Geometry & Topology
Volume 6 (2002) 657{814
Published: 18 December 2002

A compendium of pseudoholomorphic beasts in $\mathbb{R} \times (S^1 \times S^2)$

Clifford Henry Taubes

Department of Mathematics, Harvard University
Cambridge, MA 02138, USA

Email: chtaubes@math.harvard.edu

Abstract

This article describes various moduli spaces of pseudoholomorphic curves on the symplectization of a particular overtwisted contact structure on $S^1 \times S^2$. This contact structure appears when one considers a closed self dual form on a 4{manifold as a symplectic form on the complement of its zero locus. The article is focussed mainly on disks, cylinders and three-holed spheres, but it also supplies groundwork for a description of moduli spaces of curves with more punctures and non-zero genus.

AMS Classification numbers Primary: 32Q65

Secondary: 57R17, 57R57

Keywords: Pseudoholomorphic curves, moduli spaces, contact structures

Proposed: Robion Kirby
Seconded: Yasha Eliashberg, John Morgan

Received: 21 November 2002
Accepted: 18 December 2002

1 Introduction

The purpose of this article is to describe various moduli spaces of pseudoholomorphic subvarieties in the symplectization of a certain over twisted contact 1–form on $S^1 \times S^2$. This said, the motivation for such a study comes from 4–manifold differential topology using three key observations. Here is the first: As explained in [19] and [20], every compact, oriented 4–manifold with positive sum of second Betti number and signature has a closed 2–form that is symplectic where non-zero and whose zero set is a finite, disjoint union of embedded circles. Moreover, as explained in [20], this closed 2–form restricts to a certain $[0, \infty) \times (S^1 \times S^2)$ neighborhood of each of its vanishing circles as the symplectization of a particular contact 1–form, the one of interest here.

Here is the second key observation stemming from [20]: Numerical invariants of the moduli spaces of pseudoholomorphic subvarieties in the complement of the zero circles most probably contain the 4–manifold's Seiberg–Witten invariants. An optimist would hope to find novel 4–manifold invariants here as well [21].

Granted the second key observation, here is the third: Hofer [8, 9, 10]; Hofer, Wysocki and Zehnder [11, 12, 13] (see also the references in [10]); Eliashberg [2], and Eliashberg with Hofer [3] have studied the salient issues that confront the construction of numerical invariants from moduli spaces of pseudoholomorphic subvarieties on non-compact symplectic manifolds with symplectization type ends. In particular, they teach that such constructions require an understanding of the analogous moduli spaces on the corresponding symplectizations. In any event, given the 4–manifold circumstances just described, the symplectization is that of the contact form in question on $\mathbb{R} \times (S^1 \times S^2)$.

With the preceding understood, it is time to be precise about the relevant geometry. For this purpose, introduce coordinates (s, t, θ, φ) for $\mathbb{R} \times (S^1 \times S^2)$ where s is the coordinate for the $\mathbb{R}$ factor in $\mathbb{R} \times (S^1 \times S^2)$, $t \in \mathbb{R}/2\pi\mathbb{Z}$ is the coordinate for the S^1 factor and $(\theta, \varphi) \in [0, \pi] \times (\mathbb{R}/2\pi\mathbb{Z})$ are standard spherical angle coordinates for the S^2 factor. This done, the contact form in question is

$$\alpha \equiv -(1 - 3\cos^2\theta)\,dt - \sqrt{6}\cos\theta\sin^2\theta\, d\varphi. \tag{1.1}$$

The resulting symplectic form on $\mathbb{R} \times (S^1 \times S^2)$ is

$$\omega = d(e^{-\sqrt{6}s}\alpha). \tag{1.2}$$

In this regard, note that the convention here is such that the $s \to \infty$ end of $\mathbb{R} \times (S^1 \times S^2)$ is the concave side end in that $|\omega|$ drops to zero as $e^{-\sqrt{6}s}$ in this direction. Conversely, the end where $s \to -\infty$ is the convex end. Said differently, the contact form α is of concave type with $S^1\times S^2$ viewed as the boundary of $[0,\infty)\times(S^1\times S^2)$.

By the way, the factor of $\sqrt{6}$ that enters above and subsequently propagates throughout this article is a consequence of a desire to have ω define a self-dual 2–form with respect to the standard product metric, $ds^2+dt^2+d\theta^2+\sin^2\theta\, d\varphi^2$, on $\mathbb{R}\times(S^1\times S^2)$.

It proves convenient in the ensuing discussion to have introduced functions f and h on $\mathbb{R}\times(S^1\times S^2)$ defined as follows:

$$f \equiv e^{-\sqrt{6}s}(1-3\cos^2\theta) \quad\text{and}\quad h \equiv \sqrt{6}\, e^{-\sqrt{6}s}\cos\theta\sin^2\theta\,. \tag{1.3}$$

This done, we have

$$\omega = dt\wedge df + d\varphi\wedge dh. \tag{1.4}$$

The almost complex structure J used here to define the term 'pseudoholomorphic' is specified by the relations

$$J\partial_t = g\partial_f \quad\text{and}\quad J\partial_\varphi = g\sin^2\theta\,\partial_h, \tag{1.5}$$

where $g \equiv \sqrt{6}\, e^{-\sqrt{6}s}(1+3\cos^4\theta)^{1/2}$. This almost complex structure is ω–compatible. This is to say that the bilinear form

$$g^{-1}\omega(\cdot\,, J(\cdot)) \tag{1.6}$$

defines a smooth metric on $\mathbb{R}\times(S^1\times S^2)$. Infact, the metric in (1.6) is the standard product metric, $ds^2+dt^2+d\theta^2+\sin^2\theta\, d\varphi^2$ but written in terms of t, f, φ and h as

$$dt^2+g^{-2}(df^2+\sin^{-2}\theta\, dh^2)+\sin^2\theta\, d\varphi^2. \tag{1.7}$$

Note that J is not integrable. By the way, J sends the vector field ∂_s to a multiple of the Reeb vector field, $\hat{\alpha} = -g^{-1}[(1-3\cos^2\theta)\partial_t+\sqrt{6}\cos\theta\,\partial_\varphi]$, the unique vector field that contracts with α to give 1 and is annihilated by $d\alpha$. In addition, J is invariant under translations of the coordinate s on $\mathbb{R}\times(S^1\times S^2)$. Thus, J is a standard almost complex structure for the 'symplectization' of the contact structure defined by α.

As remarked, the almost complex structure in (1.5) defines the notion used here of a pseudoholomorphic subvariety. A certain subset of the latter, called here HWZ subvarieties, are of particular interest. Here is the definition:

Definition 1.1 An *HWZ subvariety*, $C\subset\mathbb{R}\times(S^1\times S^2)$, is a non-empty, closed subset with the following properties:

- The complement in C of a countable, nowhere accumulating subset is a two-dimensional submanifold whose tangent space is J–invariant.
- $\int_{C\cap K}\omega < \infty$ if $K \subset \mathbb{R}\times(S^1\times S^2)$ is an open set with compact closure.
- $\int_C d\alpha < \infty$.

The HWZ subvarieties are remarkably well behaved. As explained in the next section, each intersects the large $|s|$ portions of $\mathbb{R}\times(S^1\times S^2)$ as a finite, disjoint union of cylinders. Moreover, each such cylinder intersects the appropriate component of each large and constant $|s|$ slice of $\mathbb{R}\times(S^1\times S^2)$ transversely and the resulting s–parameterized family of circles in $S^1\times S^2$ converges pointwise to multiply cover an embedded circle whose tangent lines are annihilated by $d\alpha$.

As indicated by the preceding remarks, the closed, integral curves in $S^1\times S^2$ of the kernel of $d\alpha$ play a prominent role in this story. They are called 'closed Reeb orbits'. Here is the full list of such circles:

- There are two distinguished ones, labeled $(+)$ and $(-)$, these being the respective loci where $\theta = 0$ and where $\theta = \pi$.
- The others are labeled by data $((p,p'),\upsilon)$ where $\upsilon \in \mathbb{R}/2\pi\mathbb{Z}$ and where (p,p') are integers subject to three constraints:

(a) At least one is non-zero, if $p = 0$, then $p' = \pm 1$, if $p' = 0$, then $p = 1$ and if both are non-zero, then they are relatively prime.

(b) $|p'|/|p| > \sqrt{3}/\sqrt{2}$ when $p < 0$.

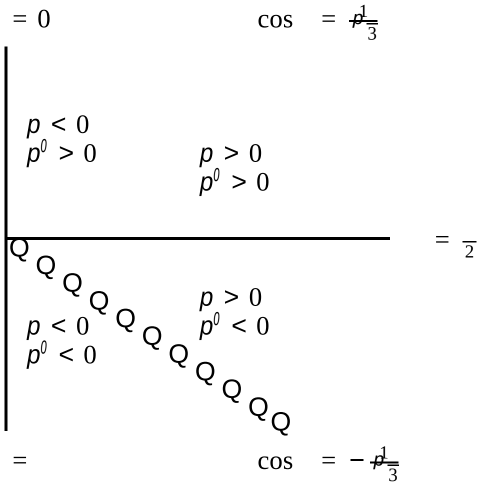
(c) $p > 0$ when $|p'|/|p| < \sqrt{3}/\sqrt{2}$.

The Reeb orbit labeled by these data are the loci where

(1) $p't - p\varphi = \upsilon$

(2) θ is constant and such that $p'(1-3\cos^2\theta) = p\sqrt{6}\cos\theta$ with $p'\cos\theta \ge 0$. (1.8)

As it turns out, the second constraint determines θ in terms of (p,p'), and vice-versa. What follows is a schematic drawing of the possible values of (p,p') as a function of the azimuthal angle θ on the sphere.



(1.9)

A momentary digression is in order here to comment on the fact that the closed Reeb orbits are not isolated in $S^1\ S^2$. In particular, this is a direct consequence of the fact that the contact form is invariant under an $S^1\ S^1$ subgroup, T, of the isometry group of $S^1\ S^2$. The convention used here takes the first factor of S^1 in T to rotate the S^1 factor in $S^1\ S^2$ via translation of the coordinate t; the second factor of S^1 in T rotates the S^2 factor of $S^1\ S^2$ via translation of the spherical angle φ. This understood, the = 0 and Reeb orbits are the only closed Reeb orbits that are invariant under the whole of T. Each of the others is preserved by no more than the product of a finite subgroup with a 1–parameter subgroup of T.

As indicated above, each end of an HWZ subvariety C determines a closed Reeb orbit by the asymptotics of its constant s slices as either $s \to \infty$ or $s \to -\infty$. This said, introduce the number, ∂_C, of such convex side ($s \to -\infty$) ends with limit Reeb orbit where is neither 0 nor . This integer ∂_C plays a key role in the subsequent discussion.

A second integer, I_C, also plays a key role here. What follows is an informal definition of I_C; the somewhat technical formalities are relegated to Section 3. The discussion starts with the observation that the full set of HWZ subvarieties has a reasonable structure of its own. As explained later in Section 3, this set enjoys a topology whereby a subvariety, C, has a neighborhood that is homeomorphic to the zero set of a smooth map from a Euclidean ball of some dimension N_C to Euclidean space of a possibly different dimension, n_C. In this regard, the difference, $I_C = N_C - n_C$ defines a locally constant function and should be thought of as formal dimension at C of the space of HWZ subvarieties.

For future reference, note that the assignment of the integer $@_C$ to an HWZ subvariety C also defines a locally constant function in this topology.

A second integer, I_C, also plays a key role here. What follows is an informal definition of I_C; the somewhat technical formalities are relegated to Section 3. The discussion starts with the observation that the full set of HWZ subvarieties has a reasonable structure of its own. As explained later in Section 3, this set enjoys a topology whereby a subvariety, C, has a neighborhood that is homeomorphic to the zero set of a smooth map from a Euclidean ball of some dimension N_C to Euclidean space of a possibly different dimension, n_C. In this regard, the difference, $I_C = N_C - n_C$ defines a locally constant function and should be thought of as formal dimension at C of the space of HWZ subvarieties.
For future reference, note that the assignment of the integer $@_C$ to an HWZ subvariety C also defines a locally constant function in this topology.

By the way, the interpretation of I_C as a dimension is based on the following observation: When n_C can be taken to be zero, then n_C can be taken zero on a neighborhood of C. This said, the subset of those HWZ subvarieties C with $n_C = 0$ forms an open subset with a natural smooth manifold structure and I_C gives the dimension of this manifold at C.

The set of HWZ subvarieties with the topology just described is called the "moduli space" of HWZ subvarieties and is denoted by $\mathfrak{M}$. Of interest in this article are the components of $\mathfrak{M}$ where $I_C \le @_C + 1$. In particular, the most prominent result of this article is a complete description of these $I_C \le @_C + 1$ components. Theorems A.1–4 summarize many of the salient conclusions. The author currently plans to discuss some of the $I_C > @_C + 1$ moduli spaces in a sequel to this article.

The focus here on the $I_C \le @_C + 1$ components of M has its ultimate justification in the proposed use of the constructs of Hofer and his coworkers to study invariants of smooth 4–manifolds using two-forms that are symplectic where non-zero. However, as such applications are dreams for now (and $I_C \le @_C + 1$ cases surely enter such dreams) take the results of the theorems here as simply answers to the following question: What do the small dimensional, HWZ moduli spaces look like?

The statements of Theorems A.1, A.2 and A.3 make explicit reference to the fact that the group T acts on $\mathfrak{M}$ as does the group $\mathbb{R}$. The former via its action on the $S^1 \times S^2$ factor in $\mathbb{R} \times (S^1 \times S^2)$ and the latter by its action on the $\mathbb{R}$ factor via translation of the coordinate s. As these group actions commute with each other, so $\mathbb{R} \times T$ acts on $\mathfrak{M}$. Moreover, this action is continuous and smooth on the smooth manifold parts of $\mathfrak{M}$.

Theorems A.1, A.2 and A.3 refer to 'irreducible' subvarieties. The term here denotes a subvariety that cannot be disconnected by the removal of any finite set of points. These theorems also use as notation $\mathfrak{M}_C$ to denote the component in $\mathfrak{M}$ of a particular HWZ subvariety C.

Theorem A.1 *Let $C \subset X$ be an irreducible, HWZ pseudoholomorphic subvariety. Then the following are true:*

- $I_C \geq \partial_C$.
- *If $I_C = \partial_C$, then $\partial_C = 0, 1$, or 2.*

(a) *$\partial_C = 0$ if and only if $C = \mathbb{R} \times \gamma$ where γ is a $\theta_0 \in \{0, \pi\}$ closed Reeb orbit from the first point in (1.8). In this case, $\mathfrak{M}_C \subset \mathfrak{M}$ consists of the point C.*

(b) *$\partial_C = 1$ if and only if $C = \mathbb{R} \times \gamma$ where γ is a closed Reeb orbit from the second point in (1.8). In this case, $\mathfrak{M}_C$ is a smooth manifold, diffeomorphic to S^1, invariant under the action of $\mathbb{R}$ and a single orbit under the action of T.*

(c) *$\partial_C = 2$ if and only if C is a cylinder that is invariant under a 1–parameter subgroup of T. Here, C is embedded and the limit Reeb orbits from C are characterized as in (1.8) by data $(\ \ (p, p');\ \ )$ where $|p'|/|p| > \sqrt{3}/\sqrt{2}$. In this case, $\mathfrak{M}_C$ is a smooth manifold, one orbit under $\mathbb{R} \times T$ and diffeomorphic to $\mathbb{R} \times S^1$.*

- *If $I_C = \partial_C + 1$, then $\partial_C = 1, 2$ or 3.*

(a) *$\partial_C = 1$ if and only if one of the following three scenarios prevail:*

(1) *C is an embedded disk invariant under the second factor of S^1 in T. Here, C is embedded and its limit Reeb orbit is a $((0,\ 1);\ )$ case from (1.8). In this case, $\mathfrak{M}_C$ is a smooth manifold, one orbit under $\mathbb{R} \times T$, and diffeomorphic to $\mathbb{R} \times S^1$.*

(2) *C is an embedded cylinder, invariant under an S^1 subgroup of T with only one convex side limiting Reeb orbit. Here, C is embedded and the convex side Reeb orbit is characterized by $((p, p');\ )$ with $p < 0$ and $|p'|$ is the least integer that is greater than $p\sqrt{3}/\sqrt{2}$. Meanwhile, the concave side Reeb orbit is characterized, as in (1.8), by either $(+)$ or $(-)$ with the sign in question that of p'. In this case, $\mathfrak{M}_C$ is a smooth manifold, one orbit under $\mathbb{R} \times T$, and diffeomorphic to $\mathbb{R} \times S^1$.*

(3) *C is an embedded cylinder, invariant under an S^1 subgroup of T with two convex side limiting Reeb orbits. Here, one is characterized by $((p, p'),\ )$ with $p > 0$, and with $|p'|$ equal to the greatest integer that is less than $p\sqrt{3}/\sqrt{2}$. Meanwhile, the second convex side limit Reeb orbit is characterized, as in (1.8), by either $(+)$ or $(-)$ with the sign in question opposite that of p'. In this case, C is also embedded. Again, $\mathfrak{M}_C$ is a smooth manifold, one orbit under $\mathbb{R} \times T$, and diffeomorphic to $\mathbb{R} \times S^1$.*

(b) *If $\partial_C = 2$ or 3, then C is an immersed, thrice punctured sphere with no limit Reeb orbits described, as in (1.8), by either $(+)$ or $(-)$. In each case, $\mathfrak{M}_C$ is a smooth manifold. Theorems A.2 and A.3, below, describe the classification and structure of $\mathfrak{M}_C$.*

All other cases have $I_C \geq \partial_C + 2$.

Note that Theorem A.4, below, describes the number of singular points in the immersed subvarieties that appear Part b of the third point.

Theorem A.1 makes reference to HWZ subvarieties that are preserved by subgroups of $\mathbb{R} \times T$. In this regard, an irreducible subvariety is preserved by an $\mathbb{R} \times S^1$ subgroup of $\mathbb{R} \times T$ if and only if it is the cylinder $\mathbb{R} \times \gamma$ where γ is a closed Reeb orbit. Such a cylinder is preserved by the whole of $\mathbb{R} \times T$ if and only if γ is either the $(+)$ or $(-)$ Reeb orbit in the notation of (1.8). Meanwhile, an irreducible HWZ subvariety C that is moved by the $\mathbb{R}$ action but preserved by an S^1 subgroup of T is either a cylinder or a disk. All such are described in great detail in Section 4 of this paper. Here is a table that summarizes the sorts of subvarieties that appear in Theorem A.1:

$\mathbb{R} \times T$ invariant cylinders: The $\theta = 0$ and $\theta = \pi$ loci.

$\mathbb{R} \times S^1$ invariant cylinders: $\mathbb{R} \times$ (Closed Reeb orbit).

S^1 invariant disks: These have $t =$ constant. The one end is on the convex side with constant s slices that converge as $s \to -\infty$ onto a closed Reeb orbit where $\cos^2\theta = 1/3$.

S^1 invariant cylinders: These can have either one or two convex side ends. In any event, each $\theta \notin \{0, \pi\}$, closed Reeb orbit determines exactly two S^1 invariant cylinders with constant s slices that converge to it as $s \to -\infty$.

Three-holed spheres with one concave side and two convex side ends.

> Three-holed spheres with no concave side and three convex side ends. (1.10)

No thrice-punctured sphere from Theorem A.1 is fixed by a non-trivial element in T. Nonetheless, much is known about these punctured spheres and the moduli space components that contain them. The next theorem summarizes what is known about the $\partial_C = 2$ cases.

Theorem A.2 *The* $\partial_C = 2$ *components of* $\mathfrak{M}$ *that appear in Part b of the third point of Theorem A.1 consist solely of immersed, thriced punctured spheres. Each such component is a smooth manifold. Moreover, these components have the following classification and structure:*

Classification *The components are classified by ordered sets of four integers having the form* $((p, p'), (q, q'))$ *subject to the following constraints:*

- (a) $pq' - p'q > 0$.
- (b) $q' - p' > 0$ *unless* $p'q' > 0$
- (c) *If* (m, m') *denotes either* (p, p') *or* (q, q'), *then* $|m'|/|m| > \sqrt{3}/\sqrt{2}$ *when* $m < 0$, *and* $m > 0$ *when* $|m'|/|m| < \sqrt{3}/\sqrt{2}$.

Structure *The component of* $\mathfrak{M}$ *that corresponds to* $((p, p'), (q, q'))$ *is diffeomorphic to* $\mathbb{R} \times T$. *Moreover, this diffeomorphism is* $\mathbb{R} \times T$ *equivariant.*

The story on the $\partial_C = 3$ cases from Theorem A.1 is provided by the next result.

Theorem A.3 *The* $\partial_C = 3$ *components of* $\mathfrak{M}$ *that appear in Part b of the third point of Theorem A.1 consist solely of immersed, thrice punctured spheres. Each such component is a smooth manifold. Moreover, these components have the following classification and structure:*

Classification *The components are in 1–1 correspondence with the unordered sets of three pair of integers that are constrained in the following way: Such a set,* L, *can be ordered as* $\{(p, p'), (q, q'), (k, k')\}$ *with*

- (a) $p + q + k = 0$ *and* $p' + q' + k' = 0$.
- (b) $|k'|/|k| > \sqrt{3}/\sqrt{2}$.
- (c) $\{(p, p'), (q, q')\}$ *obey the constraints in the first point of Theorem A.2. In this regard, a set* L *with an ordering that satisfies these three conditions has precisely two distinct orderings that satisfy the conditions.*

Structure *The component of* $\mathfrak{M}$ *that is labeled by* L *is a smooth manifold that is diffeomorphic to* $(0,1) \times \mathbb{R} \times T$*. Moreover, this diffeomorphism is* $\mathbb{R} \times T$ *equivariant for the* $\mathbb{R} \times T$ *action on* $(0,1) \times \mathbb{R} \times T$ *that fixes the* $(0,1)$ *factor and acts in the canonical fashion on the* $\mathbb{R} \times T$ *factor. Finally, the quotient,* $(0,1)$*, of this moduli space component by the* $\mathbb{R} \times T$ *action has a natural compactification as* $[0,1]$*, where the two added points label the* $\mathbb{R} \times T$ *quotient of two components of the* $\partial = 2$ *moduli space components described in Theorem A.2. In this regard, the relevant components are labeled by the first two pairs from the two possible orderings of* L *that obey the three constraints given in the preceding point.*

The final theorem in this section describes the number of singular points of the thrice punctured spheres described in Theorems A.2 and A.3. For a subvariety C, this number, m_C, ‘counts’ the number of singular points in the immersion. A precise definition of this count is given in Section 3a. In any event, $m_C = 0$ if and only if C is embedded, and m_C is the number of double points when all singularities are locally transversal intersections of pairs of disks.

The statement of Theorem A.4 implicitly views S^1 as the unit radius circle about the origin in $\mathbb{C}$.

Theorem A.4 *Suppose that* C *is either described by Theorem A.2 and its moduli space component is classified by the data* $\{(p,p'),(q,q')\}$*, or else* C *is described by Theorem A.3 and its moduli space component is described by the data set* $\{(p,p'),(q,q'),(k,k')\}$*. In either case, the integer* m_C *is one half of the number of pairs* $(\tau,\tau') \in S^1 \times S^1$ *such that* $\tau \neq \tau'$*, neither* τ *nor* τ' *equals 1, and* $\tau^p \tau'^q = \tau^{p'} \tau'^{q'} = 1$*. Thus,*

$$2m_C = |pq' - p'q| - gcd(p,p') - gcd(q,q') - gcd(p+q, p'+q') + 2$$

where $gcd(m,m')$ *denotes the greatest common divisor of* m *and* m'*. For example,* $m_C = 0$ *and so* C *is embedded if and only if one of the following conditions holds:*

- $|pq' - p'q|$ *is either 1 or 2.*
- $|pq' - p'q|$ *divides with integer remainder both members of at least one of the pairs of integers* $(p,p'), (q,q')$ *and* $(k = -p-q, k' = -p'-q')$*.*

The remainder of this article has five more sections that are organized along the following lines:

Section 2 states and proves a theorem that describes the behavior of a natural class of pseudoholomorphic subvarieties on a non-compact symplectic 4–manifold whose ends are symplectomorphic to either the $s > 0$ or $s < 0$ portions of $\mathbb{R} \times (S^1 \times S^2)$. In this regard, the symplectomorphism is required to identify the almost complex structure with that depicted in (1.5). Propositions 2.2 and 2.3 summarize the principle results of Section 2.

Section 3 considers the structure of the moduli spaces of the subvarieties from Section 2. In particular, this section defines the topology for $\mathfrak{M}$ and provides, in Proposition 3.2, a local model for neighborhoods of points in $\mathfrak{M}$. In addition, Proposition 3.6 provides an 'index theorem' that computes the analog of I_C in explicitly geometric terms.

Section 4 focuses on the explicit example of $\mathbb{R} \times (S^1 \times S^2)$ and provides a proof of Theorem A.1. By the way, the proof of Theorem A.1 derives additional constraints on the possibilities for I_C. The latter are summarized in Proposition 4.3.

Section 5 focuses on $\mathbb{R} \times (S^1 \times S^2)$ and proves Theorem A.2. This section also proves the assertions of Theorem A.4 about Theorem A.2's subvarieties.

Section 6 focuses on $\mathbb{R} \times (S^1 \times S^2)$ and contains the proof of Theorem A.3. This section also contains the proofs of the assertions in Theorem A.4 about the subvarieties from Theorem A.3.

2 Regularity

The discussion here and in the third section concerns an oriented, symplectic 4–manifold X that can be described in the following way: Start with a smooth, oriented 4–manifold with boundary, X_0, where the boundary of X_0 is a disjoint union of copies of $S^1 \times S^2$. Suppose that this boundary can be written as $\partial^- X_0 \cup \partial^+ X_0$, where each component of $\partial^- X_0$ has a neighborhood with an orientation preserving diffeomorphism to $[0, 1) \times (S^1 \times S^2)$, and each component of $\partial^+ X_0$ has one to $(-1, 0] \times (S^1 \times S^2)$. In this regard, view the latter as subsets of $\mathbb{R} \times (S^1 \times S^2)$ with its orientation defined by the symplectic form in (1.2).

Given X_0, then X is obtained by attaching $(-\infty, 0] \times (S^1 \times S^2)$ to each component of $\partial^- X_0$ and attaching $[0, \infty) \times (S^1 \times S^2)$ to each component of $\partial^+ X_0$. Meanwhile, the symplectic form, ω, on X is required to restrict to an open neighborhood of each component of $X - X_0$ as either the form in (1.2) or else as this form after passing to a suitable 2–fold cover. In this regard, the deck transformation for this cover is the fixed point-free involution, $\sigma: S^1 \times S^2 \to S^1 \times S^2$ that sends

$$(t, \theta, \varphi) \to (t + \pi, \pi - \theta, -\varphi) \tag{2.1}$$

Note that α in (1.1) is invariant under the action of σ, and thus ω is invariant under the induced involution on $\mathbb{R} \times (S^1 \times S^2)$. Therefore, both descend to the associated quotient.

Here is some terminology used below: An end of X is a component of $X - X_0$. A component of $X - X_0$ that comes from a component of $\partial^- X_0$ is called a *convex end* of X, while one that comes from a component of $\partial^+ X_0$ is called a *concave end*. Also, a component of $X - X_0$ where ω restricts directly as the form in (1.2) is said to have *orientable z-axis line bundle*. A component where passage to the double cover is required is said to have *non-orientable z-axis line bundle*.

Of course, the prime example in this paper of such an X is $\mathbb{R} \times (S^1 \times S^2)$ with the symplectic form in (1.2). In this case, there is one convex end and one concave end. Moreover both ends of X have an orientable z–axis line bundle. As explained in [20], other examples come from compact 4–manifolds with 2–forms that are symplectic where non-zero and vanish on an embedded union of circles. In the latter examples, all of the ends are concave, but there can be some with a non-orientable z–axis line bundle.

By the way, when X comes from a compact 4–manifold as just described, a result of Gompf [7] asserts that the parity of the number of ends of X with orientable z–axis line bundle is opposite that of the sum of the first Betti number, the second Betti number and the signature of the original compact 4–manifold. On the other hand, start with such a compact 4–manifold and, according to Luttinger [15], the closed form can be manipulated so that the resulting manifold X has only orientable z–axis ends.

Given X with its symplectic form as just described, there are almost complex structures on X that are ω–compatible and restrict to some open neighborhood of each end of X as follows: If the end has orientable z–axis line bundle, then J restricts as the almost complex structure in (1.5). On the other hand, if the end has non-orientable z–axis line bundle, then J should restrict as the push-forward of (1.5) via the covering map induced by σ. In this regard, note that (1.5) is σ–invariant.

Unless explicitly noted otherwise, assume that every almost complex structure that appears below has the properties just described. This said, fix such a J and Definition 1.1 has the following analog:

Definition 2.1 With X, ω and J as just described, a subset $C\subset X$ is an *HWZ subvariety* when the following conditions are met:

- C is closed, and the complement of a countable, non-accumulating set is a smooth submanifold with a J–invariant tangent space.
- Let $K\subset X$ be any open set with compact closure. Then $\int_{C\cap K}\omega<\infty$.
- Let $E\subset X-X_0$ be any component. Then $\int_{C\cap E}d\alpha<\infty$ where α is the contact form in (1.1) when E has orientable z–axis line bundle; otherwise, α is the push-forward of the form in (1.1) via the covering map defined by ψ in (2.1).

With the preceding understood, it can now be said that the purpose of this section is to state and then prove the two propositions that follow that describe the ends of an HWZ subvariety in X. With regards to the proofs, note that they introduce various constructions that are used in later portions of this article.

Proposition 2.2 *Let* $C\subset X$ *be an* HWZ *subvariety. Then:*

- C *has a finite number of singular points.*
- C *intersects each sufficiently large and constant* $|s|$ *slice of* $X-X_0$ *transversely.*
- *There is a finite union,* Γ*, of disjoint closed Reeb orbits in* ∂X_0*, thus integral curves of the distribution kernel* $(d\alpha)$*, with the following significance:*
  - (a) *Let* $U\subset\partial X_0$ *be any tubular neighborhood of* Γ*. Then* C*'s intersection with each sufficiently large and constant* $|s|$ *slice of* $X-X_0$ *lies in* U*.*
  - (b) *Fix a tubular neighborhood projection from* U *to* Γ *and then* C*'s intersection with each sufficiently large and constant* $|s|$ *slice of* $X-X_0$ *projects to each component of* Γ *as a finite to one covering map.*
- *There exists a complex curve* C_0 *with a finite set of cylindrical ends together with a proper, pseudoholomorphic map into* X *whose image is* C *and which embeds the complement of a finite set.*

The curve C_0 will be called 'the model curve' for C. The set Γ will be called the 'limit set' for C. Note that the closed Reeb orbits are listed in (1.8) for the components of ∂X_0 with orientable z–axis line bundle. The closed Reeb orbits in the unorientable z–axis line bundle components of ∂X_0 are the images of the orbits listed in (1.8) under the 2–1 covering map induced by the map ψ in (2.1).

It follows from Proposition 2.2 that the $|s| \to \infty$ limit of the constant $|s|$ slices of C on $X - X_0$ converge to some union of closed Reeb orbits. The following proposition gives a more detailed picture of this convergence:

Proposition 2.3 *Let $C \subset X$ be an* HWZ *subvariety. There exists a finite union, $\Gamma \subset \partial X_0$, of closed Reeb orbits, and, after a tubular neighborhood of Γ is identified via an exponential map with $\Gamma \times D$ with $D \subset \mathbb{R}^2$ an open disk, there exists $s_0 > 0$ such that*

- *The intersection of C with each constant $|s| \geq s_0$ slice of $X - X_0$ lies in $\Gamma \times D$.*
- *Each component of the $|s| \geq s_0$ portion of C's intersection with $X - X_0$ can be parameterized by $[s_0, \infty) \times S^1$ via a map which sends (s, τ) to $(\pm s, \gamma(m_\gamma \tau), \eta(s, \tau))$ where m_γ is a positive integer, $\gamma \in \Gamma$ and η is a smooth map from $[s_0, \infty) \times S^1$ to D. Here, the $+$ sign is used with a component in a concave end of X and the $-$ sign with a component in a convex end.*
- *There exists $\kappa > 0$ and, for each integer $k \geq 0$, a constant $c_k \geq 0$ such that the C^k–norm of η is bounded by $c_k e^{-\kappa |s|}$.*

Note that these two propositions would follow directly from Theorems 1.2 and 1.4 in [11] but for the fact that the latter assume a non-degeneracy condition on the closed Reeb orbits that is not obeyed here.

The remainder of this section is occupied with the proofs of Propositions 2.2 and 2.3.

(a) Proof of Proposition 2.2

Before getting to specifics, note that the assertions of the proposition are local to the ends of X and so no generality is lost by assuming that the ends have orientable z–axis line bundle. This is because the almost complex structure in (1.5) is ι–invariant and thus ι preserves the conditions for the appellation HWZ subvariety. Thus, the setting for a non-orientable z–axis line bundle end can be pulled up via the double cover map induced by ι in (2.1) and viewed as a ι–equivariant example of the orientable z–axis line bundle case. This understood, all ends in the subsequent discussion are implicitly assumed to have orientable z–axis line bundle.

To begin the proof, remark first that the first and fourth points follow directly from the second and third. Thus, the argument below focuses on the latter two points. This argument is broken into ten steps.

Step 1 For each integer $n \geq 2$, let C_n denote the intersection of C with the portion of $X - X_0$ where $|s| \in [n-2, n+2]$. Via the evident identification of this cylinder with $W \equiv [-2, 2] \times \partial X_0$, each C_n can be viewed as a proper, pseudoholomorphic subvariety of W. In this regard, note that $\omega = d(e^{-\sqrt{6}s}\alpha)$ is a symplectic form on W. The first claim here is that there is an n–independent upper bound to the symplectic area of C_n. This follows from the finiteness of $d\alpha$'s integral over $C \cap (X - X_0)$. Indeed the fact that $d\alpha$ has finite integral has two key implications:

- $\lim_{n\to\infty} \int_{C_n} d\alpha = 0$.
- *Fix a component of ∂X_0 and then the sequence of numbers (indexed by n) obtained by integrating α over the intersection of C with the $|s| = n$ slice of the corresponding component of $X - X_0$ is convergent.* (2.2)

Note that the second point implies the assertion about the upper bound for the symplectic area of C_n.

Step 2 Use the compactness theorem of Proposition 3.3 in [22] to conclude that any subsequence of $\{C_n\}$ has inside it, a subsequence (hence renumbered consecutively from 1) which converges to a proper, pseudoholomorphic subvariety $C' \subset W$. This convergence is in the following sense:

First, the sequence

$$\left(\sup_{x\in C'} \mathrm{dist}(x, C_n) + \sup_{x\in C_n} \mathrm{dist}(x, C')\right) \tag{2.3}$$

converges with limit zero.

Second, C' can be written as a finite union $\{C'_\gamma\}_{1\leq\gamma\leq N}$ for some integer $N \geq 1$, where each C'_γ is a proper, pseudo-holomorphic subvariety and the intersection of C'_γ with $C'_{\gamma'}$ is finite when $\gamma \neq \gamma'$. Moreover, there is a corresponding sequence of positve integers $\{m_\gamma\}$ such that the set pairs $c \equiv \{(C'_\gamma, m_\gamma)\}$ has the property that for any 2–form ν on W, the sequence

$$\int_{C_n} \nu \tag{2.4}$$

converges with limit

$$\sum_{(C',m)\in c} m \int_{C'} \nu \,. \tag{2.5}$$

Step 3 It follows from the preceding step that $d\alpha$ vanishes on C', and this implies that C' has the form $[-2, 2] \times \Gamma$, where Γ is a finite union of closed Reeb orbits. These closed Reeb orbits are listed in (1.8). Were they are isolated, it would follow from the discussion in Step 1 that the data $c = \{([-2,2] \times \gamma, m) : \gamma \in \Gamma$ and $m \in \{1, 2, \ldots\}\}$ describing the limit of $\{C_n\}$ is uniquely defined from C. However, in the present case, the closed Reeb orbits are not isolated. Even so, one can still draw this same conclusion:

Lemma 2.4 *Let $C \subset X$ be an* HWZ *subvariety. Then all limits of the corresponding sequence $\{C_n\}$ produce the same data set c as limit. In particular, there is a finite union, Γ, of closed Reeb orbits with the following significance: Given $\delta > 0$, there exists s_δ such that each point of C's intersection with any constant $|s| \geq s_\delta$ portion of $X - X_0$ has distance δ or less from Γ. Conversely, each point of Γ has distance δ or less from a point in C's intersection with any constant $|s| \geq s_\delta$ portion of $X - X_0$.*

This lemma is proved below in the final step of the proof of Proposition 2.3, so accept it for the time being.

Before continuing to Step 4, note that the remainder of the proof of Proposition 2.2 (including that of Lemma 2.4) will assume that the component of ∂X_0 in question is concave. In this regard, the arguments for the convex case are identical save for some notational changes.

Step 4 To begin, re-introduce the coordinate functions (t, f, h, φ) for $\mathbb{R} \times (S^1 \times S^2)$ as defined in (1.3). In this regard, note that the submanifolds where (t, f) are constant are pseudoholomorphic cylinders or pairs of disks (when $f = 0$), while those where (φ, h) are constant are pseudoholomorphic cylinders.

Fix attention on a component, γ of Γ and remember from (1.8) that γ can be labeled either $(+)$, $(-)$ or $((p, p'), \ )$. In all cases, the spherical angle θ is a constant, θ_0, on γ.

The value of θ_0 forces two cases to be distinguished: Case 1 has $\gamma \neq ((0, \pm 1), \ )$, which is equivalent to the condition that $\cos^2\theta_0 \neq 1/3$. Meanwhile, Case 2 has $\gamma = ((0, \pm 1), \ )$ and therefore $\cos^2\theta_0 = 1/3$. The discussion below focus on the case where $\cos^2\theta_0 \neq 1/3$ and this condition will be assumed implicitly. The discussion for the $\cos^2\theta_0 = 1/3$ case is essentially identical to that below after switching the roles which are played by the coordinates t and φ and also by f and h. For this reasons, the discussion in the $\cos^2\theta_0 = 1/3$ case is omitted.

With the preceding understood, return to the loop γ, and observe that in the case where $\theta_0 \notin \{0, \pi\}$, this loop has a parameterization by a periodic variable $\tau \in \mathbb{R}/(2\pi|p|\mathbb{Z})$ via

$$t = \tau,$$
$$\theta = \theta_0,$$
$$\varphi = \varphi_0 + \tau p'/p. \tag{2.6}$$

Here, $\varphi_0 \in \mathbb{R}/(2\pi\mathbb{Z})$ is a constant. In the case where $\theta_0 \in \{0, \pi\}$, the parameterization of γ is also given by (2.6) with the last two lines absent. Even so, take $p = -1$ when $\theta_0 \in \{0, \pi\}$. In any case, the projection from $S^1 \times S^2$ to the S^1 factor restricts to γ as an $|p|$ to 1 covering map.

Note that the functions f and h in (1.3) restricts to γ to obey

$$h = (p'/p)\sin^2(\theta_0)f. \tag{2.7}$$

Step 5 Lemma 2.4 implies that at all large values of s, the intersection of C with some fixed radius tubular neighborhood of $\{s\} \times \gamma$ lies very close to $\{s\} \times \gamma$. In particular, take the tubular neighborhood in question to have disjoint closure from the other components of Γ. With this noted, there must exist f_1 such that C has empty intersection with the boundary of the closure of this tubular neighborhood where $|f| \le f_1$. In particular, there is an unambiguous component of the intersection between C and $\mathbb{R} \times (S^1 \times S^2)$ where $|f| \le f_1$ which lies in the given tubular neighborhood. Given the preceding, agree to restrict attention to the just mentioned $|f| \le f_1$ portion of C. By the way, note that on this portion of C, the limits $|f| \to 0$ and $s \to \infty$ can be assumed equal by constraining the tubular neighborhood of γ so $|\ln((1 - 3\cos^2\theta)/(1 - 3\cos^2\theta_0))| < 1$ on it.

In the subsequent steps, this particular $|f| \le f_1$ portion of C will still be denoted by C.

With these last remarks understood, it then follows from Lemma 2.4 and (2.6) that C intersects each constant (t, f) pseudoholomorphic subvariety in $m|p|$ points counting multiplicities. Here, m is the weight which appears with γ in the set $\mathfrak{c}$ from Step 3. Note that all of the multiplicities here are positive. Moreover, because of (2.7), these intersection points are grouped in subsets of m points (counting multiplicities), where each of the $|p|$ points in the set

$$\{(\varphi = \varphi_0 + t + 2\pi k p'/p, h = (p'/p)\sin^2(\theta_0)f) : 1 \le k \le |p|\} \tag{2.8}$$

is very close to a unique such subset. Indeed, the following assertion is a direct consequence of Lemma 2.4:

Fix $\delta > 0$, and $\varepsilon > 0$ exists such that if $|f| < \varepsilon$, $t \in \mathbb{R}/(2\pi|p|\mathbb{Z})$ and ϑ is a point in (2.8), then the $m(\varphi, h)$ values near ϑ coming *from points in C with coordinates $(t; f)$ have distance or less from .* (2.9)

Agree to use to denote the map which assigns the coordinates $(t; f) \, 2 \, S^1 \quad \mathbb{R}$ to the points in $\mathbb{R} \quad (S^1 \quad S^2)$. The restriction of to C has the following key property: There exists a countable, non-accumulating set C such that restricts to $C -$ as an $mjpj$ to 1 covering map. That is, if $x \, 2 \quad (C - \ )$, then the multiplicity of each point in $^{-1}(x)$ is precisely 1. On the other hand, if $x \, 2$, then

$^{-1}(\ (x))$ *contains less than $mjpj$ points; thus some of them have multiplicty greater than 1.* (2.10)

By the way, it is important to note that each point in is either a singular point of C or a smooth point of C but a critical point of . (These last assertions and (2.10) all follow more or less directly from the fact that the constant $(t; f)$ surfaces are pseudoholomorphic. A detailed argument can be had by mimicking, almost verbatim, the discussion in Part a of the Appendix to [22].)

With regards to , the key observation now is that if is nite, then any small, positive and constant jfj slice of C projects as nite to one covering map over the analogous slice of the $(t; f)$ cylinder. This said, Steps 6{8 of the proof demonstrate that is nite; the ninth step shows how the points of the proposition then follow.

Step 6 This step introduces the space C which consists of the triples $(w; \ _-; \ _+)$ with $(w; \ ) \, 2 \, C$ and $(w; \ ) = w$. Here, consists of the 2{tuple $(' \ ; h \ )$.

The local structure of C can be analyzed by mimicking the discussion in Part a of the Appendix to [22]. What follows is a summary of some of the important features. First, C embeds in C as the diagonal where $_- = \ _+$. In what follows, C and its image in C will not be notationally distinguished. Second, C is a smooth manifold except near points $(w; \ _-; \ _+)$ where one or both of $(w; \ ) \, 2$. Meanwhile, the structure of C near one of these singular points can be described in detail as Part a of the Appendix in [22]. In any event, the projection mapping $(w; \ _-; \ _+)$ to w restricts over the complement of $(\ )$ as an $(mjpj)^2$ to 1 covering map. Third, let C denote the closure in C of $C - C$. Then $C^0 \setminus C = \quad C$. Indeed, this follows from (2.10).

There is one more crucial point to make about C, namely: If $f_1 > 0$ is su ciently small, then:

> *$\varphi_+ - \varphi_-$ is a bona fide, real-valued function on C; in fact, given $\varepsilon > 0$, there exists $\delta > 0$ such that when $|f| < \delta$, then $|\varphi_+ - \varphi_-| < \varepsilon$.* (2.11)

This assertion is an immediate consequence of (2.8) and (2.9). Thus, it is a corollary to Lemma 2.4, and this is essentially the only place in the argument for Proposition 2.2 that requires Lemma 2.4. However, the conclusion expressed in (2.11) is absolutely crucial for the subsequent arguments.

Step 7 Introduce the set $G \subset C$ which consists of the points $(w; {}_-, {}_+)$ where the function $\underline{h} \equiv h_+ - h_-$ is zero. This set is an example of an 'embedded graph'. This is to say that G is a locally compact subset with the following additional properties: First, G has a distinguished subset, G_V, which is a locally finite collection of points. Elements in G_V are called vertices. Meanwhile, $G - G_V$ is a locally finite set of properly embedded, open intervals in $C^0 - G_V$. The closure of each component of $G - G_V$ is called an edge. Finally, each vertex has a neighborhood in C whose intersection with G consists of a finite union of properly embedded images of the half open interval $[0, 1)$ by an embedding which sends 0 to the vertex in question. Moreover, these embedded intervals intersect pairwise only at the given vertex.

In the present case, G has some additional properties which are summarized below:

- *Each vertex of G is either an $\underline{h} = 0$ critical point of h in the smooth part of C or else a point $(w; {}_+, {}_-) \in C$ where one or both of $(w; \ ) \in$.*
- *Let $\underline{\varphi} \equiv \varphi_+ - \varphi_-$. The 1–form $d\underline{\varphi}$ is non-zero on the tangent space of $G - G_V$. Infact, at all points on the interior of each edge, the 2–form $d\underline{\varphi} \wedge d\underline{h}$ orients C^0 so that is an orientation preserving map.*
- *The intersection of G with some open neighborhood of each vertex is a finite union of embedded, half open arcs with endpoints lying on the vertex, but disjoint otherwise. Moreover, the tangent lines to the arcs at the vertex are well defined and disjoint. The interior of each arc is part of an edge of the graph. The number of such arcs is non-zero and even. Exactly half of the arcs are oriented by $d\underline{\varphi}$ so $\underline{\varphi}$ increases towards the vertex while half are oriented by $d\underline{\varphi}$ so that $\underline{\varphi}$ decreases towards the vertex.*
- *If $f_2 \in (0, f_1)$ is chosen to be sufficiently generic, then the $|f| = f_2$ locus in C intersects only the smooth part of C, and where $df \neq 0$. In*

addition, Γ *intersects this* $|f| = f_2$ *submanifold of* C *as a finite set of points, all in the interior of edges and this intersection is transverse.*

(2.12)

These points are proved by copying the arguments in Steps 3–5 of Part b in the Appendix to [22]. In this regard, note that the observation that $d\underline{\varphi}$ is non-zero on $G - G_v$ (where $\underline{h} = 0$) plays a starring role in the next step.

By the way, the conclusion of the second point about $d\underline{\varphi}$ orienting $G - G_v$ is a specific consequence of the fact that J from (1.5) maps dh to a φ–independent multiple of $d\varphi$. Indeed, because C is pseudoholomorphic, the pair $(\varphi; h)$ obey a Cauchy Riemann like equation on $C -$ as functions of the variables $(t; f)$. This equation can be written schematically as $d\varphi = j\ dh$. Here, j is a function of f and h only; in particular, j is independent of φ. One consequence of this equation $d\varphi = j\,dh$, is that the function $\underline{\varphi} = \varphi_+ - \varphi_-$ on $C - {}^{-1}(\ (\ ))$, when viewed as a function of $(t; f)$, obeys $\underline{d}(\varphi_+ - \varphi_-) = j(f; h_+)dh_+ - j(f; h_-)dh_-$. In particular, where $h_+ = h_-$, this reads $d\underline{\varphi} = j(f; h)d\underline{h}$; and this last equation directly implies the assertion in the second point of (2.12).

Step 8 To complete the proof that is finite (modulo the proof of Lemma 2.4), note first that each point of $C \cap C'$ is a vertex of G. Moreover, these are precisely the vertices of G where $\underline{\varphi} = 0$. Thus, it is sufficient to show that there are at most a finite number of $\underline{\varphi} = 0$ vertices of G. In fact, the claim is that the number of $\underline{\varphi} = 0$ vertices of G where $|f| < f_2$ is no greater than the number of points where $|f| = f_2$ on G. This last number is finite by virtue of the final point in (2.12)).

