel.*

Proof Claim 1 is just Corollary 2. To obtain Claim 2, observe that if $\pi_1(\varepsilon(M^n))$ is pro-equivalent to $\{G, \mathrm{id}\}$, then Lemma 2 implies that G is finitely presentable. Hence we may apply Lemma 8 to obtain the desired sequence. Claims 3 and 4 follow similarly from Lemma 8, with the necessary algebra being found in Lemma 4 and Corollary 1. □

7 Obtaining generalized $(n-3)$–neighborhoods of infinity

We now show how to obtain appropriate neat sequences of generalized $(n-3)$–neighborhoods of infinity. To do this, we begin with a neat sequence $\{U_i\}$ of generalized 1–neighborhoods of infinity and make geometric alterations to kill $\pi_j(U_i,\partial U_i)$ for $2 \le j \le n-3$. These alterations will not change the fundamental groups of the original U_i's, hence any work accomplished by Theorem 4 will be preserved.

If U_i is a generalized 1–neighborhood of infinity and $\rho : \widetilde{U}_i \to U_i$ is the universal covering projection, then $\partial\widetilde{U}_i = \rho^{-1}(\partial U_i)$ is the universal cover of ∂U_i, thus, $\pi_j(U_i,\partial U_i) = \pi_j(\widetilde{U}_i,\partial\widetilde{U}_i)$ for all j. Moreover, if U_i is a generalized $(k-1)$–neighborhood of infinity, the Hurewicz Theorem (Theorem 7.5.4 of [25]) implies that $\pi_k(\widetilde{U}_i,\partial\widetilde{U}_i) = H_k(\widetilde{U}_i,\partial\widetilde{U}_i)$. The last of these — the homology in the universal cover — is usually the easiest to work with. Throughout the remainder of this paper, the symbol "$\sim$" over a space denotes a universal cover.

When calculating homology groups we prefer cellular homology. If (X^n, Y^{n-1}) is a manifold pair with $Y^{n-1} \subseteq \partial X^n$, then a handle decomposition of X^n built on Y^{n-1} gives rise to a relative CW–complex $(K, Y^{n-1}) \simeq (X^n, Y^{n-1})$ obtained by collapsing handles onto their cores such that each j–cell of $K - Y^{n-1}$ corresponds to a unique j–handle of X^n. Then the cellular chain complex

$$0 \to C_n \to C_{n-1} \to \cdots \to C_0 \to 0 \qquad (\dagger)$$

for (K, Y^{n-1}), where each C_j is generated by the j–cells of $K - Y^{n-1}$, may be used to calculate the homology of (X^n, Y^{n-1}). We will frequently abuse terminology slightly by referring to $(\dagger)$ as the chain complex for (X^n, Y^{n-1}) and referring to the j–handles of X^n as the generators of C_j.

If $\pi_1(Y^{n-1}) \xrightarrow{\cong} \pi_1(X^n)$ and we wish to calculate $H_*(\widetilde{X}^n, \widetilde{Y}^n)$, we may use the cellular chain complex

$$0 \to \widetilde{C}_n \to \widetilde{C}_{n-1} \to \cdots \to \widetilde{C}_0 \to 0 \qquad (\ddagger)$$

of the pair $(\widetilde{K}, \widetilde{Y}^{n-1})$. This may be given the structure of a $\mathbb{Z}[\pi_1 K]$–complex, where $\widetilde{C}_j$ is a free $\mathbb{Z}[\pi_1 Y^{n-1}]$–module with one generator for each j–cell of $K - Y^{n-1}$ (see Chapter I of [7] for details). Alternatively, we will refer to $(\ddagger)$ as a chain complex for $(\widetilde{X}^n, \widetilde{Y}^n)$ where $\widetilde{C}_j$ has one $\mathbb{Z}[\pi_1 X^n]$–generator for each j–handle of X^n. The additional algebraic structure means that each $H_j(\widetilde{X}^n, \widetilde{Y}^n)$ may be viewed as a $\mathbb{Z}[\pi_1 X^n]$–module.

Another useful way to view $(\ddagger)$ is as the chain complex for the homology of (X^n, Y^{n-1}) with local $\mathbb{Z}[\pi_1 X^n]$–coefficients. Then $\tilde{C}_j = C_j \otimes \mathbb{Z}[\pi_1 X^n]$ is generated by the j–handles of X^n (with preferred base paths) and for $j > 2$ the boundary map is determined by $\mathbb{Z}[\pi_1 X^n]$–intersection numbers. In particular, if h^j is a j–handle of X^n with attaching $(j-1)$–sphere α^{j-1}, then

$$\partial h^j = \sum_s \varepsilon\left(\alpha^{j-1}, \beta_s^{n-j}\right) h_s^{j-1}$$

where $\varepsilon\left(\alpha^{j-1}, \beta_s^{n-j}\right)$ denotes the $\mathbb{Z}[\pi_1 X^n]$–intersection numbers between α^{j-1} and the belt sphere β_s^{n-j} of a $(j-1)$–handle h_s^{j-1} measured in ∂X_{j-1} where $X_{j-1} = (Y^{n-1} \times [0,1]) \cup$ (handles of index $\leq j-1$). See Chapter 6 and Appendix A of [22] for further discussion. When the chain complex is viewed in this manner, we will still denote the corresponding homology groups by $H_*\left(\tilde{X}^n, \tilde{Y}^n\right)$.

The following algebraic lemma will be used each time we attempt to improve a generalized j–neighborhood of infinity to a generalized $(j+1)$–neighborhood.

Lemma 9 *Suppose U is a generalized j–neighborhood of infinity ($j \geq 1$) in a one ended inward tame open n–manifold. Then $H_{j+1}\left(\tilde{U}, \partial\tilde{U}\right)$ is finitely generated as a $\mathbb{Z}[\pi_1 U]$–module.*

Proof Fix a triangulation of ∂U and for each $k \geq 2$, let K^k denote the corresponding k–skeleton. Let K^1 denote the corresponding 2–skeleton. Note that the inclusion $K^k \hookrightarrow U$ induces a π_1–isomorphism for all $k \geq 1$. Hence, we have universal covers $\tilde{U} \supseteq \partial\tilde{U} \supseteq \tilde{K}^k$.

Since $H_j\left(\partial\tilde{U}, \tilde{K}^j\right) = 0$, the long exact sequence for the triple $\left(\tilde{U}, \partial\tilde{U}, \tilde{K}^j\right)$ provides an epimorphism of $\mathbb{Z}[\pi_1]$–modules $H_{j+1}\left(\tilde{U}, \tilde{K}^j\right) \twoheadrightarrow H_{j+1}\left(\tilde{U}, \partial\tilde{U}\right)$. Hence, the desired conclusion will follow if we can show that $H_{j+1}\left(\tilde{U}, \tilde{K}^j\right)$ is finitely generated. This will follow immediately from Theorem A of [26] if we can show that $H_i\left(\tilde{U}, \tilde{K}^j\right) = 0$ for all $i \leq j$. Again we employ the exact sequence for $\left(\tilde{U}, \partial\tilde{U}, \tilde{K}^j\right)$:

$$\cdots \to H_i\left(\partial\tilde{U}, \tilde{K}^j\right) \to H_i\left(\tilde{U}, \tilde{K}^j\right) \to H_i\left(\tilde{U}, \partial\tilde{U}\right) \to \cdots$$

Clearly the first term listed vanishes for all $i \leq j$ and, by hypothesis, so does the third term; thus, forcing the middle term to vanish. □

The next lemma is the key to this section.

Lemma 10 *Let M^n ($n \geq 5$) be open, one ended and inward tame at infinity and let $k \leq n-3$. Then each generalized 1–neighborhood of infinity V_0 contains a generalized k–neighborhood of infinity U_0 such that $\pi_1(V_0) \xleftarrow{\cong} \pi_1(U_0)$.*

Proof If $k = 1$, there is nothing to prove; otherwise we assume inductively that $k \geq 2$ and each generalized 1–neighborhood of infinity V contains a generalized $(k-1)$–neighborhood of infinity U such that $\pi_1(V) \xleftarrow{\cong} \pi_1(U)$.

Now let V_0 be a generalized 1–neighborhood of infinity. By the inductive hypothesis, we may assume that V_0 is already a generalized $(k-1)$–neighborhood of infinity. We will show how to improve V_0 to a k–neighborhood of infinity.

As we noted earlier, $\pi_i\left(\widetilde{V}_0, \partial\widetilde{V}_0\right) = \pi_i(V_0, \partial V_0)$ for all i, $H_i\left(\widetilde{V}_0, \partial\widetilde{V}_0\right) = 0$ for $i \leq k-1$, and $H_k\left(\widetilde{V}_0, \partial\widetilde{V}_0\right) = \pi_k\left(\widetilde{V}_0, \partial\widetilde{V}_0\right)$. Furthermore, by Lemma 9, $H_k\left(\widetilde{V}_0, \partial\widetilde{V}_0\right)$ is finitely generated as a $\mathbb{Z}[\pi_1(V_0)]$–module.