To prove the preceding claim, choose a $\underline{\varphi} = 0$ vertex of G and then follow some edge out from this vertex where $\underline{\varphi}$ is increasing. The existence of such an edge is guaranteed by the third point in (2.12). Continue to travel along this edge. According to the second point of (2.12), the function $\underline{\varphi}$ continues to increase. Either this edge eventually hits another vertex of G where $|f| \le f_2$, or else it hits the $|f| = f_2$ locus of G. In this regard, note that $|f|$ is bounded away from zero on such an edge, since $|\underline{\varphi}|$ converges to zero along any path in C where $|f|$ limits to zero because of (2.11). Indeed, remember that $|\underline{\varphi}|$ started at zero and then increases along the edge.

If the edge ends in a second vertex of G, then $\underline{\varphi} > 0$ at this vertex, and there is another edge coming into this vertex on which $\underline{\varphi}$ is increasing in the outward pointing direction (by the fourth point in (2.12).) Continue out on this new edge. Note that $\underline{\varphi}$ still is increasing. Iterate this procedure. As $\underline{\varphi}$ always increases, the path so traced out remains in a compact subset of the $|f| \le f_2$

portion of C^0. Meanwhile, no vertices are hit by this piecewise smooth path in G more than once. By compactness, the path must end, and the only possible way to do so is to hit the $|f| = f_2$ locus.

With the preceding understood, let $G^{(1)} \subset G$ denote the compliment of the interiors of the edges which are traversed by the path just described. Note that $G^{(1)}$ is also described by (2.12) except that $G^{(1)}$ may have some isolated vertices. Agree to ignore these as they play no role in what follows. By the way, observe that the intersection of $G^{(1)}$ with the $|f| = f_2$ locus contains one fewer point then that of G.

Given $G^{(1)}$, repeat the procedure just described in the previous three paragraphs, but with $G^{(1)}$ replacing G. The result is a $G^{(2)} \subset G^{(1)}$ which is described by (2.12) (except maybe for some isolated vertices). Note that $G^{(2)}$ has two fewer intersection points with the $|f| = f_2$ locus as did G. Of course, one can continue in this vein, creating $G^{(3)}, \ldots,$ etc., each time reducing by one the size of the set of intersections with the $|f| = f_2$ locus. Eventually, this finite set of $|f| = f_2$ intersections is exhausted, say for $G^{(k)}$, in which case $G^{(k)}$ has no non-isolated $\underline{\varphi} = 0$ vertices. In particular, this means that the original number of $\underline{\varphi} = 0$ vertices in G is no greater than the size of the $|f| = f_2$ locus in G.

Step 9 Given that the set is finite, it follows that there exists some $f_1 > 0$ such that when $|f| < f_1$, then the constant $|f|$ slice of C defines an $m|p|$ to 1 covering map over the corresponding slice of the (t, f) cylinder. This understood, then C can be parameterized as a multiple cover of the appropriate component of the $|f| < f_1$ portion of the (t, f) cylinder. One such parameterization uses coordinates $(,)$ where $\in \mathbb{R}/(2 \ m|p|\mathbb{Z})$ and $\in (_0, \infty)$. In the case where $\in \{0, \}$, the latter parameterize the $|f| < f_1$ portion of C via the map that sends $(,)$ to

$$
\begin{aligned}
t &= \ , \\
f &= \mathrm{sign}(p)e^{-} \ , \\
\varphi &= \varphi_0 + \ p'/p + x(,), \\
h &= e^{-} \ \sin^2 {}_0(p'/p + {}^{-1}\mathrm{sign}(p)w(,)),
\end{aligned}
\tag{2.13}
$$

where $= 6^{-1/2}(1 + 3\cos^4 {}_0)^{-1/2}|1 - 3\cos^2 {}_0|$. Meanwhile, as and φ are not good coordinates near the poles of S^2, the parameterization in case when $_0 \in \{0, \}$ replaces the latter by the functions

$$a_1 \quad 6^{-1/4}|f|^{-1/2}|h|^{1/2}\cos(\varphi) \quad \text{and} \quad a_2 \quad 6^{-1/4}|f|^{-1/2}|h|^{1/2}\sin(\varphi). \tag{2.14}$$

This done, the parameterization sends (;) to ($t =$, $f = -e^{-\sqrt{6}}$, a_1(;), a_2(;)).

Here is the point of such a parameterization: Introduce the 2{component, column vector , with either top entry x and bottom entry w or top entry a_1 and bottom entry a_2 as the case may be. By virtue of (1.5) and the fact that C is pseudoholomorphic, this vector obeys a di erential equation with the schematic form

$$@ \quad + L_0 \quad + \mathfrak{R}(\; ; \quad) = 0: \qquad (2.15)$$

Here, L_0 in (2.15) denotes the operator

$$L_0 = \begin{pmatrix} - \,{}^{0} & -@ \\ @ & - \end{pmatrix} \qquad (2.16)$$

where ^{0} and are constants. Meanwhile, $\mathfrak{R}(;)$ is a ne linear in the second factor and obeys

$$j\mathfrak{R}(a; b)j \qquad (jaj^2 + jajjbj) \; ; \qquad (2.17)$$

where is independent of , a and b where is large and jaj is small. Equation (2.15) also exhibits the notation, used subsequently, where the partial derivative of a function by a parameter is denoted by the function's symbol adorned with the parameter as subscript.

As a solution to (2.15), the vector also obeys the asymptotic condition

$$\lim_{\;! \;1} j \; j = 0 \qquad (2.18)$$

by virtue of the fact that the constant and large slices of C converge to the Reeb orbit in question.

Observe now that the equation in (2.15) is, for small $j \; j$, uniformly elliptic. This noted, and given (2.18), standard elliptic regularity arguments as in Chapter 6 of [17] apply to (2.15) and nd that derivatives of to all orders converge to zero as tends to in nity. For future reference, this conclusion is stated formally as the following lemma.

Lemma 2.5 *Given a non-negative integer k and > 0, there exists such that $j r^k \; j <$ at all points with $>$.*

This observation of Lemma 2.5 implies the assertions of Proposition 2.2. For example, the transversal intersection of C with the large and constant jsj slices follows after rst pulling the di erential ds back to C while noting that $ds = -\sqrt{6}(f\,df + h\,dh)$. This done, write h in terms of the components of and the $s \to \infty$ limit of zero for $|\nabla\theta|$ implies that $\partial_s \theta \neq 0$ at large $|s|$ on C. The assertions of the third point of the proposition follow in a similar vein.

Step 10 This step is devoted to the following proof.

Proof of Lemma 2.4 A digression comes first to summarize certain observations made prior to the statement of the lemma. To begin the digression, recall that the possible limits are described by sets of the form $\{(m_\gamma; \mathbb{R} \times \gamma) : \gamma \in \Gamma\}$, where m_γ is a positive integer weight, and where Γ is a finite set of distinct closed Reeb orbits in $S^1 \times S^2$. Moreover, the function θ on S^2 has some constant value, θ_0, on each circle. In this regard, the possible θ_0 values which can appear for some $\gamma \in \Gamma$, and the corresponding weight m_γ are uniquely determined as the set of possibilities for these data is discrete. In fact, these parameters can be distinguished by intersection numbers of C with various pseudoholomorphic submanifolds of $\mathbb{R} \times (S^1 \times S^2)$. This is to say that there might be different sets Γ which appear as limits, but each such set has the same collection of θ_0 values, and the multiplicities m_γ for the elements $\gamma \in \Gamma$ depend only on these θ_0 values and so are the same for all of the possible sets Γ which could appear.

With regard to these θ_0 values, the argument for Lemma 2.4 given here are valid for the elements in Γ with $\cos^2\theta_0 \neq 1/3$. The argument for the elements with $\cos^2\theta_0 = 1/3$ are left to the reader in as much as they are essentially the same as those given below after switching the roles of $(\varphi; h)$ with those of $(t; f)$.

With the digression now complete, suppose that Γ is as above and $\gamma \in \Gamma$ has $\cos^2\theta_0 \neq 1/3$. Then $\gamma \in \Gamma$ can be parameterized as in (2.6). Of course, the key point is that the θ_0 value in (2.6) determines γ only up to the constant $\varphi_0 \in \mathbb{R}/(2\pi\mathbb{Z})$. Thus, the different possible limits are distinguished by having different values for φ_0, and it is this possibility that will now be ruled out.

To begin this task, first focus on a point in $C - K$ which is very close to $\mathbb{R} \times \gamma$ for a particular $\gamma \in \Gamma$. Here, suppose that γ is parameterized by (2.6) for some choice θ_0 and φ_0. With respect to this question of φ_0, remember that for any fixed $\delta > 0$, but small, the points of C where $|f| < \delta$ that have identical $(t; f)$ coordinates form some $|p|$ subsets, each with $m \geq m_\gamma$ members. All members of the same subset have φ coordinates which differ by a very small amount, while members of different subsets have φ coordinates which differ by $2\pi k/p$ where $k \neq 0 \bmod(p)$.

In any event, some neighborhood of the chosen point in $C - K$ has a parameterization by coordinates $(\sigma; u)$ with σ periodic and u small in absolute value and with sign that of p via

$$t = \;,$$
$$f = u,$$
$$' = '_0 + \; p^0/p + x(\;;u),$$
$$h = (p^0/p)\sin^2(\;_0)u + y(\;;u). \quad (2.19)$$

Here, $|x|$ and $|y|/|u|$ are small and, in particular, much less than $2\;/|p|$. By virtue of (1.5), the fact that C is pseudoholomorphic manifests itself in the fact that x and y obey the di erential equation

$$x_u = -g^{-2}\sin^{-2}(\;)y\;,$$
$$x\; = \sin^{-2}(\;)y_u + (p^0/p)(\sin^2(\;_0)/\sin^2(\;) - 1). \quad (2.20)$$

Here, $g = \sqrt{6}\, e^{-\sqrt{6}s}(1 + 3\cos^4\;)^{1/2}.$ Note that the subscripts ' ' and 'u' on the variables x and y indicate the partial derivative by the corresponding coordinate. This notation is used frequently in the subsequent discussions.

In this last equation, both and g are functions of the variables and u, but the dependence is only implicit, through the dependence of y in (2.19) on . Thus, view and g as functions of f and h, the former via $h/f = \sqrt{6}\cos\;\sin^2\;(1 - 3\cos^2\;)^{-1}$, and the latter via the relation $g = \sqrt{6}(f^2 + h^2\sin^{-2}\;)^{1/2}$. This understood, the right side of the rst line in (2.19) can by written as the derivative of the restriction to C of a function Q on $\mathbb{R}\;(S^1\;S^2)$. Indeed, Q can be any function of the variables f and h whose partial derivative in h is $-g^{-2}\sin^{-2}(\;)$. For example,

$$Q = -\sqrt{6}\, f^{-1}\ln(csc\; + \cot\;). \quad (2.21)$$

In any event, with Q chosen, the rst line in (2.21) reads

$$x_u = Q\;. \quad (2.22)$$

With (2.22) understood, remember that there are some m members of C which are very close to the chosen point and have the same $(t;f)$ value. Thus, C determines not just one pair of functions $(x;y)$ as in (2.19), but a set, $\{(x_j;y_j)_{1\; j\; m}\}$ of m such pairs. Note that this set can only be ordered locally near each point in the compliment of ( ). The ordering may be permuted around circles which enclose points of ( ) or around the u = constant circle. In any event, $\;_j x_j$ is a bona de function of $(\;;u)$, and

$$\underline{x}(u)\quad \int_{0}^{2\;/|p|}\;_j x_j(\;;u)\,d\; \quad (2.23)$$

is a function just of the coordinate u.

The claim here is that $\underline{x}$ is the constant function. Indeed, this claim is a consequence of the observation that $\ _jQ(y_j)$ is also a bona fide function of $(\ ;u)$; and thus it follows from (2.22) that the derivative of $\underline{x}$ is zero.

The fact that $\underline{x}$ is constant implies that the group of m points under observation must keep the same $'_0$ value no matter the size of u. Indeed, a change in $'_0$ must change $\underline{x}$, as can be seen from (2.19). As argued at the outset, the constancy of $'_0$ implies the assertion of Lemma 2.4.

(b) Proof of Proposition 2.3

The first two points of Proposition 2.3 are simply restatements from Proposition 2.2. The only point at issue here is the last one. As remarked at the outset, the last point in the proposition would follow directly from Theorem 1.4 of [11] were the closed orbits of the Reeb vector field non-degenerate in a certain technical sense which is satisfied here only for the closed orbits with $_0\ 2\ f0;\ \ g$ and, in the non-orientable z–axis line bundle case, that for which the fundamental class generates the first homology over $\mathbb{Z}$. However, the degeneracies here are due entirely to the fact that the contact form is invariant under the $T = S^1\ \ S^1$ subgroup of isometries of $S^1\ \ S^2$, those that fix the poles of S^2. This fact can be used to modify the arguments in [11] to apply here. Although such a modification is straightforward, the presentation of the details would be lengthy, and thus an alternate proof is offered below.

Before embarking on the details of the proof, there are some preliminary comments to be made. First, the proof of the last point in Proposition 2.3 is given below only for the case where the end in question is concave and has orientable z–axis line bundle. As in the proof of Proposition 2.2, the argument for the convex end case is identical in all essential aspects and thus left to the reader. Meanwhile, as the assertions are local to the ends of X, the non-orientable z–axis line bundle case can be treated as a –equivariant example of the orientable z–axis line bundle case. Moreover, as the assertions in the third point of Proposition 2.3 are local to each end of C, attention can be restricted to a single end. This said, there is but one limit closed Reeb orbit involved.

The final comment here is that the proof of the last point of Proposition 2.3 is broken into six steps, and all but the final step assume that the limit Reeb orbit for the end in question has neither $\cos^2\ _0 = 1{=}3$ nor $_0\ 2\ f0;\ \ g$. The case where the Reeb orbit has $\cos^2\ _0 = 1{=}3$ can be dealt with using simple modifications of the arguments given below; basically, the modifications involve the switching of the roles played by the pair $(t;f)$ with those of $(';h)$. As the details add nothing novel, they will be left to the reader. Likewise, the essentials of the argument in the case where $\theta_0 \in \{0, \pi\}$ are the same as those given below, however, there are some specific differences which deserve comment. In particular, these comments constitute the final step of the proof. (As remarked previously, the case of Proposition 2.3 where the Reeb orbit has $\theta_0 \in \{0, \pi\}$ is also directly a consequence of some general results in [11].)

With the preceding understood, what follows are the details of the proof of the third point of Proposition 2.3.

Step 1 To set the stage, return to the notation used in Step 9 of the proof of Proposition 2.2. Thus, the end in question of C is referred to henceforth as C; and its small $|f|$ portion is parameterized as in (2.13). This done, re-introduce the two component column vector η from (2.15) and (2.18). Concerning the equation in (2.15), note that the constant ε' that appears in L_0 from (2.16) is zero while $\lambda > 0$. In fact,

$$\lambda = 6^{1/2} \sin^2\theta_0 (1 + 3\cos^2\theta_0)(1 + 3\cos^4\theta_0)^{-1/2} |1 - 3\cos^2\theta_0|^{-1}. \tag{2.24}$$

The operator L_0 is a formally self-adjoint operator on the $\mathbb{R}^2$ valued functions on S^1 and so has a complete set of eigenvectors. Having constant coefficients, the eigenvalues and eigenvectors can be readily found. In particular, the corresponding eigenvalue set is

$$\left\{ 2^{-1}(-\lambda \pm (\lambda^2 + 4n^2/(|m|p|)^2)^{1/2}) : n = 0, 1, 2, \ldots \right\}. \tag{2.25}$$

Note that there is a single zero eigenvalue, one of the $n = 0$ cases in (2.25). The corresponding eigenvector is the constant column vector e_- with top entry 1 and lower entry 0. The other $n = 0$ eigenvalue is $-\lambda$ and it also has multiplicity one with a constant eigenvector. The $n \neq 0$ eigenvalues have multiplicity 2 and the components of the eigenvectors are linear combinations of the functions $\sin(n\psi/(|m|p|))$ and $\cos(n\psi/(|m|p|))$.

Step 2 Introduce the L^2–orthogonal projections, $\eta^{+,-,0}$, of the vector η in (2.15) onto the respective spans of the eigenvectors of L_0 with eigenvalues where are positive (η^+), negative (η^-) and zero (η^0).

Here is a key fact:

> *The L^2–orthogonal projection, η^0, of the vector η in (2.15) onto the kernel of L_0 is zero.* (2.26)

Indeed, η^0 is constant by virtue of (2.22) as its bottom entry is zero and its top entry is the average value of $x(t, \cdot)$ around the t = constant circles. This understood, the constant in question is zero by virture of (2.18).

Step 3 The purpose of this step is to prove that the function $f^+(s) = \int dt\,|\eta^+|^2(t;s)$ and the analogously defined function $f^-(s)$ both decay exponentially fast to zero as s tends to infinity. To see that such is the case, first let E denote the smallest of the absolute values of the non-zero eigenvalues of L_0. Second, use Lemma 2.5 to find ρ_0 so that the $s > \rho_0$ versions of the $\mathfrak{R}$ term in (2.15) obey $|\mathfrak{R}(\cdot;r\,\eta)| \le 10^{-2}E|\eta|$. Third, for fixed $s > \rho_0$, consider (2.15) to be an equality between $\mathbb{R}^2$–valued functions on the circle. This done, then the L^2 inner product on the t–parameterized circle of both sides of (2.15) with $\eta^+(\cdot;s)$ leads to the inequality

$$2^{-1}\partial_s^2 f^+ + Ef^+ - 10^{-2}E(f^+ + f^-) \ge 0. \tag{2.27}$$

Meanwhile, the analogous inner product of both sides of (2.15) with η^- leads to

$$2^{-1}\partial_s^2 f^- - Ef^- + 10^{-2}E(f^+ + f^-) \ge 0. \tag{2.28}$$

It now follows from these last two equations that $f \equiv f^- - 0.2f^+$ obeys the differential inequality

$$2^{-1}\partial_s^2 f - 0.97Ef \ge 0. \tag{2.29}$$

This last equation can be integrated to find that when s' is large and $s > s'$, then

$$f(s) \ge e^{2\delta(s-s')}f(s'); \tag{2.30}$$

where $\delta = 0.97E$.

There is one immediate conclusion to draw from (2.30) which is this: As f is supposed to have zero for its $s \to \infty$ limit, it follows from (2.30) that $f(s)$ is nowhere positive. This is to say that for all sufficiently large s,

$$f^- \le 0.2f^+. \tag{2.31}$$

The preceding inequality can now be inserted into (2.27) to yield

$$2^{-1}\partial_s^2 f^+ + 0.97Ef^+ \ge 0. \tag{2.32}$$

This equation can be readily integrated to see that

$$f^+(s) \le e^{-2\delta(s-s')}f^+(s') \tag{2.33}$$

whenever s' is large and $s > s'$. Here, $\delta = 0.97E$. Together, (2.26), (2.31) and (2.33) assert that the function $g(s) = \int dt\,|\eta(t;s)|^2$ has exponential decay to zero as s tends to infinity.

Step 4 This step proves the following assertion: For any integer $k \ge 0$, the function $g_k(s) \equiv \int dt\,|(\nabla^k\eta)(t;s)|^2$ has exponential decay to zero as s tends to infinity.

Here is the argument: First, remark that r commutes with both $@$ and L. Second, remark that $r^k{}^+$ is in the span of the eigenvectors of L_0 with positive eigenvalue while $r^k{}^-$ is in the span of those with negative eigenvalue. Third, di erentiate both sides of (2.27) and (2.28) k{times, and then take the respective L^2 inner products with $r^k{}^+$ and $r^k{}^-$. Fourth, let f ( ) now denote $\int d\, j r^k \, j^2$. Fifth, invoke Lemma 2.5 to conclude that when is large (with lower bound depending on k), this new f obeys (2.27) and the new f^- obeys (2.28). Fifth, repeat the argument in the preceding step to obtain the desired conclusion.

Step 5 The third point in Proposition 2.3 follows from the conclusions of the previous step using standard Sobolev inequalities.

Step 6 This step assumes that the closed Reeb orbit under consideration has $_0 \, 2 \, f0; \; g$, and infact, $_0 = 0$ as the discussion for the $_0 =$ case is identical save for some innocuous sign changes. The purpose of this step is to point out the two places in the argument for the $_0 = 0$ case where the modi cations to the just concluded argument are more than cosmetic. In particular, the argument here requires the parameterization of C using the functions $(a_1; a_2)$ as in (2.14). This done, the only other signi cant modi cation to the argument involves the constants ^{0} and that appear in (2.16). In this $_0 = 0$ case, these are $^0 = \; = \frac{p_{\overline{3}}}{p_{\overline{2}}}$. This said, then the spectrum of the operator L_0 in (2.16) is the set

$$\Big\{ -\, p_{\overline{3}} = p_{\overline{2}} + n = m : n \; 2 \; \mathbb{Z} \Big\} : \tag{2.34}$$

Here, each eigenspace is two-dimensional; and the components of the eigenvector for any given n are linear combinations of sin($n \; = m$) and cos($n \; = m$).

With the preceding understood, the rest of the argument for the $_0 = 0$ case is even simpler than that for the cases considered previously because the $_0 = 0$ version of L_0 has no zero eigenvalue.

3 Deformations

Let C be an irreducible, HWZ pseudoholomorphic subvariety in X: Here, X is as described in the introduction to the previous section, with some ends concave and others convex, some with orientable z{axis line bundle and others with the latter non-orientable. After a preliminary discussion on Cs topology, the focus here is on deformations of C which preserve both the topology and its status as an HWZ pseudoholomophic subvariety. In this regard, note that the conclusions of the subsequent discussions are summarized by Propositions 3.1, 3.2 and 3.6. In particular, Proposition 3.1 asserts an adjunction formula that generalizes a formula for compact pseudoholomorphic subvarieties that relates the self-intersection number and intersection number with the canonical divisor to the Euler characteristic. Next, Proposition 3.2 asserts that the set of HWZ subvarieties in X has a natural topology that gives C a neighborhood that is homeomorphic to the zero locus of a smooth map from a neighborhood of the origin in $\mathbb{R}^N$ to $\mathbb{R}^n$ for a suitable choice of integers N and n. Here, $\mathbb{R}^N$ and $\mathbb{R}^n$ naturally appear as kernel and cokernel of a Fredholm operator on C. Finally, Proposition 3.6 provides a geometric formula for the index, $I_C \equiv N - n$, of this Fredholm operator.

(a) The Euler characteristic of C

The Euler characteristic of any embedded, compact, connected, pseudoholomorphic submanifold $C \subset X$ is determined via the adjunction formula from the class, $[C]$ of C in $H_2(X;\mathbb{Z})$. In this regard, remember that the symplectic form pulls back without zeros to C and so endows C with a canonical orientation. In any event, the adjunction formula reads:

$$-\chi(C) = \langle e, [C]\rangle + \langle c_1, [C]\rangle, \tag{3.1}$$

where, $\langle\, ,\rangle$ denotes the pairing between cohomology and homology, $e \in H^2(X)$ is the image of the Poincare dual to $[C]$, and $c_1 \in H^2(X)$ is the first Chern class of the canonical line bundle $K \equiv \wedge^2 T^{1,0}X$. Here, $T^{1,0}X$ is the J–holomorphic part of the complexified cotangent bundle of X.

Now, suppose that C is required only to be an irreducible subvariety whose singularities are purely transversal double points with local intersection number 1. In particular, C is the image of a connected surface, C_0, via an immersion. Let m_C denote the number of double points in X. The corresponding adjunction formula in this case reads:

$$-\chi(C_0) = \langle e, [C]\rangle + \langle c_1, [C]\rangle - 2m_C. \tag{3.2}$$

In the general case where C is irreducible and has singularities other than transverse double points, there is still an adjunction formula which gives the Euler characteristic of the smooth model for C. This is to say that C is the image in X of a compact, complex curve, C_0, via a pseudoholomorphic map which is an embedding off of a finite set in C_0. And, the Euler characteristic of C_0 is given by the left-hand side of (3.2) where m_C now denotes the number of double points in a symplectic deformation of the map from C_0 into X

which immerses C_0 with only transversal double points self intersections that have local intersection number 1. Indeed, the existence of such a deformation follows from the fact that the singularities of a pseudoholomorphic variety are essentially those of a complex curve in $\mathbb{C}^2$ as shown by [16]. In any event, make such a deformation and then (3.2) applies.

The next order of business is to introduce a version of the adjunction formula that applies to a non-compact pseudoholomorphic subvariety $C \subset X$. In this case, the right-hand sides of (3.1) and (3.2) are presently meaningless as $[C]$ is in $H_2(X; X - X_0)$ while both e and c_1 are in $H^2(X)$. However, a pairing of a class $\alpha \in H^2(X;\mathbb{Z})$ with $[C]$ can be unambiguously defined with the choice of a suitably constrained section of the restriction to C of the complex line bundle $E_\alpha \to H$ with first Chern class α. In particular, the section must have a compact zero set on C. With such a section chosen, a generic, compactly supported perturbation produces a section which vanishes transversely at a finite set of points in the smooth part of C. Then, the count of these points with the standard ± 1 weights defines the pairing $\langle \alpha, [C]\rangle$. Note that this number is unchanged when the chosen section of E_α is deformed through sections whose zero sets all lie in a fixed, compact subset of C.

To put the just described count definition for $\langle \alpha, [C]\rangle$ in a slightly larger context, note first that the chosen section of $E_\alpha|_C$ can be extended as a section of E_α over the whole of X. The zero set of this extended section then carries a fiducial two-dimensional, relative homology class that represents the Poincare dual of α. For example, if the original section over C and its extension to X are chosen to have transversal zero set, then this relative two-dimensional homology class is the fundamental class of the zero set. In this case, the pairing $\langle \alpha, [C]\rangle$ is simply the intersection number of C with the zero set of the extended section.

This definition of $\langle \alpha, [C]\rangle$ needs some elaboration when $\alpha \neq e$ and $[C]$ is e's Poincare dual. To proceed with this elaboration, remark first that when C is embedded in X, then E_e is defined by its transition function over the intersection of two coordinate patches. The first coordinate patch is $X - C$. The second is the image in X via the metric's exponential map of a certain open disk bundle, N_0, in the normal bundle of C. Here, the fiber radius of N_0 varies smoothly over C to ensure that the exponential map's restriction is an embedding. This understood, identify N_0 with its exponential map image. Now, declare $E|_{X-C}$ to be the trivial bundle $(X - C) \times \mathbb{C}$ and declare $E|_{N_0}$ to be the pull-back via projection to C of C's normal bundle. Thus, $E|_{N_0 - C}$ has a canonical section and thus a canonical trivialization with which to identify it with the restriction to $(N_0 - C)$ of $(X - C) \times \mathbb{C}$.

In the case where C is not embedded, there are perturbations of C in any given neighborhood its singular points that result in an embedded, oriented submanifold. Choose such a perturbation and use the resulting submanifold to define E_e as a proxy for C.

With the preceding understood, a three part digression is in order to define the appropriate sections over C of E_e and the canonical line bundle K.

Part 1 This first part of the digression simply recalls a definition from Proposition 2.2. According to Propositions 2.2 and 2.3, each constant and large $|s|$ slices of $C \cap (X - X_0)$ consists of a finite union of embedded circles that converges in the C^∞ topology as $s \to \infty$ to a finite union of closed Reeb orbits on ∂X_0. Remember that this set of orbits, Γ, is called *the limit set* for C. In particular, each end of C defines an element in Γ and each element in Γ corresponds via the aforementioned limit to one or more ends of C.

Part 2 This part of the digression specifies the section of $E_e|_C$ to be used when defining $\langle e, [C]\rangle$. In this regard, it is enough to specify the section over C's intersection with the $|s| \geq s_0$ portion of $X - X_0$ for any $s_0 \gg 0$ and so this is the purpose of the subsequent discussion. As previously noted, once such a definition is made, then all extensions of this section to the remainder of C give the same count for the $\langle e, [C]\rangle$. Thus, the focus below is the definition of a nowhere zero section of the restriction of C's normal bundle to the large $|s|$ portion of C.

The task at hand begins with three remarks. The first remark is that Proposition 2.2 asserts that when s_0 is large, then C's intersection with $|s| \geq s_0$ portion of $X - X_0$ is an embedded submanifold with boundary. The second remark is that with s_0 so chosen, then the restriction of $E_e|_C$ to this intersection is to be identified with C's normal bundle. Here is the final remark: It follows from Propositions 2.2 and 2.3 that when s_0 is large, then a specification of a nowhere zero section of the restriction to C's intersection with the $|s| \geq s_0$ portion of $X - X_0$ of Cs normal bundle is determined, up to homotopy through non-vanishing sections, by a nowhere zero section of the normal bundle in ∂X_0 to C's limit set.

To elaborate on these last remarks, note first that a component, $M = S^1 \times S^2$, of ∂X_0 has two-dimensional homology whose generator is the image of the fundamental class, $[S^2]$, of any 2–sphere of the form $\{\text{point}\} \times S^2 \subset S^1 \times S^2$. In this regard, these 2–spheres are oriented by the form $\sin\theta\, d\theta \wedge d\varphi$. This said, then the pairing of e with $[S^2]$ determines, up to bundle equivalence, the restriction of E_e to M and thus to $[s_0, \infty) \times M$. In the context at hand, this pairing is equal to the intersection number between C and any copy of $\{\text{point}\} \times S^2$ in $[s_0, \infty) \times M$.

A more explicit formula for this intersection number is available. However, a digression is required before stating this formula. This digression explains how the large $|s|$ asymptotics of any given end, E, of C can be characterized in part by a suitable pair, (p, p'), where p is an integer and p' is either another integer, or one of the symbols $+$ or $-$. Here, the pair (p, p') is deemed suitable when the following requirement is met: If $p' = +$ or $-$, require $p < 0$. If p' is an integer and $p < 0$, require that $|p'/p| > \sqrt{3}/\sqrt{2}$; and if p' is an integer with $|p'/p| < \sqrt{3}/\sqrt{2}$, require that $p > 0$.

The meaning of (p, p') is as follows: If $p' = +$ or $-$, then the large and constant $|s|$ slices of the end in question converge to the Reeb orbit with $\theta = 0$ or $\theta = \pi$, respectively. This understood, then $m \equiv |p|$ gives the multiplicity by which these large $|s|$ circles cover the Reeb orbit. If p' is an integer, let $m \geq 1$ denote the greatest common divisor of p and p'. Then, the pair $(p/m, p'/m)$ determine a circles worth of closed Reeb orbits as dictated in (1.8), and the large $|s|$ slices of the end in question are asymptotic to one of the latter. Meanwhile, the integer m gives the multiplicity by which these large $|s|$ slices of the end cover the Reeb orbit.

With the digression complete, remark that the intersection number in question is the sum of the first components from the integer pairs (p, p') that come from the ends of C that lie in the component given by M of $X - X_0$.

Now, suppose that the large $|s|$ slices of the end, E, is asymptotic, in the manner described by Propositions 3.2 and 3.3, to a Reeb orbit $\gamma \subset M$. What follows describes how nowhere zero sections of γ's normal bundle in M produce nowhere zero sections of the large $|s|$ portion of E's normal bundle. To begin the story, let N_γ denote a small radius disk bundle in the normal bundle to γ in M. Here, the radius should be such that the metric's exponential map embeds N_γ in M. This understood, identify N_γ with its exponential map image. With these identifications made, then, Propositions 3.2 and 3.3 provide $s_1 > s_0$ and a description of the $|s| > s_1$ portion of E as the image over $(s_1, \infty) \times \gamma$ of a multi-valued section of N_γ. This is to say that there is a degree m covering map $\varphi: S^1 \to \gamma$ and a section, η, over $(s_1, \infty) \times S^1$ of $\varphi^* N_\gamma$ such that the composition of η with the tautological map $\hat{\ }: \varphi^* N_\gamma \to N_\gamma$ maps $(s_1, \infty) \times S^1$ diffeomorphically onto the $|s| > s_1$ portion of E. This composition, of η with $\hat{\ }$, identifies the normal bundle to the large $|s|$ portion of E with that of the image of η in $\varphi^* N \to (s_1, \infty) \times S^1$. Meanwhile, the latter is canonically isomorphic to $\varphi^* N$s pullback over itself via its defining projection to $(s_1, \infty) \times \gamma$. Following this chain of bundle isomorphisms produces a nowhere zero section of the large $|s|$ portion of E's normal bundle from a nowhere zero section of the Reeb orbit's normal bundle in M.

Now, to provide such a section of γ's normal bundle, consider first the case where the component M has orientable z–axis line bundle. Also, suppose that $\theta = \theta_0 \notin \{0, \pi\}$ on γ. Then, a nowhere zero section of γ's normal bundle is defined by the vector field ∂_θ along γ.

If $\theta_0 = 0$ or π on γ, the functions $(x_1 \equiv \sin\theta\cos\varphi,\ x_2 \equiv \sin\theta\sin\varphi)$ are zero on γ and the triple (t, x_1, x_2) are good coordinates for M near γ. With this understood, the vector field tangent to any line through the origin in the (x_1, x_2) plane defines along γ a nowhere vanishing section of γ's normal bundle.

Now consider the case where $M \subset \partial X_0$ defines an end of X whose z–axis line bundle is non-orientable. The first order of business is to write out the 2–1 covering map, $\hat{\psi}\colon S^1 \times S^2 \to M\ (= S^1 \times S^2)$ whose deck transformations are generated by ι in (2.1). To do so, view S^2 as the unit sphere in $\mathbb{R}^3$ and introduce Cartesian coordinates $(t, x_1 = \sin\theta\cos\varphi,\ x_2 = \sin\theta\sin\varphi,\ x_3 = \cos\theta)$ on the domain $S^1 \times S^2$. Let (t', x_1', x_2', x_3') denote the analogous coordinates for the range. Then, the map is defined so that Cartesian coordinates of $\hat{\psi}(t, x_1, x_2, x_3)$ are

$$\begin{aligned} t' &= 2t, \\ x_1' &= x_1, \\ x_2' &= \cos(t)x_2 + \sin(t)x_3, \\ x_3' &= -\sin(t)x_2 + \cos(t)x_3. \end{aligned} \tag{3.3}$$

With $\hat{\psi}$ defined, turn back to the task at hand. In this regard, each component of the inverse image of a closed Reeb orbit under the map $\hat{\psi}$ is either described as in Step 4 of the proof of Proposition 2.2, or else lies in a constant t slice where $\cos^2\theta = 1/3$. In any event, fix attention on a closed Reeb orbit $\gamma \subset M$. Here, there are two cases to consider, depending on whether $[\gamma]$ is or is not a generator of $H_1(M; \mathbb{Z})$.

Consider first the case where γ is not an integral generator of H_1. Then γ will necessarily be an even multiple of a generator. (This follows from the form of $\hat{\psi}$ in (3.3) and the fact that θ is constant on any inverse image of γ.) As an even multiple of a generator of $H_1(M; \mathbb{Z})$, the inverse image under the map in (3.3) of γ must have two components where each is mapped diffeomorphically by (3.3) onto γ. With this said, it follows that a nowhere vanishing section of γ's normal bundle in M is defined via the following two-step procedure: First, take a section of the normal bundle in $S^1 \times S^2$ of one of γ's inverse images under (3.3) by the rules which were just described for the orientable z–axis line bundle case. Second, use (3.3) to push this section down to M as a section of the normal bundle to γ.

Next, consider the case where γ generates $H_1(M;\mathbb{Z})$. In this case, the inverse image of γ under (3.3) must be connected and (3.3) maps this inverse image as a 2–1 covering map onto γ. Moreover, as θ = constant on such an inverse image, it follows that the functions x_2' and x_3' from (3.3) vanish on γ and together with t' define good coordinates for M near γ. With this understood, the vector field tangent to any line through the origin in the (x_2', x_3') plane defines along γ a nowhere zero section of γ's normal bundle.

Part 3 This part of the digression defines the desired section over C of the canonical bundle K and thus gives meaning to $\langle c_1, [C]\rangle$. In this regard, note that it is again sufficient to specify the section only over C's intersection with each component of the $|s| \geq s_0$ part of $X - X_0$ for very large s_0. In the discussion below, only the case of a concave component will be considered, as the convex case can be directly obtained from the latter.

With the preceding understood, consider first the promised section of K over C's intersection with $[s_0, \infty) \times M$ where $M \subset \partial X_0$ is a concave component with orientable z–axis line bundle. In this regard, take s so large that $C \cap ([s_0, \infty) \times M)$ is a union of cylindrical components with each defining in the limit a particular loop in C's limit set. The specification of a section of K over a given component now depends on whether the corresponding loop in the limit set has $\theta \in \{0, \pi\}$ or not. Consider first the case where $\theta \notin \{0, \pi\}$ on the loop. Near such a component, the complex valued 1–forms $dt + ig^{-1}df$ and $\sin^2\theta\, d\varphi + ig^{-1}dh$ span $T^{1,0}X$ and so their wedge product, $(dt + ig^{-1}df) \wedge (\sin^2\theta\, d\varphi + ig^{-1}dh)$ gives a section of K. (Note that this section is defined over the whole of $[0, \infty) \times M$, and its zero set is the locus where $\theta \in \{0, \pi\}$.)

To consider the case where a limit set loop of a component of $C \cap ([s_0, \infty) \times M)$ has $\theta = 0$ or π, first introduce the 'Cartesian coordinates' $(x_1 \equiv \sin\theta \cos\varphi, x_2 \equiv \sin\theta \sin\varphi)$. These are smooth coordinates near the $\theta = 0$ and $\theta = \pi$ loci. Furthermore, $(dt + ids) \wedge (dx_1 - idx_2)$ specifies a non-vanishing section of K over the $\theta = 0$ locus while $(dt + ids) \wedge (dx_1 + idx_2)$ plays the same role for the $\theta = \pi$ locus. Now, the required section of K over a component of $C \cap ([s_0, \infty) \times M)$ with limit set loop in the $\theta = 0$ locus is obtained by restriction of any smooth extension of the section $(dt + ids) \wedge (dx_1 - idx_2)$ to a tubular neighborhood of this locus. The required section of K in the case where the limit set loop lies in the $\theta = \pi$ locus is defined analogously.

Now consider the promised section of K over C's intersection with $[s_0, \infty) \times M$ where $M \subset \partial X_0$ is a concave component with non-orientable z–axis line bundle. For this purpose, it is important to keep in mind that the canonical line bundle of $[0, \infty) \times M$ pulls up via the map in (3.3) to the canonical bundle just studied for the orientable z–axis case.

With the preceding point taken, fix s large enough to insure that $C \cap ([s_0, \infty) \times M)$ is a union of cylindrical components with each defining in the limit a particular loop in C's limit set. Now, the inverse image of this loop has θ locally constant and there are two cases, as before, depending on whether $\theta \in \{0, \pi\}$ or not on this inverse image. Consider the latter case first. If the constant s slices of the end in C converge as $s \to \infty$ as a multiple cover of a closed Reeb orbit with two inverse images under the map in (3.3), use the push-forward from one of them of the section $(dt + ig^{-1}df) \wedge (\sin^2\theta\, d\varphi + ig^{-1}dh)$. If the constant s slices of the end in question converge as $s \to \infty$ as a multiple cover of a closed Reeb orbit with a single inverse image under (3.3), proceed as follows to obtain the appropriate section: Note first that the section $(dt + ig^{-1}df) \wedge (\sin^2\theta\, d\varphi + ig^{-1}dh)$ changes sign under the involution in (2.1) but its product with e^{it} does not. Therefore, $e^{it}(dt + ig^{-1}df) \wedge (\sin^2\theta\, d\varphi + ig^{-1}dh)$ is the pull-back via (3.3) of a section over $[0, \infty) \times M$ of the canonical bundle of X. Use the latter section for the definition of $\langle c_1, [C]\rangle$.

In the case where $\theta \in \{0, \pi\}$ on the inverse image of the limit loop, then the inverse image under the map in (3.3) of the component of $C \cap ([s_0, \infty) \times M)$ in question has two components, one with θ very nearly zero and the other with θ very nearly equal to π. With this point understood, a section on the component in question of $C \cap ([s_0, \infty) \times M)$ is obtained as follows: Use ψ to push forward any nowhere zero section on a tubular neighborhood of the $\theta = 0$ locus whose restriction to the $\theta = 0$ locus equals $(dt + ids) \wedge (dx_1 - idx_2)$.

With the digression now over, return to the original subject, which is an adjunction formula for the smooth model for C. In this regard, note that Proposition 2.2 asserts that C is the image of a complex curve, C_0, with cylindrical ends via a pseudoholomorphic map which is an embedding off of a finite set of points. Moreover, Propositions 2.2 and 2.3 imply that the Euler characteristic of C_0 can be computed as the usual algebraic count of the zeros of a section of C_0's tangent bundle if, on the large $|s|$ portion of C_0, this same tangent vector pushes forward to $\mathbb{R}$ via the composition of the pseudoholomorphic map with the projection $\mathbb{R} \times M \to \mathbb{R}$ as a nowhere zero multiple of the vector ∂_s. With this last point understood, it then follows from the given choice of sections of $E_e|_C$ and K for the respective definitions of $\langle e, [C]\rangle$ and $\langle c_1, [C]\rangle$ that $\chi(C_0)$ is given by (3.2) where m_C is defined as in the case where C was compact.

Indeed, the argument for (3.2) in this case amounts to an essentially verbatim recapitulation of a standard proof for the compact case as given in [5].

The following proposition summarizes the conclusions of the preceding adjunction formula discussion:

Proposition 3.1 *Let X be as described in the introduction to Section 1. Let $C \subset X$ be an irreducible, pseudoholomorphic subvariety, and let C_0 denote the smooth model curve for C. Then $\chi(C_0)$ is given by*

$$-\chi(C_0) = \langle e, [C]\rangle + \langle c_1, [C]\rangle - 2m_C,$$

where:

- *The cohomology-homology pairings are defined as described above.*
- *m_C denotes the number of double points of any perturbation of the defining map from C_0 onto C which is symplectic, an immersion with only transversal double point singularities with locally positive self-intersection number, and which agrees with the original on the complement of some compact set.*

If C is immersed, then this formula can be rewritten as

$$-\chi(C_0) = \text{degree}(N) + \langle c_1, [C]\rangle,$$

where degree(N) is the degree of the normal bundle to the immersion as defined using a section that is non-vanishing at large $|s|$ and is homotopic at large $|s|$ through non-vanishing sections to that described in Part 2, above.

Some readers may prefer the formula that writes $-\chi(C_0)$ in terms of degree(N) because the pairing $\langle e, [C]\rangle$ is not invariant under deformations of C unless C's large $|s|$ slices converge with multiplicity 1 to a set of distinct, limit Reeb orbits.

(b) A topology on the set of pseudoholomorphic subvarieties

The first task is the introduction of a certain topology on the set me, $\mathfrak{M}_{e,\chi}$ of irreducible, pseudoholomorphic subvarieties in X with fundamental class Poincaré dual to the given class $e \in H^2(X;\mathbb{Z})$ and with the given number χ equal to the Euler characteristic of the model curve C_0. The topology in question comes from the metric for which the distance between a pair $C, C' \in \mathfrak{M}_{e,\chi}$ is

$$\sup_{x\in C}\text{distance}(x, C') + \sup_{x'\in C'}\text{distance}(C, x')\,. \tag{3.4}$$

Given this definition, then the next order of business is a structure theorem for a neighborhood in $\mathfrak{M}_{e,}$ of any given subvariety. The following proposition summarizes the story:

Proposition 3.2 *Let $C \in \mathfrak{M}_{e,}$. There exists a Fredholm operator D and a homeomorphism from a neighborhood of C in $\mathfrak{M}_{e,}$ to the zero set of a smooth map from a ball in the kernel of D to the cokernel of D.*

The proof of this proposition is given in the next subsection.

The description of D is simplest when C is compact and the associated pseudoholomorphic map $\varphi\colon C_0 \to X$ is an immersion. In this case, there is a well defined 'normal bundle', $N \to C_0$ that is a real 2–plane bundle whose restriction to any open $K \subset C_0$ embedded by φ is the φ–pullback of the normal bundle to $\varphi(K)$. The almost complex structure on X endows N with the structure of a complex line bundle over C_0, and the associated Riemannian metric from X can then be used to give N the structure of a Hermitian line bundle with a holomorphic structure. The induced d–bar operator on the space of sections of N will be denoted by $\bar\partial$. (The $\bar\partial$ operator used here is twice the usual $\partial/\partial\bar z$.)

In the case at hand, the operator in Proposition 3.2 is the first order, $\mathbb{R}$–linear operator from $C^\infty(N)$ to $C^\infty(N\otimes T^{0,1}C_0)$ that sends a section ξ of N to

$$D\xi \equiv \bar\partial\xi + \nu\xi + \mu\bar\xi. \tag{3.5}$$

Here ν and μ are respective sections of $T^{0,1}C_0$ and $N^2\otimes T^{0,1}C_0$ that are determined by the 1–jet of the almost complex structure J along C. Although the kernel dimension may depend on ν and μ, the index $\equiv$ dim(kernel) – dim(cokernel) of D does not. In fact, the index is the same as that of $\bar\partial$ (as an $\mathbb{R}$–linear operator) namely: $\mathrm{index}(D) = 2\,\mathrm{degree}(N) + \chi(C_0)$. As $\chi(C_0)$ obeys (3.2) and $\mathrm{degree}(N) = \langle e,[C]\rangle - 2m_C$, this index can also be written in various equivalent ways, for example:

$$\begin{aligned}&\mathrm{index}(D) = \langle e,[C]\rangle - \langle c_1,[C]\rangle - 2m_C.\\ &\mathrm{index}(D) = -\chi(C_0) - 2\langle c_1,[C]\rangle.\end{aligned} \tag{3.6}$$

When $C \in \mathfrak{M}_{e,}$, still compact, is not immersed, Proposition 3.2's operator D is more complicated. What follows is a brief description of this new operator. The definition of D requires, as a preliminary step, the introduction of a first-order differential operator, $\underline{D}$, which sends a section of $\varphi^*T_{1,0}X$ to one of $\varphi^*T_{1,0}X \otimes T^{0,1}C_0$. Here and below, $T_{1,0}$ denotes the holomorphic part of the corresponding complexified tangent bundle. The operator $\underline{D}$ differs from the corresponding $\bar\partial$ by a zeroth order, $\mathbb{R}$–linear multiplication operator and thus has the same schematic form as depicted on the right-hand side of (3.5). This operator $\underline{D}$ is defined so that its kernel provides the vector space of deformations of the map φ which remain, to first-order, pseudoholomorphic as maps from C_0 into X. In this regard, $\underline{D}$ is not quite the sought after operator as its use in Proposition 3.2 would allow only those deformations which preserve the induced complex structure on the image curve. The point is that the cokernel of $\underline{D}$ is too large when $\chi < 0$. Meanwhile, when $\chi \geq 0$, the kernel of $\underline{D}$ is too big as it contains deformations which come from holomorphic automorphisms of C_0.

To address these problems, introduce, first of all, the usual $\bar\partial$ operator which sends sections of $T_{1,0}C_0$ to $T_{1,0}C_0 \otimes T^{0,1}C_0$. The kernel of this operator, V, is trivial when $\chi(C_0) < 0$, but not trivial otherwise. (Its dimension over $\mathbb{C}$ is $\chi + 1$.) Meanwhile, let V denote the cokernel of this same version of $\bar\partial$. The complex dimension of V is $-3\chi/2$ when $\chi < 0$, one for a torus and zero for a sphere. Fix some favorite subspace of smooth and compactly supported sections of $T_{1,0}C_0 \otimes T^{0,1}C_0$ that projects isomorphically to the cokernel of $\bar\partial$ and identify the latter with V.

Next, remark that as φ is pseudoholomorphic, its differential provides a $\mathbb{C}$–linear map, $\partial\varphi\colon T_{1,0}C_0 \to \varphi^* T_{1,0}X$ and thus one, also denoted by $\partial\varphi$, from $T_{1,0}C_0 \otimes T^{0,1}C_0$ to $\varphi^* T_{1,0}X \otimes T^{0,1}C_0$. In particular, note that the appropriate version of $\partial\varphi$ sends V and V injectively into the respective kernel and target space of $\underline{D}$. With this understood, then $\underline{D}$ induces an operator, the desired D, that maps $C^\infty(C_0; \varphi^* T_{1,0}X)/\partial\varphi(V)$ to the L^2 complement in $C^\infty(C_0; \varphi^* T_{1,0}X \otimes T^{0,1}C_0)$ of $\partial\varphi(V)$. The conclusions of Proposition 3.2 hold for this D when C is not immersed.

By the way, note that the index of this new D is still given by the formula in (3.6) where m_C is now interpreted as in Proposition 3.1.

The remainder of this subsection describes Proposition 3.2's D in the case where $C \subset X$ is a non-compact, pseudoholomorphic subvariety. The discussion has been divided into three parts.

Part 1 As the story is simplest when C is immersed, this condition will be assumed until the final part. In this regard, note that the removal of the immersion assumption requires no new technology since a pseudoholomorphic subvariety is, in any event, embedded where $|s|$ is large on $X - X_0$.

The first remark is that the operator D is formally the same as that which is described in (3.5). In particular, N is defined as before, while the hermitian structure on N and the Riemannian structure on TC_0 are both induced by the Riemannian metric on X. However, there is some subtlety with the range and domain of D. In particular, these are defined as follows: First, a very small $\delta > 0$ must be chosen. There is an upper bound to the choice which is determined by the properties of C^0s limit set. In any event, with δ chosen, the domain of D is the (Hilbert space) completion of the set of smooth sections of N for which

$$\int e^{-\delta|s|}(|\nabla\xi|^2 + |\xi|^2) \tag{3.7}$$

is finite; moreover, (3.7) defines the square of the relevant norm. Here, and below, the function s which was originally specified only on $X - X_0$ has been extended to the remainder of X as a smooth function. By the way, the integration measure in (3.7) and in subsequent integrals is the area form from the Riemannian metric on C that is induced by the metric on X. Of course, given that C is pseudoholomorphic, this measure is the same as that defined by the restriction to C of $|\omega|^{-1}\omega$.

Meanwhile, the range space for D is the completion of the set of smooth sections of N for which

$$\int e^{-\delta|s|}|\eta|^2 \tag{3.8}$$

is finite; in this case (3.8) gives the square of the relevant norm. Let $\mathfrak{L}_1$ denote the just defined domain Hilbert space for D and let $\mathfrak{L}_0$ denote the range.

With the preceding understood, here is the key lemma:

Lemma 3.3 *If δ is positive, but sufficiently small, then the operator D as just described extends as a bounded, Fredholm operator from $\mathfrak{L}_1$ to $\mathfrak{L}_0$. Moreover,*

- *The index of D as well as the dimensions of the kernel and cokernel of D are independent of δ.*
- *In fact, the kernel of D is the vector space of sections of ξ of N with $D\xi = 0$ and $\sup|\xi| < \infty$.*
- *Polarize the norm square in (3.8) to obtain an inner product on $\mathfrak{L}_0$ and represent the cokernel of D as the orthogonal complement to D's image. Then, multiplication by $e^{-\delta|s|}$ identifies cokernel(D) with the space of sections η of $N \otimes T^{0,1}C_0$ with $\int|\eta|^2 < \infty$ and which are annihilated by the formal L^2 adjoint, $D^\dagger$, of D.*
- *There exists ε_1 which is independent of δ and which has the following significance: Let P_1 be an $\mathbb{R}$–linear bundle homomorphism from $T C_0$ to $\mathrm{Hom}(N; T^{0,1}C_0)$ with norm $|P_1| < \varepsilon_1$. Also, let P_0 be an $\mathbb{R}$–linear*

bundle homomorphism from N to $N \otimes T^{0,1}C_0$. Moreover, suppose that $e^{2\delta|s|}(|P_0|+|P_1|)$ is bounded. Then, $D+P_1(\nabla)+P_0$ extends as a bounded, Fredholm operator from $\mathfrak{L}_1$ to $\mathfrak{L}_0$ whose index is the same as D's.

This lemma is also proved below.

Part 2 This part of the discussion describes the operator D in the case when C is not immersed. But keep in mind that there exists s_0 such that C's intersection with the $|s| > s_0$ portion of $X - X_0$ is nonetheless a disjoint union of embedded cylinders which intersect the constant $|s|$ slices of $X - X_0$ transversely.

As in the case where C is compact, the operator D is constructed by first introducing the operator $\underline{D}$ that is defined just as in the compact case. Thus, $\underline{D}$ maps sections of $\varphi^* T_{1,0}X$ to those of $\varphi^* T_{1,0}X \otimes T^{0,1}C_0$. By analogy with the case where C_0 is immersed, the domain for $\underline{D}$ is the completion of the space of sections of $\varphi^* T_{1,0}X$ for which the expression in (3.7) is finite; and (3.7) defines the square of the norm for this completion. Meanwhile, the range of D is the completion of the space of sections of $\varphi^* T_{1,0}X \otimes T^{0,1}C_0$ for which (3.8) is finite, and (3.8) defines the square of the norm for the completion in this case. The domain Hilbert space will be denoted by $\underline{\mathfrak{L}}_1$ and the range by $\underline{\mathfrak{L}}_0$.

Also needed in this discussion are vector spaces which play the role here that the kernel and cokernel of $\overline{\partial}\colon T_{1,0}C_0 \to T_{1,0}C_0 \otimes T^{0,1}C_0$ play in the compact case. In this regard, the two versions of the linear map $\partial\varphi$ are still available as φ is, in any event, pseudoholomorphic. With this understood, let V denote the vector space of sections $T_{1,0}C_0$ which are annihilated by $\overline{\partial}$ and for which (3.7) is finite. Meanwhile, let V denote the space of sections of $T_{1,0}C_0 \otimes T^{0,1}C_0$ for which (3.8) is finite and whose product with $e^{-\delta|s|}$ is annihilated by the formal, L^2 adjoint of $\overline{\partial}$.

The linear map $\partial\varphi$ maps V injectively into $\underline{D}$'s kernel in $\underline{\mathfrak{L}}_1$, and it maps V injectively into $\underline{\mathfrak{L}}_0$. Thus, $\underline{D}$ induces an operator, D, from $\mathfrak{L}_1 \equiv \underline{\mathfrak{L}}_1/\partial\varphi(V)$ to $\mathfrak{L}_0 \equiv \underline{\mathfrak{L}}_0^T$ where $\underline{\mathfrak{L}}_0^T$ denotes the orthogonal complement to $\partial\varphi(V)$. One then has:

Lemma 3.4 *Whether or not C is immersed, the assertions of Lemma 3.3 hold with the operator $D\colon \mathfrak{L}_1 \to \mathfrak{L}_0$ as just described in the preceding paragraph.*

The proof of this lemma will be left to the reader in as much as its proof is a straightforward marriage of the arguments given below for Lemma 3.3 with those given for the compact case. Note that were all closed Reeb orbits in C's limit set isolated, then this lemma more or less restates results from [13].

Part 3 The remainder of this subsection contains the following proof.

Proof of Lemma 3.3 Although the lemma can be proved by modifying the analysis in [13], an abbreviated argument will be provided. The argument below has two steps.

Step 1 The argument presented here for the proof is based on some general facts about elliptic differential operators on manifolds with cylindrical ends. These facts were originally established in [14], but see also [23] which analyses a first order elliptic operator in a context that has many formal analogies with the context here. What follows here is simply a summary of those facts which are relevant to the case at hand.

To set the stage for the subsequent discussion, consider a non-compact manifold, Y, which comes with an open set having compact closure, and a diffeomorphism from the complement of this open set to $[-1, \infty) \times Z$. This diffeomorphism will be used to identify $[-1, \infty) \times Z$ as a subset of Y. With this understood, let $s: Y \to [-2, \infty)$ be a smooth function which restricts to $[0, \infty) \times Z$ as the projection to the first factor.

Now, suppose that D is a first-order, elliptic operator on Y taking sections of one vector bundle, E, to those of a second, E'. Suppose further that both of these bundles are provided with fiber metrics and metric compatible connections. Parallel transport via these connections along the paths $[0, \infty) \times \{\text{point}\} \subset [0, \infty) \times Z$ then identifies these bundles with their respective restrictions to $\{0\} \times Z$, and this identification will be explicit in what follows. Note that the latter identifications make the fiber metrics s–independent on $[0, \infty) \times Z$. Use ∇ to denote the covariant derivative of either connection.

Next, suppose that s is the Euclidean coordinate on the $[0, \infty)$ factor of $[0, \infty) \times Z$ and that the restriction of D here has the form:

$$D = A_0 \partial_s + L_0 + \mathfrak{e}_1(\nabla) + \mathfrak{e}_0 \tag{3.9}$$

where:

- A_0 is a s–independent, isometric isomorphism between E and E'.
- L_0 is a s–independent, first-order operator which differentiates only along vectors which are tangent to Z. Require that $A_0^{\dagger} L_0 = L_0^{\dagger} A_0$.

- σ_1 is vector bundle homorphism from $T^*([0,\infty)\times Z)$ into $\mathrm{Hom}(E, E')$. Here, it is required that $|\sigma_1|$ should be small, where a precise upper bound, ε_1, depends on the norm of the symbol of L_0. However, $\varepsilon_1 < 1/10$ in any event.
- σ_0 is a section of $\mathrm{Hom}(E, E')$.
- There exists $\varepsilon > 0$ such that $e^{\varepsilon s}(|\sigma_1| + |\sigma_0|)$ is bounded on $[0,\infty)\times Z$. (3.10)

To define appropriate domain and range spaces for D, it is necessary to first choose a Riemannian metric on X whose restriction to $[0,\infty)\times Z$ is the product metric constructed using ds^2 on the first factor and an s–independent metric on the second.

The definition of the domain and range for D also requires the choice of $\delta \in \mathbb{R}$. With δ chosen, the domain Banach space, $\mathfrak{L}_1$, is obtained by completing the set of sections of E for which (3.7) is finite using the expression in (3.7) for the square of the norm after substituting p for s. Meanwhile, the range Banach space is obtained by completing the set of section of E' for which the corresponding version of (3.8) is finite using the latter for the square of the norm.

Under, the preceding assumptions, here is the fundamental conclusion from [14]:

Lemma 3.5 *If $\delta \neq 0$, but $|\delta|$ is sufficiently small, then D defines a Fredholm map from $\mathfrak{L}_1$ to $\mathfrak{L}_0$. Moreover, the following are true:*

- *Each element in* $\mathrm{kernel}(D)$ *has a well defined $s \to \infty$ limit on* $[0,\infty)\times Z$. *In addition:*
  - (a) *This limit is zero when $\delta < 0$, and it lies in the kernel of L_0 when $\delta > 0$.*
  - (b) *When $\delta > 0$, these limits define a homomorphism ∂:* $\mathrm{kernel}(D) \to \mathrm{kernel}(L_0)$.
  - (c) *Let $\psi \in$* $\mathrm{kernel}(D)$ *and let $k \in \{1, 2, \ldots\}$. Then $e^{|\delta|s/2}|\nabla^k\psi|$ is square integrable on Y; and if $|\psi| \to 0$ as $s \to \infty$, then $e^{|\delta|s/2}|\psi|$ is also square integrable on Y.*
- *Identify the cokernel of D with the orthogonal complement to the image of D as defined by the inner product induced by the norm in (3.8) on $\mathfrak{L}_0$. With this identification understood, let $\eta \in$* $\mathrm{cokernel}(D)$ *and set $\eta' \equiv e^{-\delta s}\eta$. Then η' is annihilated by the formal L^2 adjoint of D it has a well defined $s \to \infty$ limit on* $[0,\infty)\times Z$. *In addition:*

(a) *This limit is zero when* > 0, *and it lies in the kernel of* L_0 *when* < 0.

(b) *When* < 0, *these limits define a homomorphism* $\_ :$ cokernel$(D) \to$ kernel(L_0).

(c) *Let* e^{-} *with* $\in$ cokernel(D) *and let* $k \in \{1, 2, \ldots\}$. *Then,* $e^{|\,|/2}|\nabla^k|$ *is square integrable on* Y; *and if* $|\,| \to 0$ *as* $\to \infty$, *then* $e^{|\,|/2}|\,|$ *is also square integrable on* Y.

If > 0, *then the images of* $\_$ *and* $\_\_$ *are orthogonal, complementary subspaces in* kernel(L_0).

With this last lemma understood, the proof of Lemma 3.3 is reduced to verifying that the conditions in (3.9) and (3.10) are satisfied in the present case.

Step 2 This step in the proof verifies for certain cases that the operator D which appears in Lemma 3.3 obeys the conditions in (3.10). This is accomplished using the fact that when s_0 is large, then C's intersection with the $|s| \geq s_0$ part of $X - X_0$ is a disjoint union of cylinders where the constant s slice of each cylinder is very close to some closed Reeb orbit. Moreover, according to Proposition 2.3, as $|s| \to \infty$, this constant $|s|$ slice converges in the C^k topology for any k exponentially fast in $|s|$ to some multiple covering of the closed Reeb orbit. The implications of this observation are slightly different depending on whether the cylinder is in a component of $X - X_0$ which is concave or convex and has or does not have an orientable z–axis line bundle. In this regard, only the concave case is presented below as the argument for the convex case is identical save for changing various signs. With M now taken to define a concave end of X, the case with orientable z–axis line bundle is considered in this step, and the non-orientable z–axis line bundle case is considered in the next.

Until directed otherwise, assume that $M = S^1 \times S^2 \subset \partial X_0$ defines a concave end of X with orientable z–axis line bundle. Now, suppose that $\gamma \subset M$ is an element in the limit set for C. Here, there are three cases which are treated separately. The first case has $\theta = \theta_0$ on γ with $\theta_0 \notin \{0, \pi\}$ and with $\cos^2\theta_0 \neq 1/3$. In this case, a component of C which lies in a small radius tubular neighborhood of γ can be parameterized as (2.13). It then follows from (2.15) that D has the form of (3.9) where A_0 is the identity 2×2 matrix and where L_0 is the operator L in (2.16). Moreover, the fact that in (2.15) decays exponentially fast to zero implies that the requirements in (3.10) are met in this case.

Next, consider the case where γ is such that $\cos^2\theta_0 = 1/3$. In this case, the roles of (t, f) and (φ, h) can be switched in the discussion of Step 9 of Proposition 2.3 to find a parameterization of the part of C near $[s_0, \infty) \times \gamma$ by coordinates (τ, σ) for which D has the form in (3.9) where A_0 is the identity and L_0 is again given by (2.16), but with $\alpha' = 0$ and $\alpha = 2$. In this case, the coodinate τ takes values in an interval of the form $[\tau_0, \infty)$ while σ takes values in $[0, 2\pi m]$ where m is the number of times each large, but constant s slice of C wraps around a given tubular neighborhood of γ. In this parameterization, $h = \varepsilon e^{-2\tau}$ with ε either 1 or -1, and $\varphi = \sigma$. Meanwhile, the coordinates t and f are parameterized as $t = x(\tau, \sigma)$, $f = 3\varepsilon e^{-2\tau} w(\tau, \sigma)$ and where x and w are functions of τ and σ which are periodic in σ and which decay to zero exponentially fast as τ tends to infinity. Note that a column vector with x the top entry and w the bottom obeys a differential equation with the schematic form of (2.15).

The third case to consider is where θ_0 is either 0 or π on γ. The discussion here concerns solely the $\theta_0 = 0$ case as the $\theta_0 = \pi$ case has an identical story modulo some inconsequential sign changes. In this case, use of the parameterization in (2.14) finds that the operator D has the form in (3.9) using the version of L_0 in (2.16) that has $\alpha' = \alpha = \sqrt{3}/\sqrt{2}$. By the way, note that in this case, $\mathrm{kernel}(L_0) = 0$ because $\sqrt{6}$ is irrational.

Step 3 This step considers the form of D when the end of C in question lies in a concave end of X with non-orientable z–axis line bundle. Here, there are two cases to consider; they depend on whether the fundamental class of the corresponding element, γ, of C's limit set does not or does generate $H_1(S^1 \times S^2; \mathbb{Z})$.

In the case where the fundamental class of γ does not generate $H_1(S^1 \times S^2; \mathbb{Z})$, then the inverse image of γ via the map in (3.3) has two distinct components which differ in the sign of $\cos(\theta_0)$. Likewise, the inverse image of the relevant portion of C via the map in (3.3) has two components, one near each component of the inverse image of $[s_0, \infty) \times \gamma$. Choose one such component and parameterize it as in the previous step. The result gives an operator D of the form in (3.9) which obeys the constraints in (3.10).

Now consider the case where the fundamental class of γ does generate $H_1(S^1 \times S^2; \mathbb{Z})$. In this case, the inverse image of γ under the map in (3.3) is the circle, γ_0, with $\theta_0 = \pi/2$ and $\varphi = 0$. This is to say that the coordinates (x'_1, x'_2, x'_3) for the S^2 factor which appear in (3.3) are either $(1, 0, 0)$ or $(-1, 0, 0)$ on γ_0. Meanwhile, the corresponding inverse image of C will be very close to $[s_0, \infty) \times \gamma_0$. In the case where this inverse image has two components, choose one and parameterize it as in Step 2. The resulting expression for D will then have the form of (3.9) and obey the constraints in (3.10). In this regard, note that the inverse image here will have two components precisely when the constant $s > s_0$ circles in this end of C are even multiples of a generator of $H^1(M;\mathbb{Z})$.

Finally, suppose that the inverse image of the relevant part of C has just one component. In this case, C can be parameterized by coordinates $\;\in [\;_0; \infty)$ and $\;' \in [0; 2\;m]$ where m is an odd, positive integer. This parameterization writes the coordinates $(s; t'; x_2'; x_3')$ for this end of C as $(\;;\;'; a'; b')$ with a' and b' functions of and $'$ which are periodic in $'$ with period $2\;m$.

With these last points understood, it follows that the inverse image of C under the map in (3.3) can be parameterized by the same function and a function $\in [0; 2\;m]$ which writes the coordinates $(s; t; x_2; x_3)$ as $(\;;\;; a; b)$ where a and b are functions of and which obey

$$a(\;;\; + \;m) = -a(\;;\;) \text{ and } b(\;;\; + \;m) = -b(\;;\;): \tag{3.11}$$

Moreover, the column vector with top entry a and bottom entry b obeys an equation with the schematic form given by (2.15) with $' = 0$ and $= \sqrt{6}$. Finally, decays exponentially fast to zero as tends to infinity. By the way, note that the involution in (2.1) is realized on the inverse image curve by sending to $+ \;m$.

In any event, these last remarks imply that the operator D on C has the form given in (3.9) and satisfies the constraints in (3.10). Indeed, this is because the pullback of D to C's inverse image curve is as described in Step 3 for the $_0 = \;=2$ case. However, be forwarned that the domain and range Hilbert spaces on C, the spaces $\mathfrak{L}_0$ and $\mathfrak{L}_1$ in Lemma 3.5, pullback to the inverse image curve as subspaces of $\mathbb{R}^2$ valued functions over the inverse image curve which change sign when is changed to $+ \;m$. Here, D on C should be viewed as an operator on $\mathbb{R}^2$ valued functions by trivializing C's normal bundle using the restriction to C of the vector fields which are tangent to the x_2' and x_3' axis.

(c) The proof of Proposition 3.2

The strategy and most of the technical details for the proof follow those for the proof in the case where C is compact (see, for example, [18]). Certain special cases of the proposition also follow from the analysis in [13].

The argument given below for Proposition 3.2 has two parts. The first finds a ball B kernel(D), a smooth map, f, from B into the cokernel of D and an embedding from $f^{-1}(0)$ into a neighborhood of C in $\mathfrak{M}_{e,\Omega}$. The second part of the argument proves that this embedding of $f^{-1}(0)$ into $\mathfrak{M}_{e,\Omega}$ is onto an open set. Both parts consist mostly of fairly straightforward generalization of arguments that are used in the compact case and in [13]. Thus, the discussion below will be brief, with many of the details left for the reader. In this regard, the discussion that follows will consider only the case where C is immersed with purely transversal double point self intersections; the general case is left as an exercise. In any event, these two parts to the proof of Proposition 3.2 constitute Steps 1–6 and Step 7 of the seven steps into which the proof below is divided.

Step 1 What follows is a brief summary of the formal set up for the first part of the proof of Proposition 3.2 (as described above). In this regard, note that the basic conclusions here in Step 1 follow more or less automatically from an application of the implicit function theorem. However, there are two subtle points in this application. The first such point involves the choice of the appropriate spaces and the map between them. The second subtle point involves a reference to a particular regularity theorem in [17].

To start the summary, the basic observation is that a constant $\rho_0 > 0$ and an 'exponential' map, q, from the radius ρ_0 disk bundle $N_0 \subset N$ to X can be found with the following properties:

- *q restricts to the zero section as φ and embeds each fiber of N_0 as a pseudoholomorphic disk.*
- *At each point along the zero section, q's differential maps TN isomorphically to TX.*
- *If η is a section of N_0, then the image of $q(\eta(\cdot))$ is a pseudoholomorphic submanifold of X if and only if η obeys an equation with the schematic form*

$$D\eta + \mathfrak{R}(\eta; \nabla\eta) = 0,$$

  where $\mathfrak{R}$ is a smooth, fiber preserving map from $N_0 \times (N \otimes T^*C)$ to $N \otimes T^{0,1}C$ that is affine in the second factor and obeys

$$|\mathfrak{R}(a; b)| \le \zeta(|a|^2 + |a|\,|b|)$$

  for some constant ζ.
- *In fact, q can be constructed so that $\mathfrak{R}(a; b) = \mathfrak{R}_0(a) + \mathfrak{R}_1(a)b^{1,0}$ where $\mathfrak{R}_0$ and $\mathfrak{R}_1$ respectively map N_0 to $N \otimes T^{0,1}C$ and to $\mathrm{Hom}_{\mathbb{C}}(T^{1,0}C; N \otimes T^{0,1}C)$, and where $b^{1,0}$ denotes the projection of b onto the $(1,0)$ summand in $T^*C_{\mathbb{C}}$.* (3.12)

The map q can be constructed as described in Section 2 of [24]. Indeed, the basic point is that q maps the fibers of N_0 into X as pseudoholomorphic disks. This understood, then a local, $\mathbb{C}$–valued fiber coordinate on N_0 can be found with respect to which the q pullback of $T^{1,0}X$ is spanned by a form that annihilates vertical vectors and $e = d + $ where vanishes on the zero section and also annihilates vertical vectors. The various points in (3.12) then follow by exploiting these last remarks.

With (3.12) understood, the goal is to use the implicit function theorem coupled with various standard elliptic regularity theorems to describe the small solutions to (3.12). To broaden the subsequent discussion so as to cover certain generalizations of (3.12), it proves useful to consider the form of $\mathfrak{R}$ in the third point of (3.12) without assuming the validity of the final point. Thus, the subsequent discussion makes no reference to this final point of (3.12). It also proves useful to set up the implicit function theorem in a weighted Sobolev space of sections of N where finite norm demands local square integrability of the section and its covariant derivative, but makes no demand for an L^∞ bound. Were the purpose solely that of proving Proposition 3.2, a norm with L^∞ implications can be used. In any event, with the lack of an L^∞ bound from the norm, a trick is employed to handle the nonlinearity in $\mathfrak{R}$.

Here is the trick: First, introduce a smooth function $: [0,\infty) \to [0,2)$ whose value at $t \in [0,\infty)$ is 1 when $t < 1$, $1/t$ when $t > 2$. Given > 0, set $: [0,\infty) \to [0,2)$ to denote the function whose value at t is $(t/)$. Now, fix 1 and the trick is to replace the given $\mathfrak{R}$ in the third point of (2.14) with $\mathfrak{R} : N_0 \quad (N \quad T C) \to N \quad T^{0,1}C$ whose value on (a,b) is $\mathfrak{R}((|a|)a,b)$. Solutions to the $\mathfrak{R}$ version of (3.12) are then found via a straightforward implicit function theorem argument. This understood, and given that is small, Theorem 5.4.1 in [17] guarantees that these solutions are pointwise smaller than over the whole of C, and so they solve the desired $\mathfrak{R}$ version of (3.12).

Solutions to the $\mathfrak{R}$ version of (3.12) are found in the following way: A certain space of sections of N is split as $\mathrm{kernel}(D) \quad \mathfrak{L}$ while a corresponding space of sections of $N \quad T^{0,1}C_0$ is split as $\mathrm{cokernel}(D) \quad \mathfrak{L}'$. Then, for each in $\mathrm{kernel}(D) \quad \mathfrak{L}$, the section $D + \mathfrak{R}(; r)$ of $N \quad T^{0,1}C_0$ is projected into $\mathfrak{L}'$ to define a map from $\mathrm{kernel}(D) \quad \mathfrak{L}$ to $\mathfrak{L}'$. The differential of this last map at $= 0$ (which is formally D) can be seen to identify $\mathfrak{L}$ with $\mathfrak{L}'$. Thus, the implicit function theorem gives a ball B about the origin in $\mathrm{kernel}(D)$ and a smooth map, $: B \to \mathfrak{L}$ with the following properties:

When $\in B$, *then* $( ) = O(\| \|^2)$.

When $2 B$, *set* $+ \;(\;)$ *and then* $D \;+ \mathfrak{R}\,(\;;\,r\;)$ *projects to zero in* $\mathfrak{L}^0$. (3.13)

This ball B is the one that used in Proposition 3.2.

With (3.13) understood, de ne a map $f\colon B\;!$ cokernel(D) by taking the projection onto cokernel(D) of $\mathfrak{R}\,(\;+\;(\;),\;r(\;+\;(\;))$. This is the map to be used in Proposition 3.2. Indeed, if $j\;+\;(\;)j$ is everywhere less than , then it follows from the third point in (3.12) and the second in (3.13) that the image in X of the map $q(\;+\;(\;))\colon C_0\;!\;X$ is a pseudoholomorphic submanifold in $M_{e;}$ near to C if and only if $f(\;) = 0$. Meanwhile, standard elliptic estimates for D nd some $_D > 0$ such that the following is true: If $< \;_D$ and if B is de ned so that its elements have supremum norm less than $=2$, then $j\;+\;(\;)j$ is guaranteed to have supremum norm less than and so $= \;+\;(\;)$ solves $D\;+\mathfrak{R}(\;;\,r\;) = 0$. Thus, $f^{-1}(0)$ is truly mapped to a neighborhood of C in $M_{e;}$. Conversely, as is smooth, some relatively standard $s\;!\;1$ estimates on the sections of N in kernel(D) $\mathfrak{L}$ guarantee that the map just described embeds $f^{-1}(0)$ in $M_{e;}$.

Step 2 This step introduces the important function spaces involved. For this purpose, x > 0, but very small and x $k\;2\;f0;1;:::g$. Let E denote either N, $N\;T^{0;1}C_0$ or the tensor product of N with some multiple tensor product of $T\;C_0$. Let $L^2_{k;}(E)$ denote the completion of the set of smooth, compactly supported sections of E using the norm whose square is

$$jj\;jj^2_{k;}\qquad \int e^{\,jsj} \sum_{0\;p\;k} j\,r^k\;j^2 : \tag{3.14}$$

Note that this norm uses the growing exponential $e^{\;s}$ while those in (2.8) and (2.9) use the shrinking exponential $e^{-\;s}$.

The following is a basic fact about these spaces which is left to the reader to verify:

The covariant derivative extends to a bounded map from $L^2_{1;}$ *to* $L^2_{0;}$. (3.15)

Step 3 Reintroduce the operator D from (3.5). On each end of C, this operator has the schematic form in (3.9) and thus associated to each end of C is the kernel of the relevant version of the operator L_0. Note that each of these kernels is either zero or one-dimensional. In any event, let W denote the direct sum (indexed by the ends of C) of these kernel(L_0) vector spaces.

Fix > 0, but very small and, for the moment, consider D with its range and domain as described in Lemma 3.3. Lemma 3.5 provides the homomorphism : kernel(D) $!$ W. Note that the kernel of is the intersection of kernel(D) with $L^2_{0;}(N)$ and thus equal to the kernel of this same operator D but viewed as a map from $L^2_{1;}(N)$ to $L^2_{0;}(N \quad T^{0;1}C_0)$. Use K to denote the kernel of the latter version of D. Thus, kernel(D) can be split as

$$\text{kernel}(D) = \text{image}(\ ) \quad K: \tag{3.16}$$

Viewed as mapping $L^2_{1;}(N)$ to $L^2_{0;}(N \quad T^{0;1}C_0)$, the operator D is Fredholm, a fact implied by Lemma 3.5. Use $K \qquad L^2_{0;}(N \quad T^{0;1}C_0)$ to denote the corresponding cokernel. In this regard, note that the domain for Lemma 3.5's map _ is this vector space K . And, according to the second point of Lemma 3.5, the space K can be split as

$$K = \text{image}(\_\ ) \quad e^{-2\ jsj}\ \text{kernel}(D\ ): \tag{3.17}$$

The latter splitting is made in a canonical way by requiring the two summands to be orthogonal with respect to the inner product on $L^2_{0;}(N \quad T^{0;1}C_0)$.

Step 4 The purpose of this step is to identify W as a subspace of sections of N over C_0. For this purpose, use Lemma 3.5 to decompose W = image( ) image(_). Then, identify the image( ) summand of W with the corresponding subspace of kernel(D) in (3.16). (Note that the splitting in (2.18) is not canonical.)

To identify the image(_) summand of W as space of sections of N, rst use the coordinates on each end of C which are described in the proof of Lemma 3.3 to view image(_) as a subspace of sections of N over the $s > s_0$ portion of C. In this regard, note that each such section of N is pointwise bounded. Next, choose a smooth function, , on $\mathbb{R}$ which has values between 0 and 1, and which is zero on $(-1; 2s_0)$ and which is one on $(3s_0; 1)$. Compose with the function s on C_0 to view as a function on C_0. Finally, embed image(_) as a subspace of sections of N over the whole of C_0 by sending $w\ 2$ image(_) to w. Note that with the latter embedding understood, the vector space image(_) W has now been identi ed both as a subspace of sections of N and also, via (3.17), as a subspace of sections of $N \quad T^{0;1}C_0$. These two versions of image(_) will not be notationally distinguished.

Step 5 Let $K^1 \qquad L^2_{1;}(N)$ denote the orthogonal complement of K as de- ned using the inner product on $L^2_{0;}(N)$. (Note that this is meant to be the analog of the L^2 inner product as opposed to that of the L^2_1 inner product.)

Likewise, define $K^\perp \subset L^2_{0;\delta}(N \otimes T^{0,1}C_0)$. Also, introduce the corresponding $L^2_{0;\delta}$–orthogonal projection $\prod: L^2_{0;\delta}(N \otimes T^{0,1}C_0) \to K^\perp$.

The discussion in Step 1 referred to spaces $\mathfrak{L}$ and $\mathfrak{L}'$. In this regard, define $\mathfrak{L}$ to be $\mathfrak{L} \cap \text{image}(\_) \cap K^\perp$ and define $\mathfrak{L}'$ to be $\mathfrak{L}' \cap \text{image}(\_) \cap K^\perp$. Note that $\mathfrak{L}'$ sits in $L^2_{0;\delta}(N \otimes T^{0,1}C_0)$, and let $\prod: L^2_{0;\delta}(N \otimes T^{0,1}C_0) \to \mathfrak{L}'$ denote the $L^2_{0;\delta}$–orthogonal projection.

Step 6 With $\epsilon > 0$, but small, define a map from $\text{kernel}(D) \oplus \mathfrak{L}$ to $\mathfrak{L}'$ by sending $(\ ;w)$ in the former space to

$$\prod (Dw + \mathfrak{R}(\ + w; r(\ + w)) \tag{3.18}$$

It is a straightforward task (which is left to the reader) to check that this map has surjective differential at $(0;0)$ along the $\mathfrak{L}$ summand of $\text{kernel}(D) \oplus \mathfrak{L}$. This understood, then the implicit function theorem finds a ball $B \subset \text{kernel}(D)$ and a smooth map $: B \to \mathfrak{L}$ such that when $\in B$, then $(\ ;\ (\ ))$ solves (3.18). Moreover, the implicit function theorem also finds $_1 > 0$ such that any pair $(\ ;w)$ solving (3.18) with $\in B$ and $\|w\|_{1;} < {}_1$ has $w = (\ )$. This understood, define $f: B \to \text{cokernel}(D)$ by sending to $f(\ ) \equiv (1 - \prod)(D\ (\ ) + \mathfrak{R}(\ +\ (\ ); \ +\ (\ ))$. Thus, $+\ (\ )$ solves the $\mathfrak{R}$ version of (3.12) when $\in f^{-1}(0)$.

Here is one other automatic consequence of the implicit function theorem: The graph of in $\text{kernel}(D) \oplus \mathfrak{L}$ is homeomorphic to a neighborhood of $(0;0)$ in the space of solutions to (3.18). Thus, $f^{-1}(0)$ is homeomorphic to a neighborhood of 0 in the space of $(\ ;w) \in \text{kernel}(D) \oplus \mathfrak{L}$ for which $+\ (\ )$ obeys the $\mathfrak{R}$ version of (3.18).

Step 7 Theorem 5.4.1 in [17] now finds some $_3 \in (0;\ )$ such that when lies in the centered, radius $_3$ ball in B, then $+\ (\ )$ is everywhere bounded in norm by . Thus, solves the $\mathfrak{R}$ version of (3.12) and, as a consequence, the map that sends $\in f^{-1}(0)$ to $+\ (\ )$ properly embeds $f^{-1}(0)$ in a neighborhood of C in $\mathfrak{M}_{e;}$.

Meanwhile, an application of Proposition 2.3 finds $_2 \in (0;\ )$ such that if $\in \mathfrak{L}_1$ obeys the original, $\mathfrak{R}$ version of (3.12) and has $|\ | < {}_2$ everywhere, then $= +\ (\ )$ with $\in f^{-1}(0)$. Thus, the aforementioned map from $f^{-1}(0)$ to $\mathfrak{M}_{e;}$ is a homeomorphism onto an open neighborhood of C.

(d) The index of D

It is useful to have a formula for the index of D that generalizes those in (3.6). Such a formula is given in Proposition 3.6 at the end of this subsection. The derivation of the analog of (3.6) is a six step affair which follows this preamble. However, before starting, note that the concave and convex end discussions are not identical, but are completely analogous. Thus, the discussion below focuses on the concave case while only summarizing the modifications that are required when convex ends are present. In particular, the first four steps below involve only the case where X has solely concave ends. Note here that this assumption in Steps 1–4 about X is mostly implicit. Also, except for Step 6, the operating assumption is that C is immersed in X with only transversal, double point self-intersections.

Before starting, it is pertinent to remark on the absence in the subsequent discussion of reference to a Maslov or Conley-Zehnder index as in [13]. The first point is that the index formula for a Cauchy-Riemann operator on some given completion of the space of sections of a bundle over a surface must have the following schematic form: Index = 2 degree + χ + 'boundary correction'. Here, χ is the Euler characteristic of the surface, degree is a first Chern number of the bundle in question and 'boundary correction' is just what it says. In this regard, the definition of the degree requires some choice of trivialization of the bundle along the ends of the surface and a different choice might change the degree. Of course, if it does, it will also change the 'boundary correction' term to compensate. In this regard, an index formula with a formal Conley-Zehnder index term correction to the basic '2 degree + χ' simply reflects a particularly natural choice for the trivialization of the normal bundle of the pseudoholomorphic surface on its ends. In particular, these Conley-Zehnder trivializations are trivializations that are induced from certain trivializations of the normal bundles to the limiting closed Reeb orbits.

The preceding understood, remark that the formula given below in Proposition 3.6 also uses certain natural trivializations of the normal bundle to the surface on its ends that are induced from those of the limiting closed Reeb orbits. In fact, the latter were described previously in Part 2 of Section 3a. However, the formal introduction of a 'Conley-Zehnder' index in this case is not done here for three reasons. First, an integer valued Conley-Zehnder index is awkward to define in the present circumstances because the first Chern class of $T^{1,0}X$ evaluates with absolute value 2 on the S^2 factors of the ends of X. Second, the index formula for the non-isolated Reeb orbits necessarily has terms with no analog in the index formula [13]. Third, the formula given in Proposition

3.6 are, in any event, reasonably easy to use without the additional burden of a Conley-Zehnder index computation.

Step 1 Consider first an end of C which lies in an end of X with orientable z–axis line bundle. By virtue of Proposition 2.3, the large s part of the corresponding end of C_0 has a product structure with special coordinates (,) such that J maps ∂ to ∂. Here, $\in \mathbb{R}/(2\ m|p|\mathbb{Z})$ while $\in [\ _0, \infty)$ with $m > 0$, $p \neq 0$ and both integers. In this regard, the integer m should be viewed as the multiplicity by which the large and constant $|s|$ slices of this end of C wrap around the limiting Reeb orbit. This interpretation of m requires taking $p = -1$ when the limiting Reeb orbit for this end is characterized as in (1.8) by one of the symbols $+$ or $-$. Note that these coordinates differ from those in (2.13) or its $\cos^2\ _0 = 1/3$ analog; however, they differ only by functions whose derivatives to any order decay to zero at an exponential rate as $s \to \infty$.

There is also a special trivialization of the normal bundle over the end in question as the product $\mathbb{R}^2$ bundle for which J sends the column vector $\binom{1}{0}$ to $\binom{0}{1}$ With respect to this trivialization, the operator D defines an operator on the $\mathbb{R}^2$ valued functions that has the form

$$D\ = \partial\ + L_0\ + P\ . \tag{3.19}$$

Here, is a 2–component column vector, L_0 is described by (2.16) and P is a 2×2 matrix valued function that is small and drops to zero exponentially fast as $\to \infty$. Meanwhile, $'$ and are the constants in (2.16), and thus one of the following hold:

$$' = 0 \text{ and } > 0.$$
$$' = \ > 0. \tag{3.20}$$

In this regard, the second case occurs only when the given end has its corresponding $_0$ in $\{0, \}$.

Because the coefficients of P drop to zero exponentially fast at large , the index of D is the same as that of $D_0 \equiv D - (\ )P$, where $(\ ) : [\ _0, \infty) \to [0, 1]$ is 0 on a certain interval of the form $[\ _0,\ _1]$ and $(\ )$ is 1 on $[2\ _1, \infty)$. Here, $_1 > \ _0$ can be taken to be very large.

This operator D_0 is introduced because its coefficients are constant on $[2\ _1, \infty) \times S^1$. In particular, this implies (see [1]) that D_0 has the same index as the identical operator on the domain in C_0 where $\leq 4\ _1$ but with appropriate spectral boundary conditions imposed on the $= 4\ _1$ boundary of this domain. By way of motivation, this translation to a spectral boundary condition problem is made so as to facilitate the comparison of the index of D_0 with that of a certain complex linear operator with known index.

In any event, there is one key observation that is used to determine the appropriate spectral boundary conditions on the $\quad = 4 \ _1$ circle: Any bounded element in the kernel of D_0 must restrict to the half cylinder $S^1 \quad [2 \ _1; \infty)$ in the span of the set

$$e^{-E} \ _E(\ ) : E \quad 0 \text{ and } L_0 \ _E = E \ _E \ : \tag{3.21}$$

The eigenvalues E which appear above have the form

$$E = -2^{-1}(\ ' + \ - [(\ ' - \ )^2 + 4n^2/(m|p|)^2]^{1/2}) : \tag{3.22}$$

Here, $n \in \{0, 1, \ldots\}$ is such as to make $E \quad 0$. In particular, with regard to this last concern, note that when the first point of (3.20) holds, then $E > 0$ for all positive n and $E = 0$ for $n = 0$. On the other hand, when the second point of (3.20) holds, then $E > 0$ if and only if $n > m\sqrt{3}/\sqrt{2}$ because $\ ' = \ = \sqrt{3}/\sqrt{2}$ and $|p| = 1$ in this case.

Concerning the eigenspace for the eigenvalue E, it is important to note that this is a two-dimensional vector space over $\mathbb{R}$ if the integer n in (3.22) is strictly positive. Such is also the case when $n = 0$ and the second point in (3.20) holds. However, when $\ ' = 0$ in (3.20), then each of the $n = 0$ values of E has 1–dimensional eigenspace. In any case, the components of the eigenfunctions $\ _E$ in (3.21) are certain linear combinations of $\cos(n \ /(m|p|))$ and $\sin(n \ /(m|p|))$.

It now follows directly from (3.21) that the index of D_0 is the same as that of the identical operator on the domain where $\quad 4 \ _1$ in C_0 with spectral boundary conditions on the $\quad = 4 \ _1$ boundary of this domain which restrict the sections of the normal bundle under consideration to lie in the set spanned by $[\ _E : E \quad 0$ is given by (3.22)].

Step 2 Here and in Step 3, assume that C only intersects components of $X - X_0$ that have orientable z–axis line bundle. With this assumption understood, the remainder of this step considers not D_0, but a particular $\mathbb{C}$–linear operator version of the $\bar{\partial}$ operator on the sections of the normal bundle. The reason for this digression should be apparent by the end of the next step.

The first order of business is to specify the version of $\bar{\partial}$ to be used. In particular, the operator is constrained only on the ends of C_0; and on an end, in terms of the coordinates $(\ ;\ )$ and the aforementioned trivialization of the normal bundle, this operator sends an $\mathbb{R}^2$ valued function $\quad$ to

$$\bar{\partial} \quad + \begin{pmatrix} 0 & -\partial \\ \partial & 0 \end{pmatrix} \quad : \tag{3.23}$$

By the way, the di erence of two $\bar\partial$ operators for the same complex line bundle is, as a $\mathbb{C}$–linear operator, the tensor product with a section of $T^{0,1}C_0$. Were, C_0 compact, then the two operators would have the same index, but such an event is not guaranteed when C_0 is non-compact.

The index of the version of $\bar\partial$ in (3.23) is readily computed with the help of the Riemann-Roch theorem, and the computation yields the identity: $\text{index}(\bar\partial) = 2\,\text{degree}(N) + \chi(C_0) + \partial_T$. Here, $\text{degree}(N)$, $\chi(C_0)$ and ∂_T denote the following: First, $\text{degree}(N)$ is the degree of the line bundle N as determined via the sections over the end of C_0 which are described in Part 2 of Section 3a; meanwhile χ and ∂_T respectively denote the Euler characteristic and the number of ends of C_0.

The formula just described for the index of $\bar\partial$ can be rewritten using Proposition 2.1 and the fact that $\text{degree}(N) = \langle e, [C]\rangle - 2m_C$. For example, here is an equivalent formula: $\text{index}(\bar\partial) = \langle e, [C]\rangle - \langle C_1, [C]\rangle - 2m_C + \partial_T$.

Note that the kernel of $\bar\partial$ on an end of C_0 can also be written as in (3.21) except that $E = n/(m|p|)$ replaces the formula for E in (3.22). Here, n can be any non-negative integer. For the case of $\bar\partial$, all of the eigenspaces are two-dimensional.

As a constant coe cient operator on the ends of C_0, the index of the $\bar\partial$ operator can also be identi ed with the index of the same operator on the $4\ {}_1$ domain in C_0 with spectral boundary conditions on each $= 4\ {}_1$ circle. In particular, these boundary conditions restrict the sections of N on the $= 4\ {}_1$ circle to be linear combinations from those column vectors with top component $\cos(n\ /(m|p|))$ and bottom component $\sin(n\ /m|p|)$ or else top component $-\sin(n\ /(m|p|))$ and bottom component $\cos(n\ /(m|p|))$. Here, $n \in \{0, 1, \dots\}$.

Step 3 This step compares the spectral boundary conditions for D_0 and for $\bar\partial$. In this regard, note that for both operators, the relevant boundary condition for each boundary component restricts the sections of N to lie in a certain direct sum of nite dimensional spaces. In the case of $\bar\partial$, these summands are indexed by integers $n \in \{0, 1, 2, \dots\}$ and each is two-dimensional. Meanwhile, the summands in the case of D_0 are subspaces of those for $\bar\partial$.

In particular, on an end of C_0 for which the corresponding element in C's limit set has $_0 \notin \{0, \}$, the $n > 0$ summands for D_0 are the same as those for $\bar\partial$, while the $n = 0$ summand for D_0 is a 1–dimensional subspace of that for $\bar\partial$. On the other hand, if the corresponding element in C's limit set has $_0 \in \{0, \}$, then there is some integer $m_0 \ \ 1$ such that the $n \ \ m_0$ summands for D_0

and for $\bar\partial$ agree, while the $n < m_0$ summands for D_0 are all trivial (zero-dimensional). Here, m_0 is the least integer that is greater than $m\sqrt{3}/\sqrt{2}$ with m defined as follows: The large and constant s slices of the end in question define a class in $H_1(S^1\times S^2;\mathbb{Z})$. Then m is the absolute value of the pairing between this class and a generator of $H^1(S^1\times S^2;\mathbb{Z})$.

With this last paragraph understood, it now follows from the analysis in [1] that the index of D_0 is less than that of $\bar\partial$, with the deficit, δ, accounted for by a contribution from each end of C_0. Here, an end $E\subset C_0$ with $\theta_0\notin\{0,\pi\}$ contributes 1 to δ, while one with $\theta_0\in\{0,\pi\}$ has the non-zero contribution $2m_0(E)$. Thus, with ∂_0 denoting the sum, indexed by the ends $E\subset C_0$ with $\theta_0\in\{0,\pi\}$, of the corresponding quantities $(1-2m_0(E))$, the following formulae hold: $\operatorname{index}(D) = 2\cdot\operatorname{degree}(N) + \chi(C_0) + \partial_0$. Equivalently, $\operatorname{index}(D) = \langle e,[C]\rangle - \langle c_1,[C]\rangle - 2m_C + \partial_0$, and also $\operatorname{index}(D) = -\chi(C_0) - 2\langle c_1,[C]\rangle + \partial_0$.

Step 4 Now consider the possibility that C_0 has some ends which lie in $[0,\infty)\times M$ where $M\subset\partial X_0$ is a component with non-orientable z–axis line bundle. In this case, the index of D is computed via the route just used, first through a deformation on the ends of C_0 to a constant coefficient operator, next through a reinterpretation of the latter as an operator with spectral boundary conditions, and finally via a comparison with the index of the analogous spectral boundary condition interpretation of a certain $\bar\partial$ operator.

With regard to this $\bar\partial$ operator, note that the latter will be Fredholm with $\operatorname{index}(\bar\partial) = 2\cdot\operatorname{degree}(N) + \chi(C_0) + \partial_T$. (As before, $\operatorname{index}(\bar\partial) = \langle e,[C]\rangle - \langle C_1,[C]\rangle - 2m_C + \partial_T$ is an equivalent formula.) The $\bar\partial$ operator used here is, once again, restricted on the ends of C_0. The restrictions described above for $\bar\partial$ are used for the ends of C_0 which map to components of $X - X_0$ where the corresponding z–axis line is orientable. The restrictions on the remaining ends of C_0 are described below.

In any event, the resulting analysis finds $\operatorname{index}(D) = \operatorname{index}(\bar\partial) - \delta$, where δ is again a sum of contributions from each end of C_0. An end of C_0 which maps to a component of $X - X_0$ where the corresponding z–axis line bundle is orientable makes the same contribution to δ as before. Meanwhile, the description of this contribution for an end of C_0 which lies where the z–axis line bundle is non-orientable requires the distinction between two cases. The first case discusses those ends where the constant $s > s_0$ circles define even multiples of a generator of $H^1(M;\mathbb{Z})$. Note that this case occurs automatically when the corresponding element of C's limit set is not a generator of $H^1(M;\mathbb{Z})$. The second case discusses the situation when these circles are odd multiples of a generator.

In the first case, the inverse image in $[s_0, \infty) \times (S^1 \times S^2)$ of the end in question under the map in (2.3) has two components, and so determines a particular value of θ_0 up to the replacement $\theta_0 \to \pi - \theta_0$. Moreover, after the identification of the end with either of these components, the analysis for the orientable z–axis line bundle case repeats here hitch free. In this regard, please note that the $\overline{\partial}$ operator for use on C_0 is restricted so as to yield (3.23) after the identification of the given end with one of its inverse images in $[s_0, \infty) \times (S^1 \times S^2)$

With the preceding understood, here is the result of the analysis: The end in question contributes 1 to the deficit δ if the corresponding $\theta_0 \in \{0, \pi\}$, and otherwise the contribution is twice the value of the associated integer m_0.