We break the remainder of the proof into overlapping but distinct cases:

Case 1 $2 \leq k < \frac{n}{2}$

Choose a finite collection of disjoint embeddings $(D_j, \partial D_j) \hookrightarrow (V_0, \partial V_0)$ of k–cells representing a generating set for $\pi_k(V_0, \partial V_0)$ viewed as a $\mathbb{Z}[\pi_1 V_0]$–module. Let Q be a regular neighborhood of $\partial V_0 \cup (\bigcup D_j)$ in V_0. Notice that $\pi_1(\partial V_0) \to \pi_1(Q)$ and $\pi_1(Q) \to \pi_1(V_0)$ are both isomorphisms. (If $k > 2$ this is obvious. If $k = 2$ notice that each ∂D_j already contracts in ∂V_0 since V_0 is a generalized 1–neighborhood.) Thus, $\widetilde{Q} = p^{-1}(Q)$ is the universal cover of Q.

Let $U_0 = \overline{V_0 - Q}$. Since the D_j's have codimension greater than 2, then $\pi_1(U_0) \to \pi_1(V_0)$ is an isomorphism. It remains to show that U_0 is a k–neighborhood of infinity.

To see that $\pi_1(\partial U_0) \to \pi_1(U_0)$ is an isomorphism, recall from above that $\pi_1(Q) \xrightarrow{\cong} \pi_1(V_0)$. Then observe that the pair (V_0, Q) may be obtained from the pair $(U_0, \partial U_0)$ by attaching $(n-k)$–handles (the duals of the removed handles), which has no effect on fundamental groups.

To see that $\pi_i(U_0, \partial U_0) = 0$ for $i \leq k$, we will show that the corresponding $H_i\left(\widetilde{U}_0, \partial\widetilde{U}_0\right)$ are trivial. By excision, it suffices to show that $H_i\left(\widetilde{V}_0, \widetilde{Q}\right) = 0$ for $i \leq k$.

For $i < k$, the triviality of $H_i(\widetilde{V}_0, \widetilde{Q})$ can be deduced from the following portion of the long exact sequence for the triple $(\widetilde{V}_0, \widetilde{Q}, \partial\widetilde{V}_0)$:

$$\to H_i(\widetilde{V}_0, \partial\widetilde{V}_0) \to H_i(\widetilde{V}_0, \widetilde{Q}) \to H_{i-1}(\widetilde{Q}, \partial\widetilde{V}_0) \to$$

The first term is trivial because V_0 is a generalized $(k-1)$–neighborhood, and the last term is trivial (when $i-1 < k$) because $\widetilde{Q}$ is homotopy equivalent to a space obtained by attaching k–cells to $\partial\widetilde{V}_0$. Thus the middle term vanishes.

In dimension k, we use a portion of the same long exact sequence:

$$\to H_k(\widetilde{Q}, \partial\widetilde{V}_0) \to H_k(\widetilde{V}_0, \partial\widetilde{V}_0) \to H_k(\widetilde{V}_0, \widetilde{Q}) \to H_{k-1}(\widetilde{Q}, \partial\widetilde{V}_0) \to$$

The last term above is again trivial for the reason cited above. Furthermore, the map α is surjective by the construction of Q; hence β is trivial, so $H_k(\widetilde{V}_0, \widetilde{Q})$ vanishes.

Case 2 $2 < k \le n-3$

The strategy in this case is similar to the above except that when $k \ge \frac{n}{2}$ we cannot rely on general position to obtain embedded k–disks. Instead we will use the tools of handle theory.

By the inductive hypothesis and the fact that $\pi_k(V_0, \partial V_0)$ is finitely generated as a $\mathbb{Z}[\pi_1(V_0)]$–module, we may choose a generalized $(k-1)$–neighborhood $V_1 \subset V_0$ so that, for $R = V_0 - V_1$, the map $\pi_k(R, \partial V_0) \to \pi_k(V_0, \partial V_0)$ is surjective. Applying VanKampen's theorem to $V_0 = R \cup_{\partial V_1} V_1$ shows that $\pi_1(R) \to \pi_1(V_0)$ is an isomorphism, and it follows that $\pi_1(\partial V_0) \to \pi_1(R)$ is also an isomorphism. Hence $p^{-1}(R)$ is the universal cover $\widetilde{R}$ of R.

Claim $H_i(\widetilde{R}, \partial\widetilde{V}_0) = 0$ *for* $i \le k-2$

We deduce this claim from the long exact sequence of the triple $(\widetilde{V}_0, \widetilde{R}, \partial\widetilde{V}_0)$:

$$\to H_{i+1}(\widetilde{V}_0, \widetilde{R}) \to H_i(\widetilde{R}, \partial\widetilde{V}_0) \to H_i(\widetilde{V}_0, \partial\widetilde{V}_0) \to$$

The third term listed above is trivial for $i \le k-1$, therefore it suffices to show that the first term vanishes when $i \le k-2$. Let $\widehat{V}_1 = p^{-1}(V_1) \subset \widetilde{V}_0$. Since $\pi_1(V_1) \to \pi_1(V_0)$ needn't be an isomorphism, $\widehat{V}_1$ needn't be the universal cover of V_1. In fact, $\widehat{V}_1$ will be connected if and only if $\pi_1(V_1) \to \pi_1(V_0)$ is

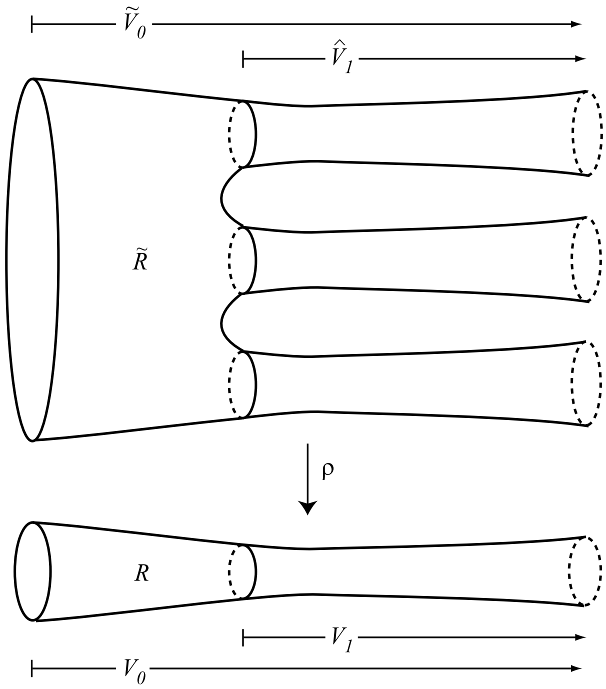

Figure 2

surjective. In general, $\widehat{V}_1$ has path components $\left\{\widehat{V}_1^\alpha\right\}_{\alpha\in A}$ (one for each element of $\mathrm{coker}(\pi_1(V_1) \to \pi_1(V_0))$) each of which is a covering space for V_1. See Figure 2. Moreover, $\partial\widehat{V}_1 = \rho^{-1}(\partial V_1) = \bigsqcup_{\alpha\in A}\partial\widehat{V}_1^\alpha$, and for each α we have $\pi_1(\partial\widehat{V}_1^\alpha) \xrightarrow{\cong} \pi_1(\widehat{V}_1^\alpha)$. Thus each $\left(\widehat{V}_1^\alpha, \partial\widehat{V}_1^\alpha\right)$ is a "covering pair" for $(V_1, \partial V_1)$. It follows that $\pi_i\left(\widehat{V}_1^\alpha, \partial\widehat{V}_1^\alpha\right)$ is trivial for all $i \le k-1$, so by the Hurewicz Theorem, $H_i\left(\widehat{V}_1^\alpha, \partial\widehat{V}_1^\alpha\right) = 0$ for all α and for all $i \le k-1$. Therefore $H_i\left(\widehat{V}_1, \partial\widehat{V}_1\right) = 0$ for $i \le k-1$, implying (via excision) that $H_{i+1}\left(\widetilde{V}_0, \widetilde{R}\right)$ vanishes for $i \le k-2$, thus completing the proof of the claim.

We now have a cobordism $(R, \partial V_0, \partial V_1)$ with $\pi_1(\partial V_0) \xrightarrow{\cong} \pi_1(R)$ and $H_i\left(\widetilde{R}, \partial\widetilde{V}_0\right) = 0$ for $i \le k-2$ (where $k-2 \le n-4$). By Chapter 6 of [22], there is a handle decomposition of R built upon ∂V_0 which contains no handles of index $\le k-2$ and so that the existing handles have been attached in order of increasing index.