Now turn to the case of an end of C_0 where the corresponding large, constant $|s|$ circles is an odd multiple of a generator of $H_1(M; \mathbb{Z})$. Let $C_{01} \subset [s_0, \infty) \times (S^1 \times S^2)$ denote the inverse image of C_0 via the map in (3.3). As noted earlier, C_{01} is a pseudoholomorphic cylinder with $\theta_0 = \pi/2$ and with φ asymptotic as $s \to \infty$ to either 0 or π. Thus, C_{01} has the previously described parameterization by coordinates $(\rho, \tau) \in [\rho_0, \infty) \times [0, 2\pi m]$ where m is an odd, positive integer. In this regard, note that the assumptions in play here force the relevant element in the limit set of C_{01} to have $\theta_0 = \pi/2$, and this implies that the integer p is equal to one. Use coordinates $(\rho', \tau') \in [\rho_0, \infty) \times [0, 2\pi m]$ for C_0 so that the map in (3.3) restricts to C_{01} as that which writes $\rho' = \rho$ and $\tau' = 2\tau$.

Under the previously described identification between the normal bundle of C_{01} and the pull back via (3.3) of the normal bundle of C_0, sections of C_0's normal bundle pull up as $\mathbb{R}^2$ valued functions on C_0' which obey $\eta(\rho, \tau + m\pi) = -\eta(\rho, \tau)$. This follows from (3.3) and the comments in the final paragraph of Part 2 of Section 3a. The operator D on C_0 then pulls up to be the corresponding D on C_{01}, but with the domain and range restricted to the functions which are odd under the involution $\tau \to \tau + m\pi$.

Given these last points, the analysis done for the other ends of C_0 can be repeated to find that the index of D on C_0 is the same as that of this differential operator on the $\rho \leq 4\rho_1$ portion of C_0 with spectral boundary conditions given as follows: The sections on the $\rho = 4\rho_1$ boundary should pull up to C_{01} as a linear combination of two column vectors where one has top component $\cos(n\tau/m)$ and bottom component $\sin(n\tau/m)$ while the other has top component $-\sin(n\tau/m)$, and bottom component $\cos(n\tau/m)$. However, here n must be a positive, odd integer.

Now consider the constraint for the $\overline{\partial}$ operator on this end of C_0. To guarantee that this operator has index equal to 2 degree$(N) + \chi(C_0) - \partial_T$, the constraint is as follows: Use the coordinates (ρ', τ') on the end of C_0 and trivialize the normal bundle over this end using the vectors tangent to the x_2^0 and x_3^0 coordinates which appear in (3.3). With these coordinates and the bundle trivialization understood, the $\overline{@}$ operator acts on $\mathbb{R}^2$ valued functions. In this guise, the operator should be that which sends a column vector function to 2^{-1} $_0 + L^0$ where L^0 is the matrix operator given by the second term in (3.23) with ^{0} derivates replacing those of .

To compare this $\overline{@}$ operator with D, it proves convenient to pull the former up to C_{01} and express it as an operator on the normal bundle to the latter. In particular, the resulting operator di ers from the operator in (3.23) by the zero'th order operator which sends to the operator which is given by (3.23) but with $@$ replaced by $@$ $+ 1$. As with the pullback to C_{01} of D, the range and domain of this operator must be restricted to those $\mathbb{R}^2$ valued functions which obey $(; + m) = - (;)$.

With the preceding understood, the analysis previously done can be repeated to nd that the index of $\overline{@}$ on C_0 is the same as that of the same operator on the 4 $_1$ portion of C_0 with spectral boundary conditions given as follows: The sections on the $= 4$ $_1$ boundary should pull up to C_{01} as linear combinations of two column vectors, one whose top component is $\cos(n =m)$ and whose bottom component is $\sin(n =m)$ and the other whose top component is $-\sin(n =m)$ and whose bottom component is $\cos(n =m)$. In this case, n here must be an odd integer and no less than $-m$.

Given now all of the preceding, a comparison of the spectral boundary conditions just described for D and for $\overline{@}$ nds that the given end of C_0 contributes $m + 1$ to the de cit . Now, each end E C_0 where the z{axis line bundle is non-orientable and where the constant $s > s_0$ circles de- ne odd multiples of a generator of $H_1(M;\mathbb{Z})$ supplies a positive, odd integer m $m(E)$ and the sum, indexed by such ends of C_0, of $-m(E)$ de nes an integer $@_1$. Meanwhile, each end E C_0 where the corresponding $_0$ is in $f0; g$ (whether or not the ambient z{axis line bundle is orientable) de nes an non-zero integer $m_0(E)$ (as described above). Use $@_0$ to denote the sum, indexed by this last set of ends, of $(1 - 2m_0(E))$. Then, the index of D is given by $\text{index}(D) = 2\ \text{degree}(N) + (C_0) + @_0 + @_1$, which is no di erent than saying that $\text{index}(D) = he;[C]i - hc_1;[C]i - 2m_C + @_0 + @_1$ or $\text{index}(D) = - (C_0) - 2hc_1;[C]i + @_0 + @_1$.

Step 5 This step considers the possibility that X has some convex ends that are intersected by C. The rst point to make is that the analysis for the convex end case is completely analagous to that for the concave ends. Indeed, the di erence amounts to essentially considering the eigenvalues E of the operator

L_0 in (3.19) which are non-positive rather than non-negative. The result of this analysis (whose details are left to the reader) follows.

To begin, let $E \subset C$ be an end which lies in a convex end of X with orientable z–axis line bundle. Then E adds a factor 1 to the previous index formula, $2\,\mathrm{degree}(N) + \chi(C_0) + \partial_0 + \partial_1$, when the corresponding element in C's limit set has $\theta_0 \notin \{0, \pi\}$. On the other hand, when $\theta_0 \in \{0, \pi\}$, then E contributes $2m_0(E) - 1$ to the integer ∂_0, where $m_0(E)$ is defined as in Step 3.

Next, let $E \subset C$ denote an end which lies in a convex end of X with non-orientable z–axis line bundle. In this case, the contribution of E to the index formula again depends on whether the large, but constant s circles in E give an even or an odd multiple of a generator of $H_1(S^1 \times S^2; \mathbb{Z})$. In the case where these circles give an even multiple of a generator, then E contributes 1 to the previous index formual when the components of the inverse image of the end under the map in (3.3) define limiting Reeb orbits with $\theta_0 \notin \{0, \pi\}$. On the other hand, where such lifts produce limiting Reeb orbits with $\theta_0 \in \{0, \pi\}$, then E contributes $2m_0(E) - 1$ to the integer ∂_0, where $m_0(E)$ is defined as in Step 3 by either of the lifts.

In the case where the constant s circles in E give an odd multiple of a generator of the first homology of $S^1 \times S^2$, then E contributes $m(E)$ to the integer ∂_1. Here, $m(E)$ is the absolute value of this multiple.

Step 6 This step considers the general case and so drops the assumption that C is immersed in X. For such C, the operator D is the one that Lemma 3.4 considers. But for a change of interpretation, its index is again $\mathrm{index}(D) = \langle e, [C]\rangle - \langle c_1, [C]\rangle - 2m_C + \partial_0 + \partial_1$. Here, the change concerns the integer m_C which now must be interpreted as in the second point of Proposition 3.1.

To justify the preceding formula, first remember that the given map from C_0 to X has arbitrarily small, C^1 perturbations for which the result satisfies the requirements in the second point of Proposition 3.1. The perturbation can be done so that the perturbed map is pseudoholomorphic for an appropriate almost complex structure on X which differs from the original on a compact subset of X. The operator D of Lemma 3.4 is defined for this new map, and if the perturbation is small in the C^2 topology, it will differ from the original by an operator with small norm; thus, the new and old versions of Lemma 3.4's D will have the same index. Meanwhile, the new map, being an immersion, has an associated operator, call it D', which is given by (3.5) and is the one discussed in Steps 1–4, above. It is left as an exercise to check that the operator D of Lemma 3.4 and the operator in (3.5) have the same index when C is immersed.

The following proposition summarizes the results of the preceding discussion:

Proposition 3.6 *Let* $C\in\mathfrak{M}_{e;}$ *and define the operator* D *either by (3.5) when* C *is immersed, or as in Lemma 3.4. Let* $\mathfrak{L}_0$ *and* $\mathfrak{L}_1$ *be as defined in either Lemma 3.3 or Lemma 3.4 as the case may be and view* D *as a Fredholm operator from* $\mathfrak{L}_0$ *to* $\mathfrak{L}_1$*. Then, the following formulae hold:*

$$\mathrm{index}(D) = \langle e,[C]\rangle - \langle c_1,[C]\rangle - 2m_C + \partial_0 + \partial_1 + \partial.$$

$$\mathrm{index}(D) = -\chi(C_0) - 2\langle c_1,[C]\rangle + \partial_0 + \partial_1 + \partial.$$

Here, the integers m_C*,* ∂_0*,* ∂_1 *and* ∂ *are defined as follows:*

m_C *denotes the number of double points of any perturbation of the defining map from* C_0 *onto* C *which is symplectic, an immersion with only transversal double-point singularities with locally positive self-intersection number, and which agrees with the original on the complement of some compact set.*

∂_0 *is the sum of the contributions of the form* $\varepsilon(1-2m_0(E))$ *from each end* E *of* C_0 *for which the corresponding limiting Reeb orbit has* $\theta_0\in\{0,\pi\}$*. In this regard,* $\varepsilon=+1$ *when* E *lies in a concave end of* X *and* $\varepsilon=-1$ *otherwise. Meanwhile,* $m_0(E)$ *is a positive integer, and here is its definition when the corresponding end of* X *has orientable* z*–axis line bundle: Let* $m(E)$ *denote the absolute value of the pairing between a generator of* $H_1(S^1\times S^2;\mathbb{Z})$ *and any constant,* $s>s_0$ *circle in* E*. Then,* $m_0(E)$ *is the least integer which is greater than* $m(E)\sqrt{3}/\sqrt{2}$*. In the case when the corresponding component of* X *has non-orientable* z*–axis line bundle, use this last formula but with* E *replaced by either component of its inverse image under the map in (3.3).*

∂_1 *is the sum of the contributions of the form* $\varepsilon m(E)$ *from each end* E *of* C_0 *that satisfies the following criteria:*

(a) *The end* E *lies in an end of* X *with non-orientable* z*–axis line bundle.*

(b) *The absolute value of the pairing between a generator of the first cohomology of the corresponding end of* X *and any constant,* $s>s_0$ *circle in* E *is odd. The preceding understood, take* $\varepsilon=-1$ *when* E *lies in a concave end of* X *and take* $\varepsilon=++1$ *otherwise. Meanwhile, take* $m(E)$ *to be the absolute value of the cohomology pairing just described.*

∂ *is the number of ends of* C_0 *that satisfy one of the following two criteria:*

(a) *The end* $E\subset C_0$ *lies in a convex end of* X *with orientable* z*–axis line bundle. In addition,* E*'s corresponding limit Reeb orbit has* $\theta_0\in\{0,\pi\}$*.*

(b) *The end $E \subset C_0$ lies in a convex end of X that has non-orientable z–axis line bundle. In addition, the inverse image via (3.3) of E has 2–components and neither defines a limiting Reeb orbit with $\theta_0 \in \{0, \pi\}$.*

Finally, when C is immersed, the index formula can also be written as

$$\text{index}(D) = \text{degree}(N) - \langle c_1, [C]\rangle + \partial_0 + \partial_1 + \partial,$$

where N denotes the normal bundle of C, and where its degree is defined using a section that is non-vanishing on the ends of C and there homotopic through non-vanishing sections to the sections described in Part 2 of Section 3a.

4 Subvarieties in $\mathbb{R} \times (S^1 \times S^2)$

This section turns away from the general discussion of the previous two sections to concentrate on the special case where $X = \mathbb{R} \times (S^1 \times S^2)$ with the symplectic structure and associated almost complex structure as described in (1.1)–(1.4). In particular, the discussion here highlights general features of the moduli space of HWZ subvarieties in $\mathbb{R} \times (S^1 \times S^2)$ and results in a proof of Theorem A.1. In fact, given Proposition 3.2, the latter follows directly from Proposition 4.2, below. Other features deemed of particular interest are summarized in Propositions 4.3 and 4.8.

(a) Examples

Before diving into generalities, it proves useful to introduce various explicit examples of HWZ pseudoholomorphic subvarieties as these cases will be referred to in the later subsections. The examples below exhaust the set of HWZ subvarieties that are invariant under some 1–parameter subgroup of the subgroup $T = S^1 \times S^1$. In this regard, remember that T is generated by the vector fields ∂_t and ∂_φ and it is the component of the identity of the subgroup of the isometry group of $S^1 \times S^2$ that preserves the contact form α.

Example 1 The simplest examples are the cylinders $C = \mathbb{R} \times \gamma$, where $\gamma \subset S^1 \times S^2$ is a closed Reeb orbit. Remember that these closed Reeb orbits are described in (1.8). In particular, each such γ is characterized, in part, by some constant value, θ_0, for the polar angle θ on S^2. For example, if $\theta_0 \notin \{0, \pi\}$, then the closed Reeb orbit determines a pair of relatively prime integers (p, p') and if $p \neq 0$, then the corresponding cylinder is parameterized by a periodic coordinate $\tau \in \mathbb{R}/(2\pi|p|\mathbb{Z})$ and u with $|u| \in (0, \infty)$ and $\mathrm{sign}(u) = \mathrm{sign}(p)$ by the rule

$$(t = \tau, \; f = u, \; \varphi = \varphi_0 + p'/p\,\tau, \; h = p'/p \sin^2(\theta_0) u)\,. \tag{4.1}$$

Note that the condition $p \neq 0$ is equivalent to the assertion that $\cos^2\theta_0 \neq 1/3$. Meanwhile, if $p = 0$, then $p' = \pm 1$ and $\cos\theta_0 = p'/\sqrt{3}$. In this case, the parameterization of the corresponding cylinder is by $\tau \in \mathbb{R}/(2\pi\mathbb{Z})$ and u with $|u| \in (0, \infty)$ and $\mathrm{sign}(u) = p'$ according to the formula

$$(t = t_0, \; f = 0, \; \varphi = \tau, \; h = u), \tag{4.2}$$

where t_0 is a constant.

Finally, in the case where θ_0 is 0 or π, the parameterization of the corresponding cylinder is by $\tau \in \mathbb{R}/(2\pi\mathbb{Z})$ and $u \in (0, \infty)$ according to the formula

$$(t = -\tau, \; f = -u)\,. \tag{4.3}$$

Note that in this case, the function s along C is given by $s = -6^{-1/2}\ln(u/2)$.

Example 2 This example has t = constant and $f = -\sigma < 0$ with σ being constant. Each such example is a once punctured sphere — a plane — whose limit set lies in the $(s \to -\infty)$ and thus convex end of $\mathbb{R} \times (S^1 \times S^2)$. Here, the limit set is a ($p = 0$, $p' = \pm 1$) and so $\cos^2\theta_0 = 1/3$ closed Reeb orbit. Given that the limit closed Reeb orbit in question has $\cos\theta_0 = p'/\sqrt{3}$, such a subvariety can parameterized by polar coordinates on the plane, $(\tau, u) \in \mathbb{R}/(2\pi\mathbb{Z}) \times [0, \infty)$ according to the rule

$$(t = t_0, \; f = -\sigma, \; \varphi = \mathrm{sign}(p')\tau, \; h = \mathrm{sign}(p')u). \tag{4.4}$$

The maximum value of s achieved by this plane occurs at the plane's origin, where $u = 0$. At this point, $s_{\max} = -6^{-1/2}\ln(\sigma/2)$ and θ is either 0 or π depending on whether $p' = +1$ or -1, respectively.

Example 3 In this case, t = constant and $f = \sigma > 0$ with σ again being constant. Each such variety is an embedded cylinder whose limit set lies in the convex end of $\mathbb{R} \times (S^1 \times S^2)$. In this regard, the limit set consists of both $\cos^2\theta_0 = 1/3$ closed Reeb orbits. Such a subvariety can be parameterized by $\tau \in \mathbb{R}/(2\pi\mathbb{Z})$ and $u \in \mathbb{R}$ according to the formula

$$(t = t_0, \; f = \sigma, \; \varphi = \tau, \; h = u)\,. \tag{4.5}$$

In this case, the maximum value of s is achieved when $u = 0$ with value $s_{\max} = -6^{-1/2}\ln(\sigma/2)$. This maximum value occurs where $\theta = \pi/2$.

Example 4 These next examples are all cylinders where φ and h are both constant, but with $h \neq 0$. Consider first the case where $\varphi = \varphi_0$ and $h = \eta \neq 0$. The limit set for such a cylinder consists of one $(p = 1;\ p' = 0)$ closed Reeb orbit (so $\theta_0 = \pi/2$, $\varphi = \varphi_0$) and either the closed Reeb orbit with $\theta = 0$ or that with $\theta = \pi$. The $\theta = 0$ closed Reeb orbit appears when $\eta > 0$ and the $\theta = \pi$ closed Reeb orbit appears when $\eta < 0$. In any case, both of ends of the cylinder lie in the convex end of $\mathbb{R} \times (S^1 \times S^2)$. The cylinder is parameterized by coordinates $\tau \in \mathbb{R}/(2\pi\mathbb{Z})$ and $u \in \mathbb{R}$ by the rule

$$(t = \tau;\ f = u;\ \varphi = \varphi_0;\ h = \eta)\,. \tag{4.6}$$

The maximum value of s on this orbit is $s_{\max} = -6^{-1/2}\ln(3\eta/(2\sqrt{2}))$; it occurs where $u = 0$ and so $\cos^2\theta = 1/3$.

The next three examples describe 1–parameter families of embedded, pseudo-holomorphic cylinders. However, the subsequent discussion requires the preliminary digression that follows.

To start the digression, remark that the families considered below are labeled in part by a pair (p, p') of relatively prime integers with $p > 0$ and an angle $\varphi_0 \in \mathbb{R}/(2\pi\mathbb{Z})$. In this regard, the combination $p\varphi - p't$ is constant on a cylinder with label $((p, p'),\ \varphi_0)$ and so such a cylinder is invariant under the subgroup of T generated by the vector field $p\partial_t + p'\partial_\varphi$.

A cylinder labeled by $((p, p'); \varphi_0)$ can be parameterized by $\tau \in \mathbb{R}/(2\pi|p|\mathbb{Z})$ and $u \in \mathbb{R}$. The parameterization is given as

$$(t = \tau;\ f = u;\ \varphi = \varphi_0 + (p'/p)\tau;\ h = h(u)) \tag{4.7}$$

where the function $h(\cdot)$ is constrained to obey the differential equation

$$h_u = (p'/p)\sin^2\theta\,. \tag{4.8}$$

Here, the subscript u signifies differentiation with respect to u. Note that $\sin^2\theta$ in (4.8) should be viewed as an implicit function of u and h through its dependence on f and h. Indeed, the identity

$$h/f = \sqrt{6}\cos\theta\sin^2\theta/(1 - 3\cos^2\theta) \tag{4.9}$$

can be inverted locally to write $\theta = \theta(h/f)$.

Note that (4.8), being a first-order, ordinary differential equation for one unknown function, has a single constant of integration which parameterizes its solution set. This constant of integration determines, and is completely determined by fixing the value of the coordinate s in some fiducial manner.

It is illuminating to use (4.8) and the condition $f = u$ to derive equivalent differential equations for both θ and s. In particular, the latter read:

$$\theta_u = 6^{-1/2} e^{\sqrt{6}s}(1 + 3\cos^4\theta)^{-1}\{\sqrt{6}\cos\theta - p'/p(1 - 3\cos^2\theta)\}\sin\theta .$$
$$s_u = -6^{-1/2} e^{\sqrt{6}s}(1 + 3\cos^4\theta)^{-1}\{1 - 3\cos^2\theta + \sqrt{6}p'/p\cos\theta\sin^2\theta\}. \quad (4.10)$$

For any given functional dependence of s on u, the top equation in (4.10) has constant θ solutions only for $\theta = 0, \pi, \theta_0$ and, in the case when $|p'|/p > \sqrt{3}/\sqrt{2}$, also at a fourth angle, θ'_0. Here, θ_0 and θ'_0 are determined as follows: In the case where $p' = 0$, set $\theta_0 = \pi/2$. For the other cases, digress momentarily to note that

$$(\sqrt{6}\tau)^{-1}\{-1 + (1 + 2\tau^2)^{1/2}\} \quad (4.11)$$

has absolute value less than 1/3 for any choice of $\tau \neq 0$. In particular, for any such τ, the expression in (4.11) can be written as $\cos\theta_0$ for precisely one $\theta_0 \in (0, \pi)$. Meanwhile, if τ is such that $|\tau| > \sqrt{3}/\sqrt{2}$, then

$$(\sqrt{6}\tau)^{-1}\{-1 - (1 + 2\tau^2)^{1/2}\} \quad (4.12)$$

has absolute value less than 1. Thus, this last expression equals $\cos\theta'_0$ for precisely one angle $\theta'_0 \in (0, \pi)$. Note that the expressions in (4.11) and (4.12) have opposite signs, and thus $\cos\theta_0$ and $\cos\theta'_0$, have opposite signs.

With the preceding understood, end the digression and define θ_0 and θ'_0 for (p, p') when p' is non-zero by replacing τ in the preceding digression by p'/p. As remarked, the angles θ_0, θ'_0 (when $|p'|/p > \sqrt{3}/\sqrt{2}$) and the angles $\theta = 0$ and $\theta = \pi$ provide the only h = constant solutions to (4.9). In this regard, the constant θ solutions to (4.9) provide the pseudoholomorphic cylinders that are already described in Example 1 above.

With the constant solutions to the top equation in (4.10) understood, consider next the non-constant solutions. These solutions have the property that θ_u is either always positive or always negative. Thus, in the case where $|p'|/p < \sqrt{3}/\sqrt{2}$, either θ ranges without critical points between 0 and θ_0 or between θ_0 and π, with these particular values giving the inf and sup of θ as the case may be. On the other hand, when $p'/p > \sqrt{3}/\sqrt{2}$, then there are three possible ranges for θ. The first has θ ranging without critical values between 0 and θ_0, the second between θ_0 and θ'_0 and the third between θ'_0 and π. The analgous situation exists when $p'/p < -\sqrt{3}/\sqrt{2}$, for here θ ranges without critical points either between 0 and θ'_0, or between θ'_0 and θ_0, or between θ_0 and π.

What ever the range of , as long as the latter is not constant, then it is a simple matter to change variables from u to in the second line of (4.10) and thus see $s = s(\ )$ as an anti-derivative of the function

$$-\{1 - 3\cos^2 \ + \sqrt{6}\ \cos\ \sin^2\ \}/(\sqrt{6}\cos\ -\ (1 - 3\cos^2\ ))\sin\ . \qquad (4.13)$$

Note that any pair of anti-derivatives for (4.13) di er by an additive constant, and the latter can be taken as another parameter which distinguishes the cylinders in any the families that are considered below.

With $s = s(\ )$ now viewed as an anti-derivative of (4.13), the dependence of u can be obtained by either solving the algebraic equation $u = e^{-\sqrt{6}s}(1 - \cos^2\ )$, or by using $s(\ )$ in the top line of (4.10) to view the latter autonomous equation for as a function of u. In any event, the second approach does freely identify the sign of $_u$ along the cylinder.

With the digression now over, consider the examples.

Example 5 In this example, all of the cylinders have both components of the limit set in the convex end of $\mathbb{R}\ \ (S^1\ \ S^2)$. To describe these cylinders, rst choose (p, p') and $'_0$ as constrained in the preceding digression. Then, each of the cylinders under consideration is parameterized by (4.7) as described in this same digression.

If $|p'|/p > \sqrt{3}/\sqrt{2}$, then p'/p and $'_0$ determines a 1–parameter family of examples where the parameter can be taken to be the maximum value that the function s achieves on the given example. If $p'/p > \sqrt{3}/\sqrt{2}$, then the limit set consists of the $= 0$ closed Reeb orbit and the closed Reeb orbit with $(\ = \ _0, ' = '_0 + p'/pt)$. Meanwhile, the coordinate varies along the pseudoholomorphic cylinder without critical points between 0 and $_0$. If $p'/p < -\sqrt{3}/\sqrt{2}$, then the limit set consists of the $=$ closed Reeb orbit and the $(\ = \ _0, ' = '_0 + p'/pt)$ closed Reeb orbit while varies on the cylinder without critical points between these two extremes.

In the case where $|p'|/p < \sqrt{3}/\sqrt{2}$, then each such pair $((p, p'), '_0)$ determines two single parameter families of pseudoholomorphic cylinders. On each such family, the parameter can still be taken to be the maximum value achieved by the function s. For both of these families, the closed Reeb orbit with $(\ = \ _0, ' = '_0 + p'/pt)$ comprises one of the components of the limit set. But, the families are distinguished by the other component of the limit set, which is either the closed Reeb orbit where $= 0$ or $=$. As before, the function varies without critical points with its supremum and in mum given by the limit set values.

Example 6 These examples consider the cases of (4.7) where $|p'|/p > \sqrt{3}/\sqrt{2}$ and where ranges between the extreme values $_0$ and $_{-0}$. In this regard, note that cos $_0$ has the same sign as p' while cos $_{-0}$ has the opposite sign. For these cylinders, both components of the limit set again lie in the convex end of $\mathbb{R} \times (S^1 \times S^2)$. Here, the limit set consists of the two closed Reeb orbits where ($=\ _0;\ ' = '_0 + p'/pt$) and ($=\ _{-0};\ ' = '_0 + p'/pt$). Note that in this example, $_u$ is negative when $p'/p > \sqrt{3}/\sqrt{2}$ and $_u$ is positive when $p'/p < -jsqrt3/\sqrt{2}$. The parameter which distinguishes the elements in any one family can again be taken to be the value of s at its maximum.

Example 7 These examples also consider the cases of (4.7) where $|p'|/p > \sqrt{3}/\sqrt{2}$, but here ranges between $_{-0}$ and in the positive p' case, and between 0 and $_{-0}$ in the negative p' case. These examples are embedded cylinders with one component of the limit set in the convex end of $R \times (S^1 \times S^2)$ and the other in the concave end. In this regard, the orbit in the convex end is the closed Reeb orbit with ($=\ _{-0};\ ' = ' + p'/pt$). Meanwhile, the component of the limit set in the concave end is, depending on the sign of p', either the $=$ or $= 0$ closed Reeb orbit.

In any event, for fixed $(p; p')$ and $'_0$, there is, once again, a 1–parameter family of such examples. However, in this case, the function s restricts to the cylinder with neither maxima nor minima, and so the value of s at some specified value can be taken as the parameter.

(b) The Index of the Operator D

The purpose of this subsection is to describe certain aspects of the kernel, cokernel and index for the operator D of Propositions 3.2 and 3.6 for an HWZ subvariety in $X = \mathbb{R} \times (S^1 \times S^2)$.

The next proposition summarizes the index story by restating Proposition 3.6 in this special case. With regard to the statement of the subsequent proposition, remember that (C_0) denotes the Euler characteristic of the smooth model curve, C_0, for a given pseudoholomorphic subvariety C. Also, note that the integer $\langle c_1; [C]\rangle$ is to be defined as described in Section 3a.

Proposition 4.1 *Let $C \subset X$ be an irreducible, HWZ pseudoholomorphic subvariety, and use C to define the operator D as described in Section 3b. Then,*

$$\mathrm{Index}(D) = -(C_0) - 2\langle c_1; [C]\rangle + @ + @_+ + @_- ; \qquad (4.14)$$

where $@$, $@_+$ and $@_-$ are defined as follows:

- ∂ denotes the number of ends of C which lie in the convex end of X and which approach an element of C's limit set with $\theta_0 \notin \{0, \pi\}$.
- ∂_+ is the sum of contributions of the form $(1 - 2m_0(E))$ from each end of C_0 which lies in the concave end of X and for which the corresponding element in C's limit set has $\theta_0 \in \{0, \pi\}$. Here, $m_0(E)$ is the positive integer which is defined as follows: Let $m(E)$ denote the absolute value of the pairing between a generator of $H^1(S^1 \times S^2; \mathbb{Z})$ and any sufficiently large, but constant s circle in E. Then, $m_0(E)$ is the least integer which is greater than $m(E)\sqrt{3}/\sqrt{2}$.
- ∂_- is the sum of contributions of the form $2m_0(E) - 1$ from each end of C_0 for which the corresponding element in C's limit set has $\theta_0 \in \{0, \pi\}$ and lies in the convex end of X. Here, $m_0(e)$ is defined as above.

It is important to note that (4.14) places serious constraints on the subvarieties with a given index for the operator D. The following proposition lists the constraints on the subvarieties with index$(D) \leq \partial + 1$.

Proposition 4.2 *Let $C \subset X$ be an irreducible, HWZ subvariety. Then the following is true:*

- *Index$(D) \geq \partial$.*
- *If* index$(D) = \partial$, *then* $\partial = 0$, *1, or 2.*
  - (a) *If $\partial = 0$, then C is a $\theta_0 \in \{0, \pi\}$ case from Example 1.*
  - (b) *If $\partial = 1$, then C is a $\theta_0 \notin \{0, \pi\}$ case from Example 1.*
  - (c) *If $\partial = 2$, then C comes either from Example 3 or 6.*
- *If* index$(D) = \partial + 1$, *then* $\partial = 1$, *2 or 3.*
  - (a) *C is the plane from Example 2, so $\partial = 1$.*
  - (b) *C is a cylinder from Example 5 where one closed Reeb orbit in the limit set is characterized by a pair (p, p') where $|p'|$ is the greatest integer that is less than $(\sqrt{3}/\sqrt{2})|p|$. The other closed Reeb orbit has $\theta_0 = 0$ if $p' < 0$ and $\theta_0 = \pi$ if $p' > 0$. Here, $\partial = 1$.*
  - (c) *C is a cylinder from Example 7 where one closed Reeb orbit in the limit set is characterized by a pair (p, p') where $|p'|$ is the least integer that is greater than $(\sqrt{3}/\sqrt{2})|p|$. Here also, $\partial = 1$.*
  - (d) *C is an immersed, thrice punctured sphere with $\partial = 2$, or 3. Moreover, C has no intersections with the $\theta \in \{0, \pi\}$ locus and none of its limit set closed Reeb orbits have $\theta_0 \in \{0, \pi\}$.*
- *All other cases have* index$(D) \geq \partial + 2$.

The question of existence and the classification of the thrice punctured spheres from Part d of the propositions third point is deferred to Sections 5 and 6. Sections 5 and 6 also provide a formula for the number of double points for these immersed spheres. Note that Theorem A.1 in the Introduction follows from Propositions 3.2 and 4.2.

The following proposition elaborates on the inequality in the final point of Proposition 4.2.

Proposition 4.3 *Let* $C \subset X$ *be an irreducible, HWZ subvariety. Then*

$$\mathrm{Index}(D) \ge 2(-1 + g + Q + \partial + \partial_0^{cc} + \partial_0^{cv}) + \partial^c$$

where

- g *is the genus of* C_0.
- Q *is the number, counted with multiplicity, of intersections of* C *with the* $\theta \in \{0, \pi\}$ *locus.*
- ∂_0^{cc} *is the number of concave side ends of* C *where the* $s \to \infty$ *limit of* θ *is either 0 or* π.
- ∂_0^{cv} *is the number of convex side ends of* C *where the* $s \to \infty$ *limit of* A *is either 0 or* π.
- ∂^c *is the number of concave side ends of* C *where the* $s \to \infty$ *limit of* θ *is not 0 nor* π.

With regard to the organization of the remainder of this section, the next subsection, 4c, contains the first of three parts to the proof of Proposition 4.2. Subsection 4d constitutes a digression that proves Proposition 4.3. With aspects of the latter proof then available, Subsection 4e resumes the proof of Proposition 4.2 and contains the latters second part. The final part of the proof of Proposition 4.2 is in Subsection 4f. There is an extra subsection that considers the cokernel dimension of the operator D when the relevant subvariety is any from an example in Section 4a or any thrice punctured sphere from Part d of the third point of Proposition 4.2. In particular, Proposition 4.8 in this subsection asserts that this cokernel is trivial in all these cases.

(c) Proof of Proposition 4.2, Part 1

Consider in this part of the proof solely the case where no closed Reeb orbit in Cs limit has $\theta_0 \in \{0, \pi\}$. The first point to make is that the preceding assumption implies that

$$\langle c_1, [C]\rangle \ge 0 . \tag{4.15}$$

Indeed, $\langle c_1, [C]\rangle$ computes a sum of non-zero integer terms, where each corresponds to a zero along C of the section over X of the line bundle K given by $(dt + ig^{-1}df) \wedge (\sin^2\theta\, d\varphi + ig^{-1}dh)$. Since neither dt nor df vanish on X, this section vanishes only where $\sin^2\theta\, d\varphi$ and dh vanish, which is along the pair of cylinders where $\theta \in \{0, \pi\}$. Thus, $\langle c_1, [C]\rangle$ counts, with the appropriate integer weight, the intersections of C with these half cylinders. It is left to the reader to verify that the integer weight here is negative in all cases.

Now, under the given assumptions, $\partial_+ = 0$, $\partial_- = 0$ and $\partial \geq 1$. Moreover, as C_0 is connected, $\chi(C_0) \leq 1$ so (4.15) implies that $\mathrm{Index}(D) \geq \partial - 1$ with equality if and only if C_0 is a plane, (and thus $\partial = 1$) and $\langle c_1, [C]\rangle = 0$. To rule out this possibility, remember that a plane has one end, and so there is just one closed Reeb orbit in the limit set. The latter either has $\cos^2\theta_0 = 1/3$ or not. If not, then, as will be argued momentarily, the zeros of the 1–form dt count with appropriate weights to give $\chi(C_0)$, and all of these weights are negative. In particular, this means that $\chi(C_0) \leq 0$ which rules out the $\cos^2\theta_0 \neq 1/3$ possibility.

The claim that the zeros of dt count with negative weights to compute $\chi(C_0)$ is valid for any model curve C_0 for an irreducible, HWZ subvariety with no $\cos^2\theta_0 = 1/3$ closed Reeb orbits in its limit set. Here is a digression to explain why: First, because no limit set closed Reeb orbit has $\cos^2\theta_0 = 1/3$, the gradient on C_0 of the pull-back of the function f is not tangent to any constant, but sufficiently large $|s|$ circle in C_0. This implies that the zeros of df count, with the usual weights, $\chi(C_0)$. Here, one should be careful to count degenerate zeros appropriately. However, the zeros of df are the same as those of dt and all are isolated and all count with negative weight. To see that such is the case, introduce the real and imaginary parts, (x_1, x_2), of a complex parameter on a plane in C_0. Then, by virtue of (1.5), the functions (t, f) obey a version of the Cauchy-Riemann equations,

$$
\begin{aligned}
g t_1 &= f_2, \\
g t_2 &= -f_1.
\end{aligned} \tag{4.16}
$$

This last equation implies via fairly standard elliptic equation techniques that dt and df have the same zeros, that t and f are real analytic functions on C_0, that their zeros are isolated and that all count with negative weights to give $\chi(C_0)$.

By the way, note that the pair (φ, h) obey an analogous equation:

$$
g \sin^2\theta\, \varphi_1 = h_2,
$$

$$g \sin^2\theta\, \varphi_2 = -h_1. \tag{4.17}$$

In particular, (4.17) implies that the zeros of $d\varphi$ or dh count with negative weights to give $-\chi(C_0)$ when C_0 is the model curve for an irreducible, HWZ subvariety which lacks both intersections with the $\theta_0 \in \{0, \pi\}$ cylinders and $\theta_0 \in \{0, \pi/2, \pi\}$ closed Reeb orbits in its limit set.

With the digression complete, consider now the only remaining possibility in the case under consideration, which is for the closed Reeb orbit in C's limit set to have $\cos^2\theta_0 = 1/3$. In this case, the 1–form dj pulls back without zeros on the constant but large $|s|$ circles in C_0. Moreover, as C does not intersect either the $\theta = 0$ or $\theta = \pi$ locus, this 1–form pulls back as a smooth 1–form on C_0. As remarked above, this pull-back also count with negative weights to give $-\chi(C_0)$. Thus, $-\chi(C_0) \geq 0$ and in no case can C_0 be a plane.

Now consider the possibilities when Index$(D) = \partial$. Since $\partial - \chi = 0$, this can happen only if $\chi(C_0) = 0$ and $\langle c_1, [C]\rangle = 0$. As $\chi(C_0) = 0$, the surface C_0 is a cylinder and so has two ends. Thus, either $\partial = 1$ or $\partial = 2$.

Meanwhile, C has no intersections with the $\theta = 0$ and $\theta = \pi$ loci because $\langle c_1, [C]\rangle = 0$. As before, this implies that the 1–form dj pulls back to C_0 as a smooth 1–form. Of course, dt always pulls back to C_0 as a smooth 1–form. Since each is closed, the integral of each along the constant s circles in C_0 must be independent of s. The latter assertion demands that the closed Reeb orbits at the ends of C have the same value for the integers p and p' in (1.8). In particular, $p't + p\varphi$ pulls back to C_0 as a bona fide function, moreover, one which approaches a constant value asymptotically on the ends of C_0.

Now, dt and also $d\varphi$ are either identically zero on C_0 or nowhere zero as $-\chi(C_0)$ would be negative otherwise. The argument is the same as given previously. Moreover, both cannot vanish identically, so at least one is nowhere zero. Either being nowhere zero implies that C is immersed.

If $dt \equiv 0$, then C_0 is a $\cos^2\theta_0 = 1/3$ case from Example 1 and index$(D) = \partial = 1$, or else C is described in Example 3 and index$(D) = \partial = 2$. If $dt \neq 0$, then C can be parametrized by a periodic coordinate $\tau \in \mathbb{R}/(2\pi|p|\mathbb{Z})$ and a linear coordinate u as in (2.19). Here, the pair of functions (x, y) obey (2.20). Note also that x is just the restriction of $p't + p\varphi$ to C. In any event, as argued subsquently to (2.20), the function x in (2.23) must be constant, and this implies that the asymptotic values of $p't+p\varphi$ on the ends of C are identical. Meanwhile, as is demonstrated momentarily, (2.20) can be employed with the maximum principle to prove that the function x has neither local maxima nor local minima on C. Thus, $p't + p\varphi$ is constant on C. This implies that C is either a case from Example 1 with $_0 \in \{0; \ \}$, or else a case from Example 6. The former has index$(D) = @ = 1$ and the latter has index$(D) = @ = 2$.

To complete the index$(D) = @$ story, return now to the postponed part of discussion of local maxima and minima for the function x in (2.19). For this purpose, note that (2.20) implies that x obeys the second-order di erential equation

$$(g^2 \sin^2 \ x_u)_u + (\sin^2 \ x \ ) \ + 2\frac{p'}{p}(\sin \ \cos \ ) \ = 0: \qquad (4.18)$$

Meanwhile, the identity $h/f = \sqrt{6}\cos \ \sin^2 \ /(1 - 3\cos^2 \ )$ can be inverted where $\notin \{0; \ \}$ to write q as a function of h/f, and thus the restriction of to C can be viewed as a function of h/u. Hence, the dependence of this function comes via the dependence of on h. In particular, one can write $= \ _h h$ and then employ the top line in (2.20) to rewrite (4.18) as

$$(g^2 \sin^2 \ x_u)_u + (\sin^2 \ x \ ) \ - 2\frac{p'}{p}(g^2 \sin^3 \ \cos \ ) \ _h x_u = 0: \qquad (4.19)$$

The strong form of the maximum principle applies directly to the latter equation and precludes x from having local maxima and minima.

Finally, this part of Proposition 4.2's proof ends by considering the possibilities when index$(D) = @ + 1$. Here, (4.14) allows only two possibilities for the pair and $\langle c_1; [C]\rangle$; the rst has $(C_0) = 1$ and $\langle c_1; [C]\rangle = -1$, while the second has $(C_0) = -1$ and $\langle c_1; [C]\rangle = 0$. In the rst case, C_0 is a plane which intersects the union of the $= 0$ and $=$ loci exactly once. Moreover, the fact that C_0 is a plane implies that the coordinate t must restrict to C_0 as an $\mathbb{R}$–valued function, and thus t must restrict as an $\mathbb{R}$–valued function to the closed Reeb orbit which comprises C's limit set. The only closed Reeb orbits with this property have $\cos^2 \ _0 = 1/3$, and thus t is constant. Therefore, t approaches a constant value on the end of C_0 which implies, via the maximum principle, that t is constant on the whole of C_0. Hence, C is described by Example 2 in Section 4a.

The other possibility has $\langle c_1; [C]\rangle = 0$ and $(C_0) = -1$. Now, by virtue of the de nition of the pairing $\langle c_1; \ \rangle$ in Section 3a, this rst condition makes C disjoint from the $\in \{0; \ \}$ locus. Meanwhile, with Euler characteristic 1, C_0 is either a once punctured torus with $@ = 1$ or else a thrice punctured sphere in which case the possibilities for $@$ are 1, 2 or 3. The torus case is ruled out by the following argument: If the one end has $p \neq 0$, then $|f|$ increases uniformly with increasing $|s|$ when the latter is su ciently large. Thus, if $p > 0$, the function f would have a global minimum on C and if $p < 0$, then f would have a global maximum. As neither can happen, no such torus exists. In the case $p = 0$, then $p' \neq 0$ and the same argument applies with h substituted for f.

Consider now the case for punctured spheres. As Proposition 4.7, below, asserts that the cases that arise in Proposition 4.2 are immersed, all that is left to say about these $\chi(C_0) = -1$ and $\langle c_1, [C]\rangle = 0$ cases is summarized by the following assertion:

$$\textit{There are no cases with } \partial = 1. \tag{4.20}$$

The argument for this claim is a somewhat more sophisticated version of the preceding argument that ruled out punctured tori. In fact, the argument that follows proves (4.20) with no preconditions on the number of concave side ends or the genus. The only precondition is that C has no intersections with the $\theta = 0$ and $\theta = \pi$ loci. The claim in (4.20) is an immediate corollary to the following assertion:

> *If C has no intersections with the $\theta = 0$ and $\theta = \pi$ loci, and C is not an $\mathbb{R}$–invariant cylinder then the pullback of the function θ to C_0 has neither local maxima nor local minima.*
>
> *In addition, θ's restriction to any concave side end takes values at arbitrarily large s that are both larger and smaller than its $s \to \infty$ limit on the end.* (4.21)

To prove the first point, use (4.16) and (4.17) to derive a second-order differential equation for θ to which the maximum principle can be applied. In this regard, it is important to note that the latter equation has the form

$$\Delta\theta + \sigma_1\,\theta + \sigma_2\, d\theta = 0, \tag{4.22}$$

where Δ is the Laplacian on C_0, and where σ_1 and σ_2 are well defined provided that $\theta \notin \{0, \pi\}$.

To argue for the second point, it proves useful to introduce the function $\phi \equiv h/f$ which is defined where $\cos^2\theta \neq 1/3$. In particular, the latter is a monotonic function of θ on each of the three $\cos^2\theta \neq 1/3$ components of $(0, \pi)$ and it obeys a version of (4.22). To discuss ends where the $s \to \infty$ limit of θ satisfies $\cos^2\theta = 1/3$, the function $\phi^{-1} = f/h$ will be used instead. The latter is a monotonic function of θ on each $\theta \neq \pi/2$ component of $(0, \pi)$ and also obeys a version of (4.22).

To proceed, suppose that E is a concave side end of C_0 whose corresponding closed Reeb orbit has angle $\theta = \theta_0$ with $\cos^2\theta_0 \neq 1/3$. The argument for the case where $\cos^2{}_0 = 1/3$ is left to the reader in as much as it is essentially identical to the one given below after changing the roles of the pair (f, t) and (h, φ). With the $\cos^2{}_0 \neq 1/3$ assumption understood, note that the second point in (4.21) follows if the assertion in question holds for the function instead of . In this regard, note that the $s \to \infty$ limit of is $\sqrt{6}\cos{}_0 \sin^2{}_0(1 - 3\cos^2{}_0)^{-1}$. Use $_0$ to denote the latter. To continue, remark that it is enough to prove the following: Given $s_0 > 0$, then $_0$ is neither the in num nor supremum of the values of on the $s \geq s_0$ portion of the end E. Since the arguments in either case are essentially the same, only the argument .for the in mum will be given. To start the latter, suppose that $_0$ were 's in mum. The argument that follows derives a contradiction from this assumption.

To obtain the contradiction, note rst that for large values of the coordinate s, the end of C in question can be parameterized as in (2.13) where the functions (x, w) obey (2.15). Furthermore, $|x|$ and $|w|$ tend to zero as $\to \infty$. Also, as asserted in Lemma 2.5, the derivatives of x and w tend to zero as $\to \infty$. Keep all of this in mind. The focus here is on the function w since up to a positive multiple, w is $- {}_0$. Thus, under the given assumption, $w \geq 0$ for large s on E and thus $w > 0$ at large s as cannot have a local minimum. However, this possibility is precluded by (1.23). Indeed, were w positive everywhere, then at all large values of the prameter in (2.13), the bottom component of (2.15) would force the di erential inequality

$$w + x - 2^{-1} w \quad 0. \tag{4.23}$$

Here, > 0 is the constant that appears in (2.16). To see why (4.23) holds, rst note that the term $\mathfrak{R}(a, b)$ as it appears in (2.17) depends on its rst entry, a, only through a's bottom component. Indeed, this follows from the lack of φ dependence in the complex structure in (1.5). This point understood, then the $\mathfrak{R}$ term in (2.15) is bounded by a constant multiple of $|w|(|w| + |w| + |x|)$ and then (4.23) follows from Lemma 2.5's guarantee that the derivatives of x and w vanish in the limit as $\to \infty$.

Now, (4.23) implies that the function, $\underline{w}()$, of which is obtained by averaging w over the = constant circles would, per force, be greater than zero and obey

$$\underline{w} - 2^{-1} \underline{w} \quad 0 \tag{4.24}$$

at large values of . Of course, the latter inequality forces the growth of w as gets large; and this last conclusion provides the promised contradiction.

(d) Proof of Proposition 4.3

Given that the Euler characteristic $\chi(C_0)$ is equal to $2 - 2g -$ (the number of ends), the asserted inequality follows from the inequality for $-2\langle c_1, [C]\rangle + \partial_+ + \partial_-$ asserted by the following result.

Lemma 4.4 *Let C be an irreducible, HWZ-subvariety. Then*

$$-2\langle c_1, [C]\rangle + \partial_+ + \partial_- \geq 2Q + \partial_0^{cc} + \partial_0^{cv} . \tag{4.25}$$

The remainder of this subsection contains the following proof.

Proof of Lemma 4.4 As the proof of this lemma is long, it is broken into seven steps.

Step 1 It is the computation of $\langle c_1, [C]\rangle$ that complicates the proof; the complexity of this computation stems from the fact that the defining section of the canonical bundle K near the $\theta_0 = \{0, \pi\}$ ends is different from that used near the other ends. This complication is addressed via a decomposition of $\langle c_1, [C]\rangle$ as a sum of terms, one of which algebraically counts the zeros on C of the section $(dt + ig^{-1}df) \wedge (\sin^2\theta\, d\varphi + ig^{-1}dh)$ of K, while the others, one for each $\theta_0 \in \{0, \pi\}$ end of C, are 'correction factors'. This decomposition of $\langle c_1, [C]\rangle$ is provided momentarily. Coming first is a lemma with proof which simplifies the definition of this decomposition.

Lemma 4.5 *An irreducible, HWZ-pseudoholomorphic subvariety in* $\mathbb{R} \times (S^1 \times S^2)$ *that is not a $\theta \in \{0, \pi\}$ cylinder from Example 1 intersects such a cylinder a finite number of times.*

Proof of Lemma 4.5 Since the $\theta = 0$ and $\theta = \pi$ loci are pseudoholomorphic cylinders, and isolated in the sense of [12], this lemma is a version of Proposition 4.1 in [12]. Even so, a proof is given below since various portions of it are used subsequently. In any event, the proof that follows is different from that offered in [12]. The proof of this lemma constitues Steps 2 and 3 of the proof of Lemma 4.4. In this regard, Step 2 establishes that there are at most finitely many intersections on the concave side of $\mathbb{R} \times (S^1 \times S^2)$, while Step 3 does the same for the convex side.

Step 2 This step rules out the possibility of infinitely many intersections between C and the $\theta \in \{0, \pi\}$ subvarieties where $s > 0$. To be more precise, only intersections with the $\theta = 0$ cylinder will be discussed here since the analogous discussion for the intersections with the cylinder are identical but for insigni cant notational changes.

To start the story, pick some large and positive value, s_0, for s with the property that C is disjoint from both the 0 cylinder and the = =2 and ' = 0 cylinder on the slice $s = s_0$. In this regard, note that either C coincides with one of these subvarieties or else the intersections with either have no accumulation points. By assumption, C does not coincide with the former, and if C coincides with the latter, then there is nothing further to discuss. Thus, it is safe to assume that such an s_0 exists. With s_0 chosen, let denote the minimum distance between C's intersection with the $s = s_0$ slice and the intersection with this slice of the 0 and (=2; '(' = 0) cylinders.

The discussion proceeds from here by assuming that C does, in fact, have an in nite number of intersections with the 0 cylinder where $s > s_0$; some unacceptable foolishness is then derived from this assumption. In particular, a contradiction arises by considering C's $s > s_0$ intersection number with certain pseudoholomorphic cylinders from Example 4 with h > 0 and ' 0.

In this regard, there are some preliminary facts to recall about intersection numbers. First, the local intersection numbers between pairs of pseudoholomorphic subvarieties are strictly positive. Second, the local intersection numbers between such subvarieties are invariant under su ciently small perturbations of the maps of the model, smooth curves. Third, if a subvariety has compact intersection with the $s > s_0$ portion of $\mathbb{R}$ (S^1 S^2) and no intersections with C in the $s = s_0$ slice, then it has a well de ned $s > s_0$ intersection number with C, and this intersection number is invariant under compact deformations of the subvariety which avoid C on the $s = s_0$ slice.

Next, some comments are in order concerning the (h ; ' 0) cylinders. First, each such cylinder intersects the $s > s_0$ portion of $\mathbb{R}$ (S^1 S^2) in a compact set. Second, there exists $_0$ such that if 0 < < $_0$, then the intersection of the (h ; ' 0) cylinder and the $s = s_0$ slice occurs in a radius =10 tubular neighborhood of the union of the cylinders where 0 and where (=2; ' 0). Thus, as long as 0 < < $_0$, no (h ; ' 0 cylinder intersects C where $s = s_0$. Note that these last two facts imply that each such (h ; ' 0) cylinder has a nite number of intersections with C where $s > s_0$, and thus nite $s > s_0$ intersection number with C. Moreover, this intersection number is *independent* of as long as 2 (0; $_0$). Third, x $s_1 > s_0$ and a tubular neighborhood of the s_0 s s_1 portion of the 0 cylinder. Then, there exists some $_1$ such that for 0 < < $_1$, each (h ; ' 0) cylinder intersects this tubular neighborhood as a graph over the $s_0 \le s \le s_1$ portion of the $\theta = 0$ cylinder, and as $\delta \to 0$, these graphs converge in a smooth manner to the trivial graph, the $\theta = 0$ cylinder itself.

Note that this third comment, plus the remarks about positivity of local intersection numbers and their invariance under small perturbations implies the following: Fix $n > 0$ and there exists $\delta[n] > 0$ such that when $\delta \in (0, \delta[n])$, then the $(h = \delta, \varphi = 0)$ cylinder has $s > s_0$ intersection number at least n with C. Of course, this conclusion is ridiculus, because the δ–invariance of this intersection number (for $0 < \delta < \delta_0$) implies that the $(h = \delta_0/2, \varphi = 0)$ cylinder has infinite intersection number with C.

Step 3 This step rules out the possibility of C intersecting the $\theta = 0$ and $\theta = \pi$ cylinders infinitely many times where $s < 0$. Here again, only intersections with the $\theta = 0$ cylinder will be discussed. The strategy here is similar to that used in Step 2: Assume that C has infinitely many negative s intersections with the $\theta = 0$ cylinder and find a ridiculous conclusion. In this case, the untenable conclusion is that C has infinitely many $s < 0$ intersections with certain cylinders from Example 7.

To start the story, choose s_0 so that C is disjoint from the $\theta = 0$ cylinder where $s = s_0$. Now, let ε denote the minimum distance between C's intersection with the $s = s_0$ copy of $S^1 \times S^2$ and that of the $\theta = 0$ cylinder. Next, choose p' to be an integer more negative than -1 and such that with θ_0 determined by p' as in Example 7, no θ_0 closed Reeb orbits lie C's limit set.

Example 7 describes a 1–parameter family of pseudoholomorphic cylinders all labeled by θ_0 and some fixed choice for an angle $\varphi_0 \in [0, 2\pi]$. In this regard, fix $\varphi_0 = 0$ and then the resulting 1–parameter family of cylinders can be labeled as $\{\Sigma_r\}_{r \in \mathbb{Z}}$ where the distinguishing feature of Σ_r is that its $s = r$ slice has distance $\varepsilon/2$ from the $s = r$ slice of the $\theta = 0$ cylinder. What follows are some relevant facts to note about Σ_r. First, if $r < s_0$, then there are no $s = s_0$ intersections between Σ_r and C, and there are at most a finite number of such intersections where $s < s_0$. Thus, there is a well defined $s < s_0$ intersection number between each $r \le s_0$ version of Σ_r and C. Moreover, this intersection number is independent of $r \le s_0$. In this regard, note that there are no very negative s intersections between any Σ_r and C since the limit sets for C and Σ_r are disjoint.

Here is a second crucial fact: Fix $s_1 < s_0$ and a tubular neighborhood of the portion of the $\theta = 0$ cylinder where $s \in [s_1, s_0]$. Then, there exists r_1 such that for $r < r_1$, each Σ_r intersects this tubular neighborhood as a graph over the $s_1 \le s \le s_0$ portion of the $\theta = 0$ cylinder, and as $r \to -\infty$, these graphs converge in a smooth manner to the trivial graph, the $\theta = 0$ cylinder itself.

Now, as before, this last observation implies that given any positive n, the $s < s_0$ intersection number between $_r$ and C is at least n if r is su ciently small. However, the invariance of this intersection number with variations of r in $(-1;s_0]$ implies the silly conclusion that C has in nite $s < s_0$ intersection number with any $_r$.

Step 4 With the proof of Lemma 4.5 complete, here is the advertised decomposition of $hc_1;[C]i$:

$$hc_1;[C]i = - {}_0 + \quad {}_{E 2 V} \quad (E) \tag{4.26}$$

where $_0$ and the sum are de ned as follows: First, $_0$ is the intersection number between C and the 0 and cases from Example 1. This is to say that $- {}_0$ counts the zeros of the section $(dt + ig^{-1}df) \wedge (\sin^2 \ d' + ig^{-1}dh)$ of K with each zero contributing the usual weight to the count. As explained above, these weights are all negative. By the way, Lemma 4.5 insures that there are at most a nite number of terms which enter into the de nition of $_0$. Meanwhile, the sum in (4.26) is indexed by the elements of the collection, V, of ends of C which correspond to the $_0\ 2\ f0;\ g$ closed Reeb orbits in C's limit set. The weight (E) of an end $E\ 2\ V$ accounts for the fact that the section in $(dt + ig^{-1}df) \wedge (\sin^2 \ d' + ig^{-1}dh)$ of K is not the correct section to use on E. The preferred section over E has the form

$$(dt + ids) \wedge (dx_1 - idx_2) + O(j\sin \ j); \tag{4.27}$$

where $x_1 = \sin \ \cos\ '$ and $x_2 = \sin \ \sin\ '$. Actually, any section of Kj_E can be used as long as it is homotopic to that in (4.27) through sections which do not vanish at large values of jsj on E. This freedom to use homotopic sections simpli es the computation for (E).

Given (4.26) and with $_0$ understood to be non-positive and zero if and only if C avoids the $\ 2\ f0;\ g$ loci, then Lemma 4.4 becomes an immediate corollary to the following lemma.

Lemma 4.6 *The number* (E) *in (4.26) is constrained to obey*

$v(E) \quad -m_0(E)$ *when* E *is in the concave side of* $\mathbb{R} \quad (S^1 \quad S^2)$.

$v(E) \quad m0(e) - 1$ *when* E *is in the convex side of* $\mathbb{R} \quad (S^1 \quad S^2)$.

Here, $m_0(E)$ *is de ned as in Proposition 4.1.*

The remaining Steps 5{7 of the proof of Lemma 4.4 are devoted to the following proof.

Proof of Lemma 4.6 Note that the discussions below consider only the case where $\theta_0 = 0$, for the case $\theta_0 = \pi$ is identical save for some notation and sign changes.

Step 5 The end E can be parameterized by coordinates (ρ,τ) as in (2.14). Remember that this parametrization has $\tau \in \mathbb{R}/(2\pi m\mathbb{Z})$ and r is either in $(-\infty, 1]$ or $[-1, \infty)$ depending on whether E sits in the convex or concave side of $\mathbb{R}\times(S^1\times S^2)$. Here, m is the same as the number $m(E)$ that is defined in Proposition 4.1. The parametrization then writes

$$(t = \tau,\ f = -e^{-\sqrt{6}\,\rho},\ a_1 = a_1(\rho,\tau),\ a_2 = a_2(\rho,\tau)) \tag{4.28}$$

where $(a_1,a_2) = 6^{-1/4}e^{-\sqrt{6}\,\rho/2}h^{1/2}(\cos\varphi,\sin\varphi)$. In this regard, note that the pair of functions (x_1,x_2) which appear in (4.27) is related to (a_1,a_2) via

$$(a_1,a_2) = (x_1,x_2) + O(x_1^2 + x_2^2). \tag{4.29}$$

Step 6 To continue with the definition of $\delta(E)$, it proves useful to select a function χ on $\mathbb{R}$ with the following properties:

$$\begin{aligned} &\chi(\rho) = 1 \text{ for } \rho \ge 1\\ &\chi(\rho) = 0 \text{ for } \rho \le -1\\ &\chi' \ge 0. \end{aligned} \tag{4.30}$$

Then, for $R \in \mathbb{R}$, set $\chi_R(\rho) \equiv \chi(\rho - R)$.

A section of K on E can be written using χ which interpolates between the 'wrong' section, $(dt + ig^{-1}df)\wedge(\sin^2\theta\, d\varphi + ig^{-1}dh)$, near $\rho = 0$ and the preferred section in (4.27) where $|\rho| \gg 1$. Doing so finds that the number $\delta(E)$ in (4.26) is equal to the usual algebraic count of the number of the zeros of a certain complex valued function. In particular, by virtue of (4.29) and the previously mentioned homotopy flexibility, the following functions suffice:

- $\chi_R(a_1 - ia_2) - (1 - \chi_R)$, with any $R \gg 1$ when E is in the concave side of $\mathbb{R}\times(S^1\times S^2)$.
- $(1 - \chi_R)(a_1 - ia_2) - \chi_R$, with any $R \ll -1$ when E is in the convex side of $\mathbb{R}\times(S^1\times S^2)$. (4.31)

A straightforward homotopy argument will verify that the algebraic counting of the zeros of the functions in (4.31) depends only on the winding number of the large $|\rho|$ version of the map from the circle $\mathbb{R}/(2\pi m\mathbb{Z})$ to $\mathbb{C} - \{0\}$ which sends τ to $(a_1 - ia_2)|_{(\rho,\tau)}$. In particular, if this function winds like $e^{-ik\tau/m}$, with $k \in \mathbb{Z}$, then

$v(E) = -k$ *when* E *is in the concave side of* $\mathbb{R} \ \ (S^1 \ \ S^2)$.

$v(E) = k$ *when* E *is in the convex side of* $\mathbb{R} \ \ (S^1 \ \ S^2)$. (4.32)

The verification of (4.32) is left as an exercise save for the warning to remember that the orientation of E is defined by the restriction of $-d \ \wedge d$. Note also that (4.32) constitutes a special case of some more general conclusions in Section 4 of [12].

Step 7 This step reports on the possibilities for the winding numbers of $a_1 - ia_2$ on the large, constant $| \ |$ circles in E. In particular, the possibilities for the winding number of $a_1 - ia_2$ are constrained by the fact that $(a_1; a_2)$ satisfy (2.15) and in addition

$$\lim_{| \ | \to \infty} (a_1^2 + a_2^2) = 0. \tag{4.33}$$

Note also that Lemma 4.5 and equation (4.29) insure that $(a_1^2 + a_2^2)$ is never zero when $| \ |$ is sufficiently large.

To see how these constraints arise, use the positivity at large $| \ |$ of $a_1^2 + a_2^2$ to write the complex number $a_1 - ia_2$ as $a_1 - ia_2 = e^{-v-i(w+k \ /m)}$; here v and w are smooth, real valued functions of $(\ ; \ )$ where $| \ |$ is large; while $k \in \mathbb{Z}$ is the winding number in question. Note that v has no limit as $| \ |$ tends to infinity as (4.33) implies that

$$\lim_{| \ | \to \infty} \inf v(\ ; \ ) = \infty. \tag{4.34}$$

In addition, the top component of (2.15) implies that

$$v \ = w \ + k/m - \sqrt{3}/\sqrt{2} + c, \tag{4.35}$$

where $|c| \ \ e^{-v}$ at large values of $| \ |$ by virtue of (4.33) and Lemma 2.5.

To apply (4.35), first introduce $\underline{v}(\ )$ to denote the average of v over a large, but constant circle. Likewise, introduce $\underline{c}$. Both are smooth function of where $| \ |R$ is large, and (4.35) asserts that

$$\underline{v} \ = k/m - \sqrt{3/2} + \underline{c}. \tag{4.36}$$

In order to use (4.36) to obtain the concave side constraints in Lemma 4.6, first pick > 0 and then some large $_0$ so that both $\underline{v}$ and $\underline{c}$ are defined where $_0$ and so that $|\underline{c}|$ for such . Then, take $R \ \ 1$ and integrate both sides of (4.36) from $_0$ to $_0 + R$. The result provides the inequality

$$\underline{v}(_0 + R) - \underline{v}(_0) \ \ (k/m - \sqrt{3}/\sqrt{2} + \ )R. \tag{4.37}$$

Now, as k/m is rational, $\sqrt{3}/\sqrt{2}$ is irrational, m is fixed by the end E and can be made as small as desired by taking $_0$ large, the inequality in (4.37) is compatible with condition in (4.34) only when $k > \sqrt{3}m/\sqrt{2}$. As $m_0(E)$ is defined to be the smallest integer that obeys this inequality, it follows that $k > m_0(E)$ which, together with the first line in (4.32), gives the first assertion of Lemma 4.6.

To consider the convex side constraints in Lemma 4.6, pick > 0 but small as before, and then some very negative $_0$ so that now both $\underline{v}$ and $\underline{c}$ are defined where $_0$ and so that $|c|$ for such . Then, take R 1 and integrate both sides of (4.36) from $_0 - R$ to $_0$. The result provides the inequality

$$\underline{v}(\ _0) - \underline{v}(\ _0 - R) \quad (k/m - \sqrt{3}/\sqrt{2} - \ )R. \tag{4.38}$$

Arguing as before finds (4.38) compatible with (4.34) only when $k < \sqrt{3}m/\sqrt{2}$. Thus, k is at most one less than $m_0(E)$ which, together with the second line in (4.32), implies the second assertion of Lemma 4.6.

(e) Proof of Proposition 4.2, Part 2

This part of the proof of Proposition 4.2 considers the cases where there is a closed Reeb orbit in C's limit set which has $_0 \in \{0, \ \}$. In this regard, note first that such a subvariety has (C_0) 0 since the pullback of dt to C_0 is not exact on an end which approaches a $_0 \in \{0, \ \}$ orbit. The same argument implies that C must have one or more limit set closed Reeb orbits with $\cos^2 \ _0 \neq 1/3$.

The next restriction is simply that if @ in (4.14) is zero, then C is a cylinder which is described by (4.2). To show that such is the case, note first that if all elements of C's limit set have $_0 \in \{0, \ \}$, then $f < 0$ on the ends of C and the maximum principle requires the $f < 0$ condition to hold on the whole of C. In particular, f is not zero on C, and so h/f is well defined on C. Moreover, $h/f \ \ -\sqrt{3}/sqrt2 \sin^2$ as nears either 0 or , so tends to zero on all ends of C. Meanwhile, the maximum principle applies to (4.21) and forbids non-zero local maxima or minima. Thus 0 and so h 0 and C is described by (4.2). Now consider that Lemma 4.4 and the observation that (C_0) 0 imply the following:

> *An irreducible, HWZ subvariety with a* $_0$ $\{0, \ \}$ *closed Reeb orbit in its limit set has index* $(D) >$ @ $+ 1$ *unless it is an immersed cylinder which avoids both the* $_0 \in \{0, \ \}$ *loci.* (4.39)

With regard to (4.39), here is the explanation for the assertion that C is immersed: In the case at hand, (C_0) must be zero for index(D) to equal @ $+ 1$.

Thus, C_0 is a cylinder and so C has exactly two ends. Moreover, as dt is homologically nontrivial on the end where θ tends to either 0 or π, it must be so on the other. Therefore, the other end can not have $\cos^2\theta_0 = 1/3$, and so dt pulls back to sufficiently large, but constant $|s|$ circles without zeros. It then follows from (4.16) that the number of zeros of dt's pullback to C_0 is equal to $-\chi(C_0)$, and is thus zero. This last point implies that C is immersed.

To summarize the preceding, the cylinder C must have one of its limiting closed Reeb orbit with $\theta_0 \notin \{0, \pi\}$ and $\cos^2\theta_0 \neq 1/3$. This 'other' closed Reeb orbit is therefore characterized in part by the condition that $\varphi - \frac{p'}{p}t$ is constant, where p' and p are relatively prime integers and $p \neq 0$. This last point is important because if $\varphi - \frac{p'}{p}t$ is constant on C itself, then C comes from Example 7 when the $\theta_0 \in \{0, \pi\}$ closed Reeb orbit corresponds to end of C from the concave side of $\mathbb{R} \times (S^1 \times S^2)$; otherwise C comes from Example 5. In this regard, only the cases which are described in Proposition 4.2 have $\text{index}(D) = \partial + 1 = 2$; all of the others have $\text{index}(D) > 2$. This last assertion follows from Propositions 4.1 and (4.32).

Thus, a demonstration that $\varphi - \frac{p'}{p}t$ is constant on C completes the proof of Proposition 4.2 but for the immersion remark in Part d of the third point. The proof of the latter is deferred to the next subsection while the remainder of this subsection demonstrates that $\varphi - \frac{p'}{p}t$ is indeed constant on C.

The constancy of $\varphi - \frac{p'}{p}t$ is considered below only for the case where C has a $\theta_0 = 0$ closed Reeb orbit. As before, the considerations for the $\theta_0 = \pi$ case are identical in all essential aspects.

The demonstration starts with the annunciation of a maximum principle:

The restriction to C of the multivalued function $\varphi - \frac{p'}{p}t$ has neither local maxima nor minima. (4.40)

To prove this claim, note first that dt pulls back to C without zeros. Indeed, this follows from a combination of three facts: First, C is a cylinder so has zero Euler characteristic. Second, dt pulls back without zeros to all sufficiently large and constant $|s|$ circles so its zeros with the appropriate integer weight count $\chi(C)$. Finally, all such weights are negative by virtue of (4.16). To complete the proof of (4.40), note that as dt pulls back without zeros to C, so df does too and this allows C to be parameterized as in (2.19) in terms of functions (x, y). In this regard, the constant φ_0 together with p' and p label the $\theta_0 \notin \{0, \pi\}$ closed Reeb orbit in C's limit set. In particular, up to a constant, $\varphi - \frac{p'}{p}t$ is the function x. The latter, with y, satisfies (2.20) and

(2.20) implies the second-order differential equation for x in (4.18) to which the maximum principle applies.

Now, the function x tends to zero on the end of C that corresponds to the $_0 \in \{0, \pi\}$ closed Reeb orbit, so if x tends to zero on the other end of C as well, then (4.40) establishes that the combination $\varphi - \frac{p'}{p} t$ is constant on C. Thus, the demonstration now focuses on the behavior of x on the end of C which is near the $\theta = 0$ cylinder. In this regard, remember that this end of C is characterized in part by the pair of integers (k, m), where m is the absolute value of the degree of the fundamental class of a sufficiently large, but constant $| \ |$ circle in $H_1(S^1 \times S^2; \mathbb{Z})$ and k is the winding number defined so at all sufficiently large and constant $| \ |$, the $\mathbb{C}$–valued function $a_1 - ia_2$ is homotopic to $e^{-ik \ /m}$ as a map from $\mathbb{R}/(2\pi m\mathbb{R})$ to $\mathbb{C} - \{0\}$. Here, C is parametrized as in (4.28).

The integers (k, m) can be identified as follow: The fact that C is a cylinder and dt is closed implies that $m = |p|$; and the fact that C has no intersections with the $\in \{0, \}$ cylinders and $d\varphi$ is closed implies that $k = \text{sign}(p)p'$.

Next, write $a_1 - ia_2 = e^{-v-i(w+k \ /m)}$ so that v and w are smooth, real valued functions of $(\, , \,)$ where $| \ |$ is large. Note that $w = x + \varphi_0$. Also, v satisfies (4.34). Now introduce a new function, $z(\, , \,)$, by writing $v = (k/m - \sqrt{3}/\sqrt{2}) \ + z(\, , \,)$ and note that (2.15) implies the Cauchy-Riemann like equations

$$
\begin{aligned}
z \ &= w \ + c, \\
w \ &= -z \ + c'.
\end{aligned}
\tag{4.41}
$$

Here, c appears already in (4.35) while c' is a smooth function which satisfies similar bounds as c. In particular, by virtue of (2.17), Lemma 2.5 and Proposition 2.3's insurance for the exponential decay at large $| \ |$ of $(a_1^2 + a_2^2)$, these functions at sufficiently large $| \ |$ obey

$$
|c| + |c'| \le e^{-\delta | \ |}, \tag{4.42}
$$

with $\delta > 0$ some independent constant.

As is demonstrated below, it is a moment's investment to establish from (4.34), (4.41) and (4.42) that w (and also v) have constant limits as $| \ |$ goes to infinity. Given that such is the case, it follows from the fact that $\underline{x}$ in (2.23) is constant that the limit of w is φ_0. As $w = x + \varphi_0$, this means that x vanishes asymptotically on the end of C near the $\theta = 0$ cylinder, as required.

The argument that (z, w) has a constant limit as $| \ |$ goes to infinity is given here for the concave side case only, as the other case is settled with the identical argument up to a sign change or two. To proceed in this case, rst note that the average values on the constant r circles of z and w have limits as $\to \infty$ as can be seen by integrating both sides of (4.41) over the constant circles. Thus, the case is settled with a proof that the remaining parts of x and w decay to zero at large . For this purpose, introduce the $' = \;\; = 0$ version of the operator L_0 in (2.16) and let f^+ and f^- denote the functions of (for large) whose values are the respective L^2 norm of the L^2 orthogonal projection of the column vector with top component z and bottow component w onto the span of the eigenvectors of L_0 with positive (+) and negative ($-$) eigenvalues. Then (4.41) and (4.42) imply that

$$f^- \quad f^-/N - e^- \quad \text{and} \quad f^+ \quad -f^+/N + e^- \; : \tag{4.43}$$

Integration of the left most equation in (4.43) nds the dichotomy: Either $f^-(\;) \quad ^-e^{/N}$ with $^- > 0$ or else $f^-(\;) \quad ^-e^-$. Meanwhile, integration of the right most equation in (4.43) nds that f^+ has no choice but to decay exponentially fast as tends to in nity. However, exponential growth of f^- is forbidden by (4.34) as the part of $(v; w)$ that contributes to f^- integrates to zero around any constant circle and thus exponential growth of f^- forces exponentially large values on $-v$ at places on each large r circle.

(f) Proof of Proposition 4.2, Part 3

The proof of Proposition 4.2 is completed here with a proof of the following formal restatement of a portion of Part d of Proposition 4.2's third point:

Proposition 4.7 *An HWZ pseudoholomorphic subvariety C whose model curve, C_0 is a thrice-punctured sphere with index(D) = @ + 1 is the image of C_0 via an immersion.*

Proof of Proposition 4.7 To begin, consider the zeros of the pullback to C_0 of $\;_t dt - \;_{'} d'$ in the case where $\;_t$ and $\;_{'}$ are constant real numbers. Of course, this pullback has a zero at any local singular point of the tautological map to X since all pullbacks vanish at such points. In any event, each zero of the pullback of $\;_t dt - \;_{'} d'$ counts with a negative weight when used in an 'Euler class' count. A proof of this last assertion uses (4.16) and (4.17) to derive a second order di erential equation for the pullback to C_0 of $\;_t t - \;_{'} '$ to which the maximum principle applies and rules out local extrema. By the way, an argument near points where $dt \neq 0$ can also be made directly from (4.18) since the corresponding x in the case of (4.18) where $p'/p = \;_t / \;_{'}$ di ers locally by an additive constant from a non-zero multiple of $\;_t t - \;_{'} '$.

Not all values of α_t and α_φ provide a form whose zero count produces $\chi(C_0)$, hence the quotes in the preceding paragraph around the words 'Euler class.' Indeed, the equality with $\chi(C_0)$ of the algebraic counting of the zeros of the pullback of $\alpha_t dt - \alpha_\varphi d\varphi$ can be guaranteed only when $\alpha_t/\alpha_\varphi \neq p'/p$ for all pairs $(p; p')$ which come via (1.8) from the closed Reeb orbits in C's limit set. To explain, this constraint on α_t/α_φ arises precisely because the form $p'/p dt - d\varphi$ pulls back as zero on the closed Reeb orbit which supplied the pair $(p; p')$. In particular, the condition $\alpha_t/\alpha_\varphi \neq p'/p$ for all such pairs guarantees that the pullback of $\alpha_t dt - \alpha_\varphi d\varphi$ to all sufficiently large, constant $|s|$ circles in C_0 has no zeros. When such is the case, a standard argument proves that a multiplicity weighted count of the zeros of $\alpha_t dt - \alpha_\varphi d\varphi$ yields $\chi(C_0)$.

Having digested the preceding, fix $\delta \neq 0$ but small and take $(p'/p + \delta)dt - d\varphi$ where the pair $(p; p')$ comes from one of the closed Reeb orbits in C's limit set where $\cos^2\theta_0 \neq 1/3$. The pullback of this form has norm $O(\delta)$ on an end of C which approaches a closed Reeb orbit that supplies the pair $(p; p')$, and it is relatively large, $O(1)$, on other ends of C. In any event, the argument from the preceding paragraph applies here and explains why the pullback of $(p'/p + \delta)dt - d\varphi$ to C_0 has exactly one zero provided that δ is small in absolute value but not zero. Thus, if the tautological map from C_0 to X has a singular point, then the pullback of $(p'/p + \delta)dt - d\varphi$ vanishes only at this point. In particular, this pullback cannot have a zero on any end of C. However, just such a zero is exhibited below, and so the tautological map from C_0 to X lacks local singular points.

To exhibit the asserted zero, focus on an end $E \subset C$ which approaches a closed Reeb orbit that supplies $(p; p')$, and parameterize said end by coordinates $(\sigma; u)$ as in (2.19) and (2.20). Thus, $\sigma \in \mathbb{R}/(2\pi m|p|)$ and either $|u| \in [u_0; \infty)$ or $|u| \in (0; u_0]$ depending on whether E is on the convex or concave side of $\mathbb{R} \times (S^1 \times S^2)$. In terms of this parameterization, $(p'/p + \delta)dt - d\varphi$ pulls back as

$$(\delta - x_\sigma)d\sigma - x_u du; \tag{4.44}$$

and thus it vanishes on E only at points where $x_u = 0$ and where $x_\sigma = \delta$.

The constraint $x_u = 0$ is satisfied at two or more points on every constant u circle since the first line in (2.20) identifies these points with the critical points of y's pullback to such a circle. Meanwhile, $|x_\sigma|$ limits to zero as $|s| \to \infty$ on E. Therefore, as δ can be as small as desired and chosen either positive or negative as desired, the vanishing of $-\delta + x_\sigma$ occurs on the $x_u = 0$ locus for arbitrarily small but non-zero choices of δ. (Keep in mind here that the simultaneous zeros of x_σ and x_u are isolated, otherwise x would be constant and C_0 would be a cylinder from one of the examples in Section 4a.) The preceding argument proves that there is exactly one solution to (4.44).

(g) The cokernel of D

Future constructions with Proposition 4.2's subvarieties may simplify with the knowledge that the corresponding operator D has, in all cases, trivial cokernel. A formal statement of this assertion appears below, and its proof occupies the remainder of this subsection.

Proposition 4.8 *The cokernel of D is trivial if C is an irreducible, pseudo-holomorphic HWZ subvariety with* index(D) @ + 1.

The remainder of this subsection is occupied with the following proof.

Proof of Proposition 4.8 All cases save the thrice-punctured spheres mentioned in Part d of the third point follow directly from

Lemma 4.9 *Let C be an irreducible, HWZ subvariety that is invariant under some* 1–*parameter subgroup of the group T. Then, the corresponding operator D has trivial cokernel.*

Proof of Lemma 4.9 If C is fixed by a circle subgroup in T, then the corresponding operator D is equivariant and its analysis can be simplified with the help of a separation of variables strategy. This is to say that D preserves the character eigenspaces of the circle's action on the domain and range, and the restriction of D to such an eigenspace reduces the partial differential equation D = 0 to a first-order ODE. This last reduction, plus some timely applications of the maximum principle prove the asserted triviality of D's cokernel in each of the cases in Section 4a. The details here are straightforward and left to the reader.

With this last lemma in hand, the only remaining cases for Proposition 4.8 are the thrice-punctured spheres from Part d of the third point in Proposition 4.2. In this regard, remember that these spheres are immersed; and remember that an immersed HWZ subvariety has a well-defined normal bundle and that the operator D is a differential operator, as in (3.5), on the space of sections of said normal bundle.

The preceding understood, the argument for a thrice-punctured sphere case given below is a generalization of the argument introduced by Gromov [6] when considering pseudoholomorphic planes and spheres, and it also has antecedents in some of HWZ's work. The argument begins with the observation that for such C, there exist vectors $\in \ker(D)$ whose restriction to each end of C_0 does not have limit zero as $|s| \to \infty$. Indeed, to find such an element, note that C is not preserved by any 1–parameter subgroup of T, and thus the infinitesimal version of the T action through its generators produces a two-dimensional subspace, V, in the kernel of D. Moreover, as only the closed Reeb orbits with ${}_0 \in \{0, \}$ are completely T invariant, and as no such orbit appear in C's limit set, so the generic element in V has the desired property.

In fact, V has the following 'universal limit' property: Given $\in$ kernel(D) and an end $E \subset C_0$, there exists a unique element $v \in V$ such that $-v$ has limit zero as $|s| \to \infty$ on E. Indeed, this follows because each closed Reeb orbit with ${}_0 \notin \{0, \}$ has precisely a one-dimensional family of deformations and the latter is the orbit of a 1–parameter subgroup of T.

The next observation is that a vector $\in$ kernel(D) with non-zero limit on all three ends of C_0 as $|s| \to \infty$ must have exactly one zero on C_0, and a non-degenerate one at that. Indeed, such a vector is, *a priori*, a section of the normal bundle, N, of C. Moreover, as can be approximated at large $|s|$ on each end of C by a vector from V, so at large $|s|$, is both non-zero and homotopic through non-zero vectors to the section from Part 2 of Section 3a which is used to define the expression $\langle e, [C] \rangle$ in Proposition 3.1. Thus, a count of the zero's of with the appropriate multiplicities computes the expression $\langle e, [C] \rangle - 2m_C$ which appears in Proposition 3.1. Moreover, each such zero counts with positive weight, its order of vanishing, as a consequence of its annihilation by D in (3.5). Given all of the above and the fact that $(C_0) = -1$, the formula in Proposition 3.1 can hold if and only if has precisely one zero, and this zero has multiplicity one.

With the preceding understood, suppose now that the dimension of the cokernel of D is positive. This implies that the dimension of the kernel of D is at least $\partial + 2$, and, as is shown next, such a condition leads to the absurd conclusion that this kernel has a vector with non-zero $|s| \to \infty$ limit on each end of C_0 and with at least two zeros. To view this vector in the $\partial = 3$ case, start with the observation that D's kernel has at least five linearly independent vectors when its cokernel is nontrivial. Choose any as in the preceding paragraph and let $z \in C_0$ denote its one zero. As dim(kernel(D)) $\geq$ 5, there is a two-dimensional subspace in kernel$(D)/(\mathbb{R}\ )$ of vectors which vanish at z. Let $W \subset$ kernel(D) project isomorphically onto such a subspace. As W is two-dimensional, there is, given any $z' \neq z$, a vector ${}_{z'} \in W$ such that $+ {}_{z'}$ vanishes at both z

and z'. So, to avoid the desired contradiction, each such $\eta + \eta_{z'}$ must have zero $|s| \to \infty$ limit on some end of C_0.

To see that such behavior is absurd, note that a non-vanishing limit is an open condition on the kernel of D and so there must be an end of C_0 with the property that each such $\eta + \eta_{z'}$ has zero $|s| \to \infty$ limit on this end. This last conclusion cannot occur unless W has a basis, $\{\eta_1, \eta_2\}$, such that η can be written as $\eta_1 + \alpha\eta_2$ where $\eta_1 - \eta$ and η_2 have zero $|s| \to \infty$ limit on this end. Here, α is a smooth, real-valued function on C_0. Moreover, as the operator D annihilates η, η_1 and η_2, the function α must obey the equation $\bar\partial\alpha = 0$ on C_0. However, the only real valued functions with this property are the constants, and these are ruled out since $\eta \notin W$.

Consider next the case where $\partial = 2$. In this case, the non-triviality of the cokernel of D implies that the kernel of D has dimension 4 and so now the analogous vector space W may only be one-dimensional. In any event, choose η as before, to have non-vanishing $|s| \to \infty$ limit on each end of C_0, and let η' be a non-trivial section of W. Let $E \subset C_0$ denote the concave side end and the following is true:

> *There exists a set where all points are accumulation points, s is unbounded, and η is proportional to η'.* (4.45)

Accept (4.45) and the $\partial = 2$ case follows unless η' is a constant multiple of η along this set. Of course the latter would imply that $\eta \in W$ since a non-trivial element in the kernel of D has isolated zeros. (In fact, the set in question can be shown to be a piecewise smooth curve.)

The proof of (4.45) begins with the observation that the operator D at large s on C_0 can be viewed as an operator on $\mathbb{C}$–valued functions of coordinates θ and s with θ periodic which has the form in (3.9) with $\tau = s$ and with A_0 the identity matrix. By viewing D in this way, both η and η' become $\mathbb{C}$–valued functions. In this regard, η at large s has the form

$$\eta = r_0 e_0 + \zeta; \tag{4.46}$$

where r_0 is a constant, non-zero real number and $|\zeta| \le e^{-\delta s}$ at large s with δ a positive constant.

Meanwhile, η' can be written as

$$\eta' = r_0' \eta + \zeta'; \tag{4.47}$$

where r_0' is a constant real number and where $|\zeta'| \le e^{-\delta' s}$ at large values of s with $d' > 0$ being constant. By assumption, ζ' is not identically zero.

Now, can vanish only on a finite or countable set where s 0 since it can have only a finite number of zeros on any compact set. Thus, there is some $s_0 > 0$ and a countable or finite set U $[s_0, \infty)$ such that maps the constant $s \in [s_0, \infty) - U$ circles in the $(\ , s)$ cylinder to $\mathbb{C} - \{0\}$. As such, has a winding number, the latter defines a locally constant function on $[s_0, \infty) - U$ and the only way (4.45) can fail is if this winding number is zero on all components of $[s_0, \infty) - U$ where s is sufficiently large. Indeed, if has non-zero winding number on a sufficiently large, but constant s circle, then it follows from (4.46) that and must be colinear at no fewer than two points on such a circle. This last fact implies (4.45).

Thus, to prove (4.45), it is enough to prove that has non-zero winding number on all sufficiently large and constant $s \in [s_0, \infty) - U$ circles. In this regard, it can be proved that U is infact finite since cannot have an infinite number of zeros without vanishing all together. However, as the proof of this last assertion is much longer than the proof that an infinite number of positive s zeros of implies (4.45), the proof of the latter claim follows. For this purpose, assume for the moment that actually has an infinite number of s 0 vanishing points.

To see how this last assumption leads to (4.45), note first that there is in this case at most one connected component of $[s_0, \infty) - U$ where the winding number is zero. This is because the zeros of all occur with positive multiplicity and thus the winding number changes in a monotonic fashion between consecutive components of $[s_0, \infty) - U$.

In particular, there exists some s_1 s_0 such that the winding number on every constant $s \in [s_1, \infty) - U$ circle is non-zero.

Now assume that has only a finite set of positive s zeros. In this case, the winding number of is defined on all sufficiently large and constant s circles. Then, for the sake of argument, suppose that this winding number is zero. Under this assumption, view as a $\mathbb{C} - \{0\}$–valued function and introduce real-valued functions v and w, defined at large s on the $(\ , s)$ cylinder by writing $= e^{-v-iw}$. These functions then obey (4.41) where s and where c and c' obey (4.42). In addition, the condition that $| \ |$ $e^{- \ s}$ at large s forces the condition

$$\lim_{\ \to \infty} \inf v(\ , \ ) = \infty. \tag{4.48}$$

However, as demonstrated using (4.43), this last condition is incompatible with (4.41) and (4.42). Thus, cannot have zero winding number on all sufficiently large and constant s circles.

5 The structure of the $\partial = 2$, thrice-punctured sphere moduli space

The purpose of this section is to give a complete description of the moduli space of thrice-punctured spheres with two convex side ends that arise in Part d of the third point in Proposition 4.2. In particular, the arguments given here establish Theorem A.2. The following proposition provides a restatement of Theorem A.2:

Proposition 5.1 *The components of the moduli space of thrice-punctured, $\partial = 2$ spheres which arise in Part d of the third point in Proposition 4.2 can be put in $1-1$ correspondence with the sets of two ordered pairs of integers, $\{(p, p'), (q, q')\}$, which obey*

- $pq' - qp' > 0.$
- *$q' - p' > 0$ unless both are non-zero and have the same sign.*
- *If $(m, m') \in \{(p, p'), (q, q')\}$ and if $|m'/m| < \sqrt{3}/\sqrt{2}$, then $m > 0$. On the other hand, if $m < 0$, then $|m'/m| > \sqrt{3}/\sqrt{2}$.*

Moreover, the component that corresponds to a given set $I \equiv \{(p, p'), (q, q')\}$ is a smooth manifold which is $\mathbb{R} \times T$ equivariantly diffeomorphic to $\mathbb{R} \times T$.

Subsection 5a, below, explains how such sets I of integers are associated to the moduli space components and derives the constraints on those sets which arise. The second subsection proves that each set I of four integers can be associated to at most one moduli space component. A proof is also given for the assertion that the associated moduli space, for a given I is either empty or diffeomorphic to $\mathbb{R} \times T$. Subsections 5c-g are occupied with the proof of the assertion that every set of $I = \{(p, p'), (q, q')\}$ that satisfies the constraints has an associated moduli space component.

With the proof of Proposition 5.1 complete, the final subsection provides a formula for the number of double points of Proposition 5.1's subvarieties in terms of the corresponding set I. Proposition 5.9 summarizes the latter and its assertions directly imply the part Theorem A.4 that concerns Theorem A.2's subvarieties.

(a) Constraints on thrice-punctured spheres with one concave side end

A pseudoholomorphic, thrice-punctured sphere $C \subset \mathbb{R} \times (S^1 \times S^2)$ with one concave side end and no intersections with the $\theta \in \{0, \pi\}$ locus determines a set of three pair of integers, $\{(p, p'), (q, q'), (k, k')\}$, in a manner that will now be described.

To start, remember that the large and constant $|s|$ slices of C consist of a disjoint union of three embedded circles, with one on the concave side of $\mathbb{R} \times (S^1 \times S^2)$ (where s is positive) and two on the convex side. And, as $|s|$ tends to infinity, these constant $|s|$ circles converge pointwise as multiple covers of the closed Reeb orbits which comprise C's limit set. In particular,

$$m\varphi - m' t = \text{constant} + O(e^{-|s|/\rho}) \mod (2\pi) \tag{5.1}$$

on each such circle; here $\rho > 0$ is constant while m and m' are integers that are associated to the given end of C. In fact, the limiting closed Reeb orbit in question is determined in part using the pair of integers in (1.8) provided by the quotient of (m, m') by their greatest common divisor. The multiplicity of covering over the closed Reeb orbit is then equal to this greatest common divisor of m and m'. Finally, the signs of m and m' are fixed by the following convention: Take the signs of m and m' (when non-zero) to equal the signs of the respective restrictions of f and h to the closed Reeb orbit in question. In this regard, note that $m = 0$ if and only if f restricts as zero to the closed Reeb orbit, and likewise $m' = 0$ if and only if h restricts as zero to the closed Reeb orbit. Also, note that (2.7) guarantees the compatibility of this sign determination with (5.1). The constraint for the third point in Proposition 5.1 arises from this use of integer pairs (m, m') to parameterize the closed Reeb orbits in $S^1 \times S^2$.

In this way, C determines the set $\{(p, p'), (q, q'), (k, k')\}$ of three pair of integers. The convention here is that the third pair listed, (k, k'), comes from the concave side end of C. Meanwhile, the order of appearance of the first two pair has, as yet, no intrinsic significance since this order corresponds to an arbitrary labeling of the convex side ends of C. However, the first two pair will be ordered shortly.

With the preceding understood, this subsection finds necessary conditions for a set $\{(p, p'), (q, q'), (k, k')\}$ to arise from a pseudoholomorphic, thrice-punctured sphere with one concave side end and no intersections with the $\theta \in \{0, \pi\}$ locus.

To begin, note that there is an evident first constraint:

Constraint 1 *No pair in* $\{(p, p'), (q, q'), (k, k')\}$ *can vanish identically.*

There is also a second constraint which comes from (5.1) and the fact that d' and dt both restrict to C as smooth, closed forms:

Constraint 2 *$k = p + q$ and $k' = p' + q'$.*

(The reader is left with the task of verifying that this constraint is consistent with the given sign conventions.) Note that the constraints listed in Proposition 5.1 for $\{(p, p'), (q, q')\}$ imply the third constraint for $(m, m') = (k, k')$. Thus, no new constraints appear with the association of (k, k') to an end of an $\partial = 2$, thrice-punctured sphere.

There are additional constraints. The next one involves the integer

$$pq' - qp'. \tag{5.2}$$

and asserts:

Constraint 3 $\ne 0$.

Indeed, suppose, to the contrary, that $= 0$. Now, both p and q can't vanish as then all ends of C would have $\cos^2{}_0 = 1/3$ closed Reeb orbit limits and an argument from Section 4 proved this impossible. In addition, the vanishing of implies that $p'/p = q'/q = (p' + q')/(p + q)$ so none of p, q or $p + q$ can vanish. Moreover, the equality of these ratios implies that there are at most two values for the $|s| \to 1$ limits of on C. However, as seen in Section 4, this cannot happen unless C is a cylinder.

By the way, given that $\ne 0$ and that changes sign upon interchanging (p, p') with (q, q'), an ordering of these pairs is unambiguously defined by requiring to be positive. This last convention is implicit in all that follows.

Of course, with > 0 there are obvious constraints on the relative signs between the four integers p, p', q and q' that arise just from the definition of . However, a less obvious constraint is:

Constraint 4 *$q' - p' > 0$ unless both are non-zero and have the same sign.*

The proof of this constraint is quite lengthy, so is broken into nine steps.

Step 1 This first step proves the following assertions:

If $p' + q' = 0$, then $p' < 0$.

If $p' = 0$, then $q' > 0$.

If $q' = 0$, *then* $p' < 0$.

To see why the first point holds, suppose, to the contrary that $p' > 0$. Then, as α in this case is $-p'(p+q)$ and is positive, so $k = p + q < 0$. This implies that $f < 0$ on the concave end of C_0. However, if this is the case, then h cannot vanish on the concave end of C_0 because h is zero and f is negative if and only if $\theta \in \{0, \pi\}$. Thus, because h tends to zero as $s \to \infty$ but is nowhere zero at large s, there are, given $R > 0$, compact components of constant h level sets that lie where $s > R$. However, this is absurd since (4.17) guarantees that $d\varphi$ has non-zero integral over such a level set while the $p' + q' = 0$ condition guarantees that $d\varphi$ is exact where s is large on C.

To see why the second point above holds, note that $\alpha = pq'$ in this case. Thus, positivity of α requires positivity of q' or else p would be negative. If p were negative, then f would be negative on the $(p; p')$ end of C and as $p' = 0$, this end of C would be asymptotic as $|s| \to \infty$ to a component of the $\theta = \{0, \pi\}$ locus.

A similar argument establishes that p' is negative when $q' \in 0$.

Step 2 This step proves Constraint 4 with the extra assumption that p and q cannot both be either strictly positive or strictly negative. Indeed, if both are strictly positive, $p' > 0$ and $q' < 0$, then $\alpha < 0$. On the other hand, if both are strictly negative, then $f < 0$ on C_0 while h changes sign. Thus, the $h = 0$ locus is non-empty and occurs where f is negative, which is precluded since C does not intersect the $\theta \in \{0, \pi\}$ locus.

Step 3 This step proves Constraint 4 with the added assumption that one of p, q or $p + q$ is zero. To start, assume that $p = 0$, $p' > 0$ and $q' < 0$. Positivity of α then requires that $q < 0$ and so f is negative at large s on the $(q; q')$ end of C and also at large $|s|$ on the concave (that is, $(k; k')$) end of C. In fact, as is argued momentarily, f is strictly negative on C. Granted this claim, the assumed violation of Constraint 4 results in the following absurdity: As h is positive on the $(p; p')$ end of C_0 and negative on the $(q; q')$ end, so the $h = 0$ locus is non-empty. But, as $f < 0$, this means that C intersects the $\theta \in \{0, \pi\}$ locus.

To see that $f < 0$ on C, suppose not. Then f has non-trivial, positive regular values (by the maximum principle). Moreover, as f is negative on the $(q; q')$ and concave ends of C, there exists, given $R \gg 1$, such a regular value whose level set sits entirely where $|s| \geq R$ in the $(p; p')$ end of C. Because dt restricts as a non-zero form on such a level set, by virtue of (4.16), and because dt is exact at large s on the (p, p') end of C when $p = 0$, so this level set cannot be compact. In fact, it must be a properly embedded copy of $\mathbb{R}$ with each end leaving via the (p, p') end of C. But this last conclusion is absurd as dt is nowhere zero on such a level set yet (5.1) asserts that t tends to a constant as $|s|$ tends to infinity on the (p, p') end of C.

Now assume that Constraint 4 is violated when $q = 0$. As $q' < 0$ and $\alpha > 0$, this requires $p < 0$ and the preceding argument applies with insignificant changes to find $f < 0$ on C. Again, as h must change sign on C, such a conclusion is absurd.

Finally, assume that Constraint 4 is violated when $p + q = 0$. Now, the $\alpha > 0$ condition requires that p and $p' + q'$ have the same sign. The argument given below takes this sign to be positive. But for some straightforward sign changes, the same argument also handles the case where this sign is negative.

To start, note that when $p > 0$, then f is positive on the (p, p') end of C and negative on the (q, q') end. Moreover, f has no finite limits on either end, as it tends uniformly to ∞ because $|s| \to \infty$ on the (p, p') end and to $-\infty$ on the (q, q') end. Meanwhile, even as f limits to 0 as $s \to \infty$ on the concave side end of C, this function must take negative and positive values at arbitrarily large values of s. Indeed, were it strictly negative at very large s, then a component of the level set of some very small, negative regular value of f would be compact and lie entirely where s is very large. And, as the integral of dt over such a level set could not be zero (due to (4.16)), the existence of such a level set would run afoul of (5.1). An analogous argument explains why f cannot be strictly positive at large s.

With the preceding understood, suppose that $-\varepsilon$ is a regular value of f with ε positive and very small. For tiny ε, a component of this level set will extend far down the concave side end of C. In particular, as $p' + q' > 0$, when ε is small, the function h will be positive on some of this component. Then, this component sits entirely where $h > 0$ in C because $\theta \notin \{0, \pi\}$ on C. Now, consider increasing ε and viewing the behavior of the $f = -\varepsilon$ level set. In particular, when $\varepsilon \gg 1$, then this level set necessarily sits entirely in the (q, q') end of C where $h < 0$. Thus, as no component of this level set is null-homologous, there is some intermediate values of ε where a component of the $f = -\varepsilon$ level set intersects both the $h > 0$ region and the $h < 0$ region. Of course, such an event is absurd for θ would take value 0 or π on C.