This give rise to a cellular chain complex for the pair $\left(\tilde{R}, \partial\tilde{V}_0\right)$ of the form:

$$0 \longrightarrow \tilde{C}_n \xrightarrow{\partial_n} \tilde{C}_{n-1} \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_{k+1}} \tilde{C}_k \xrightarrow{\partial_k} \tilde{C}_{k-1} \longrightarrow 0$$

where each $\tilde{C}_i$ is a finitely generated free $\mathbb{Z}[\pi_1(R)] = \mathbb{Z}[\pi_1(V_0)]$–module with one generator for each i–handle of $(R, \partial V_0)$. For $[c] \in H_k\left(\tilde{R}, \partial\tilde{V}_0\right)$ write $c = \sum \alpha_i e_i$ where each $\alpha_i \in \mathbb{Z}[\pi_1(R)]$ and each e_i is a k–handle of R with a preferred base path. Let $R_{k-1} \subset R$ denote $S_i \cup (k-1)$–handles, where $S_0 \approx \partial V_0 \times [0,1]$ is a closed collar on ∂V_0 in V_0. We may represent $[c]$ with a single k–handle as follows: introduce a trivial cancelling $(k, k+1)$–handle pair $\left(h^k, h^{k+1}\right)$ to ∂R_{k-1}, then do a finite sequence of handle slides of h^k over the other k–handles until h^k is homologous to c. (Again see [22].) Now, since $\partial_k c = 0$, we may apply the Whitney Lemma in ∂R_{k-1} to move the attaching $(k-1)$–sphere of h^k off the belt spheres of all the $(k-1)$–handles.

Note In the case $k-1 = 2$, the belt spheres of the $(k-1)$–handles have codimension 2 in ∂R_{k-1}, so a special case of the Whitney Lemma (p. 72 of [22]) is needed. In particular we need to know that the belt spheres are π_1–negligible in ∂R_{k-1}, ie, that $\pi_1(\partial R_{k-1} - \{\text{belt spheres}\}) \xrightarrow{\cong} \pi_1(\partial R_{k-1})$. Since $\pi_1(\partial V_0) \xrightarrow{\cong} \pi_1(R_{k-1})$ (attaching the 2–handles does not kill any π_1), this condition is satisfied. See Lemma 16 for the dual version of this fact.

We may now assume that h^k was attached directly to S_0. By repeating this for each element of a finite generating set for $H_k\left(\tilde{V}_0, \partial\tilde{V}_0\right)$ we obtain a finite set $\left\{h_1^k, \cdots, h_t^k\right\}$ of k–handles attached to S_0, so that if $Q = S_0 \cup \left(\bigcup h_j^k\right)$, then $H_k\left(\tilde{Q}, \partial\tilde{V}_0\right) \to H_k\left(\tilde{V}_0, \partial\tilde{V}_0\right)$ is surjective. The same argument used in Case 1 will now show that $U_0 = V_0 - Q$ is a generalized k–neighborhood of infinity. $\square$

Combining Lemma 10 with the Generalized 1–Neighborhoods Theorem gives:

Theorem 5 (Generalized $(n-3)$–Neighborhoods Theorem) *Let M^n ($n \geq 5$) be a one ended, open n–manifold that is inward tame at infinity. Then*

(1) *M^n contains a neat sequence $\{U_i\}$ of generalized $(n-3)$–neighborhoods of infinity,*

(2) *if $\pi_1(\varepsilon(M^n))$ is stable, we may arrange that the U_i's are strong $(n-3)$–neighborhoods of infinity,*

(3) *if* $\pi_1(\varepsilon(M^n))$ *is semistable, we may arrange that each* $\pi_1(U_i) \leftarrow \pi_1(U_{i+1})$ *is surjective, and*

(4) *if* $\pi_1(\varepsilon(M^n))$ *is perfectly semistable, we may arrange that each* $\pi_1(U_i) \leftarrow \pi_1(U_{i+1})$ *is surjective and has perfect kernel.*

8 Obtaining generalized $(n-2)$–neighborhoods of infinity

Much like Siebenmann's original collaring theorem, the crucial step to obtaining a pseudo-collar neighborhood of infinity is in improving generalized $(n-3)$–neighborhoods of infinity to generalized $(n-2)$–neighborhoods of infinity. Lemma 12 shows that, for manifolds with semistable fundamental group systems at infinity, if we succeed our task is complete.

Lemma 11 *Suppose* M^n $(n \geq 5)$ *contains generalized* $(n-3)$*–neighborhoods of infinity* $U_1 \supseteq U_2$ *such that* $\pi_1(U_1) \leftarrow \pi_1(U_2)$ *is surjective, and let* $R = U_1 - U_2$*. Then* R *admits a handle decomposition on* ∂U_1 *containing handles only of index* $(n-3)$ *and* $(n-2)$*. Hence,* $(R, \partial U_1)$ *has the homotopy type of a relative CW pair* $(K, \partial U_1)$ *such that* $K - \partial U_1$ *contains only* $(n-3)$*– and* $(n-2)$*–cells.*

Proof Consider the cobordism $(R, \partial U_1, \partial U_2)$. Since $\pi_1(U_1) \twoheadleftarrow \pi_1(U_2)$ it is easy to check that $\pi_1(R) \twoheadleftarrow \pi_1(\partial U_2)$. Hence, $\pi_i(R, \partial U_2) = 0$ for $i = 0, 1$ so we may eliminate all 0– and 1–handles from a handle decomposition of R on ∂U_2. Then the dual handle decomposition of R on ∂U_1 has handles only of index $\leq n-2$ and, by arguing as in the Claim of Lemma 10, we see that $\pi_i(R, \partial U_1) = H_i\left(\tilde{R}, \partial \tilde{U}_1\right) = 0$ for $i \leq n-4$, so we may eliminate all handles of index $\leq n-4$ from this handle decomposition. (In the process we increase the numbers of $(n-3)$– and $(n-2)$–handles.) Collapsing the remaining handles to their cores gives us $(K, \partial U_1)$. □

Lemma 12 *Suppose* M^n $(n \geq 5)$ *contains a neat sequence* $\{U_i\}_{i=1}^{\infty}$ *of* $(n-3)$*–neighborhoods of infinity with the property that* $\pi_1(U_i) \leftarrow \pi_1(U_{i+1})$ *is surjective for all* i*. Then*

(1) *each pair* $(U_i, \partial U_i)$ *is homotopy equivalent to a (probably infinite) relative CW pair* $(K_i, \partial U_i)$ *such that* $K_i - \partial U_i$ *contains only* $(n-3)$*– and* $(n-2)$*–cells;*

(2) *if some U_k is an $(n-2)$–neighborhood of infinity, then $\partial U_k \hookrightarrow U_k$ is a homotopy equivalence, ie, U_k is a homotopy collar.*

Proof Roughly speaking, the first assertion is obtained by applying Lemma 11 to each $R_i = U_{i+1} - U_i$. Since this process is infinite, there are some technicalities to be dealt with. We refer the reader to [24] for details.

If U_k is a generalized $(n-2)$–neighborhood of infinity then we already know that $\pi_i(U_k, \partial U_k) = 0$ for $i \leq n-2$. Moreover, our first assertion guarantees that $H_i(\widetilde{U}_k, \partial\widetilde{U}_k)$ is trivial for $i > n-2$. Thus, $\pi_i(\widetilde{U}_k, \partial\widetilde{U}_k) = \pi_i(U_k, \partial U_k)$ is trivial for $i > n-2$, so by a theorem of Whitehead (see Section 7.6 of [25]) $\partial U_k \hookrightarrow U_k$ is a homotopy equivalence. □

With the end goal now clear, we begin the task of improving generalized $(n-3)$–neighborhoods of infinity to generalized $(n-2)$–neighborhoods. Even in the ideal situation where $\pi_1(\varepsilon)$ is stable and U is a strong $(n-3)$–neighborhood of infinity this may not be possible. Siebenmann recognized that, in this ideal situation, the problem was captured by the Wall finiteness obstruction of U. In our more general situation ($\pi_1(\varepsilon(M^n))$ semistable and U a generalized $(n-3)$–neighborhood of infinity) we will confront the same issue along with some new problems caused by the lack of π_1–stability.