Step 4 The subsequent steps prove the remaining cases of Constraint 4 by establishing the following:

If none of $p, p', q, q', p+q,$ *and* $p'+q'$ *are zero and if the signs of* p' *and* q' *are opposite, then* $p' < 0.$ (5.3)

To begin the justification for (5.3), use Step 2 to conclude that if both p and q are non-zero, then one is positive and the other negative. Thus, f changes sign on C. Moreover, where s is very negative (thus, on the convex side of $\mathbb{R} \times (S^1 \times S^2)$), the function f tends to infinity on one end of C and minus infinity on the other. In particular, when $R \gg 1$, then the $f = R$ locus is an embedded circle in the convex side end of C that corresponds to the positive member of the pair (p, q). Meanwhile, the $f = -R$ locus is likewise a circle on the convex side end of C that corresponds to the negative member of the pair (p, q). Now, (4.16) implies that f and t have the same critical points, and so Section 4f proves that there is only one critical point of f on C, and thus only one critical value. The sign of this critical value depends on the sign of $p+q$. In particular, if $p+q > 0$, then this critical value is positive, and if $p+q < 0$, then this critical value must be negative. The maximum principle is involved here, since each non-critical level set of f is either a single embedded circle or a pair of embedded circles. Indeed, when $p+q > 0$, then the $f = 0$ and the very negative and constant f loci must be isotopic in C to accommodate the maximum principle. In this case, the $f \leq 0$ portion of C is an infinite half cylinder. On the other hand, if $p+q < 0$, then the $f = 0$ locus and the very positive and constant f loci in C must be isotopic and so now the $f \geq 0$ portion of C is a half infinite cylinder. The analogous conclusions hold for the $h = 0$ locus with p and q replace by p' and q'.

Step 5 With the preceding understood, consider the possible location of the $h = 0$ locus in a hypothetical case where (5.3) is violated. When $p+q < 0$ this locus lies in the half cylinder where $f \geq 0$, and this implies that either the $h \geq 0$ locus or the $h \leq 0$ locus is a subcylinder of the $f \geq 0$ locus. The former can happen only if $p'+q' < 0$ and $p > 0$ and the latter only if $p'+q' > 0$ and $p < 0$.

If $p' > 0, q' < 0$ *and* $p'+q < 0,$ *while* $p'+q', p$ *and* q *are each non-zero then* $p'+q'$ *and* p *have opposite signs.* (5.4)

Now suppose that $p+q > 0$ with p' still positive and q' negative. Assume that neither p nor q is zero so that one is positive and the other negative. If $p > 0$, then positivity of requires that $p'+q' > 0$. On the other hand, if $p < 0$, then the positivity of requires that $p'+q' < 0$. To summarize:

If $p' > 0; q' < 0$ and $p' + q > 0$; while $p' + q'; p$ and q are each non-zero then $p' + q'$ and p have the same signs. (5.5)

Step 6 Steps 6 and 7 argue that there are no cases of (5.4) with $p + q < 0$. To see why, remark that in this case, both f and h are negative on the concave side end of C_0. With this understood, let $C_+ \subset C_0$ denote the connected component of the $h/f > 0$ locus which contains this concave side end. Moreover, since h/f has neither local maxima nor minima, C_+ must also contain the (q, q') end of C_0. On the other hand, as $h^{-1}(0)$ must lie where $f > 0$, C_+ does not intersect the sufficiently large $|s|$ portion of the (p, p') side end of C_0. (The boundary of the closure of C_+ is the $f = 0$ locus.) They are also both negative on the convex side end corresponding to (q, q') and both are positive on the (p, p') convex side end. However, as $h^{-1}(0)$ occurs where $f > 0$, it follows that C_+ contains both the concave side (q, q') end and the concave side end of C_0.

Given the preceding, remark that h/f tends to $(k'/k)\sin^2\theta_{0K}$ as $s \to \infty$ on the concave side end of C_0, while it approaches $(q'/q)\sin^2\theta_{0Q}$ on the (q, q') side end. Here, θ_{0K} is the value of θ_0 on the closed Reeb orbit that is determined by the concave side end and θ_{0Q} is the value of θ_0 that is determined by the corresponding closed Reeb orbit for the (q, q') end. With this last point understood, the next claim is that

$$(k'/k)\sin^2\theta_{0K} < (q'/q)\sin^2\theta_{0Q}. \tag{5.6}$$

By way of justification, note first that $k'/k < q'/q$ because $\Delta = kq' - k'q$ and $\Delta > 0$. Meanwhile, the assignment of $\theta_0 \to \sqrt{6}\cos(\theta_0)/(1 - 3\cos^2\theta_0)$ defines a smooth function on the (connected) subset of $\theta_0 \in (0, \pi)$ where $\cos\theta_0 < -1/\sqrt{3}$. As the derivative of this function is negative on this interval, the fact that $k'/k < q'/q$ implies that $\theta_{0K} > \theta_{0Q}$. Therefore, $\sin^2\theta_{0K} < \sin^2\theta_{0Q}$ and (5.6) follows.

Step 7 As remarked in the previous section, the restriction to C_+ of h/f has neither local maxima nor minima. It thus follows from (5.6) that the infimum of h/f is the limiting value, $(k'/k)\sin^2\theta_{0K}$, on the concave side end of C_0. To see that such an event is absurd, introduce coordinates (t, u) on this end as in (2.19) where t is periodic and u is identified with the pullback of f and so ranges through $(-\varepsilon, 0)$ for some $\varepsilon > 0$. Then, parameterize this end of C_0 as in (2.19) in terms of functions (x, y) of the variables (t, u). In particular, the function y must be non-positive if h/f has infimum $(k'/k)\sin^2\theta_{0K}$. The latter constraint is inconsistent with (2.20) for the following reasons: First, it follows from Proposition 2.3 that $|x|$, $|u|^{-1}|y|$ and $|y_u|$ all tend to zero as $|u|$ tends to zero. This last fact implies that x and $w \equiv \sin^{-2}\theta\, y$ obey an equation which has the schematic form

$$x = w_u + (\alpha_0 + r)u^{-1}w\,; \tag{5.7}$$

where $\alpha_0 = (2/3)^{1/2}(k'/k)\cos\theta_0(1 + (k'/k)^2\sin^2\theta_0)^{-1}$ and $r = r(u,w)$ is a smooth function with $\lim_{u\to 0} r = 0$. (In fact, $|r| = O(|u|^{-1}|w|)$.) With regard to α_0, note that $\alpha_0 < 0$ since its sign is that of k. However, it is crucial to note that $\alpha_0 > -1$.

With (5.7) in hand, suppose now that $w \leq 0$ for all sufficiently small values of $|u|$. It then follows that from (5.7) that there exists a positive constant $\delta_1 < 1$ such that for all sufficiently small $|u|$,

$$x \geq w_u - \delta_1 u^{-1}w\,. \tag{5.8}$$

And, with this last point understood, let $\bar{w}(u)$ denote the average of w over the constant u circles. The latter function is negative on circles where w is not identically zero. Moreover, by virtue of (5.8), this function obeys $0 \geq \bar{w}_u - \delta_1 u^{-1}\bar{w}$ from which it follows that $\bar{w} \leq -c|u|^{\delta_1}$ where $c > 0$. (Since C_0 is not a cylinder, w cannot vanish identically on any open set; thus under the assumption that $w \leq 0$, the function $\bar{w}$ cannot vanish identically.) Thus, as $\delta_1 < 1$, so $|\bar{w}||u|^{-1} \geq c|u|^{\delta_1 - 1}$; and so $|\bar{w}||u|^{-1}$ diverges as $|u| \to 0$. This last conclusion is absurd because the divergence of this ratio is precluded by Proposition 2.3.

Step 8 This step eliminates the case of (5.4) where $p' + q' > 0$. To start the argument, note that in this case, f is negative on the concave side end of C_0 and also negative on the end that corresponds to (p, p'). On the other hand, f is positive on the end that corresponds to (q, q'). This is the end where h is negative, but h is positive on the concave side end and the end that corresponds to (p, p'). As $h^{-1}(0)$ lies where $f > 0$, there is a component, $C_- \subset C_0$, of the locus where $h/f < 0$ which contains both the concave side end and the (p, p') end of C_0. Furthermore, the closure of C_- has the $f = 0$ locus as its boundary, and $h/f \to -\infty$ as this locus is approached from C_-. Meanwhile, h/f converges to $p'/p\,\sin^2\theta_{0P}$ as $|s| \to \infty$ on the (p, p') end of C_0, and it converges to $k'/k\,\sin^2\theta_{0K}$ as $s \to \infty$ on the concave side end of C_0.

With the preceding understood, then the argument just completed in Step 6 adapts to this case with essentially no modifications given that

$$k'/k\,\sin^2\theta_{0K} > p'/p\,\sin^2\theta_{0P}\,. \tag{5.9}$$

To justify this last claim, note first that k'/k is less negative than p'/p since > 0. Thus, (5.9) follows directly if $\sin^2\theta_{0K} < \sin^2\theta_{0P}$. To see the latter inequality, note first that both θ_{0K} and θ_{0P} lie in the subinterval of $(0, \pi)$ where $1 > \cos\theta > 1/\sqrt{3}$. In this interval, the assignment to θ_0 of $\sqrt{6}\cos(\theta_0)/(1 - 3\cos^2\theta_0)$ defines a monotonically decreasing function. As the assignment to such θ_0 of $\sin^2\theta_0$ defines an increasing function of θ_0 on this same interval, the desired conclusion follows.

Step 9 This step rules out any examples of (5.5). The first case to consider here has $p' + q' < 0$. In this case, $p < 0$ but both $p + q$ and q are positive. Thus, $f > 0$ on both the concave side end and the (q, q') side end of C_0, but $f < 0$ on the (p, p') end. Meanwhile, $h > 0$ on the (p, p') end of C_0 and $h < 0$ on the other two ends. In this case, there is a connected component, $C_- \subset C_0$ of the $h/f < 0$ locus which contains both the concave side end and the (q, q') end of C_0. The boundary of the closure of C_- is the $h = 0$ locus again, and $h/f \to 0$ as this locus is approached from the C_- side. Then, with little change, the previous arguments apply to rule this case out given that (5.6) holds.

To see (5.6) in this case, note first that $k'/k < q'/q$ since $\ > 0$. In addition, note that both θ_{0K} and θ_{0Q} lie in the subinterval of $(0, \pi)$ where $-1/\sqrt{3} < \cos\theta < 0$. On this interval, the expression $\sqrt{6}\cos(\theta_0)/(1 - 3\cos^2\theta_0)$ defines a decreasing function of θ_0 and so $\theta_{0K} > \theta_{0Q}$. Now, k'/k and q'/q are both negative, so the inequality $\sin^2\theta_{0K} < \sin^2\theta_{0Q}$ does not imply (5.5). However, as $(k'/k)\sin^2\theta_{0K}$ and $(q'/q)\sin^2\theta_{0Q}$ are the values of

$$\sqrt{6}\cos\theta\,\sin^2\theta\,(1 - 3\cos^2\theta)^{-1} \tag{5.10}$$

at $\theta = \theta_{0K}$ and θ_{0Q}, the inequality in (5.6) does follow from the fact that (5.10) is a decreasing function of θ on the interval in question. Indeed, the θ–derivative of (5.10) is

$$-\sqrt{6}\sin\theta\,(1 + 3\cos^4\theta)(1 - 3\cos^2\theta)^{-2}. \tag{5.11}$$

Finally, consider the possibility that (5.5) holds with $p' + q' > 0$. Now $p > 0$ so if $q < 0$, then $f > 0$ on the concave side end of C_0 and also on the (p, p') side end. However, as $q < 0$, so $f < 0$ on the (q, q') side end. Thus, there is a component, $C_+ \subset C_0$ of the locus where $h/f > 0$ which contains both the concave side end and the (p, p') end of C_0. The closure of C_+ has the $h = 0$ locus for its boundary and h/f tends to zero as this boundary is approached in C_+. With this point understood, then the previously used argument applies given that (5.9) holds.

To see (5.9) in this case, first note that $k'/k > p'/p$ since $\ > 0$. Now, as both θ_{0K} and θ_{0P} lie where $0 < \cos\theta < 1/\sqrt{3}$, and as the function $\theta_0 \to \sqrt{6}\cos(\theta_0)/(1 - 3\cos^2\theta_0)$ is decreasing on this subinterval, it follows that $\theta_{0K} < \theta_{0P}$. With this understood, (5.9) follows from the fact that (5.10) is also decreasing, by virtue of (5.11), on this same interval.

(b) Moduli space components

Fix an ordered set $I = \{(p, p'), (q, q')\}$ of integers subject to the constraints listed in the statement of Proposition 5.1. As noted at the outset of the preceding subsection, this set labels those components of the moduli space of pseudoholomorphic, $\partial = 2$, thrice-punctured spheres in $\mathbb{R} \times (S^1 \times S^2)$ with ends that are characterized by the set of three integer pairs $\{(p, p'), (q, q'), (k = p + q, k' = p' + q')\}$. Use H_I to denote this subspace of $\mathfrak{M}$. The question arises as to the number of components H_I. Here is the answer:

Proposition 5.2 *Let $I = \{(p, p'), (q, q')\}$ denote a set of pairs of integers that obeys the constraints listed in the statement of Proposition 5.1. Then, the space $\mathfrak{M}_I$ of pseudoholomorphic, $\partial = 2$, thrice-punctured spheres from Proposition 4.2 with ends characterized by the set $\{(p, p'), (q, q'), (p + q, p' + q')\}$ has at most one connected component. Moreover, if non-empty, the latter is a smooth manifold that is $\mathbb{R} \times T$ equivariantly diffeomorphic to $\mathbb{R} \times T$.*

The remainder of this subsection is occupied with the following proof.

Proof of Proposition 5.2 The subsequent discussion for the proof of Proposition 5.2 treats the case where the integer $k = p + q$ is non-zero and positive. The proof when $k < 0$ is identical to that for $k > 0$ except for some judicious sign changes. Meanwhile, if $k = 0$, then $k' \neq 0$ and the discussion below applies after the roles of the pair (t, f) are interchanged with those of (φ, h). Thus, assume throughout that $k > 0$. Also, assume until further notice that neither p nor q is zero.

Each component of $\mathfrak{M}_I$ is a smooth manifold by virtue of Propositions 3.2 and 4.8. Moreover, as $\dim(\mathfrak{M}_1) = 3$, the subgroup $\mathbb{R} \times T$ of $\mathrm{Isom}(\mathbb{R} \times (S^1 \times S^2))$ acts transitively on each component of $\mathfrak{M}_1$, and so each is $\mathbb{R} \times T$ equivariantly diffeomorphic to $\mathbb{R} \times T$. In this regard, remember that the T action on $S^1 \times S^2$ is generated by the vector fields ∂_t and ∂_φ, while the action of $\mathbb{R}$ on $\mathbb{R} \times (S^1 \times S^2)$ is generated by ∂_s. By the way, the T action on $\mathfrak{M}_I$ must be a free action since the Riemann sphere has no complex automorphisms that fix three given points.

In any event, if $C \in \mathfrak{M}_I$ and a component $H' \subset \mathfrak{M}_I$ have been specified, there exists $C' \in H'$ with two special properties:

The $(t; f)$ *coordinates of the critical point of* f*'s restriction to* C *are the same as those of the analogous critical point on* C^0.

The constant term on the right-hand side of the concave side end version of (5.1) for C *is the same as that for* C^0.

With regard to the rst point here, remember that the restriction of f to any thrice-punctured sphere from Proposition 4.2 with each of p, q and k non-zero has precisely one critical point so precisely one critical value. Moreover, this critical value is non-zero with sign that of k. Indeed, the latter conclusions follow from the maximum principle since f's pullback to C_0 obeys (4.16) and (4.16) implies a second-order equation for f with the schematic form $d\ df + v\ df = 0$. Finally, given that the critical values of f on C and C^0 have the same sign, then a suitable translation of C^0 along the $\mathbb{R}$ factor of $\mathbb{R}\ (S^1\ S^2)$ makes them equal. Meanwhile, a suitable rotation of the S^1 factor moves C^0 so that f's critical point on the resulting subvariety has the same t coordinate as that of f's critical point on C. Such a rotation does not change the value of f at its critical point.

With regard to the second point above, by virtue of the fact that $k \neq 0$, there is an equatorial rotation of the S^2 factor moves any C^0 so that the resulting subvariety obeys the desired condition. Note that such a rotation will not change the $(t; f)$ coordinates of f's critical point.

Now let C_0 denote the model thrice-punctured sphere. As noted in the preceding sections, C_0 comes with a pseudoholomorphic immersion, , into $\mathbb{R}\ (S^1\ S^2)$ whose image is C. There is a similar immersion with image C^0. The latter is denoted by $^0_0$ since a subsequent modi cation, ^{0}, is needed for later arguments. The following lemma describes the salient features of ^{0}:

Lemma 5.3 *There exists an immersion* $^0\colon C_0 \to \mathbb{R}\ (S^1\ S^2)$ *with image* C^0 *such that* $^0(t; f) = (t; f)$.

What follows is a digression for the proof of this lemma.

Proof of Lemma 5.3 Let f_0 denote the critical value of f's restrictions to C and to C^0. Now, let $C_f\ C^0_0$ denote the portion where $f \neq f_0$. Likewise, de ne $C^0_f\ C^0_0$ as the portion where $^0_0 f \neq f_0$. There are three components of C_f, each is a cylinder and each corresponds to an end of C_0 and hence a pair from $\{(p; p^0); (q; q^0); (p + q; p^0 + q^0)\}$. As the same assertions hold for C^0_f, there is a canonical 1–1 correspondence between the components of C_f and those of C'_f: Components correspond if they correspond to the same pair from $\{(p, p'), (q, q'), (p + q, p' + q')\}$.

Meanwhile, the assignment of (t, f) to each component of C_f defines a proper covering map to some subcylinder of the (t, f) coordinate cylinder. The same is true for the components of C'_f, and corresponding components have the same image. With a component of $C - C_f$ fixed, the covering map to the appropriate (t, f) cylinder is a cyclic covering which is determined by the pair from $\{(p, p'), (q, q'), (p + q, p' + q')\}$. Thus, the analogous covering map from its partner component, $C' - C'_f$, is isomorphic. As a consequence, there exists a diffeomorphism $': C \to C'$ which intertwines the projection maps to the relevant (t, f) sub-cylinder. Note that this diffeomorphism is determined up to composition with the group of deck transformations.

This freedom with the group of deck transformations can be used to insure that the three versions of from the components of C_f patch together along $(f)^{-1}(f_0)$ to define a dieormorphism, , from C_0 to C_0. To explain, let t_0 now denote the value of the t–coordinate of f's critical points on C and C'. Consider some (t_1, f_0) in the (t, f) cylinder that with $t_1 \neq t_0$. Take a small disk about this point that is disjoint from the images of the critical point of f and of $'_0 f$. This done, then each component of C can be extended by adding the –inverse image disks. Meanwhile, each component of C' can similarly be extended with the addition of the $'_0$–inverse image disks. As these extensions remain proper covering maps over their images in the (t, f) cylinder, so the corresponding maps can be extended as well. This understood, consider a point $z \in C_0$ that lies in a component, say C_1, of C and also in the extension of another component, C_2. Then $_1$ is defined near z and so is the extended $_2$. As both compose with $_0$ to give the same (t, f) values, so $_2$ differs from $_1$ on a neighborhood of z by at most a deck transformation of C_1.

Now, local agreement of one with the extended version of another implies global agreement for the three. Indeed, this all follows from the geometry of the critical locus. In particular, because the critical point of f is non-degenerate, the closures of two components of C have piece-wise smooth circle boundary, and that of one component has a figure eight boundary. Here, the bad point in the figure eight is the critical point of f. Moreover, the critical point is the only point where the two circle boundary components intersect. These last points understood, the deck transformations of the two components of C_f with circle boundary can be used independently to create a smooth map that is defined on the whole complement in C_0 of the f critical point.

Now, some further checking should be done to insure that all is well with this map near the critical point of f. The latter task is left to the reader save for the following remark: Let g_0 denote the value of the function g at the critical point of f. Then, the behavior of the complex function $(g_0^{-1} f - it)$ near the critical point of f can be analyzed using (4.16). In particular, $=$ $_0 + az^2 + O(|z|^3)$ with respect to a holomorphic coordinate z centered at the critical point of f. Here, $_0$ is a constant and a is a non-zero constant. This local form for follows from (4.16). See, Appendix A in [22] where a completely analogous assertion is proved. Also, note that the rst-order vanishing of d follows from the observations in Section 4f that the standard algebraic count of dt's zeros is -1, and that all zero's of dt count with negative weight.

In any event, with in hand, the lemma's map $'$ is the composition of $_0$ with .

With Lemma 5.3 proved, the digression is over. To continue the proof of Proposition 5.2, introduce the 1{form $\underline{d\varphi}$ $d\varphi - {}' d\varphi$. This is a smooth, closed 1{form on C_0. It is also exact since C' and C determine the same set I. Thus, the di erence $\underline{\varphi} = \varphi - {}'\varphi$ can be viewed as a bona de function on C_0. Note that in principle, there is a choice involved in so viewing $\underline{\varphi}$, but any two choices di er by an integer multiple of 2 . In any event, by virtue of the fact that C and C' have the same constant term on the concave end version of (5.1), there is a unique choice for, $\underline{\varphi}$ that limits to zero on the concave end of C_0. This said, then $\underline{\varphi}$ has nite limits on the two convex ends of C_0 by virtue of (5.1). Thus, $\underline{\varphi}$ is a bounded function on C_0.

Now let $\underline{h}$ $h - {}' h$, which is automatically a smooth function on C_0. It then follows from (4.16) and (4.17) that the pair $(\underline{\varphi}; \underline{h})$ obeys an elliptic, rst-order di erential equation which has the schematic form:

$$
\begin{aligned}
g \sin^2 \ \underline{\varphi}_1 &= \underline{h}_2 + \ \underline{h}, \\
g \sin^2 \ \underline{\varphi}_2 &= -\underline{h}_1 + {}'\underline{h}.
\end{aligned}
\qquad (5.12)
$$

Here, g and are identi ed with their {pullbacks, while and $'$ are smooth functions.

Equation (5.12) is employed to justify certain remarks that follow about the locus, $\underline{G}$ C_0, where $\underline{h} = 0$. In particular, either $\underline{h}$ is identically zero, in which case so is $\underline{\varphi}$ and $C' = C$, or else $\underline{G}$ has the structure of an 'embedded graph' as de ned in Step 7 of the proof in Section 2 of Proposition 2.2. In this regard, the vertices of $\underline{G}$ are the $\underline{h} = 0$ critical points of $\underline{h}$. As with the directed graph which appears in Proposition 2.2's proof, this graph is naturally oriented. Its orientation is de ned by the pullback of $\underline{d\varphi}$ to each edge; the latter is non-zero by virtue of (5.12). It is also a fact that $\underline{G}$, as with its Section 2 counterpart, has a non-zero, even number of incident edges impinging on each vertex; and of these, half are point towards the vertex and half point away. (All of these last remarks are proved by copying the arguments in Steps 3–5 of Part b of the Appendix to [22].)

Now, an argument along the lines of that used in Step 8 of Proposition 2.2's proof establishes the following: If $C' \neq C$ and $\underline{G} \neq \emptyset$, and if the extreme values of $\underline{\varphi}$'s restriction to $\underline{G}$ are not its limiting values on the ends of $\underline{G}$ then the aforementioned properties of $\underline{G}$ cannot hold. Thus, Proposition 5.2 follows via *reductio ad absurdem* with an argument that proves when $C' \neq C$, then $\underline{G}$ is non-empty and neither of $\underline{\varphi}$'s extreme values on $\underline{G}$ are its limiting values on the ends of $\underline{G}$.

For the purposes of establishing that $\underline{G} \neq \emptyset$, consider (5.12) where $s \gg 1$, thus, far down the concave side end of C_0. Now parameterize this portion of C_0 by coordinates (τ, u) where $\tau \in \mathbb{R}/(2\pi|k|\mathbb{Z})$ and $u \in (0, \delta)$ or $u \in (-\delta, 0)$ depending on whether $k > 0$ or $k < 0$. Here, $\delta > 0$ is very small. The argument that follows considers first the case where $k > 0$. Thus, with $u > 0$ understood, the relevant portion of C is parameterized as in (2.19) in terms of functions x and y. In particular, the pair (x, y) obey (2.20) and of particular interest here is the second equation in (2.20). In this regard, view h as a function of u and y and thus view θ as function of u and y. Now, introduce the function $w \equiv y \sin^{-2}\theta$ and then the second equation in (2.20) implies an equation for x and w that has the schematic form as in (5.7). Note that in this version of (5.7), the constant $\lambda_0 > 0$ since $k > 0$ and λ_0 has the same sign as k.

Meanwhile, the analogous part of C' also has a parameterization as in (2.19) in terms of functions (x', y'). Then, x' and the primed analog, w', of w obey the analog of (5.7). With this understood, subtract the primed version of (5.7) from the original to obtain the following equation for $\underline{\varphi} = x - x'$ and $\underline{w} = w - w'$:

$$\underline{\varphi}_\tau = \underline{w}_u + (\lambda_0 + \underline{r})u^{-1}\underline{w}, \tag{5.13}$$

where r is a smooth function with $\lim_{u \to 0} |\underline{r}| = 0$

An equation such as (5.13) for $\underline{w}$ is useful for two reasons. First, when $u > 0$ but very small, then, as is demonstrated below, $\underline{w} = 0$ if and only if $\underline{h} = 0$. Given that such is the case, it is sufficient for the purposes of proving Proposition 5.1 to establish the existence of a zero of $\underline{w}$ along each sufficiently constant but small u circle. In this regard, remember that $\lim_{u \to 0} \underline{\varphi} = 0$ on the concave side end of C_0. Second, (5.13) does indeed imply the existence of zeros of $\underline{w}$. To see (5.13) lead to this last conclusion, suppose to the contrary that $\underline{w} > 0$ for all sufficiently small u. It then follows from (5.13) that the average, $\overline{w} \equiv \overline{w}(u)$

of $\underline{w}$ around all constant but small and positive u circles obeys the differential inequality

$$\overline{w}_u + 2^{-1}\ {}_0 u^{-1}\overline{w} < 0\,. \tag{5.14}$$

The latter implies that $\overline{w} > cu^{-\ {}_0/2}$ as $u \to 0$ with c a positive constant. This conclusion is ludicrous as both y and y' tend to zero as u tends to zero. Likewise, if $\underline{w} < 0$ for all sufficiently small u, then (5.14) implies that $\overline{w} < -cu^{-\ {}_0/2}$ as $u \to 0$ with c a positive constant, which is an equally ludicrous conclusion. Thus, (5.14) is consistent only with the conclusion that $\underline{w} = 0$ at some point on each constant, but small u circle.

With the preceding understood, return to the claim that $\underline{w} = 0$ if and only if $\underline{h} = 0$ when u is very small. In this regard, note first that $\underline{h} = 0$ requires $y = y'$ and $\ = \ '$, and so $\underline{w} = 0$. To prove the converse, note that the dependence on the coordinates f and h of is such that $(f; h = k'/k \sin^2\ {}_0 f + \ ) =\ {}_0 + y(\ /f)$ when $|\ |/f$ is small. Moreover, y is $O(\ /f)$. Thus, as both y/u and y'/u tend to zero as $u \to 0$ (as attested by Proposition 2.3), so both $w = y\sin^{-2}\ {}_0(1 + O(y/u))$ and $w' = y'\sin^{-2}\ {}_0(1 + O(y'/u))$ when u is small. Therefore, when u is small, $w = w'$ forces $y = y'$ and so $h = h'$.

With the $k > 0$ argument understood, it can be said that the argument for the $k < 0$ case is similar although not identical. In particular, the only substantive difference arises in the argument for the vanishing of $\overline{w}$ because the constant ${}_0$ which appears in (5.7) is negative when $k < 0$. To argue that $\overline{w} = 0$ when $k < 0$, remark first that though negative, ${}_0 > -1$. Thus, if $\underline{w} > 0$ where $|u| > 0$ is sufficiently small, then $\overline{w}$ obeys the following analog of (5.14):

$$\overline{w}_v - \ {}_1 v^{-1}\overline{w} < 0\,; \tag{5.15}$$

where $v \ \ -u > 0$ is small. Here, ${}_1$ is positive, but ${}_1 < 1$. This last equation implies that $\overline{w} \ \ c|u|^{\ 1}$ where c is a positive constant. In particular, $|u|^{-1}\overline{w}$ is unbounded as $|u| \to 0$ which is impossible since, as previouly noted, both $y/|u|$ and $y'/|u|$ tend to zero as $|u|$ tends to zero. With some judicial sign changes, the preceding argument also rules out the possibility that $\underline{w}$ is strictly negative where $|u|$ is sufficiently small.

With it now established that $\underline{G} \neq \emptyset$, turn now to the question of whether $\underline{\varphi}$'s extreme values on $\underline{G}$ are its limiting values on $\underline{G}$'s ends. For this purpose, suppose that $\underline{G}$ has non-compact intersection with the closure, $C_1 \ \ C_0$, of one of the three components of the complement of the $f = f_0$ locus. Now, fix some $s_0 \ \ 1$, a regular value of $|s|$ on C_1 and such that the $|s| = s_1$ locus in C_1 is a circle having transversal intersection with $\underline{G}$. As $\underline{G}$ divides the C_1 into the portion where $\underline{h} > 0$ and where $\underline{h} < 0$, so $\underline{G}$ must have an even number of intersections with the $|s| = s_0$ circle in C_1 and these points alternate upon a circumnavigation of this circle between points where φ is increasing and decreasing in the direction of increasing $|s|$.

Remark now that a compact, oriented path in the $|s| \geq s_0$ part of $\underline{G}$ with both boundary points on the $|s| = s_0$ circle pairs up two $|s| = s_0$ boundary points. This understood, remove from $\underline{G}$ a maximal set of such paths, no two sharing edges, to obtain a new embedded and directed graph, $\underline{G}_1$, in the $|s| \geq s_0$ portion of C_1. Note that $\underline{G}_1$ cannot be empty (by assumption). In addition, each $|s| > s_0$ vertex has an even number of impinging edges, half oriented by $d\underline{\varphi}$ to point outward and half to point inward. Moreover, $\underline{G}_1$ must intersect the $|s| = s_0$ locus. Indeed, otherwise there would be a properly embedded path in $\underline{G}_1$ with $|s|$ unbounded at both ends and this is ruled out by the fact that $\underline{\varphi}$ has a unique, $|s| \to \infty$ limit on C_1. Now, given that $\underline{G}_1$ interstects the $\underline{s} = s_0$ locus, it does so in an even number of points, and again, these alternate between those where $\underline{\varphi}$ is increasing with $|s|$ and those where $\underline{\varphi}$ is decreasing. Now, by assumption, no pair of these points comprise the boundary of a compact, oriented path in $\underline{G}_1$, and so each is the sole boundary point of a path in $\underline{G}_1$ on which $|s|$ is unbounded. This the case, then $\underline{\varphi}$ is increasing on half of these paths in the unbounded direction and decreasing on the other half. In particular, $\underline{\varphi}$'s limit as $|s| \to \infty$ on $\underline{G} \cap C_1$ is not an extreme value of $\underline{\varphi}$ on $\underline{G}$.

Now consider the case where one of p, q vanishes. In this regard, the argument below considers the case where $q = 0$, $p > 0$. The argument for the $p = 0$, $q < 0$ is virtually identical and is left to the reader. The following lemma is needed:

Lemma 5.4 *Let r be a smooth function of the angle θ, defined on some open interval $\underline{Z} \subset [0,\pi]$ where it is everywhere distinct from $-f/h$. Then there exists a function σ on $\underline{Z}$ and a function u of the variables (f,h), defined where $\theta \in \underline{Z}$, with $du = e^{\sigma}(df + r\,dh)$.*

Proof of Lemma 5.4 First, the f and h derivatives of θ are related via $\theta_f = -h/f\,\theta_h$. With this understood, then σ is determined up to an additive constant by the requirement that

$$(1 + rh/f)\,\sigma' + r\,(h/f)' = 0\,. \tag{5.16}$$

As $(h/f)' = \sqrt{6}\cos\theta\sin^2\theta\,(1 - 3\cos^2\theta)^{-1}$, this last equation reads

$$\sigma' = -r\sqrt{6}\cos\theta\sin^2\theta\,[(1 - 3\cos^2\theta) + r\sqrt{6}\cos\theta\sin^2\theta]^{-1}\,. \tag{5.17}$$

As $r \neq -f/h$ on its domain of definition, the latter equation can be integrated to obtain .

To employ this lemma, remark that if $C \in \mathfrak{M}_I$, then $q' > 0$ as both p and are positive and $q = 0$. Thus, one of the convex side ends of C corresponds to a closed Reeb orbit with $\cos\ _0 = 1/\sqrt{3}$. Meanwhile, $f > 0$ on the other two ends of C. As argued in the previous subsection, this implies that $f > 0$ on the whole of C.

With f everywhere positive on C, it follows that the range in $\mathbb{R}$ of the restriction to C of the function $-\sin^2\ f/h$ is disjoint from an interval of the form $[0;)$. Given such that is less than the minimum of $p/|p'|$ and $p/|p' + q'|$, let $\in (0; /2)$ be a rational number. Now employ Lemma 5.4 using $r = \ \sin^{-2}\ $. It then follows that any choice for the resulting function u is defined on a neighborhood of every $C \in \mathfrak{M}_I$. By the way, as any two choices for u differ by an additive constant, so it follows from (1.3) that there is a unique choice which limits to zero as $s \to \infty$ along every $C \in \mathfrak{M}_I$. This particular choice for u should be taken in what follows.

Concerning this function u, note first that $|u|$ tends uniformly to infinity as $s \to -\infty$ along each of the convex side ends of any $C \ \mathfrak{M}_I$. Moreover, with $C \in \mathfrak{M}_I$ specified, the function u pulls back via the defining pseudoholomorphic immersion from C_0 as a function with only one critical point, the latter being non-degenerate and hyperbolic. Indeed, this follows from the following two observations: First, J maps du to a nowhere zero multiple of $dt + \ d\varphi$. Meanwhile, as is neither 0, p/p' nor $p/(p' + q')$, the arguments from the proof of Proposition 4.7 apply to prove that $dt + \ d\varphi$ pulls back to C_0 with but one zero, which is hyperbolic.

With the preceding as background, suppose that $C \in \mathfrak{M}_I$ has been specified as well as a component $H' \ \mathfrak{M}_I$. Once again, H' is a smooth manifold and equivariantly diffeomorphic to $\mathbb{R} \ T \ \mathrm{Isom}(\mathbb{R} \ (S^1 \ S^2))$. In particular, this implies that there exists $C' \in H'$ with the following properties: First, the pair $(t + \ \varphi; u)$ at the critical point of u's restriction to C is identical to that at the critical point of u's restriction to C'. Second, the constant term on the right-hand side of the concave version of (5.1) for C is the same as that for C'.

To proceed, let $\ : C_0 \to \mathbb{R} \ (S^1 \ S^2)$ denote the defining pseudoholomorphic immersion with image C, and let $'_0$ denote the corresponding immersion with image C'. The arguments given above for the proof of Lemma 5.2 can be modified in a minor way to find a diffeomorphism $\ : C_0 \to C_0$ for which the pullbacks of $(t + \ \varphi; u)$ by $'\ \ '_0$ and are identical. This understood, introduce $\underline{\varphi}\ \ \varphi - \ '\ \varphi$ and $\underline{h} = \ h - \ '\ h$ as functions on C_0. In this regard, note that $\underline{\varphi}$ can be viewed as a bona fide $\mathbb{R}$ valued function on C_0 that is bounded, limits to zero as $s \to \infty$ on C_0 and has finite limits on the other ends of C_0. Moreover, as demonstrated momentarily, the functions $\underline{\varphi}$ and $\underline{h}$ enjoy the following properties:

> *If* $C \neq C'$*, then the* $\underline{h} = 0$ *level set is an embedded graph,* $\underline{G} \subset C_0$*; whose vertices are the* $\underline{h} = 0$ *critical points of* $\underline{h}$*. Moreover,* $d\underline{\varphi}$ *pulls back without zeros to the edges of* $\underline{G}$ *and, provided* > 0 *is sufficiently small,* $\underline{G}$ *has a non-empty set of edges.* (5.18)

As before, these properties are inconsistent and so $\underline{h} = 0$, $\underline{\varphi} = 0$ and $C = C'$.

The validity of (5.18) is easiest seen (perhaps) by using the pullbacks of the pair $(\quad t + \varphi; u)$ as local coordinates away from the critical point of u. In terms of these coordinates, the pull back of $(\varphi; h)$ obeys, by virtue of C being pseudoholomorphic, an equation with the schematic form

$$\varphi = \sin^{-2} \quad e\, h_u \quad \text{and} \quad \varphi_u = -\sin^{-2} \quad g^{-2} e^{-} \quad h \; . \qquad (5.19)$$

Here, and are the pullbacks via of their namesakes on $\mathbb{R} \times (S^1 \times S^2)$. Meanwhile, $\varphi' \quad \varphi$ and $h' \quad h$ obey (5.19) but with and replaced by $'$ and $'$, respectively. As φ and are implicit functions of f and h, it follows from (5.19) and its primed analog that the pair $(\underline{\varphi}; \underline{h})$ obey an equation with the schematic form of (5.12). And, with this last point understood, the arguments given about $\underline{G}$ in the case where none of p, q, and $p + q$ vanish can be used with only minor modifications to prove (5.18). As these modifications are slight, their details are left to the reader.

(c) Thrice-punctured spheres in $\mathbb{C} \times \mathbb{C}$

The argument for the existence of Proposition 5.1's thrice-punctured spheres starts here and runs through Subsection 5g. By way of a beginning, this subsection first considers holomorphic, triply-punctured spheres in the complex manifold $\mathbb{C} \times \mathbb{C}$ where $\mathbb{C} = \mathbb{C} - \{0\}$. The punctured spheres are then used to construct symplectic, thrice-punctured spheres in $\mathbb{R} \times (S^1 \times S^2)$ with prescribed asymptotics. The subsequent subsections explain how to deform the latter to obtain those predicted by Proposition 5.1.

To start the $\mathbb{C} \times \mathbb{C}$ discussion, identify the thrice-punctured sphere, C_0, as a complex manifold with the complement of the points 0, 1 and ∞ in the Riemann sphere, $\mathbb{P}^1 = \mathbb{C} \cup \infty$. Alternately, $C_0 = \mathbb{C} - \{0; 1\}$. Note that C_0 comes with the order six subgroup G of the complex automorphism group $PSL(2; \mathbb{C})$ which permutes the points $\{0, 1, \infty\}$. This group is generated by the automorphisms that send z to $1 - z$ and z to $1/z$.

Now, consider a holomorphic map $\varphi : C_0 \to \mathbb{C}^* \times \mathbb{C}^*$ which has the form

$$\varphi = (a r^{p+q} z^{-p} (1 - z)^{-q}, a' r^{p'+q'} z^{-p'} (1 - z)^{-q'}) , \tag{5.20}$$

where p', q, and q' are integers, a and a' are unit length, complex numbers, and $r \geq 1$ is a constant whose lower bound will be specified shortly. Here, $I \equiv \{(p, p'), (q, q')\}$ is constrained so that $\delta \equiv pq' - qp'$ is non-zero. In this regard, note that φ factors through a map to $\mathbb{C}^*$ if and only if $\delta = 0$, so this is a reasonable constraint on I.

It is important to point out that with r fixed, each φ in (5.20) has five companions with the same image in $\mathbb{C}^* \times \mathbb{C}^*$. These companions are obtained by composing φ with the non-trivial elements of G. However, in the subsequent constructions, the puncture at ∞ plays a special role and only the involution $z \to (1 - z)$ in G preserves this puncture. As this involution preserves the set of maps having the form in (5.20), it sends the four integers $\{(p, p'), (q, q')\}$ to another such set, namely $\{(q, q'), (p, p')\}$ and thus changes the sign of δ. In particular, no generality is lost in the subsequent discussions by restricting to the $\delta > 0$ case.

For future reference, make a note that the map φ is an immersion. Indeed, a singular point of φ can occur only where both $(p/z - q/(1 - z))$ and $(p'/z - q'/(1 - z))$ vanish. Since $\delta \neq 0$, this happens nowhere on the thrice-punctured disk. It is also important to note that φ has only a finite number of double points and thus embeds neighborhoods in C_0 of the punctures.

(d) The map

Fix a set $I = \{(p, p'), (q, q')\}$ of integers which obey the constraints in the three points of Proposition 5.1. An embedding in $\mathbb{R} \times (S^1 \times S^2)$ of a neighborhood of the image in $\mathbb{C}^* \times \mathbb{C}^*$ (5.20) that is proper on this image defines a properly immersed, triply-punctured sphere in $\mathbb{R} \times (S^1 \times S^2)$. This subsection describes a map that provides an embedding of this sort and that sends C_0 to a subvariety, C', with certain favorable properties. In particular, C' is pseudoholomorphic for a T–invariant almost complex structure, J', that tames ω and is asymptotic to (1.5)'s fiducial almost complex structure J as $|s| \to \infty$ on $\mathbb{R} \times (S^1 \times S^2)$. Also, C' is constructed with prescribed asymptotics. This is to say that the components of the large and constant $|s|$ slices of C' converge to multiple covers of closed Reeb orbits where the multiplicities and the closed Reeb orbits are suitably determined by the set I. Finally, C' avoids the $\theta \in \{0, \pi\}$ locus precisely when the set I obeys Proposition 5.1's constraints. The preceding properties play key roles in subsequent arguments.

The desired embedding, ψ, is described in this subsection simply as a map from a neighborhood of $\phi(C_0)$ into $(S^1\times\mathbb{R})\times(S^1\times\mathbb{R})$. Subsequently ψ is interpreted as a map into $\mathbb{R}\times(S^1\times S^2)$ by the identification the former space with the complement of the $\theta\in\{0,\pi\}$ locus in the latter via the coordinates (t,f,φ,h). Of course, such an interpretation is possible only when the initial map into $(S^1\times\mathbb{R})\times(S^1\times\mathbb{R})$ avoids $(S^1\times 0)\times(S^1\times 0)$. The next subsection provides the verification that such is the case.

To begin the discussion, take (t,f,φ,h) as coordinates on $(S^1\times\mathbb{R})\times(S^1\times\mathbb{R})$. Here, t and φ are $\mathbb{R}/(2\pi\mathbb{Z})$ valued while f and h are $\mathbb{R}$ valued. Meanwhile, introduce $\mathbb{R}/(2\pi\mathbb{Z})$ valued functions (t,φ) and $\mathbb{R}$–valued functions (u,v) for $\mathbb{C}^*\times\mathbb{C}^*$ by writing the complex coordinates ζ and ζ' as $\zeta=e^{u-it}$ and $\zeta'=e^{v-i\varphi}$. The map ψ, sends a neighborhood of ϕ's image into $(S^1\times\mathbb{R})\times(S^1\times\mathbb{R})$, so that both the coordinates t and φ on $(S^1\times\mathbb{R})\times(S^1\times\mathbb{R})$ are respectfully identified via ψ's pullback with their namesake functions on $\mathbb{C}^*\times\mathbb{C}^*$. Meanwhile, ψ respectively identifies f and h with u and v except near the ϕ image of neighborhoods of 0, 1 and ∞. Near the ϕ image of neighborhoods of the punctures, the definition of the map ψ is completed below by writing f and h as suitable functions of u and v.

To see what is involved here, observe that ϕ near $z=0$ has the form

$$\phi(z)=(ar^{p+q}z^{-p}(1+O(|z|)),a'r^{p'+q'}z^{-p'}(1+O(|z|)))\,. \tag{5.21}$$

Thus, the ϕ image of a neighborhood of $z=0$ has

$$\begin{aligned}&p\varphi-p't=p'\arg(a)-p\arg(a')+O(|z|)\mod(2\pi),\\&pv-p'u=\kappa\ln r+O(|z|).\end{aligned}\tag{5.22}$$

Even with the $O(z)$ and $\ln r$ terms absent, the identification $(f=u,h=v)$ does not make the locus in (5.22) close to a J–pseudoholomorphic subvariety when both p and p' are non-zero. In any event, the plan is to define ψ near the ϕ–image of a neighborhood of 0 so that the composition $\psi\circ\phi$ sends a neighborhood of $z=0$ close to the J–pseudoholomorphic locus that is defined by

$$\begin{aligned}&p\varphi-p't=p'\arg(a)-p\arg(a')\mod(2\pi),\\&ph-p'\sin^2\theta_0 f=0.\end{aligned}\tag{5.23}$$

Here, θ_0 is determined as follows: First, θ_0 obeys $p'(1-3\cos^2\theta_0)=p\sqrt{6}\cos\theta_0$. In this regard, remember that there is one solution when $|p'/p|<(3/2)^{1/2}$ and two otherwise. In the former case, the condition $p'(1 - 3\cos^2\theta_0) = p\sqrt{6}\cos\theta_0$ completely determines θ_0. Meanwhile, when $|p'/p| > (3/2)^{1/2}$, the sign of $\cos\theta_0$ differs for the two solutions; and with this understood, the angle θ_0 is chosen so that the signs of p' and $\cos\theta_0$ agree. The third constraint in Proposition 5.1 guarantees the existence of such a θ_0.

The subsequent definition of u and v in terms of f and h depends on whether or not one of p and p' is zero, and when both are non-zero, it depends on their signs. The case where both are non-zero and positive is presented immediately below; the other cases where both are non-zero can be obtained from this one by suitable notational changes. The case where one of p or p' is zero is presented afterwards.

In the case where both p and p' are positive, u is chosen to be an increasing function of f and v an increasing function of h. In this regard, keep in mind that with p positive, small $|z|$ means large u and thus large f; and with p' positive, the same conclusion holds for the pair v and h. In particular, the small $|z|$ region of D is to be mapped by ϕ to the convex end of $\mathbb{R} \times (S^1 \times S^2)$, that where $s \to -\infty$.

To define ϕ near the ϕ image of a neighborhood of $z = 0$, first introduce

$$\alpha_0 \equiv \sqrt{6}(p^2 + p'^2 \sin^2\theta_0)^{1/2}. \tag{5.24}$$

Next, select a function $\chi : \mathbb{R} \to [0, 1]$ that equals 1 on $(-\infty, 1]$ and 0 on $[2, \infty)$. With χ chosen and for each $R > 1$, introduce $\chi_R(\cdot) \equiv \chi((\cdot)/(4R))$ and $\beta_R(\cdot) \equiv 1 - \chi((\cdot)/2R)$.

Now, take $R_0 \gg 1 + r$, set $R \equiv R_0$ to define χ_R and β_R, and declare the pullback of the function f to be

$$f(u) = \chi_R(p^{-1}u)u + \beta_R(p^{-1}u)(p'\sin^2\theta_0)^{-1}e^{\alpha_0(u - q\ln(r))/p}. \tag{5.25}$$

Meanwhile, declare the ϕ pullback of the h to be

$$h(v) = \chi_R(p'^{-1}v)v + \beta_R(p'^{-1}v)p^{-1}e^{\alpha_0(v - q'\ln(r))/p'}. \tag{5.26}$$

Concerning this definition of these ϕ pullbacks, note in particular that both $f_u > 0$ and $h_v > 0$ when R_0 is sufficiently large as $\chi_R + \beta_R \geq 1$ everywhere, $|d\chi_R|$ and $|d\beta_R|$ are $O(1/R)$, and $\chi_R = 1$ where $d\beta_R \neq 0$.

Also, as $f(u) = u$ when $u < 2pR_0$ and $f(v) = v$ when $v < 2p'R_0$, the definition of ϕ via (5.25)–(5.26) is consistent (when R_0 is large) with the previous definition for ϕ's restriction to a neighborhood of a particular compact subset of C_0. Moreover, by virtue of (5.22), the fact that $f(u) = (p'\sin^2\theta_0)^{-1}e^{\alpha_0(u - q\ln(r))/p}$

when $u > 8pR_0$ and the fact that $h(v) = p^{-1} e^{\,0(v - q' \ln(r))/p'}$ when $v > 8p'R_0$, the composition embeds a neighborhood in C_0 of the puncture 0 as a locus in $\mathbb{R} \times (S^1 \times S^2)$ which is defined by equations of the following sort:

$$p\varphi - p't = p' \arg(a) - p \arg(a') + r_1 \mod (2\pi) .$$
$$ph - p' \sin^2 \theta_0 f = r_2 . \tag{5.27}$$

Here, the terms r_1 and r_2 come from the $O(z)$ terms in (5.22). In particular, they can be viewed as functions which are defined on an open set which contains the image of a neighborhood of $z = 0$ in D. In this regard, such an open set can be chosen so that each of r_1, $f^{-1} r_2$ and their derivatives are $O(e^{-\sqrt{6}|s|/\theta_0})$.