Lemma 13 *Suppose M^n $(n \geq 5)$ contains a neat sequence $\{U_i\}_{i=1}^{\infty}$ of $(n-3)$–neighborhoods of infinity with the property that $\pi_1(U_i) \twoheadleftarrow \pi_1(U_{i+1})$ for all i. Then each $H_{n-2}(\widetilde{U}_i, \partial\widetilde{U}_i)$ is a finitely generated projective $\mathbb{Z}[\pi_1 U_i]$–module. Moreover, as elements of $\widetilde{K}_0(\mathbb{Z}[\pi_1 U_i])$, $[H_{n-2}(U_i, \partial U_i)] = (-1)^n \sigma(U_i)$ where $\sigma(U_i)$ is the Wall finiteness obstruction for U_i.*

Proof Finite generation of $H_{n-2}(\widetilde{U}_i, \partial\widetilde{U}_i)$ follows from Lemma 9. For projectivity, consider the cellular chain complex for the universal cover $\left(\widetilde{K}_i, \partial\widetilde{U}_i\right)$ of the CW pair $(K_i, \partial U_i)$ provided by assertion 1 of Lemma 12

$$0 \to \widetilde{C}_{n-2} \xrightarrow{\partial} \widetilde{C}_{n-3} \to 0.$$

Triviality of $H_{n-3}(\widetilde{U}_i, \partial\widetilde{U}_i)$ implies that ∂ is surjective, so we have a short exact sequence

$$0 \to \ker\partial \to \widetilde{C}_{n-2} \xrightarrow{\partial} \widetilde{C}_{n-3} \to 0$$

which splits since $\widetilde{C}_{n-3}$ is a free $\mathbb{Z}[\pi_1 U_i]$–module. Thus $\widetilde{C}_{n-2} = \ker\partial \oplus \widetilde{C}_{n-3}$, so $H_{n-2}(\widetilde{U}_i, \partial\widetilde{U}_i) = \ker\partial$ is a summand of a free module, and is therefore projective.

The identity $[H_{n-2}(\tilde{U}_i, \partial \tilde{U}_i)] = (-1)^n \sigma(U_i)$ now follows immediately from Theorem 8 of [27]. An alternative argument which relies only on [26] can be found in [23]. □

Remark 6 In the above proof it is essential that $\tilde{C}_{n-1}$ is trivial, hence, the assumption that $\pi_1(U_i) \twoheadleftarrow \pi_1(U_{i+1})$ for all i is crucial. By the Generalized $(n-3)$–Neighborhoods Theorem this may be arranged whenever $\pi_1(\varepsilon(M^n))$ is semistable (and M^n is inward tame at infinity).

Lemma 14 *Suppose M^n $(n \geq 5)$ is a one ended open n–manifold that is inward tame at infinity, and that $\pi_1(\varepsilon(M^n))$ is semistable. Then the following are equivalent:*

(1) *M^n contains arbitrarily small clean neighborhoods of infinity having finite homotopy types,*

(2) *$\sigma_\infty(M^n)$ is trivial,*

(3) *M^n contains a neat sequence $\{U_i\}_{i=0}^{\infty}$ of generalized $(n-3)$–neighborhoods such that $\pi_1(U_i) \twoheadleftarrow \pi_1(U_{i+1})$ for all i and each $H_{n-2}(\tilde{U}_i, \partial \tilde{U}_i)$ is a finitely generated stably free $\mathbb{Z}[\pi_1 U_i]$–module,*

(4) *M^n contains a neat sequence $\{V_i\}_{i=0}^{\infty}$ of generalized $(n-3)$–neighborhoods such that $\pi_1(V_i) \twoheadleftarrow \pi_1(V_{i+1})$ for all i and each $H_{n-2}(\tilde{V}_i, \partial \tilde{V}_i)$ is a finitely generated free $\mathbb{Z}[\pi_1 V_i]$–module.*

Proof The equivalence of (1)–(3) follows immediately from Lemma 13 and our earlier discussion of σ_∞. Since 4) $\Longrightarrow$ 3) is obvious, we need only show how to "improve" a given U_i with stably free $H_{n-2}(\tilde{U}_i, \partial \tilde{U}_i)$ to a generalized $(n-3)$–neighborhood V_i with free $H_{n-2}(\tilde{V}_i, \partial \tilde{V}_i)$. This is easily done by carving out finitely many trivial $(n-3)$–handles as described below.

Fix i, and let F_k be a free $\mathbb{Z}[\pi_1 U_i]$–module of rank k so that $H_{n-2}(\tilde{U}_i, \partial \tilde{U}_i) \oplus F_k$ is a finitely generated free $\mathbb{Z}[\pi_1 U_i]$–module. Let $S_i \subset U_i$ be a closed collar on ∂U_i and let $\{(h_1^{n-3}, h_1^{n-2}), (h_2^{n-3}, h_2^{n-2}), \cdots, (h_k^{n-3}, h_k^{n-2})\} \subset U_i - U_{i+1}$ be trivial $(n-3, n-2)$–handle pairs attached to S_i. Set $Q = S_i \cup \left(\bigcup_{j=1}^{k} h_j^{n-3}\right)$, and let $V_i = \overline{U_i - Q}$. VanKampen's Theorem and general position show that each of the inclusions: $\partial U_i \hookrightarrow Q$, $Q \hookrightarrow U_i$, $V_i \hookrightarrow U_i$ and $\partial V_i \hookrightarrow V_i$ induce π_1–isomorphisms. Thus, V_i is a generalized 1–neighborhood of infinity, moreover, we have a triple $\left(\widetilde{U}_i, \widetilde{Q}, \partial\widetilde{U}_i\right)$ of universal covers. Clearly

$$H_\ell(\widetilde{Q}, \partial\widetilde{U}_i) = \begin{cases} 0 & \text{if } \ell \neq n-3 \\ F_k & \text{if } \ell = n-3 \end{cases},$$

so for $j \leq n-3$ the long exact sequence for triples yields:

$$\begin{array}{ccccc} H_j\left(\widetilde{U}_i, \partial\widetilde{U}_i\right) & \to & H_j\left(\widetilde{U}_i, \widetilde{Q}\right) & \to & H_{j-1}\left(\widetilde{Q}, \partial\widetilde{U}_i\right) \\ \| & & & & \| \\ 0 & & & & 0 \end{array}$$

Hence, $H_j\left(\widetilde{U}_i, \widetilde{Q}\right) = 0$ for $i \leq n-3$, and by excision, V_i is a generalized $(n-3)$–neighborhood of infinity.

In dimension $n-2$ we have:

$$\begin{array}{ccccccc} 0 & & & & & & \\ \| & & & & & & \\ H_{n-2}\left(\widetilde{Q}, \partial\widetilde{U}_i\right) & \to & H_{n-2}\left(\widetilde{U}_i, \partial\widetilde{U}_i\right) & \to & H_{n-2}\left(\widetilde{U}_i, \widetilde{Q}\right) & \to & \end{array}$$

$$\begin{array}{ccc} H_{n-3}\left(\widetilde{Q}, \partial\widetilde{U}_i\right) & \to & H_{n-3}\left(\widetilde{U}_i, \partial\widetilde{U}_i\right) \\ \| & & \| \\ F_k & & 0 \end{array}$$

Since F_k is free this sequence splits, so

$$H_{n-2}\left(\widetilde{V}_i, \partial\widetilde{V}_i\right) \cong H_{n-2}\left(\widetilde{U}_i, \widetilde{Q}\right) \cong H_{n-2}\left(\widetilde{U}_i, \partial\widetilde{U}_i\right) \oplus F_k$$

as desired. □

We now begin working towards a proof of our main theorem. In order to make the role of each hypothesis clear (and to provide additional partial results), we begin with a minimal hypothesis and add to it only when necessary.

Initial hypothesis *M^n $(n \geq 5)$ is one ended, open, and inward tame at infinity and $\pi_1(\varepsilon(M^n))$ is semistable.*

Then by the Generalized $(n-3)$–Neighborhoods Theorem we may begin with a neat sequence $\{U_i\}_{i=0}^{\infty}$ of generalized $(n-3)$–neighborhoods of infinity such that $\pi_1(U_i) \twoheadleftarrow \pi_1(U_{i+1})$ for all i. For each i, let $R_i = U_i - U_{i+1}$, $p_i\colon \widetilde{U}_i \to U_i$ be the universal covering projection, and $\widehat{U}_{i+1} = p_i^{-1}(U_{i+1}) \subseteq \widetilde{U}_i$.

Since each $H_{n-2}\left(\widetilde{\Theta}_i, \partial\widetilde{\Theta}_i\right)$ is finitely generated as a $\mathbb{Z}[\pi_1 U_i]$–module, we may (by passing to a subsequence and relabelling) assume that $H_{n-2}\left(\widetilde{R}_i, \partial\widetilde{\Theta}_i\right) \to H_{n-2}\left(\widetilde{\Theta}_i, \partial\widetilde{\Theta}_i\right)$ is surjective for all i. Consider the following portion of the long exact sequence for the triple $\left(\widetilde{\Theta}_i, \widetilde{R}_i, \partial\widetilde{\Theta}_i\right)$.

$$\begin{array}{l}
\overset{\displaystyle 0}{\overset{\|}{H_{n-1}\left(\widetilde{\Theta}_i, \widetilde{R}_i\right)}} \to H_{n-2}\left(\widetilde{R}_i, \partial\widetilde{\Theta}_i\right) \to H_{n-2}\left(\widetilde{\Theta}_i, \partial\widetilde{\Theta}_i\right) \xrightarrow{0} \\
\qquad\qquad\qquad H_{n-2}\left(\widetilde{\Theta}_i, \widetilde{R}_i\right) \xrightarrow{y} H_{n-3}\left(\widetilde{R}_i, \partial\widetilde{\Theta}_i\right) \to \underset{\displaystyle 0}{\underset{\|}{H_{n-3}\left(\widetilde{\Theta}_i, \partial\widetilde{\Theta}_i\right)}}.
\end{array}$$

Triviality of the middle homomorphism follows from surjectivity of . The first term in the sequence vanishes because it is isomorphic (by excision) to $H_{n-1}\left(\widehat{\Theta}_{i+1}, \partial\widehat{\Theta}_{i+1}\right)$ which is 0 by an application of Lemma 12. The last term is trivial since U_i is a generalized $(n-3)$–neighborhood. After another application of excision we obtain the following $\mathbb{Z}[\pi_1 U_i]$–module isomorphisms:

$$H_{n-2}\left(\widetilde{R}_i, \partial\widetilde{\Theta}_i\right) \cong H_{n-2}\left(\widetilde{\Theta}_i, \partial\widetilde{\Theta}_i\right) \tag{1}$$

$$H_{n-3}\left(\widetilde{R}_i, \partial\widetilde{\Theta}_i\right) \cong H_{n-2}\left(\widehat{\Theta}_{i+1}, \partial\widehat{\Theta}_{i+1}\right) \tag{2}$$

By Lemma 11, we may choose a handle decomposition of R_i containing only $(n-3)$– and $(n-2)$–handles. Furthermore, we assume that all $(n-3)$–handles are attached before any of the $(n-2)$–handles. Thus the homology of $\left(\widetilde{R}_i, \partial\widetilde{\Theta}_i\right)$ is given by a chain complex of the form:

$$0 \to \widetilde{C}_{n-2} \xrightarrow{\partial} \widetilde{C}_{n-3} \to 0 \tag{3}$$

where $\widetilde{C}_{n-2}$ is a free $\mathbb{Z}[\pi_1 U_i]$–module with one generator for each $(n-2)$–handle of R_i and $\widetilde{C}_{n-3}$ is a free $\mathbb{Z}[\pi_1 U_i]$–module with one generator for each $(n-3)$–handle of R_i. From this sequence we may extract the following short exact sequences.

$$0 \to im(\partial) \to \widetilde{C}_{n-3} \to H_{n-3}\left(\widetilde{R}_i, \partial\widetilde{\Theta}_i\right) \to 0 \tag{4}$$

$$0 \to H_{n-2}\left(\widetilde{R}_i, \partial\widetilde{\Theta}_i\right) \to \widetilde{C}_{n-2} \to im(\partial) \to 0 \tag{5}$$

Lemma 9 (slightly modified to apply to the pair (Θ_i, R_i)) and an argument like that used in proving Lemma 13 show that $H_{n-2}\left(\widehat{\Theta}_{i+1}, \partial\widehat{\Theta}_{i+1}\right)$ is a finitely generated projective $\mathbb{Z}[\pi_1 U_i]$–module. Hence, by identity (2), the first of these sequences splits. We abuse notation slightly and write

$$C_{n-3} = \operatorname{im}(\partial) \oplus H_{n-3}\left(R_i, \partial\Theta_i\right). \tag{6}$$

This implies that $\operatorname{im}(\partial)$ is also finitely generated projective, and

$$[\operatorname{im}(\partial)] = -\left[H_{n-3}\left(R_i, \partial\Theta_i\right)\right] \in \widetilde{K}_0[\mathbb{Z}\pi_1 U_i] \tag{7}$$

Then the second short exact sequence also splits, so we may write

$$C_{n-2} = H_{n-2}\left(R_i, \partial\Theta_i\right) \oplus \operatorname{im}(\partial)' \tag{8}$$

where $\operatorname{im}(\partial)'$ denotes a copy of $\operatorname{im}(\partial)$ lying in C_{n-2} (whereas $\operatorname{im}(\partial)$ itself lies in C_{n-3}). This shows that

$$[\operatorname{im}(\partial)] = -\left[H_{n-2}\left(R_i, \partial\Theta_i\right)\right] \in \widetilde{K}_0(\mathbb{Z}[\pi_1 U_i]). \tag{9}$$

Remark 7 Combining (1), (2), (7) and (9) shows that

$$\left[H_{n-2}\left(\Theta_i, \partial\Theta_i\right)\right] = \left[H_{n-2}\left(\widehat{\Theta}_{i+1}, \partial\widehat{\Theta}_{i+1}\right)\right] \in \widetilde{K}_0(\mathbb{Z}[\pi_1 U_i]).$$

In the special case that $\pi_1(\varepsilon(M^n))$ is stable and U_i and U_{i+1} are strong $(n-2)$–neighborhoods of infinity, this shows that

$$\left[H_{n-2}\left(\Theta_i, \partial\Theta_i\right)\right] = \left[H_{n-2}\left(\Theta_{i+1}, \partial\Theta_{i+1}\right)\right].$$

This was one of the arguments used by Siebenmann in [23] to show that his end obstruction is well-defined. One can also obtain this result by using the Sum Theorem for Wall's finiteness obstruction (see Ch. VI of [23] or [13]).

Identities (6) and (8) allow us to rewrite (3) as

$$0 \to H_{n-2}\left(R_i, \partial\Theta_i\right) \oplus \operatorname{im}(\partial)' \xrightarrow{\partial} \operatorname{im}(\partial) \oplus H_{n-3}\left(R_i, \partial\Theta_i\right) \to 0 \tag{3$'$}$$

where $\ker\partial = H_{n-2}\left(R_i, \partial\Theta_i\right)$ and $\partial|_{\operatorname{im}(\partial)'} : \operatorname{im}(\partial)' \xrightarrow{\cong} \operatorname{im}(\partial)$.

We are now ready to add to our Initial Hypothesis.

Additional Hypothesis I *From now on we assume that* $\sigma_\infty(M^n) = 0$.

Then by Lemma 14 we may assume that $H_{n-2}\left(\widetilde{C}_i, \partial\widetilde{C}_i\right) = H_{n-2}\left(\widetilde{R}_i, \partial\widetilde{C}_i\right)$ are finitely generated free $\mathbb{Z}[\pi_1 U_i]$–modules, and by Identity (8), that $im(\partial)' = im(\partial)$ are stably free. We may easily "improve" $im(\partial)'$ and $im(\partial)$ to free $\mathbb{Z}([\pi_1 U_i])$–modules by introducing trivial $(n-3, n-2)$–handle pairs to our handle decomposition of R_i. Indeed, if we introduce a trivial handle pair $\left(h^{n-3}, h^{n-2}\right)$, then $im(\partial)'$ is increased to $im(\partial)' \oplus \mathbb{Z}[\pi_1 U_i]$ and $im(\partial)'$ is increased to $im(\partial)' \oplus \mathbb{Z}[\pi_1 U_i]$ with the new factors being generated by h^{h-2} and h^{n-3}, respectively. Moreover, the new boundary map (properly restricted) is $\partial \oplus id_{\mathbb{Z}[\pi_1 U_i]}$. By doing this finitely many times we may arrange that $im(\partial)' = im(\partial)$ are free.

At this point we have a free $\mathbb{Z}[\pi_1 U_i]$–module $\widetilde{C}_{n-2}$ with a natural (geometric) basis $\left\{h_1^{n-2}, h_2^{n-2}, \cdots, h_r^{n-2}\right\}$ consisting of the $(n-2)$–handles of R_i. We also have a direct sum decomposition of $\widetilde{C}_{n-2}$ into free submodules $\widetilde{C}_{n-2} = H_{n-2}\left(\widetilde{R}_i, \partial\widetilde{C}_i\right) \oplus im(\partial)'$; hence there exists another basis $\{a_1, \cdots, a_s, b_1, \cdots, b_{r-s}\}$ for $\widetilde{C}_{n-2}$ such that $\{a_1, \cdots, a_s\}$ generates $H_{n-2}\left(\widetilde{R}_i, \partial\widetilde{C}_i\right)$ and $\{b_1, \cdots, b_{r-s}\}$ generates $im(\partial)'$. We would like the geometry to match the algebra — in particular we would like one subset of handles, say $\{h_1^{n-2}, h_2^{n-2}, \cdots, h_s^{n-2}\}$, to generate $H_{n-2}\left(\widetilde{R}_i, \partial\widetilde{C}_i\right)$ with the remaining handles $\{h_{s+1}^{n-2}, h_{s+2}^{n-2}, \cdots, h_r^{n-2}\}$ generating $im(\partial)'$. This may not be possible at first, but by introducing even more trivial $(n-3, n-2)$–handle pairs and then performing handle slides, it may be accomplished. Key to the proof is the following algebraic lemma.