By the way, as the image of in $\mathbb{C} \times \mathbb{C}$ is holomorphic, it is pseudoholomorphic with respect to the integrable almost complex structure which sends ∂_t to ∂_u and ∂_φ to ∂_h. As a consequence, the image of C_0 in $\mathbb{R} \times (S^1 \times S^2)$ by 's composition with , as defined above by t, φ and (5.25–26), is pseudoholomorphic for the almost complex structure J' that is defined near the image of a neighborhood of $z = 0$ by

$$J'\partial_t = f_u \partial_f \quad \text{and} \quad J'\partial_\varphi = h_v \partial_h . \tag{5.28}$$

Here, f_u is evaluated at $u(f)$ and likewise, h_v at $v(h)$. Note specifically that (5.25), (5.26) and (5.27) imply that

$$f_u = p^{-1}\theta_0 f = g(1 + O(e^{-\sqrt{6}|s|/\theta_0})) ,$$
$$h_v = p'^{-1}\theta_0 h = g \sin^2\theta (1 + O(e^{-\sqrt{6}|s|/\theta_0})) \tag{5.29}$$

near the image of a neighborhood of $z = 0$. Here, $g = \sqrt{6}(f^2 + h^2 \sin^{-2}\theta)^{1/2}$. Thus, the almost complex structure J' and the fiducial almost complex structure J in (1.5) are exponentially close,

$$|J' - J| = O(e^{-\sqrt{6}|s|/\theta_0}); \tag{5.30}$$

on an open set which contains the image of a neighborhood of $z = 0$ in D.

Now consider when $p = 0$ and $p' < 0$. The $p' > 0$ case is obtained from this one by switching various signs and is left to the reader. In the $p = 0$ and $p' < 0$ case, v tends to $-\infty$ as $|z|$ tends to zero. With this understood, again take $R \gg R_0$ very large and choose $h = h(v)$ to be a favorite function of v with the following properties:

- $h_v > 0$.
- $h = v$ where $v \geq -R$.

$$h = -e^{-\sqrt{6}v} \quad \text{where } v < -R^2. \tag{5.31}$$

With h now chosen, take f to be a favorite function of both u and v that satisfies

$$f_u > 0\,.$$
$$f = u \quad \text{where } v \quad -R\,.$$
$$f = -6^{1/2}h(v)(u - q\ln r) \quad \text{where } v \quad -R^2\,. \tag{5.32}$$

Given the preceding, it follows that a neighborhood in C_0 of $z = 0$ is mapped by as the large $-s$ portion of a locus that is defined by equations of the form

$$' = - \arg(a') + r_1 \mod (2\ )\,,$$
$$f = r_2\,, \tag{5.33}$$

where r_1 and r_2 are functions on a neighborhood of this portion of the image of . Here, as with their namesakes in (5.27), each of r_1, $|h|^{-1}r_2$, and their derivatives are $O(e^{-|s|/})$ where 1 is a constant.

In this case, the locus image of is pseudoholomorphic for the almost complex structure J' which is defined so that

$$J'\partial_t = f_u\partial_f \quad \text{and} \quad J'\partial_{'} = f_v\partial_f + h_v\partial_h. \tag{5.34}$$

In particular, this last equation implies that J' and J again obey (5.12) on an open set which contains the image of a neighborhood of $z = 0$ in C_0.

As remarked, there is an analogous description of in the $p = 0$, $p' > 0$ case, and also in the cases where $p' = 0$. The details here differ from those just given only in the notation so they won't be given.

Of course, a corresponding description of can be simultaneously made near the image of a neighborhood of $z = 1$ as well. The discussion is identical to that just ended save for some signs and the interchange of p with q and p' with q'. By the way, as the definition of near the image of a neighborhood in C_0 of 0 required the choice of the constant R_0, so the definition near the image of a neighborhood in C_0 of 1 requires the choice of a constant, R_1. The freedom to make these choices is exploited in a subsequent argument.

The final order of business in this subsection completes 's definition by describing this map near the image of a neighborhood in C_0 of $z = \infty$. Here, is defined so that the composition properly embeds the complement of a large radius disk in C in the concave end of $\mathbb{R} \times (S^1 \times S^2)$. In particular, with as specified below, the constant s slices of the ( )-image converge as $s \to -\infty$ to the closed Reeb orbit whose θ_0 parameter obeys $(p' + q')(1 - 3\cos^2\theta_0) = (p + q)\sqrt{6}\cos\theta_0$ and is such that $\cos\theta_0$ and $p' + q'$ have the same sign when the latter are non-zero.

The details of the definition of ϕ near the ϕ image of a neighborhood of ∞ depends here on whether $k \equiv p + q$ or $k' \equiv p' + q'$ are zero and on their signs when they are non-zero. The discussion that immediately follows assumes that both are positive. This discussion also covers the other possibilities after some straightforward sign changes. The cases where one of k or k' is zero is discussed subsequently. In what follows, α_0 denotes $\sqrt{6}(k^2 + k'^2\sin^2\theta_0)^{1/2}$.

To see what is involved in the k and k' both positive case, note that when $k > 0$, then the function $u = \ln|\eta|$ on $\mathbb{C}^* \times \mathbb{C}^*$ tends to $-\infty$ as $|z| \to 0$. Likewise, so does $v = \ln|\eta'|$ when $k' > 0$. Moreover, the ϕ image of a neighborhood of ∞ has the form

$$\begin{aligned} &k\varphi - k't = k'\arg(a) - k\arg(a') + \ldots + r_1 \mod (2\pi), \\ &kv - k'u = r_2. \end{aligned} \tag{5.35}$$

Here, r_1, r_2 and their derivatives are $O(|z|^{-1})$. Of course, this is by virtue of the fact that where $|z| > 2$,

$$\eta = (ar^k z^{-k}(1 + O(z^{-1})), a'r^{k'}z^{-k'}(1 + O(z^{-1}))). \tag{5.36}$$

To proceed, assume that $r \geq e^{40}\alpha_0^4$, take $R \geq R_\infty \gg 1$, and then take f to be a favorite function of the variable u which has the following properties: First, $f(u)$ is as previously defined near the ϕ image of the $|z| < 10$ portion of C_0. On the remaining portion of C_0, require that

$$\begin{aligned} &f_u > 0. \\ &f = u \qquad \text{where } u > 2\sin^{-2}\theta_0 R^{-1}. \\ &f = (k'\sin^2\theta_0)^{-1}R^{-1}e^{\alpha_0 u/k} \qquad \text{where } u \leq 0. \end{aligned} \tag{5.37}$$

Meanwhile, take h to be a favorite function of the variable v which is as previously defined near the ϕ image of the $|z| < 10$ portion of C_0 and is contrained near the remainder of $\phi(C_0)$ to obey

$$\begin{aligned} &h_v > 0. \\ &h = v \qquad \text{where } v > 2R^{-1}. \\ &h = k^{-1}R^{-1} \qquad \text{where } v < 0. \end{aligned} \tag{5.38}$$

Note that the constraints in (5.37–38) are consistent with the previously specified constraints on u and v because $|u|$ and $|v|$ are $o(\ln r)$ where $10 < |z| < \sqrt{r}$.

By virtue of (5.35), these definitions of f and h imply that the ϕ–image of the $|z| > 10$ portion of C_0 has the form

$$k\varphi - k'\,t = k'\arg(a) - k\arg(a') + \ \ + r_1 \mod (2\pi),$$
$$k\sin^{-2}\theta_0 h - k'\sin^2\theta_0 f = r_2\,. \tag{5.39}$$

Here, r_1 and r_2 can be viewed as functions where are defined in a neighborhood of the {image of the $|z| > 10$ portion of C_0. In this regard, such a neighborhood can be defined so that r_1, $f^{-1}r_2$ and their derivatives are $o(e^{-|s|/\delta})$ for some positive constant δ. (Note that r_2 is not the same function as that which appears in (5.35).)

As before, since the image of C_0 in $\mathbb{C} \times \mathbb{C}$ is holomorphic for the almost complex structure which maps ∂_t to ∂_u and ∂_φ to ∂_v, so the image of the $|z| > 10$ portion of C_0 is pseudoholomorphic for an almost complex structure which has the same form as that depicted in (5.28). Moreover, as (5.29) holds for this almost complex structure, so does (5.30).

Now consider the definition of near the image of a neighborhood of $z = \infty$ in the case where $k > 0$ and $k' = 0$. The definition of when $k < 0$ and $k' = 0$, or when $k = 0$ and $k' \neq 0$ is omitted since the definition in these cases is obtained from the $k > 0$, $k' = 0$ definition by changing notation.

To begin, remember that $\sin^2\theta_0 = 1$ and $\alpha_0 = \sqrt{6}k$ because $k' = 0$. Also, note that the function u tends to $-\infty$ as $|z|$ tends to ∞ on C_0 because $k > 0$. However, when r is very large, then u is on the order of $2^{-1}k\ln r$ on the circle where $|z| = \sqrt{r}$. With the preceding understood, define the embedding near the image of the $|z| > 10$ portion of C_0 by taking f to be any function of the coordinate u with the following properties:

$$\begin{aligned} &f_u > 0.\\ &f = u \quad \text{where } u > 2R^{-1}.\\ &f = R^{-1}e^{\sqrt{6}u} \quad \text{where } u < 0. \end{aligned} \tag{5.40}$$

Here, $R \gg R_1 \gg 1$. Then, with $f(u)$ chosen, take h to be any function of both u and v which obeys

$$\begin{aligned} &h_v > 0.\\ &h = v \quad \text{where } u > 2R^{-1}.\\ &h = 6^{1/2}f(u)v \quad \text{where } u < 0. \end{aligned} \tag{5.41}$$

These definitions of f and h complete the definition of near the {image of a neighborhood of $z = \infty$ in C_0. Note that the image of a neighborhood in C_0 of ∞ obeys the $k' = 0$ version of (5.39). Moreover, the image of such a neighborhood is pseudoholomorphic as defined by an almost complex structure J' that sends ∂_t to the vector field $f_u\partial_f + h_u\partial_h$ and ∂_φ to $h_v\partial_h$. In this regard, note that (5.35), (5.40) and (5.41) insure that J and J^0 are nearly identical where s is large near C_0. Indeed, the just referenced equations imply that $jJ^0 - Jj$ $e^{-ss=}$ at large s where 1 is a constant.

(e) as a map into $\mathbb{R}$ $(S^1$ $S^2)$

The rst task for this subsection is to verify that can be constructed so that its image avoids the locus where both f and h vanish. Having veri ed that such is the case, is then interpreted as a map into $\mathbb{R}$ $(S^1$ $S^2)$. The subsection next summarizes the salient features of as a map into the latter space. In particular, as > 0 and $q^0 - p^0 > 0$ unless both p^0 and q^0 are non-zero and have the same sign, the map can be de ned as above so that it and the resulting C^0 ($)(C_0)$ $\mathbb{R}$ $(S^1$ $S^2)$ have the properties listed below:

is an embedding on a neighborhood of (C_0).

There exists > 0 such that sin $>$ on C^0.

C^0 has two ends on the convex side of $\mathbb{R}$ $(S^1$ $S^2)$ and one on the concave side. Moreover, one of the convex side ends is described by (5.27) where jsj is large, and the other by (5.27) with $(q; q^0)$ replacing $(p; p^0)$. Meanwhile, the concave side end of C^0 is described by (5.39) where s is large.

C^0 is symplectic and it is pseudoholomorphic for an almost complex structure, J^0, on $\mathbb{R}$ $(S^1$ $S^2)$ with the following list of features:

(a) J^0 tames $!$.

(b) $J^0 = J$ where sin $<$ $=10$.

(c) $J^0 @_t = a_t @_f + b_t @_h$ and $J^0 @ = a @_f + b @_h$; moreover, J^0 is T{invariant so the coe cients $(a_r; b_t; a ; b)$ depend only on the coordinates f and h.

(d) There exists 1 such that $jJ^0 - Jj$ $e^{-jsj=}$ on $\mathbb{R}$ $(S^1$ $S^2)$. (5.42)

By the way, to say that J^0 tames $!$ is to assert that $!(; J^0( ))$ is a positive de nite, bilinear form on $T(\mathbb{R}$ $(S^1$ $S^2))$.

The proof of the assertions of (5.42) rounds out the discussion in this subsection. In this regard, the arguments for (5.42) and also the argument that justi es 's interpretation as a map into $\mathbb{R}$ $(S^1$ $S^2)$ make certain requirements on the parameters r, R_0, R_1 and R_1, the rst being that they should all be large.

The rst task is to prove that the image in $(S^1 \quad \mathbb{R}) \quad (S^1 \quad \mathbb{R})$ via of a neighborhood of (C_0) in avoids the $f = h = 0$ locus. For this purpose, and for use in the discussion on (5.42), it proves useful to describe C_0 as the union of four open sets, U, U_0, U_1 and U_∞. Here, $U = \{z \in C_0 : |z| > r^{-1/2}; |z-1| > r^{-1/2}$ and $|z| < r^{1/2}\}$. Meanwhile, U_0 is the subset of C_0 on which $|z| < 2r^{-1/2}$, U_1 is where $|z-1| < 2r^{-1/2}$ and U_∞ is where $|z| > 2^{-1}r^{1/2}$.

To continue, note that one or both of the {pullbacks h and f have large absolute value in (U_0) and in (U_1) so the image of both of these sets avoids the $h = f = 0$ locus. Meanwhile, on (U), the {pullbacks of f and h are identi ed with u and v. Since $\neq 0$, these are both zero only when $|z| = |1-z| = r$. In particular, when $r > 100$, the pullbacks of f and h by are not simultaneously zero on (U). Finally, the de nition of near (U_∞) insures that the pullbacks of f and h are non-zero. For example, with the case where both k and k' are positive, f is de ned by (5.37) to be positive as long as $u \quad 2\sin^{-2} \ {}_0R_\infty^{-1}$ and f is de ned to equal u when $u > 2\sin^{-2} \ {}_0R^{-1}$. In this regard, note that the $u = 2\sin^{-2} \ {}_0R_\infty^{-1}$ locus is an embedded circle where $|z| = r\exp(-2(k\sin^{-2} \ {}_0R_\infty)^{-1})(1+O(1/r))$. Moreover, as the derivative of u with respect to $|z|$ on this circle is equal to $-k/|z| + O(1/|z|^2)$, the function u and thus f only increase on the inside of this circle.

With the preceding understood, should henceforth be viewed as a map from a neighborhood of (C_0) into $\mathbb{R} \quad (S^1 \quad S^2)$ and then the next order of business is to verify that the assertions in (5.42) are correct. For this purpose, note that the third point of (5.42) follows directly from the de nition of . Likewise, Parts a) and d) of the fourth point follow directly from the de nition as does Part c) if the asserted T{invariance of J' is disregarded. Meanwhile, Part b) of the fourth point follows by standard arguments from the second point of (5.42). Thus, the subsequent discussion concerns only the rst and second points of (5.42) and the assertion of T{invariance in part c) of the fourth point. In this regard, note that the rst point in (5.42) and the T{invariance assertion are both consequences of the following lemma:

Lemma 5.5 *When r is large, the parameters R_0, R_1 and R_∞ can be chosen for the de nition of with arbitrarily large minimum and so that the set U contains all distinct pairs z, $z' \in C_0$ that are mapped to the same point in $\mathbb{R}^2$ by the pullback of $(f; h)$.*

Note that the ability to make the minimum of R_0, R_1 and R_∞ large ensures that de nes a map into $\mathbb{R} \quad (S^1 \quad S^2)$.

The proof of Lemma 5.5 is supplied momentarily. Granted this lemma, the first point in (5.42) can be argued as follows: First, as the map is both proper on C_0 and a local diffeomorphism, it is enough to demonstrate that all distinct pairs $\{z, z'\} \in C_0$ which are sent to the same point by lie in U and are thus already identified by . This last property is, of course, guaranteed by Lemma 5.5.

To argue for the T–invariance assertion in part c) of the fourth point of (5.42), note first that one strategy for the construction of the required J' would take J' as given near C' by (5.28) or (5.34) and rotate the latter via the T–action on $\mathbb{R} \times (S^1 \times S^2)$. In particular, this strategy would exploit the explicit lack of (t, φ) dependence of the functions which multiply the vector fields on the right-hand side of (5.28) or (5.34). However, as the functions in (5.28) and (5.34) are defined *a priori* only near C', such a strategy has the following prerequisite for success: When a T orbit intersects C' more than once, then the versions of J' given by (5.28) or (5.34) at the various intersection points must all agree. Lemma 5.5 provides just this prerequisite.

Proof of Lemma 5.5 Let $P \equiv (f, h)\colon (C_0) \to \mathbb{R} \times \mathbb{R}$. By virtue of its very definition, this map P embeds the subset of of (C_0) where $(f, h) = (u, v)$. This means, in particular, that P embeds (U). Also, P separately embeds the images of U_0, U_1 and U_∞ since the pullbacks via of the functions f and h have been constructed to change monotonically with u and v, near these images. Thus, after the introduction of $V_{0,1,\infty} \subset U_{0,1,\infty}$ to denote the subset where $(\ ) (f, h)$ differs from (u, v), the question at issue here is whether any of $P \ (V_{0,1,\infty})$ intersect $P \ (U)$, whether $P \ (V_\infty)$ intersects any of $P \ (U_{0,1})$ and whether $P \ (V_0)$ intersects $P \ (U_1)$ or $P \ (V_1)$ intersects $P \ (U_0)$.

To address this question, note first that with R_∞ very large, both f and h will be nearly zero on (V_∞). Meanwhile, as $(f^2 + h^2)$ is bounded away from zero on (U), (U_0) and (U_1) by an r–dependent, but R_∞–independent constant. Thus, when r is large and then R_∞ is very large, the P images of (V_∞) and each of (U), (U_0) or (U_1) are necessarily disjoint. Likewise, when R_0 and R_1 are very large, then at least one of the pullbacks of f or h has large absolute value on (V_0) and (V_1) where these pullbacks differ from (u, v). Meanwhile, both f and h are uniformly bounded on (U) once r has been specified. Thus, when R_0 and R_1 are sufficiently large, there are no intersections between the P images of (U) and those of (V_0) and (V_1).

The final possibility for trouble lies with the P images of (V_0) and (V_1). However, this issue is nontrivial only when p and q are both non-zero and have the same sign while p' and q' are also both non-zero and have the same sign. With these sign equalities now understood, the > 0 condition implies that $q'/q > p'/p$. The argument when $q'/q < 0$ is the same but for cosmetic changes as that for the case when $q'/q > 0$, so only the $q'/q > 0$ case is discussed below. In any event, given that $q'/q > 0$, (5.22), (5.25) and (5.26) imply that the pullback of the ratio h/f obeys

$$(h/f) = p'/p + O(r^{1/2}) \text{ on } (U_0 - V_0).$$

$$p'/p + O(R_0^{-}) \qquad (h/f) \qquad p'/p \sin^2 {}_{0P} - O(R_0^{-}) \text{ on } (V_0). \qquad (5.43)$$

Here, ${}_{0P}$ is the value of on the closed Reeb orbit that is associated to the end (V_0) of C'. Also, > 0 is determined by the integers $\{(p, p'), (q, q')\}$. Meanwhile, change p to q, p' to q' and P to Q in (5.43) and the resulting inequalities then describes the pullback of h/f to (U_1).

Now, as argued in Step 9 of the proof of Constraint 4 in Subsection 5b, when q'/q and p'/p are both positive and q'/q is greater than p'/p, then $q'/q \sin^2 {}_{0Q} > p'/p \sin^2 {}_{0P}$. With this point understood, and given both (5.43) and its (q, q') analog, the P images of both the pairs $\{ (V_0), (U_1)\}$ and $\{ (V_1), (U_0)\}$ are necessarily disjoint when r is rst chosen to be very large, then R_0 is chosen to be much larger than r and nally, R_1 is chosen to obey $R_1 \quad 256(p/q)R_0$. Indeed, such a choice insures that the ratios of (h/f) di er on (V_0) and (U_1), as do the analogous ratios on (V_1) and (U_0).

With the proof of Lemma 5.5 complete, only the second point of (5.42) is unspoken for. Here is the strategy for proving that such exists: The rst step proves that > 0 exists so that sin $>$ on the images of U_0, U_1 and U_1. Having done so, the remaining step considers whether sin is ever zero on the image of U. As U has compact closure, these two steps su ce to establish the second point of (5.42). By the way, in order to prove that sin $\neq 0$ on a given set, it is enough to prove that f is positive where h vanishes on the set in question.

To begin, consider the case with U_1. Here, it follows from (5.35) and (5.39) that when both k and k' are non-zero and both r and R_1 are large, then the required exists. Thus, the only troublesome case has $p' + q' = 0$ and $p + q < 0$. However, as > 0, this case requires $q' < 0$ and $p' > 0$ and so is excluded from consideration.

Now consider whether the required exists for U_0 when R_0 is large. For this purpose, note that the case where $p' = 0$ and $p < 0$ can be ignored as the conditions > 0 and $q' - p' > 0$ would otherwise be violated. Thus, one can assume that $p' \neq 0$ or else $p' = 0$ and $p > 0$. As explained momentarily, the existence of the required then follows from (5.22), (5.25) and (5.26). Indeed, to obtain , note rst that when R_0 is large, (5.26) provides a non-zero lower bound for sin on the image of V_0 U_0 when $p' > 0$. Meanwhile, if $p' = 0$, then $p > 0$ and so (5.25) insures that $f > 0$ on (V_0). Thus, it remains only to verify that sin > 0 on the closure in C_0 of the subset of U_0 where $f = u$ and $h = v$. For this purpose, note that $v = 0$ on (C_0) only on the image of points z where

$$r^{p'+q'} = |z|^{p'}|1 - z|^{q} \tag{5.44}$$

and so such a zero occurs with $|z|$ 1 if and only if p' and $p' + q'$ have opposite signs. This requires q' and p' to have opposite signs and thus p' is negative since $q' - p'$ must be positive. But, if r is large and $p' < 0$, then (5.22) requires that $u > 0$ where $v = 0$.

An analogous argument nds a non-zero lower bound for sin on the image of U_1.

With the preceding understood, it remains only to verify that the zeros of the pullback of h on (U) occur where the pullback of f is positive. In this regard, the straightforward case to verify occurs when either one of p' or q' is zero, or when both p' and q' are non-zero and they have the same sign; for in this case, h has no zeros at all on (U). Indeed, as $h = v$ on (U) so at issue are the zeros of v. As p' and q' have the same sign if one is not zero, the latter locus is placed by (5.44) where $|z| = r + O(1)$ which is not in U.

The next case to verify has $p' + q' = 0$. In this case, $h = v$ is zero on (U) only on the image of the line where $|z| = |z - 1|$. On this line, $f = u = (p + q)\ln(r/|z|)$. Now, the positivity of and the vanishing of $p' + q' = 0$ and the positivity of q' requires the positivity of $p + q$. Thus, as $|z|$ $r^{1/2}$ on U, so $f > 0$ on the zero locus in (U) of h.

Next, assume that neither p', q' or $p' + q'$ are zero but p' and q' have oppositive signs. Thus, $p' < 0$ and $q' > 0$. In this case, the locus in (5.44) has two components. One component has $|z| = r + O(1)$ so doesn't sit in U. The other component has $|z|$ or $|1 - z|$ small and can lie in U. In particular, if $p' + q' > 0$, this second component occurs where $|z| = r^{(p'+q')/p'}$; and if $p' + q' < 0$, this second component occurs where $|1 - z| = r^{(p'+q')/q'}$. In any event, the sign of $u = f$ on the –image of this other component can be determined with the help of one or the other of two identities that follow from (5.20):

$$qv - q'u = - \ln(r/|z|) \quad \text{and} \quad pv - p'u = \ln(r/|1 - z|). \tag{5.45}$$

In particular, as > 0, it follows from (5.45) that $u > 0$ on these small $|z|$ or $|1 - z|$ components of the $v = 0$ locus.

(f) Deformations

Suppose that $I \equiv \{(p,p),(q,q')\}$ are integers such that $\Delta > 0$ and $q' - p' > 0$ unless both p' and q' are non-zero and have the same sign. Take $r \gg 1$ to be large, and choose large R_0, R_1 and R_{-1} so that the map ψ is defined as in the preceding two subsections near $\psi(C_0)$ as an embedding into $\mathbb{R} \times (S^1 \times S^2)$ as described in (5.42). As in (5.42), use C' to denote $\psi(C_0)$. This C' is the starting member of a set of symplectic subvarieties which is parameterized by a non-trivial subinterval $[0,T_1] \subset [0,1]$ whose end member, C is in the space $\mathfrak{M}_I$ of Proposition 5.2. Thus, C is a J–pseudoholomorphic, thrice-punctured sphere that is parameterized by the same set I as is C', and whose existence is asserted in Proposition 5.1. This subsection describes the relevant parameterized set of symplectic subvarieties and it provides a proof that the end member of the set is a thrice-punctured sphere whose existence is predicted by Proposition 5.1.

The promised parameterized set of symplectic subvarieties is constructed with the help of a set of almost complex structures on $\mathbb{R} \times (S^1 \times S^2)$. If $\hat{J}$ is an almost complex structure in this set, then $\hat{J}$ is constrained to obey the following four conditions:

(a) $\hat{J}$ tames ω.

(b) $\hat{J} = J$ where $\sin\theta < \delta$.

(c) $|\hat{J} - J| \le \delta^{-1} e^{-\delta|s|}$ on X.

(d) $\hat{J}\partial_t = \alpha_t\partial_f + \beta_t\partial_h$ and $\hat{J}\partial_\varphi = \alpha_\varphi\partial_f + \beta_\varphi\partial_h$ where the coefficients $(\alpha_r, \beta_t, \alpha_\varphi, \beta_\varphi)$ depend only on the coordinates f and h; thus $\hat{J}$ is T–invariant. (5.46)

Here, $\delta > 0$ is determined by I. The particular set of almost complex structures under consideration is a certain, continuous, 1–parameter family $\{J_r\}_{r\in[0,1]}$ with $J_1 = J$. Note that each $\hat{J} \in \{J_r\}_{r\in[0,1]}$ obeys

$$\hat{J} = J \quad \text{where} \quad |s| > 1/\delta, \tag{5.47}$$

which is a stronger condition than (5.46c). (The condition in (5.47) is not imposed by necessity but to simplify subsequent arguments.)

The complex structure J_r is defined by the requirement that

$$J_r v = (1-r)(\chi J' v + (1-\chi)Jv) + rJv \tag{5.48}$$

when $v = \partial_t$ or ∂_φ. Here, χ is a smooth function of $|s|$ that equals 1 where $|s| < 1/(2\delta)$, vanishes where $|s| > 1/\delta$ and has derivative bounded by 4δ. The equality in (5.47) follows from the presence of in (5.48). In any event, J_r satisfies (5.46) because both J' and J do.

As stated above, the parameter, r, for the parameterized set, $\{ {}_r\}$, of subvarieties takes values in $[0,1]$ where $T \in (0,1]$. For each such r, the corresponding ${}_r$ is a J_r–pseudoholomorphic, immersed, thrice-punctured sphere that avoids the $\in \{0, \}$ locus and whose ends are constrained by the set I exactly as those of C' are via the third point in (5.42). The construction of ${}_r$ is facilitated by the following:

Lemma 5.6 *Let $\hat{J}$ denote an almost complex structure on $\mathbb{R}$ $(S^1$ $S^2)$ which obeys (5.46). Suppose that $I = \{(p,p')(q,q')\}$ is a set of pairs of integers with $pq' - qp' > 0$ and such that $q' - p' > 0$ when both p' and q' are non-zero and do not have the same sign. Given I, let $\mathbb{R}$ $(S^1$ $S^2)$ denote a subvariety that is the image via a $\hat{J}$–pseudoholomorphic map of a thrice-punctured sphere whose ends are constrained by I as those of C' are by the third point of (5.42). In addition, require that is disjoint from the $\{0, \}$ locus. Then has the following properties:*

- *is immersed and so the deformation operator D as described by (2.6) is well defined.*
- *This operator D has three-dimensional kernel and trivial cokernel.*

This lemma is proved below, so accept its validity for now. In particular, given the first two points of the lemma, a straightforward extension of the arguments in Section 3c for Proposition 3.2 establish the following two key facts:

- *There exists ${}_0 > 0$ such that J' and J obey (5.46) with ${}_0$; and for each such , the almost complex structure J_0 from (5.48) admits a J_0–pseudoholomorphic map from C_0 into $\mathbb{R}$ $(S^1$ $S^2)$ whose image, ${}_0$ is disjoint from the $\in \{0, \}$ locus and has its ends constrained by I as those of C' are by the third point of (5.42).*
- *Fix $< {}_0$ so that each J_r in (5.48) obeys the constraints in (5.46) and (5.47). Let $[0,1]$ denote the set of points r such that there exists a J_r–pseudoholomorphic map C_0 into $\mathbb{R}$ $(S^1$ $S^2)$ whose image, ${}_r$ is disjoint from the $\in \{0, \}$ locus and has its ends constrained by I as those of C' are by the third point of (5.42). This set is non-empty and open.* (5.49)

Indeed, the assertions in (5.49) are simply perturbation theoretic consequences of the vanishing of the cokernel of D for the subvariety C'. The arguments are essentially identical to those for Proposition 3.2 that appear in Section 3c.

By the way, note that J' in (5.42) obeys (5.46) but not (5.47). However, a small perturbation at very large values of $|s|$ on $\mathbb{R}$ $(S^1$ $S^2)$ gives an almost complex structure J_0 that obeys both (5.46) and (5.47) with as small as desired. This understood, perturbation theory using the invertibility of the C' version of D produces the required $_0$ as a small deformation of C'.

Now, let $T \in [0;1]$ denote the least upper bound of those $r \in$. Of prime importance is the limiting behavior as $r \to T$ of the set $\{ _r : r \in \}$. The nature of this limit, as given in the next lemma, explains how this parameterized set determines a subvariety C in Proposition 5.2's moduli space $\mathfrak{M}_I$.

Lemma 5.7 *Let be as just described and let T denote the least upper bound of . Then there exists a J–pseudoholomorphic, triply punctured sphere C in the moduli space $\mathfrak{M}_I$ of Proposition 5.2, a countable, increasing set $\{r(i)\}_{i=1;2;:::}$ and a corresponding set $\{w(i)\}_{i=1;2;:::}$ $\mathbb{R}$ with the following properties:*

- $\lim_{i \to \infty} r(i) = T$,
- *Let $_i$ denote the push-forward of $_{r(i)}$ via the diffeomorphism of $\mathbb{R}$ $(S^1$ $S^2)$ defined as translation by $w(i)$ on the $\mathbb{R}$ factor. Then*

$$\lim_{i \to \infty} (\sup_{x \in (C \cap K)} \mathrm{dist}(x; \ _i) + \sup_{x \in (_i \cap k)} \mathrm{dist}(C;x)) = 0$$

  for all compact sets K $\mathbb{R}$ $(S^1$ $S^2)$.

By the way, the arguments below imply that the sequence $\{w(i)\}$ has no convergent subsequences when $T < 1$.

Note that the assertion in Proposition 5.1 that $\mathfrak{M}_I$ is non-trivial follows directly from this last lemma.

The remainder of this subsection is occupied with the proof of Lemma 5.6, while the next subsection gives that of Lemma 5.7.

Proof of Lemma 5.6 Modulo two observations, the argument for the assertion that is immersed is identical to the argument in Section 4f for the proof of Proposition 4.7. The first observation asserts that the form $_t dt -$ $_' d'$ pulls back to with at most a single non-degenerate, hyperbolic zero when $(_t; \ _')$, are constants that define a vector which is not a multiple of either $(p';p)$, $(q';q)$ or $(k';k)$. Indeed, to see why this must be the case, note first that the form of $\hat{J}$ in (5.46) implies that the $\hat{J}$–pseudoholomorphic map sending C_0 to pulls back $_t t -$ $_' '$ as a multivalued function, , that obeys an equation with the schematic form $+ r \ d = 0$. Here, is a Laplacian on C_0 and r is an appropriate vector field. The maximum principle applies to this last equation and rules out local maxima and minima for . Thus, non-degenerate zeros of d are hyperbolic. Moreover, this same equation for implies that degenerate zeros of d provide a count of -2 or more to the Euler characteristic of C_0. Thus, d has exactly one zero, which is non-degenerate and hyperbolic.

The second observation needed concerns the part of the proof of Proposition 4.7 that follows (4.44). In particular, it follows from (5.47) that this portion of the argument applies here with no changes provided that it is applied to an end of where the corresponding closed Reeb orbit has $\cos^2 {}_0$ different from both 1/3 and 1. Since > 0, such an end is always present.

With the first point of Lemma 5.6 understood, the argument for the second point is identical in all essential aspects to that used in Section 4g to prove the $\partial = 2$ case of Proposition 4.8.

(g) The limits of the deformations

The purpose of this subsection is to provide the proof of Lemma 5.7. Being lengthy, the proof is divided into eleven steps. The first nine steps comprise what might be termed Part 1 of the proof. These steps give the proof when none of p, q and $p+q$ is zero. The last three steps provide the proof in the remaining cases. In any event, keep in mind that Steps 1–9 implicitly assume that each of p, q and $p+q$ is non-zero.

Step 1 Choose an increasing sequence $\{r(i)\}_{i=1,2,\dots}$ with limit T. This step defines a corresponding sequence $\{w(i)\}_{i=1,2,\dots} \in \mathbb{R}$. To start, remember that the proof of Lemma 5.6 established that dt's pull back to C_0 via the ${}_r$–defining, J_r pseudoholomorphic immersion has a single zero, one that is non-degenerate and hyperbolic. As neither p, q nor $p+q$ is zero, the almost complex structure J' from Section 5f that is described in (5.42) has $J' dt$ proportional to df. Therefore, this is also the case for J_r as defined in (5.48) and so df also pulls back to C_0 by the ${}_i$–defining map as a 1–form with but one zero, also non-degenerate and hyperbolic. Let x_r denote the image of this zero on ${}_r$. With x_r understood, take $w(i)$ to equal minus the value of the coordinate s at x_{ri}. Thus, the action of $w(i)$ on $\mathbb{R} \times (S^1 \times S^2)$ by translation of the $\mathbb{R}$ factor sends $x_{r(i)}$ to a point, $\underline{x}(i)$, which lies on the zero locus of the function s. (Although certain arguments in this Part 1 of Lemma 5.7's proof rely on this critical point structure of df, there are alternative arguments that do not. In fact, some of the latter are used for Part 2 of Lemma 5.7's proof.)

If the sequence $\{r(i)\}$ has a subsequence for which the corresponding sequence $\{w(i)\}$ is unbounded, then replace $\{r(i)\}$ by a subsequence for which the corresponding $\{w(i)\}$ sequence is unbounded and either strictly increasing or strictly decreasing. Agree to relable this new $\{r(i)\}$ sequence and its corresponding $\{w(i)\}$ sequence by consecutive integers starting at 1.

Let $\underline{\ }_i$ denote the image of $\ _{r(i)}$ under the translation isometry de ned by this same number $w(i)$. Likewise, let $\underline{J}_i$ denote the push-forward by this isometry of the almost complex structure $J_{r(i)}$. Thus, $\underline{\ }_i$ is $\underline{J}_i$–pseudoholomorphic. Also, $\underline{J}_i$ obeys the conditions in (5.46), albeit with the functions $(\ _t;\ _r;\ ';\ ')$ replaced by their translated versions.

Step 2 This step and Step 3 argue that the integral of ω over the intersection of $\underline{\ }_i$ with a given compact set enjoys an upper bound which is independent of the index i. In particular, this step considers the case when neither p' nor q' are zero. In this regard, remember that p and q are already assumed to be non-zero. The subsequent argument makes no use of the assumption that $p+q\neq 0$.

In any event, when p' and q' are non-zero, then a compact set K intersects $\underline{\ }_i$ where $-f^0 \quad f \quad f^0$ for suitable $f^0>0$. Likewise, a compact set K intersects $\underline{\ }_i$ where $-h^0 \quad h \quad h^0$ for suitable h^0. Having said this, consider rst the subcase where the sequence $\{w(i)\}$ is bounded. Under this extra assumption, the constants f^0 and h^0 can be large to guarantee that $\underline{J}_i = J$ where $|f| \quad f^0$ and where $|h| \quad h^0$. On this same portion of C, $dt\wedge df$ and $d'\wedge dh$ are non-negative and so

$$\int_{K\cap\underline{\ }_i} \omega \quad \int_{\{-f^0\ \ f\ \ f^0\}\cap\underline{\ }_i} dt\wedge df + \int_{\{-h^0\ \ h\ \ h^0\}\cap\underline{\ }_i} d'\wedge dh. \tag{5.50}$$

With this last inequality in hand, apply Stokes' theorem to nd

$$\int_{K\cap\underline{\ }_i} \omega \quad f^0 \left(\int_{\{f=-f^0\}\cap\underline{\ }_i} dt - \int_{\{f=f^0\}\cap\underline{\ }_i} dt\right) + h^0\left(\int_{\{h=-h^0\}\cap\underline{\ }_i} d' - \int_{\{h=-h^0\}\cap\underline{\ }_i} d'\right). \tag{5.51}$$

With regard to the derivation of (5.51), note that the concave side end of $\underline{\ }_i$ makes no contribution to the boundary terms in Stokes' theorem since both f and h limit to zero on this end as s limits to in nity. In any event, the inequality in (5.51) provides the promised upper bound for the integral of ω since the four path integrals which appear above are determined *a priori* by the set I.

In the case where $\{w(i)\}$ has no upper bound, the preceding argument still applies with no extra comments required. With one small revision, this same argument also applies to in the case where $\{w(i)\}$ has no lower bound. As before, there exists such f' and h' so that the set where $|f| < f'$ as well as that where $|h| < h'$ contain K. Moreover, given that $\{w(i)\}$ has no lower bound, the part of Σ_i where either $|f| < f'$ or where $|h| < h'$ has $\underline{J}_i = J$ when i is sufficiently large. Thus, the inequalities in (5.50) and (5.51) still hold provided that i is sufficiently large. In particular, (5.51) again provides an i–independent bound for ω's integral over $K \cap \Sigma_i$.

Step 3 Now consider the finiteness of the integral of ω in the case where either p' or q' is zero. Here, the argument is very similar to the one just given in Step 2. Indeed, the argument in this case requires but one substantial novelty. The novel part of the argument exploits the fact that the function h has precisely one non-compact level set when either p' or q' is zero. Only the $p' = 0$ case is considered below as the considerations for the $q' = 0$ case are essentially the same.

As the existence of a single non-compact h–level set is proved momentarily, accept it for now and let h_0 denote the value of h on this set. Then, there exists $h' > 0$ which depends only on K, and given i, there exists $\varepsilon > 0$ such that the integral of ω over $K \cap \Sigma_i$ is no more that twice that of ω over the subset of points in Σ_i where either $-h' < h \le h_0 - \varepsilon$ or $h_0 + \varepsilon \le h < h'$. This last conclusion constitutes the novel input. (Even with p' or q' zero, there exists $f' > 0$ such that K is contained in the subset where $|f| \le f'$.)

With the preceding understood, turn to the promised justification of the assertion that h has but one non-compact level set. For this purpose, it is necessary to return to $\Sigma_{r(i)}$. Having done so, the first point is that h is bounded on the $(p, p' = 0)$ end of $\Sigma_{r(i)}$ because φ could not have an $|s| \to \infty$ limit otherwise. Indeed, as h has no limits as $|s| \to \infty$ on (q, q') end of $\Sigma_{r(i)}$ and limits to zero on the concave side end, were h unbounded on the end where $(p, p' = 0)$, this end would contain whole components of regular value level sets of h where $J_r = J$ and where $|s|$ is everywhere large. But, $d\varphi$ pulls back without zeros to such a level set, so these properties preclude the existence of φ's $|s| \to \infty$ limit.

Note that the preceding argument also shows that h must have at least one non-compact level set, for otherwise there would be a component of a compact level set which sat entirely at large $|s|$ in the $(p, p' = 0)$ end of $\Sigma_{r(i)}$.

Now, as h tends to zero on the concave side of $\Sigma_{r(i)}$, tends uniformly to infinity on the (q, q') end of $\Sigma_{r(i)}$ and is bounded on the $(p, p' = 0)$ end of $\Sigma_{r(i)}$, it follows that the non-compact level set of h must have their ends in the $(p; p' = 0)$ end of $\Sigma_{r(i)}$. Thus, to prove that there is at most one non-compact level set of h, it is sufficient to establish the following lemma:

Lemma 5.8 *The function h has a unique $|s| \to \infty$ limit on the $(p; p' = 0)$ end of $\Sigma_{r(i)}$. In fact the restrictions of h to the large but constant $|s|$ circles in this end converge as $|s| \to \infty$ in the C^∞ topology to the constant function.*

Proof of Lemma 5.8 Where $|s|$ is large on the $(p; p' = 0)$ end of $\Sigma_{r(i)}$, $J_r = J$. Thus, by virtue of the asymptotics asserted in Proposition 2.3, this part of $\Sigma_{r(i)}$ can be parameterized as in (2.13) so that the column vector η with top component x and bottom component w obeys the analog,

$$\partial_\tau \eta + L_0 \eta = w\,\eta\,; \tag{5.52}$$

of (2.15). Here, $L_0 \equiv \begin{pmatrix} 0 & \partial \\ -\partial & \sqrt{6} \end{pmatrix}$ and w is a vector with $|w|e^{\varepsilon\tau}$ bounded as $\tau \to \infty$ for some positive constant ε. (The entries of w are functions of w, w_t and w_{tt} with affine dependence on the latter two.) Note that it is permissable to assume that x limits to zero here as $\tau \to \infty$.

Now, the point of writing (5.52) is to employ a simplified version of the analysis that was used in Steps 2–5 of the proof of Proposition 2.3 as given in Section 2. For this purpose, note that the operator L_0 has a zero eigenvalue with multiplicity 1, the eigenvector has constant top component x and bottom component $w = 0$. Meanwhile, L_0's smallest positive eigenvalue is $\sqrt{6}$ and the eigenvector has $x = 0$ and constant w. Let $E_+ > \sqrt{6}$ denote the next smallest positive eigenvalue of L_0 and let $-E_-$ denote the largest negative eigenvalue of L_0. Next, let $f^+(r)$ denote the L^2 norm, defined by integration over the circle parameterized by t, of the L^2 orthogonal projection of $\eta(\cdot; r)$ onto the span of the eigenvectors of L_0 with eigenvalue greater than $\sqrt{6}$. Define $f^-(r)$ in an analogous fashion using the span of the eigenvectors of L_0 with negative eigenvalue. Likewise, define $f^{\sqrt{6}}(r)$ to denote the L^2 norm of the L^2 orthogonal projection of η onto the span of the eigenvector of L_0 with eigenvalue $\sqrt{6}$. According to Lemma 2.5, η has zero component along the zero eigenvalue eigenvector of L_0.

Here is a key point with regard to the functions $f^{\pm;\sqrt{6}}$: As y is bounded, so each of $f^{\pm;\sqrt{6}}$ is bounded by $e^{-\sqrt{6}\tau}$.

With the definitions of $f^{\pm;\sqrt{6}}$ in hand, then (5.52) implies that for all sufficiently large τ,

$$\partial_\tau f^+ + E_+ f^+ \le \varepsilon (f^- + f^{\sqrt{6}})\,,$$

$$
\begin{aligned}
&\partial f^- - E_- f^- \quad - (f^+ + f^{\sqrt{6}}) \,, \\
&|\partial f^{\sqrt{6}} + \sqrt{6} f^{\sqrt{6}}| \quad (f^+ + f^- + f^{\sqrt{6}}) \,,
\end{aligned}
\tag{5.53}
$$

where ( ) is a positive, integrable function of on a domain of the form $[\ _0, \infty)$. Now, given the bounds $f^{\ ,\sqrt{6}} < e^{-\sqrt{6}}$ and the integrability of ( ), the inequalities in (5.53) can be integrated to yield

$$
f_+ + f_- \quad {}_1(\ )e^{-\sqrt{6}} \quad \text{and} \quad |e^{\sqrt{6}} f^{\sqrt{6}} - c| \quad {}_1(\ ) \tag{5.54}
$$

where c is a constant and ${}_1(\ )$ is a positive function of on a domain of the form $[\ _1, \infty)$ which limits to zero as $\to \infty$.

These last bounds imply that the restrictions, $\{h(\ ,\ )\}\ _1$, of h to the large but constant circles in the $(p, p' = 0)$ end of $_r$ limit in the L^2 sense to a constant as $\to \infty$. This L^2 statement can be readily bootstrapped as in Steps 4 and 5 of Section 2's proof of Proposition 2.3 to establish that the restrictions, $h(\ ,\ )$ of h to the large and constant circles limit pointwise and uniformly to a constant when $\to \infty$. Moreover, these same arguments prove that the derivatives of $h(\ ,\ )$ to all orders converge to zero as $\to \infty$.

Step 4 With the results of the previous steps in hand, Proposition 3.3 in [22] can be invoked to conclude the following: First, there is a finite, non-empty set $\{(S_k, m_k)\}$ where $\{S_k\}$ is a set of distinct, irreducible, J_T–pseudoholomorphic subvarieties and each m_k is a positive integer. Second, after passing to a subsequence of $\{\_i\}$ (and renumbering consecutively from 1), this set converges pointwise to $C_T \quad \cup S_k$ in that

$$
\lim_{i \to \infty} (\sup_{x \in (C_T \cap K)} \mathrm{dist}(x, \ _i) + \sup_{x \in (\_i \cap K)} \mathrm{dist}(C_T, x)) = 0. \tag{5.55}
$$

for all compact sets $K \subset \mathbb{R} \times (S^1 \times S^2)$. Third, and here is where the integers $\{m_k\}$ enter, the sequence of currents defined by $\{\_i\}$ converges to the current $\sum_k m_k S_k$ in the sense that if is any smooth, compactly supported 2–form on $\mathbb{R} \times (S^1 \times S^2)$, then

$$
\lim_{i \to \infty} \int_{\_i} \quad = \sum_k m_k \int_{S_k} \quad . \tag{5.56}
$$

By the way, with regard to the conclusion that $C_T \neq \emptyset$, this follows from (5.55) since $\_i$ contains a point on the fixed, compact submanifold in $\mathbb{R} \times (S^1 \times S^2)$ where $s = 0$.

Step 5 Steps 5–9 establish that C_T is the image of the thrice-punctured sphere C_0 by a J_T–pseudoholomorphic map, is disjoint from the $\in \{0, \ \}$

locus, and has its ends constrained by I as those of C^0 are by the third point of (5.42). Given the above, then Lemma 5.7 follows immediately when the sequence $\{w(i)\}$ is unbounded as then $J_T = J$. In the case where $\{w(i)\}$ is bounded, then $T = 1$ since is open; thus $J_T = J$ and Lemma 5.7 again follows.

This step starts the task by verifying that C_T is an HWZ subvariety. In this regard, remember that an HWZ subvariety is characterized by the requirement that the exterior derivative of the contact form for $S^1 \times S^2$ has nite integral over both the concave side and the convex side ends. Here, the contact form is $-(1 - 3\cos^2)dt - \sqrt{6}\cos\sin^2 d\varphi$, and its exterior derivative is integrable over each end of $\__i$. Indeed, the value of this integral is bounded independent of the index i, for an application of Stokes' theorem identi es it only as a function of the integers in the set I. Now, as the exterior derivative of the contact form restricts to any J–pseudoholomorphic subvariety as a non-negative form, it follows that this exterior derivative is a non-negative form on the ends of C_T. Thus, (5.56) and the dominated convergence theorem guarantee the niteness of the integral of this 2–form over each end of C_T.