Lemma 15 (See Lemma 5.4 of [23]) *Let F be a finitely generated free Λ–module with bases $\{x_1, \cdots, x_r\}$ and $\{y_1, \cdots, y_r\}$ and F' be another free module of rank r with basis $\{z_1, \cdots, z_r\}$. Then the basis $\{x_1, \cdots, x_r, z_1, \cdots, z_r\}$ of $F \oplus F'$ may be changed to a basis of the form $\{y_1, \cdots, y_r, z_1', \cdots, z_r'\}$ by a finite sequence of elementary operations of the form $x \mapsto x + \lambda y$.*

Proof If A is the matrix of the basis $\{y_1, \cdots, y_r\}$ in terms of $\{x_1, \cdots, x_r\}$, then the matrix of $\{y_1, \cdots, y_r, z_1, \cdots, z_r\}$ in terms of $\{x_1, \cdots, x_r, z_1, \cdots, z_r\}$ is $\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}$. Now

$$\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} A^{-1} & 0 \\ 0 & A \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & A \end{pmatrix},$$

where

$$\begin{pmatrix} A^{-1} & 0 \\ 0 & A \end{pmatrix} = \begin{pmatrix} I & A^{-1} \\ 0 & I \end{pmatrix}\begin{pmatrix} I & 0 \\ I-A & I \end{pmatrix}\begin{pmatrix} I & -I \\ 0 & I \end{pmatrix}\begin{pmatrix} I & 0 \\ I-A^{-1} & I \end{pmatrix}$$

is a product of matrices obtained by elementary moves. □

To apply this lemma, we introduce r trivial $(n-3, n-2)$–handle pairs

$$\left\{(k_i^{n-3}, k_i^{n-2})\right\}_{i=1}^{r}$$

into R_i thus giving us a geometric basis

$$B_1 = \left\{h_1^{n-2}, \cdots, h_r^{n-2}, k_1^{n-2}, \cdots, k_r^{n-2}\right\}$$

for $\widetilde{C}_{n-2}$. (In the process $\widetilde{C}_{n-2}$, $im(\partial)'$ and $im(\partial)$ are expanded, but $H_{n-2}\left(\widetilde{R}_i, \partial\widetilde{U}_i\right)$ remains the same.) According to Lemma 15 we can change B_1 to a basis of the form

$$B_2 = \left\{a_1, \cdots, a_s, b_1, \cdots, b_{r-s}, k_1', \cdots, k_r'\right\}$$

using only elementary operations which may be imitated geometrically with handle slides. Hence we arrive at a handle decomposition of R_i for which each element of B_2 corresponds to a single $(n-2)$–handle, and a subset of these handles generates the submodule $H_{n-2}\left(\widetilde{R}_i, \partial\widetilde{U}_i\right)$.

Given the above, we revert to our original notation in which $\widetilde{C}_{n-2}$ is generated by $\left\{h_1^{n-2}, \cdots, h_r^{n-2}\right\}$; assuming in addition that the subset $\left\{h_1^{n-2}, \cdots, h_s^{n-2}\right\}$ generates $H_{n-2}\left(\widetilde{R}_i, \partial\widetilde{U}_i\right)$.

Most of the work that remains to be done involves handle theory in the cobordism $(R_i, \partial U_i, \partial U_{i+1})$. For convenience, we label certain subsets of R_i. Let $S_i \subset R_i$ be a closed collar on ∂U_i, $T_i = S_i \cup ((n-3)$–handles), and $\partial_+ T_i = \partial T_i - \partial U_i$. Then $R_i = T_i \cup \left(h_1^{n-2} \cup \cdots \cup h_r^{n-2}\right)$. See Figure 3. For each h_j^{n-2} let $\alpha_j \subset \partial_+ T_i$ be its attaching $(n-3)$–sphere, and for each $(n-3)$–handle h_k^{n-3} let β_k be its belt 2–sphere.

To complete the proof, we would like to proceed as follows:

- slide the handles $h_1^{n-2}, \cdots, h_s^{n-2}$ (the ones which generate $H_{n-2}\left(\widetilde{R}_i, \partial\widetilde{U}_i\right)$) off the $(n-3)$–handles of R_i so that they are attached directly to S_i,
- then carve out the interiors of $h_1^{n-2}, \cdots, h_s^{n-2}$ to obtain the desired $(n-2)$–neighborhood.

Unfortunately, each of these steps faces a significant difficulty. For the first step, we would like to employ the Whitney Lemma to remove $\bigcup_{j=1}^{s} \alpha_j$ from $\bigcup \beta_k$. Since $\partial h_j^{n-2} = 0$ for $i = 1, \cdots, s$ the relevant $\mathbb{Z}[\pi_1 U_i]$–intersection numbers $\varepsilon(\beta_k, \alpha_j)$ are trivial as desired; however, since the α_j's are codimension 2 in

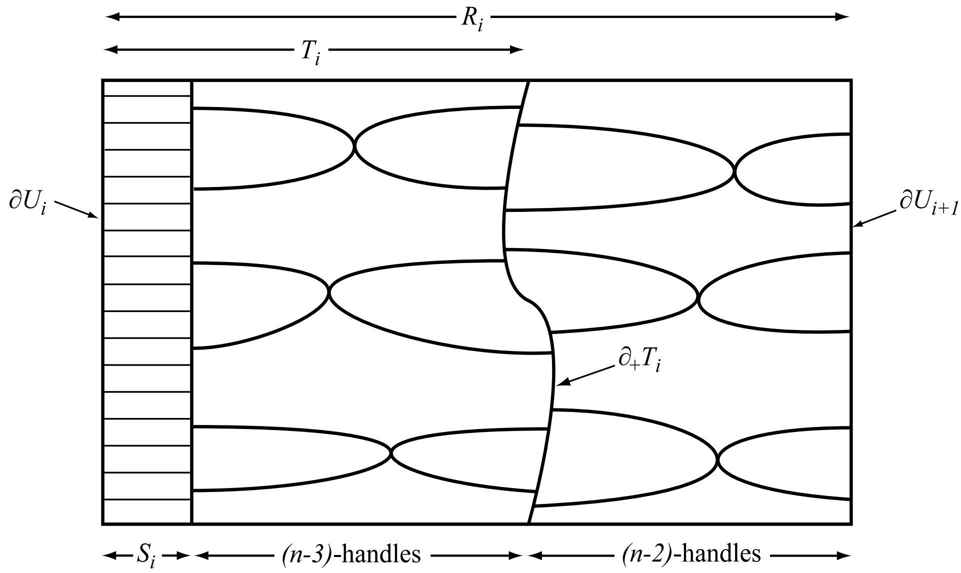

Figure 3

$\partial_+ T_i$ we also need π_1–negligibility for the α_j's. As we will soon see, this is very unlikely.

The difficulty at the second step is similar. Assume for the moment that we succeeded at step 1 — so each h_j^{n-2} ($j = 1, \cdots, s$) is attached directly to S_i. Since the cores of the h_j^{n-2}'s have codimension 2 in U_i, and the α^{n-3}'s have codimension ≤ 2 in ∂S_i, the removal of interiors of the h_j^{n-2}'s is likely to change the fundamental groups of these spaces — a situation we cannot tolerate at this point in the proof.

Both of the above problems can be understood through the following easy lemma, whose proof is left to the reader. In it, the term "π_1–negligible" is used as follows: a subset A of a space X is π_1–*negligible* provided $\pi_1(X - A) \to \pi_1(X)$ is an isomorphism.

Lemma 16 *Suppose $(W^n, \partial_- W, \partial_+ W)$ is a compact cobordism ($n \geq 5$) obtained by attaching $(n-2)$–handles $h_1, \cdots, h_q$ to a collar $C = \partial_- W \times [0,1]$. Let $\partial_+ C$ denote $\partial_- W \times \{1\}$, and let $\alpha_1, \cdots, \alpha_q \subset \partial_+ C$ be the attaching $(n-3)$–spheres and $N(\alpha_1), \cdots, N(\alpha_q) \subset \partial_+ C$ the attaching tubes for the* *handles. Then we have the following commutative diagram.*

$$\begin{array}{ccc} \pi_1(W^n) & \twoheadleftarrow & \pi_1(\partial_+ W) \\ \cong & & \cong \\ \pi_1(\partial_+ C) & \twoheadleftarrow & \pi_1(\partial_+ C - \bigcup_{i=1}^{q} N(\alpha_i)) \end{array}$$

Hence, the collection $\alpha_1, \cdots, \alpha_q$ *of attaching* $(n-3)$*–spheres is* π_1*–negligible in* $\partial_+ C$ *if and only if* $\pi_1(W^n) \leftarrow \pi_1(\partial_+ W)$ *is an isomorphism.*

Remark 8 Applying this lemma to the project at hand shows that in the special case that $\pi_1(\partial R_i) \leftarrow \pi_1(\partial U_{i+1})$ is an isomorphism for each i, the program outlined above may be carried out when $n \geq 6$. Hence, when $\pi_1(\varepsilon(M^n))$ is stable, we have a proof of Theorem 1.

Lemma 16 shows that difficulties with fundamental groups are unavoidable when $\pi_1(\varepsilon(M^n))$ is not stable — specifically, when $\pi_1(U_i) \leftarrow \pi_1(U_{i+1})$ has non-trivial kernel, the corresponding attaching $(n-3)$–spheres will not be π_1–negligible. Thus we need a new strategy for improving the U_i's to generalized $(n-2)$–neighborhoods. Instead of "carving out" the unwanted $(n-2)$–handles in U_i, we will "steal" duals for these handles from below. Our strategy is partially based on Quillen's "plus construction" (see [21] or Section 11.1 or [15]). We will require some additional hypotheses.

Additional Hypothesis II $\pi_1(\varepsilon(M^n))$ *is perfectly semistable and* $n \geq 6$.

Then by The Generalized $(n-3)$–Neighborhoods Theorem we could have chosen our original sequence $\{U_i\}_{i=0}^{\infty}$ of generalized $(n-3)$–neighborhoods of infinity so that

$$\ker(\pi_1(U_i) \twoheadleftarrow \pi_1(U_{i+1})) \text{ is perfect for all } i. \tag{10}$$

With the exception of passing to subsequences (which is permitted by Corollary 1), fundamental groups have not been changed during the current stage of the proof, hence we may simply add Property (10) to the conditions already achieved.

Fix an $i > 0$ and return to the cobordism $(R_i, \partial U_i, \partial U_{i+1})$ under discussion.

Claim 1 *There exists a pairwise disjoint collection* $\{y_j\}_{j=1}^{r}$ *of embedded* 2*–spheres in* $\partial_+ T_i$ *which are algebraic duals for the collection* $\{\alpha_j\}_{j=1}^{r}$ *of attaching* $(n-3)$*–spheres of the* $(n-2)$*–handles of* R_i*. This means that for each* $0 \leq j, k \leq r$,

$$\varepsilon(\alpha_j, y_k) = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k. \end{cases}$$

Note Technically $\lambda(\alpha_j, \gamma_k)$ denotes $\mathbb{Z}[\pi_1(\partial_+ T_i)]$–intersection number. Since $\pi_1(U_i) \cong \pi_1(\partial_+ T_i)$ we think of it as a $\mathbb{Z}[\pi_1(U_i)]$–intersection number.

Proof Let $p: \widetilde{U}_i \to U_i$ be the universal covering projection. Then $p^{-1}(\partial_+ T_i) = \partial_+ \widetilde{T}_i$ is the universal cover of $\partial_+ T_i$. Also, $\partial_+ \widetilde{T}_i - p^{-1}(\bigcup_{j=1}^{r} \alpha_j)$ covers $\partial_+ T_i - \bigcup_{j=1}^{r} \alpha_j$ and $\pi_1\left(\partial_+ \widetilde{T}_i - p^{-1}(\bigcup_{j=1}^{r} \alpha_j)\right) \cong \ker\left(\pi_1(\partial_+ T_i) \leftarrow \pi_1(\partial_+ T_i - \bigcup_{j=1}^{r} \alpha_j)\right)$ which is perfect by an application of Lemma 16.

For a fixed $1 \le j_0 \le r$, we will show how to construct γ_{j_0}. Let $\widetilde{\alpha}_{j_0}$ be a component of $p^{-1}(\alpha_{j_0})$, and let $\widetilde{D}_{j_0}$ be a small 2–disk in $\partial_+ \widetilde{T}_i$ intersecting $\widetilde{\alpha}_{j_0}$ transversely in a single point. Since $\pi_1\left(\partial_+ \widetilde{T}_i - p^{-1}(\bigcup_{j=1}^{r} \alpha_j)\right)$ is perfect, then $H_1\left(\partial_+ \widetilde{T}_i - p^{-1}(\bigcup_{j=1}^{r} \alpha_j)\right)$ is trivial; so $\partial \widetilde{D}_{j_0}$ bounds a surface $\widetilde{E}_{j_0}$ in $\partial_+ \widetilde{T}_i - p^{-1}(\bigcup_{j=1}^{r} \alpha_j)$. Let $\widetilde{D}_{j_0} \cup \widetilde{E}_{j_0}$ represent an element of $H_2\left(\partial_+ \widetilde{T}_i\right)$ and apply the Hurewicz isomorphism to find a 2–sphere $\widetilde{\gamma}_{j_0}$ in $\partial_+ \widetilde{T}_i$ representing the same element. Since they are invariants of homology class, the $\mathbb{Z}$–intersection number of $\widetilde{\gamma}_{j_0}$ with $\widetilde{\alpha}_{j_0}$ is 1; while the $\mathbb{Z}$–intersection number of $\widetilde{\gamma}_{j_0}$ with any other component of $p^{-1}(\bigcup_{j=1}^{r} \alpha_j)$ is 0. Thus, with an appropriately chosen arc to the basepoint, the $\mathbb{Z}[\pi_1(\partial_+ T_i)]$–intersection numbers of $\gamma_{j_0} = p(\widetilde{\gamma}_{j_0})$ with the α_j's are as desired. If necessary, use general position to ensure that γ_{j_0} is embedded. □

By general position, we may assume that the γ_j's miss the belt 2–spheres of each of the $(n-3)$–handles of R_i — and hence, that they miss the $(n-3)$–handles altogether. Thus, the γ_j's lie in the upper boundary component $\partial_+ S_i$ of the collar S_i. The collar structure gives us a parallel copy $\gamma_j' \subset \partial U_i$ of each γ_j. We would like to arrange for each of these γ_j''s to bound a 3–disk in R_{i-1}. To make sure this is possible, we add our last additional hypotheses.

Additional Hypotheses III *$\pi_2(\varepsilon(M^n))$ is semistable and $n \ge 7$.*

Then, in addition to all of the above, we may assume the existence of a diagram of the form:

$$\begin{tikzcd}
\pi_2(U_0) & & \pi_2(U_1) \arrow[ll, "\lambda_1"'] \arrow[dl, two heads] & & \pi_2(U_2) \arrow[ll, "\lambda_2"'] \arrow[dl, two heads] & & \pi_2(U_3) \arrow[ll, "\lambda_3"'] \arrow[dl, two heads] & & \cdots \arrow[ll, "\lambda_4"'] \\
 & \operatorname{im}\lambda_1 \arrow[ul, hook] & & \operatorname{im}\lambda_2 \arrow[ul, hook] \arrow[ll, two heads] & & \operatorname{im}\lambda_3 \arrow[ul, hook] \arrow[ll, two heads] & & \cdots \arrow[ll, two heads] &
\end{tikzcd}$$

Since each U_i is a generalized $(n-3)$–neighborhood, we have isomorphisms $\pi_2(\partial U_i) \xrightarrow{\cong} \pi_2(U_i)$, $\pi_2(R_i) \xrightarrow{\cong} \pi_2(U_i)$ and $\pi_2(\partial U_i) \xrightarrow{\cong} \pi_2(R_i)$ for all $i \ge 0$.

Assume again that i has been fixed. By including the arcs to a common basepoint (and abusing notation slightly) we view each γ_j' as representing $\left[\gamma_j'\right] \in \pi_2(U_i)$. Then, for $1 \leq j \leq r$, the above diagram guarantees the existence of a 2–sphere $\Omega_j \subset \partial U_{i+1}$ so that $\lambda_i \mu_{i+1}([\Omega_j]) = \lambda_i \left[\gamma_j'\right]^{-1}$ in $\pi_2(U_{i-1})$. By general position (as before) we may assume that Ω_j misses the $(n-3)$– and $(n-2)$–handles of R_i, and thus lies in $(\partial_+ T_i) \cap (\partial_+ S_i)$ where it does not intersect any of the α_j's. By forming the connected sum $\gamma_j \# \Omega_j$ (along an appropriate arc in $\partial_+ S_i$) we obtain a new 2–sphere in $\partial_+ T_i$ with $\varepsilon(\alpha_k, \gamma_j \# \Omega_j) = \varepsilon(\alpha_k, \gamma_j)$ for all k, and the additional property that its parallel copy $(\gamma_j \# \Omega_j)'$ in ∂U_i contracts in U_{i-1}. Note then that $(\gamma_j \# \Omega_j)'$ may be contracted in R_{i-1}.

In order to simplify notation, we replace each γ_j with $\gamma_j \# \Omega_j$ and assume that, in addition to the properties of Claim 1, we have chosen the γ_j's to satisfy:

Each γ_j is contained in $\partial_+ S_i$ and its parallel copy $\gamma_j' \subset \partial U_i$ contracts in R_{i-1}.

By general position (here we use $n \geq 7$), we may select a pairwise disjoint collection $\{D_j\}_{j=1}^{s}$ of properly embedded 3–disks in R_{i-1} with $\partial D_j = \gamma_j'$ for each $1 \leq j \leq s$.

Note We have selected bounding disks only for the duals to the attaching spheres $\alpha_1, \dots, \alpha_s$ of the handles $h_1^{n-2}, \dots, h_s^{n-2}$ which generate $H_{n-2}(U_i, \partial U_i)$.

Now let Q_i be a regular neighborhood in R_{i-1} of $\partial U_i \cup \bigcup_{j=1}^{s} D_j$ and let $V_i = Q_i \cup U_i$. See Figure 4. Our proof of the Main Existence Theorem will be complete when we prove the following.

Claim 2 *V_i is a homotopy collar.*

Notice that $\pi_1(U_i) \to \pi_1(V_i)$ is an isomorphism and $U_{i-1} \supset V_i \supset U_{i+1}$; so V_i may be substituted for U_i as part of a π_1–surjective system of neighborhoods of infinity. Hence, by Lemma 12, it suffices to show that V_i is a generalized $(n-2)$–neighborhood of infinity.

Consider the cobordism $(Q_i, \partial V_i, \partial U_i)$. Since Q_i may be obtained by attaching 3–handles (one for each D_j) to a collar on ∂U_i, it may also be constructed by attaching $(n-3)$–handles to a collar on ∂V_i. In this case the 2–spheres $\gamma_1, \dots, \gamma_s$ become the belt spheres of the $(n-3)$–handles, which we label as $k_1^{n-3}, \dots, k_s^{n-3}$, respectively. Since we already know that U_i admits an infinite

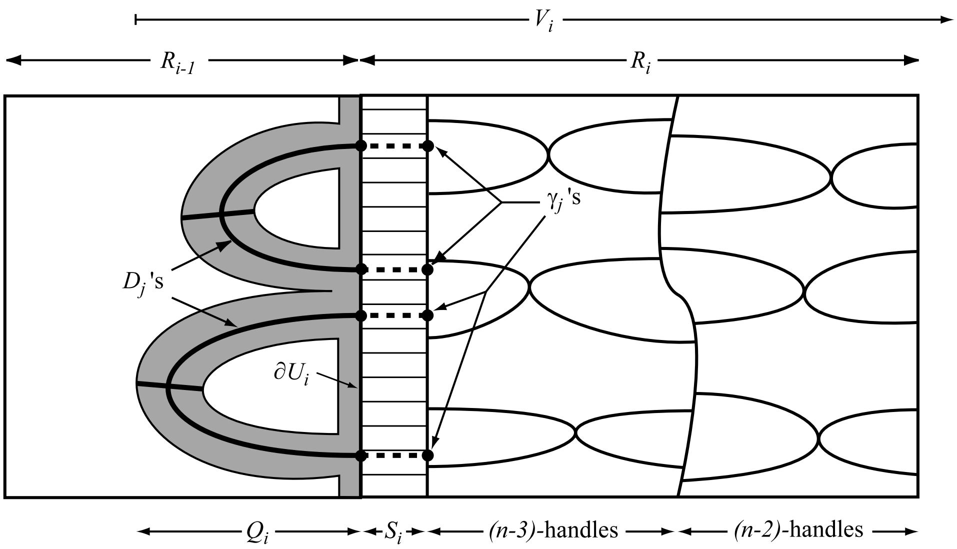

Figure 4

handle decomposition containing only $(n-3)$– and $(n-2)$–handles, this shows that V_i also admits a handle decomposition with handles only of these indices. It follows from general position that V_i is a generalized 1–neighborhood and from the usual argument that $\pi_k(V_i, \partial V_i) = \pi_k(\widetilde{V}_i, \partial \widetilde{V}_i) = H_k(\widetilde{V}_i, \partial \widetilde{V}_i) = 0$ for $k \leq n-4$. Hence, it remains only to show that $H_{n-3}(\widetilde{V}_i, \partial \widetilde{V}_i) = 0 = H_{n-2}(\widetilde{V}_i, \partial \widetilde{V}_i)$. To do this, begin with an infinite handle decomposition of $(U_i, \partial U_i)$ which has only $(n-3)$– and $(n-2)$–handles, and which contains the handle decomposition of R_i used above. Let

$$0 \to \widetilde{D}_{n-2} \xrightarrow{\partial} \widetilde{D}_{n-3} \to 0 \tag{11}$$

be the associated chain complex for the $\mathbb{Z}[\pi_1(U_i)]$–homology of $(U_i, \partial U_i)$. Then ∂ is surjective and $\widetilde{D}_{n-2} = \ker \partial \oplus \widetilde{D}'_{n-3}$, where $\partial|_{\widetilde{D}'_{n-3}}$ is an isomorphism, and $\ker \partial = H_{n-2}(\widetilde{U}_i, \partial \widetilde{U}_i) = \left\langle h_1^{n-2}, \cdots, h_s^{n-2} \right\rangle$. Hence, (11) may be rewritten as:

$$0 \to \left\langle h_1^{n-2}, \cdots, h_s^{n-2} \right\rangle \oplus \widetilde{D}'_{n-3} \xrightarrow{0 \oplus \partial|_{\widetilde{D}'_{n-3}}} \widetilde{D}_{n-3} \to 0 \tag{11'}$$

Our preferred handle decomposition of $(V_i, \partial V_i)$ is obtained by inserting the $(n-3)$–handles $k_1^{n-3}, \cdots, k_s^{n-3}$ beneath our handle decomposition of $(U_i, \partial U_i)$. Hence, the corresponding chain complex for $(V_i, \partial V_i)$ has the form

$$0 \to \left\langle h_1^{n-2}, \cdots, h_s^{n-2} \right\rangle \oplus \widetilde{D}'_{n-3} \xrightarrow{\partial_1 \oplus \partial_2} \left\langle k_1^{n-3}, \cdots, k_s^{n-3} \right\rangle \oplus \widetilde{D}_{n-3} \to 0.$$

In the usual way, the image of an $(n-2)$–handle under the boundary map is determined by the $\mathbb{Z}\pi_1$–intersection numbers of its attaching $(n-3)$–sphere with the belt 2–spheres of the various $(n-3)$–handles. Thus it is easy to see that $\partial h_j^{n-2} = \langle k_j^{n-3}, 0\rangle \in \langle k_1^{n-3}, \ldots, k_s^{n-3}\rangle \oplus \widetilde{D}_{n-3}$ for each h_j^{n-2} $(1 \le j \le s)$; and the map $\partial_2 \colon \widetilde{D}'_{n-3} \to \langle k_1^{n-3}, \ldots, k_s^{n-3}\rangle \oplus \widetilde{D}_{n-3}$ is of the form $\langle \alpha, \partial j_{\widetilde{D}'_{n-3}}\rangle$ where α is unimportant to us and $\partial j_{\widetilde{D}'_{n-3}}$ is the isomorphism from (11$'$). It is now easy to check that $\partial_1 \oplus \partial_2$ is an isomorphism; and thus, $H_{n-3}(\widetilde{V}_i, \partial\widetilde{V}_i)$ and $H_{n-2}(\widetilde{V}_i, \partial\widetilde{V}_i)$ are trivial.

Note For those who wish to avoid the technical issues involved with infinite handle decompositions, an alternative proof that V_i is a generalized $(n-2)$–neighborhood may be obtained by analyzing the long exact sequence for the triple $(V_i, Q_i \cup R_i, \partial V_i)$. The work involved is similar, but the key calculations are now shifted to the compact pair $(Q_i \cup R_i, \partial V_i)$.

9 Questions

The results and examples discussed in this paper raise a number of natural questions. We conclude this paper by highlighting a few of them.

The most obvious question is whether Conditions 1–3 of Theorem 3 are sufficient to imply pseudo-collarability. Other possible improvements to Theorem 3 involve Condition 2. For example, it seems reasonable to hope that the assumption of "perfect semistability" can be weakened to just "semistability". Note that Condition 4 and "perfectness" were not used until very near the end of the proof Theorem 3.

Unlike the conditions just mentioned, the assumption that $\pi_1(\varepsilon(M^n))$ is semistable is firmly embedded in the proof of Theorem 3. In particular, nearly all of the work done in Section 8 depends on this assumption. However, we do not know an example of an open manifold that is inward tame at infinity which is not π_1–semistable at infinity. Therefore we ask: Is every one ended open manifold that is inward tame at infinity also π_1–semistable at infinity? Must π_1 be perfectly semistable at infinity?

Lastly, we direct attention towards universal covers of closed aspherical manifolds. As we noted in Section 4, these provide some of the most interesting examples of pseudo-collarable manifolds. Hence, we ask whether the universal cover of a closed aspherical manifold is always pseudo-collarable. Since very little is known in general about the ends of universal covers of closed aspherical manifolds, one should begin by investigating whether these examples must satisfy *any* of the conditions in the statement of Theorem 3.

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