Step 6 This step veri es that C_T avoids the $\in \{0, \}$ locus. In particular, it follows from (5.55) that $\sin > $ on C_T granted that such is the case on each $\__i$. To nd such an , remember that each $\underline{J}_i$ agrees with J where $\sin < $. With this understood, take to be less than the minimum of $; (2/3)^{1/2}$ and the values of $\sin_0$ on the closed Reeb orbits that correspond to the integers in the set I. The set of points in $\__i$ where $\sin <$ is then compact and J–pseudoholomorphic. Were it non-empty, then the function h/f on each component would have a nite minimum or maximum depending on the sign of cos . However, as previously established, this function doesn't have nite local maxima or minima on a region in a J–pseudoholomorphic subvariety which avoids the $\in \{0, \}$ locus.

Step 7 The topology of C_T is the subject of Steps 7 and 8. The story starts with a brief digression to recall that the critical point of df on each $\__i$ occurs on the $s = 0$ locus. As this locus is compact, so the critical value, f_i, of f on $\__i$ takes values in some compact and index i independent interval of $\mathbb{R}$. Thus, there is an in nite subset of indices i for which the corresponding set $\{f_i\}$ converges. Pass to this subsequence when considering $\{\__i\}$ and then renumber consecutively from 1. With this now understood, let f_0 denote the limit of the set $\{f_i\}$.

To continue, remark that each end of C_T has an associated pair of integers $(m; m^0)$ which is characterized in part by the condition that $md\varphi - m^0dt$ restricts to the end in question as an exact form. In this regard, the large and constant $|s|$ slices on the end are circles which converge as $|s| \to \infty$ to a multiple wrapping of a closed, Reeb orbit, whose θ_0 values is determined by the quotients of m and m' by their greatest common divisor. Up to the action of the group T, the sign of m or m' finishes the specification of this closed Reeb orbit. Meanwhile, the greatest common divisor of m and m' is the multiplicity by which the constant $|s|$ circles wrap the limit closed Reeb orbit.

With the preceding understood, suppose that a given convex side end of C_T is characterized by (m, m') with $m \neq 0$. Then, for all $c \geq 1$, the $|f| = c$ level set in C_T has a component in this end. It then follows from (5.55) that the $|f| = c$ level set of Σ_i lies in a tubular neighborhood of the $|f| = c$ level set of C_T when i is large, and so the integral of the 1–form $md\varphi - m'dt$ about the $|f| = c$ level set in Σ_i is zero. However, the integral of the form $md\varphi - m'dt$ about any component of a large and constant level set of $|f|$ in Σ_i must be a non-zero multiple of either $mp' - m'p$ or else $mq' - mq$. Indeed, this can be seen by using Stokes theorem to compute these integrals for Σ_i in the limit that $c \to \infty$. This implies that m'/m is either p'/p or q'/q. Moreover, it then follows from (5.56) that the signs of m and p agree in the former case, while those of m and q agree in the latter. Finally, as there are precisely two components to the $|f| = c$ locus in Σ_i when $c > |f_0|$, so (5.55) and (5.56) have the following additional consequence: There are two $m \neq 0$ convex side ends to C_T, one with $(m, m') = a^{-1}(p, p')$, the other with $(m, m') = b^{-1}(q, q')$. Here, a and b are positive integers.

Slight modifications of the preceding arguments also prove that C_T has a single $m \neq 0$, convex side end; and for this end, $(m, m') = c^{-1}(p + q, p' + q')$ where c is a positive integer.

Step 8 Now consider the possibility that C_T has an end with $m = 0$. If such an end were on the convex side of $\mathbb{R} \times (S^1 \times S^2)$ and if $h > 0$ on this end, then such an end would contain a circle component of the $|h| = \ell$ level set for all $\ell \geq 1$. Moreover, dt would restrict to this circle and thus to a neighborhood as an exact 1–form. The absurdity of this conclusion in the case where neither p', q' nor $p' + q'$ is zero can be seen as follows: Under this last assumption, the 1–form dh also has exactly one critical point on each Σ_i. This follows from from the fact that $d\varphi$ has but one zero, and that J' maps $d\varphi$ into a multiple of dh. (The latter point follows from (5.28), (5.37) and (5.38).) Thus, as long as ℓ is not the absolute value of the critical value of h on a given Σ_i, there are two components to the $|h| = \ell$ level set, both circles and dt is not exact on either one since neither p, q, or $(p + q)$ is zero. Thus, (5.55) forbids a convex side

$m = 0$ end in this case. A completely analogous argument forbids a concave side $m = 0$ end when neither p', q' nor $p' + q'$ is zero.

On the other hand, if p' were zero, then h has precisely one non-compact level set on $\_i$ because, as argued in Step 3, it has but one such on $_{r(i)}$. Now, either is less than or greater than the value of h on its non-compact level set. If either, then the $h =$ level set has a single component and $qd' - q'dt$ is zero on this component, not dt.

Modulo some straightforward notational changes, this last argument rules out the case $q' = 0$ too. Thus, the only case left to consider has $p' + q' = 0$. In this case, the locus on $\_i$ where $|h| = \; > 0$ consists of a pair of circles, and $pd' - p'dt$ is not exact on both, so dt can't be exact on either.

Step 9 With the preceding understood, it follows that C_T has two convex side ends and one concave side end, with integers $a^{-1}(p; p')$ and $b^{-1}(q; q')$ associated to the convex side ends, while $c^{-1}(p + q; p' + q')$ is associated to the concave side end. Here, a, b and c are positive integers. Moreover, the integers a, b and c are constrained by the requirement that $p/a + q/b = (p + q)/c$ and its analog using p' and q'. In particular, because $\neq 0$, these constraints require $a = b = c$. The next paragraph establishes that C_T is the image of C_0 via a J_T–pseudoholomorphic immersion, and the subsequent paragraph then argues that $a = b = c = 1$ and thus ends the proof of Lemma 5.7 where neither p, q nor $p + q$ is zero.

To argue that C_T is the image of C_0 via a J_T–pseudoholomorphic immersion, rst observe that the function f has precisely one critical value on C_T, this being f_0. Indeed, a critical value at some $f = f_1$ must change the connectivity of the constant f level sets as this critical value is crossed. When the index i is large, such a change must also occur near the $f = f_1$ level set on $\_i$. In particular, such a change happens only at the critical value of f on $\_i$, and as these critical values tend to f_0 as i tends to in nity, so it follows that $f_1 = f_0$. Moreover, the fact that there are at most two points in $\_i$ which share any given $(t; f)$ value implies that there is precisely one critical point of f on C_T, and that the latter is non-degenerate. Then, Euler characteristic considerations imply that C_T is the image of C_0 via a J_T–pseudoholomorphic map and Lemma 5.6 implies that this map is an immersion.

Now consider the assertion that $a = b = c = 1$. For this purpose, let R be very large and let $C_T^R \; C_T$ denote the subset where two constraints are obeyed: First, $|f| \; R$; and second, $|f| \; 1/R$ at the points where $s \; (2\sqrt{6})^{-1} \ln R$. For large R, this is a smooth manifold with boundary, where the boundary consists of three circles, one on each end of C_T. Let $\underline{\ }_i^R \subset \underline{\ }_i$ denote the analogous set. Being immersed, C_T has a well defined normal bundle, and then when i is large with R fixed, $\underline{\ }_i^R$ lies in the image of a disk subbundle via the exponential map. Moreover, the normal bundle lifts to C_0, and the restriction of its projection to the inverse image of $\underline{\ }_i^R$ can be used to define a proper covering map from C_0 to itself whose degree is the integer a. However, this means that $a = 1$ as Euler characteristics behave multiplicatively under coverings.

Step 10 This step and Step 11 discuss the proof of Lemma 5.7 in the case where one of p, q or $p+q$ is zero. In all of these three cases, the arguments are much like those used in the preceding steps to prove the lemma when none of p, q and $p+q$ is zero. In particular, only the significant differences are highlighted and the details are left for the most part with the reader. Only the $p = 0$ case is given below as the $q = 0$ and $p+q = 0$ discussions are completely analogous to the $p = 0$ one.

As indicated above, this step and the next assume that $p = 0$. For this case, as in Step 1, the argument begins with a definition of the sequence $\{w(i)\}_{i=1,2,\ldots}$. However, the task here requires a preliminary digression. To start the digression, choose some small, constant δ which is neither 0 nor q/q'. For such a choice, the form $dt - \delta\, d\varphi$ restricts without zeros on the closed Reeb orbits that are defined by the ends of $\underline{\ }_i$. As argued in the proof of Lemma 5.6, this form, d , has but one zero on $\underline{\ }_i$, with the latter non-degenerate and of hyperbolic type. Meanwhile, it follows from (5.28) and (5.31–32) that the form $J' \cdot (dt - \delta\, d\varphi)$ has can be written as $\alpha_0 df + \beta_0 dh$, where α_0 and β_0 are functions of f and h, and where α_0 is positive. Thus, each J_r in (5.48) also sends $dt - \delta\, d\varphi$ to a 1–form $\alpha_r df + \beta_r dh$ with both α_r and β_r functions of (f, h) and with $\alpha_r > 0$. In particular, as $\alpha_r > 0$, there exist smooth functions u_r and κ_r of f and h, defined where $\sin\theta > 0$ and such that $(\alpha_r df + \beta_r dh) = e^{\kappa_r} du_r$. Indeed, with $\alpha_r > 0$, the vector field $\alpha_r \partial_h - \beta_r \partial_f$ is nowhere zero and nowhere tangent to the constant h level sets. As a consequence, u_r can be viewed as a measure of distance along the integral curves of this vector field starting from the $f > 0$ portion of the $h = 0$ level set.

Note, by the way, that u_r varies smoothly with $r \in [0, 1]$. Moreover, up to additive constants, u_r is r–independent on the two components of $\mathbb{R} \times (S^1 \times S^2)$ where $|s|$ is so large that $J_r = J$.

With the digression now over, choose a sequence $\{r(i)\}_{i=1,2,\ldots} \subset$ with limit T. As $du_{r(i)}$ is proportional to $J_{r(i)}(dt - \delta\, d\varphi)$, so the function $u_{r(i)}$ has a single critical point on $\ _{r(i)}$, and the latter is a non-degenerate saddle point. In any event, take $w(i)$ to equal minus the value of the coordinate s at this critical point. As in Part 1, if the resulting sequence $\{w(i)\}$ is not bounded, then pass to subsequence of $\{1, 2, \ldots\}$ for which the corresponding sequence $\{w(i)\}$ is non-convergent and either strictly positive and increasing, or strictly negative decreasing. Then, re-index the subsequence so the labels i are consecutive and start at 1.

Define, as before, the corresponding translated sequence $\{\Sigma_i\}$. With but minor modifications (one such switches the roles of f and h), the argument in Step 3 establish an index i independent bound for the integral of ω over the intersection of Σ_i with any given compact subset $K \subset \mathbb{R} \times (S^1 \times S^2)$. A repeat of Step 4 now finds a subsequence of the indexing set (hence renumbered consecutively from 1) and data $\{(S_k, m_k)\}$ such that (5.55) and (5.56) hold. Here again, $\{S_k\}$ is a finite set of distinct, irreducible, J_T–pseudoholomorphic subvarieties in $\mathbb{R} \times (S^1 \times S^2)$ and each m_k is a positive integer. Once again, set $C_T \equiv \cup_k S_k$.

Step 11 This step discusses the verification that C_T is the image of the thrice punctured sphere C_0 by a J_T–pseudoholomorphic map, that C_T avoids the $\theta \in \{0, \pi\}$ locus, and that its ends are constrained by I as those of C' are by the third point of (5.42). Given the above, then Lemma 5.7 follows as argued at the beginning of Step 5.

To start, note that the arguments in Steps 5 and 6 work here to prove that C_T is an HWZ subvariety on which $\sin\theta$ enjoys a positive lower bound.

To continue the verification, pass to a subsequence of the index set (hence renumbered consecutively from 1) so that the resulting sequence of the critical values of the functions $u_{r(i)}$ on Σ_i converge, and let u_0 denote the limiting value. In this regard, remember that the sole critical point of each $u_{r(i)}$ on Σ_i sits on the compact, $s = 0$ locus while $\{u_{r(i)}\}$ converges uniformly to u_T. Thus, the associated sequence of critical values is bounded.

Given the preceding, then the arguments in Steps 7 and 9 apply in this case after some minor modifications and establish that C_T has precisely three ends, two on the convex side of $\mathbb{R} \times (S^1 \times S^2)$ and one on the concave side. Moreover, the convex side ends are characterized by integers $(0, p')$ and (q, q'), while the concave side end is characterized by $(q, p' + q')$. Furthermore, the final arguments in Step 9 also apply here after a minor change to prove that C_T is the image of C_0 via a J_T–pseudoholomorphic immersion. In this regard, the only substantive modification to the arguments in Steps 7 and 9 involves the replacement of the function f by u_T when level sets are considered on C_T, and the replacement of f by $u_{r(i)}$ when considering level sets on Σ_i.

(h) Theorem A.4 and the number of double points

This last section explains how Proposition 5.1's classifying set I determines Proposition 3.1's double point number, m_C, for any subvariety in I's component of the moduli space of $\partial = 2$, thrice-punctured spheres. The following proposition summarizes the story and the remainder of this subsection is then occupied with its proof. Note that all assertions in Theorem A.4 that concern the subvarieties from Theorem A.2 follow directly from the next proposition.

Proposition 5.9 *Suppose that* $I = \{(p,p'),(q,q')\}$ *obeys the constraints listed in Proposition 5.1. If* $C \in \mathfrak{M}$ *is parameterized by* I *as in Proposition 5.1 then* C *has only transversal double point singularities. Moreover, the double point number* m_C *from Proposition 3.1 is one half of the number of ordered pairs* $(\; , \;) \in S^1 \times S^1$ *with* $\neq '$*, neither equal to 1 and such that* ${}^{p}\,{}'^{q} = {}^{p'}\,{}'^{q'} = 1$. *For example,* $m_C = 0$ *if and only if one of the following conditions hold*

- $= 1$,
- $= 2$,
- *There exists a pair* $(m,m') \in \{(p,p'),(q,q'),(p+q,p'+q')\}$ *with both* m *and* m' *divisible by* .

For a second example, if ≥ 3 *is prime and the third condition above does not hold, then* $m_C = 2^{-1}(- 1)$.

By the way, the condition in the third point of this proposition is equivalent to the following:

$$\text{There exist distinct pairs } (a,a'),(b,b') \text{ in the indicated set and integers, } c \text{ and } c', \text{ such that } ac' - a'c = 1 \text{ and } bc' - b'c = -1. \tag{5.57}$$

Thus, if $m_C = 0$ by virtue of the third condition in Proposition 5.9, then both pairs in the indicated set that are not mentioned consist of relatively prime integers. In any event, an example where the third point holds has $p = 2$, $p' = 1$, $q = 1$, $q' = 2$ and so $= 3$. By comparison, the condition in the third point of the proposition does not hold in the $= 3$ case where $p = 4$, $p' = 1$, $q = 1$ and $q' = 1$.

Proof of Proposition 5.9 The proof starts with a computation of the double point number for the spheres in the image of the composition as described in Subsections 5c-e. The proof then explains why this double point number is preserved through the deformations of Subsections 5f and 5g. The details are presented in six steps.

Step 1 This step computes the corresponding double point number, m , for the image in $\mathbb{C} \quad \mathbb{C}$ of the map that is de ned in Section 5c via (5.20) from the set I. In particular, the discussion that follows proves the singularities of the image of C_0 are purely transversal double points and that each assertion about m_C in Proposition 5.9 also hold for m . In particular, $m \; = 0$ if and only if one of the three points in Proposition 5.9 hold, and $m \; = 2^{-1}(\; - 1)$ when 3 is prime and the third point in Proposition 5.9 does not hold.

With the preceding goal in mind, note that $z \neq w \in C_0$ are sent to the same point by if and only if

$$z^p(1-z)^q = w^p(1-w)^q \quad \text{and} \quad z^{p'}(1-z)^{q'} = w^{p'}(1-w)^{q'}: \tag{5.58}$$

The rst observation here is that these equalities admit no non-trivial rst-order deformations by virtue of the fact that $\neq 0$. Thus, the intersections of the image of small open disks in C_0 are either empty or transversal.

Here is the second observation: The two equations in (5.58) imply that $z \; = w$ and also $(1-z) \; = (1-w)$. Thus,

$$w = \; z \quad \text{and} \quad (1-w) = \; '(1-z) \tag{5.59}$$

where and $'$ are distinct complex numbers, neither are equal to 1, and $= \; - \; ' = 1$. The fact that $m \; = 0$ when is either 1 or 2 follows immediately from this last point. This noted, assume that > 2 in the remainder of this step.

With the pair $(\; ; \; ')$ given, subject to the constraints just stated, then the solutions z and w to (5.59) are given by

$$z = (1- \; ^{'-1})=(1- \; ^{'-1}) \quad \text{and} \quad w = (1- \; ')=(1- \; ^{-1} \; '): \tag{5.60}$$

In this regard, note that there are $(\; -1)(\; -2)$ choices for and $'$ that satisfy the constraints and no two choices for such a pair determine the same pair $(z;w)$. Moreover, as z is the complex conjugate of w, no two pair of , $'$ determine either the same z or the same w in (5.60). This last point implies the claim that the singularities of are purely transversal double points. Moreover, it implies that the number m is half the number of pairs $(\; ; \; ')$, neither 1, both {roots of unity, distinct, and such that $(z;w)$ in (5.60) satis es (5.58). Thus, m it is at most $2^{-1}(\; -1)(\; -2)$.

Now, not all pairs $(\; ; \; ^0)$ with neither 1, distinct and both {roots of unity produce, via (5.60), a solution to (5.58). In particular, (5.58). places the following additional constraints on $(\; ; \; ^0)$.

$$^{p} \; ^{0q} = 1 \quad \text{and} \quad ^{p^0} \; ^{0q^0} = 1: \qquad (5.61)$$

As the solutions to (5.61) are pairs of {roots of unity, the number m is thus seen equal to one half of the number solutions $(\; ; \; ^0)$ to (5.61) which are distinct and such that neither is equal to 1. Thus, the general characterization of m is identical to Proposition 5.9's characterization of m_C.

Now turn to the characterization of the $m = 0$ cases. The rst part of this task veri es that $m \neq 0$ unless one of the three points in the proposition hold. For this purpose, suppose > 2. Now, x a primitive {root of unity, . This done, then the pair $(\; = \; ^{q^0}; \; ^0 = \; ^{-p^0})$ solves (5.61) as does $(\; = \; ^{q}; \; ^0 = \; ^{-p})$. Note that the former obeys the condition $\neq \; ^0$ if $p^0 + q^0 \neq 0 \mod (\;)$ and the latter if $p + q \neq 0 \mod (\;)$. Thus, as the third point of the proposition is not operative, one of these pair, say $(\; ^{q}; \; ^{-p})$, is still a viable candidate for producing a double point. In particular, the latter fails to produce a double point only if one of q or p is divisible by . For the sake of the argument, suppose that p is divisible by . This understood, then q is not divisible by since $p + q$ is not. However, as $= pq^0 - p^0 q$ and p is divisible by and q is not, so p^0 is divisible by . But this conclusion is nonsense as the divisibility of both p and p^0 by invokes the third point of the proposition. Thus, p is not divisible by , and by symmetry, neither is q. Thus, m is non-zero when the third point of the proposition is not operative.

As $m = 0$ if either of the rst two points in the proposition hold, consider whether m is necessarily zero when the third point holds. For this purpose, suppose, for the sake of argument, that both q and q^0 are divisible by . This the case, then a solution $(\; ; \; ^0)$ to (5.61) with $\neq 1$ requires $^{p} = 1$ and $^{p^0} = 1$. However, there is no $\neq 1$ solution to these last two constraints because, as noted in (5.57), p and p^0 are relatively prime. (To see why such is the case, x integers c and c^0 such that $pc^0 - p^0 c = 1$. Then, given that both $^{p} = 1$ and $^{p^0} = 1$, so $^{pc^0} = \; ^{p^0 c}$ and thus $^{pc^0 - p^0 c} = 1$.)

Analogous arguments deal with the cases where either p and p^0 or else $p + q$ and $p^0 + q^0$ are divisible by .

To nish the story for m , consider now the value of m when $\quad 3$ is prime and the third point in the proposition is not operative. For this purpose, note rst that q and q^0 are relatively prime. Indeed, as $= pq^0 - p^0 q$, any common divisor of q and q^0 must divide . Thus, as is prime and not a common divisior of q and q', so only 1 divides both simultaneously. Likewise, p and p' are relatively prime. In any event, as q and q' are relatively prime, there exist a pair of integers, c and c', such that

$$qc' - q'c = 1. \tag{5.62}$$

Note that the pair c and c' are unique up to adding to c an integer multiple of q while simultaneously adding the same multiple of q' to c'.

Now, given such a pair, de ne a homomorphism from the group of {roots of unity to itself via the formula

$$! \quad {}'[\;] \qquad {}^{-(pc'-p'c)}. \tag{5.63}$$

Note that this homomorphism is insensitive to the choice of the pair (c, c') in (5.62).

By virtue of (5.62), the homomorphism in (5.63) has the property that both $(\;'[\;])^{q} = \;^{-p}$ and $(\;'[\;])^{q'} = \;^{-p'}$. Thus, the pair $(\;; \;'[\;])$ solves (5.61). Moreover, (5.61) together with the failure of the third point in the proposition imply that this homomorphism has trivial kernel and no xed elements. As a direct consequence, the set of solutions $(\;; \;')$ to (5.61) with $\neq 1$, $' \neq 1$ and $\neq \;'$ are parameterized by the non-trival {roots of unity via $! \;(\;; \;'[\;])$. In this regard, note that one element of a solution pair to (5.61) determines the other member in the case where is prime. In any event, as there are -1 non-trivial {roots of unity, so $m = 2^{-1}(\; - 1)$ as claimed.

Step 2 This step counts the number of intersections between the image of C_0 and the image of the map that has the form of (5.20) but with a' di erent from the choice for . For this purpose, x (a, a') for , some non-negative real number 1, and another small number > 0. Note that when neither p, q nor $p+q$ is zero, then $= 0$ is a permissable choice, but not so otherwise. In any event, with these parameters chosen, let denote the version of (5.20) where (a, a') are replaced by $(e^{i} \; a, e^{i} \; a')$. A straightforward argument along the lines of the derivation of (5.20) nds that the images of and intersect in

$$1 + 2m \; + (gcd(p, p') - 1) + (gcd(q, q') - 1) + (gcd(p+q, p'+q') - 1) \tag{5.64}$$

points. Moreover, all of these intersection points count with weight $+1$ to the intersection number between (C_0) and (C_0).

By way of an explanation of the various terms in (5.64), note that for small , the left most term, 1, is contributed by an intersection very near the single critical point of . Of course, the $2m$ term counts the intersections near the double points of . The other terms count intersections that are, for small , in the images of points that have distance d on the Riemann sphere that is $o(\ )$ from one or another of the punctures. In particular, for each such point,

$$^{-1}|\ | \quad d \quad |\ | \tag{5.65}$$

where 1 is independent of the point, (if less than 10^{-3}) and the parameters r, a and a' that appear in (5.20). However, depends on if one of p, q or $p+q$ vanishes.

For example, the contribution of the term that is third to the left, 1 less than the greatest common divisor of p and p', counts the number of intersection points whose –inverse image have distance d as in (5.65) from the puncture at the origin in $\mathbb{C}$. The next term counts points whose –inverse image has distance d from $1 \in \mathbb{C}$; and the right most term counts those whose –inverse image has distance $1/d$ from the origin in $\mathbb{C}$.

Step 3 With and chosen, make the parameter r in (5.20) very large and make the parameters R_0, R_1 and R_∞ in the definition of Section 5d's map extremely large as well. With such choices, the composition immerses C_0 in $\mathbb{R} \quad (S^1 \quad S^2)$ as described in Section 5e. Note that also immerses C_0 in $\mathbb{R} \quad (S^1 \quad S^2)$. Moreover, if the parameters r, R_0, R_1 and R_∞ are sufficiently large, then the intersection points between the and images of C_0 are the images under of the corresponding and versions in $\mathbb{C} \quad \mathbb{C}$. Indeed, this conclusion follows from (5.65) and Lemma 5.5.

With the preceding understood, choose one of the $\mathbb{R}$'s worth of subvarieties $C \in \mathfrak{M}_I$ that approaches (C_0) as $|s| \to \infty$. This done, let $C \in \mathfrak{M}_I$ denote the image of C under the action of the subgroup of T that sends the coordinate $'$ to $' +$ and the coordinate t to $t +$. Thus, the large $|s|$ asymptotics of C are identical to those of .

With C and C defined, here is the key point to take from this step:

The intersection number of C with C *is given by* (5.64). (5.66)

As explained next, (5.66) follows from homological considerations. Indeed, fix some small neighborhood, $U \quad S^1 \quad S^2$ of the closed Reeb orbits in C's limit set that is disjoint from its rotated version, U . This done, then there exists $s_0 \quad 1$ so that the constant $|s| > s_0$ slices of C and also (C_0) lie in U while those of C and (C_0) lie in U . Now, remark that for sufficiently large s_0, the homology classes defined by C and (C_0) are homologous rel the set of points $(s, w) \in \mathbb{R} \quad (S^1 \quad S^2)$ with $|s| \quad s_0$ and $w \in U$. Likewise, the classes defined by C and (C_0) are homologous rel those points (s, w) where $|s| \geq s_0$ and $w \in U$. These last two points imply (5.66).

Step 4 For each large s_0, view the number of $|s| < s_0$ intersections between C and C as a piece-wise constant function, $f(\ ; s_0)$, of the parameter . As explained below, the $\to 0$ limit of $f(\ ; s_0)$ exists for all s_0 sufficiently large and this limit is equal to $2m_C + 1$. Granted this last conclusion, then the value of m_C can be found by counting the the large $|s|$ intersections of the small versions of C and C. The latter task occupies Steps 5 and 6 below.

To verify that asserted $\to 0$ behavior of $f(\ ; s_0)$, reintroduce the C–version of the normal disk bundle $N_0 \to C_0$ and its exponential map $q\colon N_0 \to \mathbb{R} \times (S^1 \times S^2)$ as described in (3.12). Thus, C is the image via q of the zero section of N_0. Moreover, given s_0, then the $|s| \leq s_0$ portion of C is obtained as the composition of q with a section, , of the normal bundle $N \to C_0$ that lies in N_0 over the $|s| \leq s_0$ portion of C and whose norm is everywhere bounded by $|\ |$. This then implies that the number of $|s| \leq s_0$ intersections of C and C is equal to the sum of twice m_C with the algebraic count of the zeros of that lie where $|s| \leq s_0$.

To prove that the latter count equals 1, remember that $\{C\}$ is constructed by the action of the 1–parameter subgroup of T generated by $\partial_t + \partial_\varphi$. This implies that $|\ |^{-1}$ converges pointwise on C as $\to 0$ to the section of N that is obtained by projecting the vector field $\partial_t + \partial_\varphi$ onto C_0's normal bundle. In this regard, the latter vanishes only at the critical point of the 1–form $dt - d\varphi$, for then $\partial_t + \partial_\varphi$ is tangent to C. Now, as argued in Section 4f, the generic version of the 1–form $dt - d\varphi$ has a single, non-degenerate zero on C. In fact, unless one of p, q or $p + q$ is zero, such is the case for the $= 0$ version. In any event, the preceding implies that the generic and sufficiently small versions of have one zero, this with multiplicity one sitting very close to the zero of dt.

Step 5 The proof of Proposition 5.9 is completed here and in Step 6 with the verification of the following claim:

> *Let (m, m') denote either (p, p'), (q, q') or $(p+q, p'+q')$. There are $gcd(m, m') - 1$ intersections between C and the small versions of C at very large values of $|s|$ on the corresponding (m, m') end of C.* (5.67)

With regards to this claim, note that it follows immediately when $gcd(m, m') = 1$ from the identification in the previous step of the small limit of $^{-1}$. This understood, suppose that $gcd(m, m') = n > 1$. Also, suppose that $m \neq 0$. The $m = 0$ cases are treated by analogous arguments and so left to the reader.

To begin, use (2.13) to parameterize the large $|s|$ portion of the end in question using coordinates $(\tau, \sigma) \in \mathbb{Z}/(2\pi|m|\mathbb{Z}) \times [\rho_0, \infty)$ and functions $x(\tau, \sigma)$ and $w(\tau, \sigma)$. In this regard, no generality is lost by assuming that both $|x|$ and $|w|$ tend to zero as $\sigma \to \infty$, and as indicated in Proposition 2.3, their sizes are $O(e^{-\kappa\sigma})$ at large σ where $\kappa > 0$ is constant. This said, then the analogous end of the small ε version of C_ε is parameterized using coordinates (τ, σ) with the pair (x, w) replaced by $(x + \delta, w)$ where $\delta \equiv 1 + \varepsilon m'/m$ is positive when ε is very small.

Given the preceding, it then follows that the intersections at large σ on the (m, m') end of C occur at and only at the values of (τ, σ) where

$$x(\tau, \sigma) = x(\tau + 2\pi rm/n, \sigma) + \delta \quad \text{and} \quad w(\tau, \sigma) = w(\tau + 2\pi rm/n, \sigma) \tag{5.68}$$

for some $r \in \{1, \ldots, n-1\}$. Thus, (5.67) follows by demonstrating that there exists, for each such r and for small δ, precisely one pair (τ, σ) where (5.68) holds. The remainder of this step argues that there is at least one pair (τ, σ) where (5.68) holds for any given r. The subsequent step proves that there is at most one such pair for each r.

To establish that (5.68) can be solved, note first that the zero locus of the difference, $\underline{w} \equiv w(\tau + 2\pi rm/n, \sigma) - w(\tau, \sigma)$ intersects each large and constant σ circle. This follows from the fact that the average of $\underline{w}$ over such a circle is zero. Let $\underline{x}$ denote the corresponding difference of the values of x. Then, as a direct consequence of the discussion in Step 7 of Section 2a, the $\underline{w} = 0$ locus has the structure of an oriented, embedded graph. Here, the orientation is given by the restriction of the 1–form $d\underline{x}$. To elaborate, this graph has the following properties:

- Each vertex is a $\underline{w} = 0$ critical point of $\underline{w}$.
- Each edge is an embedded, open arc, disjoint from the vertices and from all other edges.
- The 1–form $d\underline{x}$ has nowhere zero pull-back to the edges.
- The intersection of the graph with some open neighborhood of each vertex is a finite union of embedded, half open arcs with endpoints lying on the vertex, but disjoint otherwise. Moreover, the tangent lines to the arcs at the vertex are well defined and disjoint. The interior of each arc is part of an edge of the graph. The number of such arcs is non-zero and even. Exactly half of the arcs are oriented by $d\underline{x}$ so $\underline{x}$ increases towards

the vertex while half are oriented by $d\underline{x}$ so that $\underline{x}$ decreases towards the vertex.

If $_1$ $_0$ is chosen to be su ciently generic, then the $=$ $_1$ locus intersects the graph only in its edges and this intersection is transversal. (5.69)

It is important to note that the conclusions in the preceding paragraph and the assertions in (5.69) hold for the $\underline{w} = 0$ locus and for the same reasons, when $\underline{w}$ and $\underline{x}$ $(x(+ 2\ rm{=}n;) - x(;)$ are de ned using any value of $r\ 2\ (0;n)$.

The fact that any $r\ 2\ (0;n)$ version of the $\underline{w} = 0$ locus has the structure just described and the fact that $\underline{x}$ is a bona de function implies a great deal about the large portion of these loci. The picture of the $\underline{w} = 0$ locus drawn below is based on two consequences of this structure: First, there are no non-trivial, closed, oriented paths in the $\underline{w} = 0$ locus. Second, there are no non-compact, oriented paths in the $\underline{w} = 0$ locus with unbounded at both ends.

These last two points imply that the $\underline{w} = 0$ locus can be described as follows: Fix su ciently large $_1 >$ $_0$ making sure that the $=$ $_1$ circle misses all vertices of the $\underline{w} = 0$ locus and such that $_1$ is a regular value of on each edge. This done, there exist three nite sets, $\#_0; \#_+$ and $\#_-$, of oriented paths in the $_1$ portion of the $\underline{w} = 0$ locus. The union of all of these paths is the whole $_1$ portion of this locus. Meanwhile, no two paths in this union share any edge, although two paths can intersect at a vertex. The set $\#_0$ is distinguished by the fact that each of its paths have both ends on the $=$ $_1$ locus. Meanwhile is unbounded on paths in either $\#_+$ or $\#_-$; each such path has but one end on the $=$ $_1$ circle. The paths in $\#_+$ are distinguished from those in $\#_-$ by the fact that is unbounded in the oriented direction on a path in $\#_+$, while $!$ $_1$ in the oriented direction on a path in $\#_-$. By the way, as $\underline{x}\ !\ 0$ as $!\ 1$, these last points imply that $\underline{x}$ is negative and increasing in the unbounded direction on the paths in $\#_+$, but positive and decreasing in the unbounded direction on the paths in $\#_-$. Finally, note that as is unbounded on the $\underline{w} = 0$ locus, neither $'_+$ nor $'_-$ is empty and these sets must have the same number of elements.

The preceding picture of the $\underline{w} = 0$ locus directly implies the following:

With $_2 >$ $_1$ *and* $r\ 2\ (0;n)$ *xed, there exists* $_1$ *such that* $0 < \ < 1$ *is su ciently generic, then each* $> {}_2$ *point where the {version of (5.68) holds lies on an edge of a path in* $'_-$. *Conversely, each path in* $\#_-$ *contains precisely one* $> {}_2$ *point where this same version of (5.68) holds.* (5.70)

Step 6 This step completes the proof of Proposition 5.9 by demonstrating that the set $\#_-$ contains at most one element. To start the demonstration, remark that the number of elements of $\#_-$ is constant as r varies in $(0; n)$. Indeed, this number is locally constant because $\underline{w}$ vanishes transversely on the edges of the graph and therefore constant on $(0; n)$ as the latter is connected.

As an aside, note that the number of paths in $\#_0$ can change as r varies. For example, a path in $\#_0$ can concatonate at some value of r with one in either $\#_+$ or $\#_-$. Conversely, a piece of a path in either $\#_+$ or $\#_-$ can become a path in $\#_0$ at some value of r. The fact that the number of elements in $\#_0$ can change is related to the fact that the number of elements in $\#_0$ depends on the choice of the parameter $\ _1$.

In any event, consider the $\underline{w} = 0$ locus when r is positive but very small. In this case, $\underline{w}(\ ; \ ) = (2 \ rn{=}m)w \ (\ ; \ ) + O(r^2)$ and so the $\underline{w} = 0$ locus converges as $r \ ! \ 0$ to the locus where $w \ = 0$. In fact, x a generic $\ _3 \quad \ _1$ and then take r very small but positive. Thus, if $\underline{w}(\ ; \ _3) = 0$ at some $\ = \ _3$, then there is precisely one $\ 2 \ (\ _3; \ _3 + 2 \ rn{=}m)$ where $w \ ;(\ ; \ _3) = 0$. Thus, the $\ = \ _3$ circle has at least as many zeros of w as elements in $\#_- \ [\ \#_+$.

Meanwhile, as $\underline{x} = (2 \ rn{=}m)x \ (\ ; \ ) + O(r^2)$, it follows that $x \ < 0$ on at least as many of the $w \ = 0$ points on the $\ = \ _3$ circle as there are elements in # and $x \ > 0$ on at least as many as there are elements in $\#_-$. This said, note that by virtue of (2.15) or, equivalently, the rst point in (2.19), the $w \ = 0$ locus is the $x_u = 0$ locus, where $u = \mathrm{sign}(p)e$ with de ned as in (2.15) from the pair m and m^0 via the corresponding angle $\ _0$ for the associated closed Reeb orbit. All of this understood, then the argument subsequent to (4.44) in Section 4g nds the absolute value of the Euler class of C_0 no less than the number of elements in $\#_-$. Thus, $\#_-$ has at most one element as claimed.

6 The structure of the @ = 3, thrice-punctured sphere moduli space

This section describes the moduli spaces of the triply punctured spheres with three convex side ends that arise in Part d of the third point of Proposition 4.2 and in so doing provides a proof of Theorem A.3. The following proposition restates the latter for use here:

Proposition 6.1 *The components of the moduli space of thrice-punctured, @ = 3 spheres which arise in Part (d) of the third point in Proposition 4.2 can be described as follows:*

(A) *The components are in 1–1 correspondence with the unordered sets of three pair of integers that are constrained in the following way: The set in question, I, can be ordered as* $\{(p, p'), (q, q'), (k, k')\}$ *with*

- $p + q + k = 0$ *and* $p' + q' + k' = 0$.
- $|k/k'| > \sqrt{3}/\sqrt{2}$.
- $\{(p, p'), (q, q')\}$ *obey the three constraints in Proposition 5.1.*

In this regard, a set I *with an ordering that satisfies these three conditions has precisely two distinct orderings that satisfy the conditions.*

(B) *This 1–1 correspondence has the following properties*

- *The component of the moduli space that is labeled by* $I \equiv \{(p, p'); (q, q'); (k, k')\}$ *is a smooth manifold that is* $\mathbb{R} \times T$ *equivariantly diffeomorphic to* $(0, 1) \times \mathbb{R} \times T$. *Here,* $\mathbb{R} \times T$ *acts on the moduli space via its isometric action on* $\mathbb{R} \times (S^1 \times S^2)$ *and it acts on* $(0, 1) \times \mathbb{R} \times T$ *by ignoring the* $(0, 1)$ *factor and acting as itself on the* $\mathbb{R} \times T$ *factor.*
- *Thus, the quotient of the moduli space by the* $\mathbb{R} \times T$ *action is* $(0, 1)$, *and an identification is provided by composing the preceding diffeomorphism with the projection map from* $(0, 1) \times \mathbb{R} \times T$.
- *This quotient has a natural compactification as* $[0, 1]$ *where the two added points label the* $\mathbb{R} \times T$ *quotient of two components of the moduli space of thrice-punctured,* $@ = 2$, *spheres that arise in Part d of the third point in Proposition 4.2. Moreover, the relevant components of the* $@ = 2$ *moduli space are labeled as in Proposition 5.1 by the first two pairs from the two orderings of* I *that obey the three constraints listed above in Part A.*

The proof of this proposition is summarized in Subsection 6d, below. The results in Subsections 6a-d provide various pieces of the proof. The final subsection explains via Proposition 6.9 how the set I determines the number of double points of any immersed sphere in its moduli space component. All assertions of Theorem A.4 about the subvarieties that appear in Theorem A.3 follows directly from Proposition 6.9.

(a) Initial constraints on the $@ = 3$, thrice-punctured sphere moduli spaces

Let $C \subset \mathbb{R} \times (S^1 \times S^2)$ denote one of the thrice-punctured spheres under consideration. As described in Section 4a, each end of C determines a unique, ordered pair (m, m') of integers. However, the three pair, $\{(p, p'); (q, q'); (k, k')\}$

determined by C are not completely independent. Some initial constraints on this set are:

Constraint 1 No pair in $\{(p, p'), (q, q'), (k, k')\}$ can vanish identically.

Constraint 2 $p + q + k = 0$ and $p' + q' + k' = 0$.

Constraint 3 $\Delta \equiv pq' - qp' \neq 0$.

Constraint 4 If $(m, m') \in \{(p, p'), (q, q'), (k, k')\}$ and $m < 0$, then $m' \neq 0$.

With regard to the third constraint, note that Δ is also equal to $pk' - kp'$ and $kq' - qk'$ and thus the $\Delta \neq 0$ assertion does not require the ordering of the set $\{(p, p'), (q, q'), (k, k')\}$. The arguments for the first three constraints are analogous to those given in Section 5a for the triply punctured spheres with $\partial = 2$, and so omitted except for the remark that the second constraint here differs by a sign from the second constraint in Section 5a. The fourth constraint follows from the observation that an $m < 0$ and $m' = 0$ end would be asymptotic to the $\theta \in \{0, \pi\}$ locus.

(b) The structure of each component

Let $I \equiv \{(p, p'), (q, q'), (k, k')\}$ denote an unordered set of integer pairs that obeys the three constraints listed. Given such I, let $\mathfrak{M}_I$ denote the space of pseudoholomorphic, $\partial = 3$, thrice-punctured spheres from Part d of the third point of Proposition 4.2 whose ends are described as in (5.1) by I.

An argument that is essentially identical to the one used in Section 5b's proof of Proposition 5.2 establishes that each component of $\mathfrak{M}_I$ is a smooth manifold, $\mathbb{R} \times T$ equivariantly diffeomorphic to $N \times \mathbb{R} \times T$; here N is either a circle or an interval. As seen below in this subsection N is, in all cases, an interval. The next proposition makes the official assertion:

Proposition 6.2 *Let $I = \{(p, p'), (q, q'), (k, k')\}$ denote an unordered set of pairs of integers that obeys the constraints listed in the statement of Proposition 6.1. Then, every component of $\mathfrak{M}_I$ is diffeomorphic to $(0, 1) \times \mathbb{R} \times T$. Moreover, there exists such a diffeomorphism that is $\mathbb{R} \times T$ equivariant when $\mathbb{R} \times T$ acts on $\mathfrak{M}_I$ via its isometric action on $\mathbb{R} \times (S^1 \times S^2)$, and on $(0, 1) \times \mathbb{R} \times T$ via its natural action on the factor $\mathbb{R} \times T$.*

The remainder of this subsection contains the following proof.

Proof of Proposition 6.2 The proof is lengthy and so divided into seven steps.

Step 1 Let $H \subset \mathfrak{M}_I$ denote a component. Suppose $C \in H$ and focus attention on an end of C characterized by integers (m, m') as in (5.1). In the case where $m \neq 0$, this end is parameterized at large values of $|s|$ by coordinates (τ, u) where $\tau \in \mathbb{R}/(2\pi|m|\mathbb{Z})$ and $\operatorname{sign}(m)u \in [R, \infty)$ for some suitably large constant R. This parameterization is given by (2.19) with $p = m$ and $p' = m'$. Thus, x and y are functions of τ and u, which obey (2.20). In the $m = 0$ case, the $(0, m')$ end of C is parameterized by (τ, u) where $\tau \in \mathbb{R}/(2\pi|m'|\mathbb{Z})$ and $\operatorname{sign}(m')u \in [R, \infty)$ via $(t = t_0 + x, f = y, \varphi = \tau, h = u)$. Here, x and y are functions of τ and u which obey the version of (2.20) where the role of (t, f) are switched with that of (φ, h).

The following lemma describes the $|u| \to \infty$ behavior of y for any pair (m, m'). For this purpose, introduce

$$\delta_0 \equiv 4|\cos\theta_0||m'/m|(1 + (m'/m)^2 \sin^2\theta_0)^{-1}. \tag{6.1}$$

Note that δ_0 is determined by the pair (m, m').

Lemma 6.3 *The restrictions $|u|^{-\delta_0} y(\tau, u)$ to the constant u circles converge in the C^∞ topology as $|u| \to \infty$ to the constant function. Moreover, the assignment to each $C \in H$ of the resulting constant defines a smooth function on H.*

Proof of Lemma 6.3 The convergence of $y(\tau, u)$ in the case where $m' = 0$ is stated and proved as Lemma 5.8 in the previous section. Save for notational changes, the $m' \neq 0$ cases have the identical argument. For example, in the cases where $m \neq 0$, one should set $w \equiv 6^{-1/2}(1 + (m'/m)\sin^2\theta_0)^{-1/2} y/(u\sin^2\theta)$ and $\sigma \equiv (1 + (m'/m)^2 \sin^2\theta_0)^{-1/2} 6^{-1/2} \ln(|u|)$. This done, then (5.52) holds for w provided that the $\sqrt{6}$ in the lower right entry in L_0 is replaced $\delta_1 = (1 + (m'/m)\sin^2\theta_0)^{1/2}(\sqrt{6} + \delta_0)$. The modifications to the argument given subsequent to (5.52) are of a similar nature.

With the preceding understood, let $Y : H \to \mathbb{R}$ denote the function which assigns to C the $|u| \to \infty$ limit of the C version of $|u|^{-\delta_0} y(\tau, u)$. To prove that Y has bounded first derivative on H, consider that a tangent vector to C in H can be represented where $|s|$ is large on the (m, m') end of C by functions (x', y') of τ and u which obey the linearized version of (2.20). Moreover, $|x'|$ and $|y'/u|$ are bounded as $|u| \to \infty$. With this understood, essentially the same argument that proved Lemma 5.8 proves that $|u|^{-\delta_0} y'$ also has a unique limit as

$|u| \to \infty$. The latter conclusion implies that the function Y is at least Lipschitz on H. The behavior of the higher order derivatives of Y can be analyzed in a similar manner using the parameterization of a neighborhood of C in H that is provided by Proposition 3.2 and Steps 5 and 6 of its proof. The details of this analysis is straightforward and so omitted.

Step 2 The purpose of this step is to highlight and then prove the following:

Lemma 6.4 *The set* $I = \{(p, p'), (q, q'), (k, k')\}$ *has a canonical element for which the corresponding function* $Y : H \to \mathbb{R}$ *described by Lemma 6.3 is never zero. In this regard, if* (k, k') *denotes the canonical element, then* $k < 0$ *and* $k' \neq 0$.

Proof of Lemma 6.4 As is demonstrated below, this constraint is a consequence of C's avoidance of the $\theta \in \{0, \pi\}$ locus. To start the argument, consider an end of C where the integer m is not zero. Then the ratio h/f is defined at large $|s|$ on this end and for large $|s|$, behaves as

$$h/f = (m'/m)\sin^2\theta_0 + \text{sign}(m)|u|^{-\alpha_0^{-1}}(Y + o(1)) .$$

In particular, when m' and Y have the same sign, then $|h/f| > |m'/m|\sin^2\theta_0$ at large $|s|$ on this end; and when their signs differ, then $|h/f| < |m'/m|\sin^2\theta_0$ at large $|s|$ on the end in question.

To see where this leads, note that at least one of p, q, and k is negative. If one, denote it by k. If two, then, as argued in the eighth step of Section 5's proof of Constraint 4, the number $|m'/m|\sin^2\theta_0$ is smallest for precisely one, so denote the latter by k. In either case, let $C_- \subset C$ denote the component of the subset of points where $f < 0$ which contains the (k, k') end. Note that $k' \neq 0$ because of the fourth constraint in Subsection 6a, above.

With the preceding understood, suppose that the sign of the (k, k') end's version of Y is opposite that of k'. Then $|h/f|$ at large $|s|$ on this end is less than its $|s| \to \infty$ limit. Meanwhile, h/f cannot change sign in the closure of C_- since a zero of h on C where $f \leq 0$ is precluded when C is disjoint from the $\theta \in \{0, \pi\}$ locus. Thus, $|h/f|$ is nowhere zero in C_-, diverges as the boundary of C_- is approached, and its infimum is not its $|s| \to \infty$ limit on C_-. This can happen only if h/f has a local extreme point in C_- which is a forbidden event as (4.21) makes h/f subject to the maximum principle. In this regard, a point must be added in the case where the $f = 0$ locus is non-compact. For such to happen, then one of p and q must be zero, say p. This implies that the large $|s|$ portion of the $f = 0$ locus sits on the $(p = 0, p')$ end of C. However, $|h|$

diverges at large $|s|$ on this end, while Lemma 6.3 guarantees that f limits to zero as $|s| \to \infty$ on this end. Therefore, $|h/f|$ diverges on the $(p = 0, p')$ end of C as well.

Now consider the possibility that $Y = 0$ for this same (k, k') end of C. Were such the case, the argument given below establishes that there are still points at large $|s|$ on this end where $|h/f| < |k/k'| \sin^2 \theta_0$. Thus, the argument from the previous paragraph applies and yields the same contradictory conclusions. With the preceding understood, suppose that $Y = 0$ on an end of C characterized by integers (m, m') with $m \neq 0$.

For this purpose, return to the (m, m') version of (2.20). Set $v = \mathrm{sign}(m)u$ and view $w \equiv \sin^{-2} \theta\, y$ as a function of τ and v. With these changes, the second of the equations in (2.20) has the schematic form

$$x_\tau = w_v + \alpha_0 v^{-1} w + r(w), \tag{6.2}$$

where $|r(w)| \leq \zeta v^{-1} w^2$ with ζ some constant. This last equation yields an impossible conclusion under the assumption that y and thus w have a definite sign at large values of v. To see how, introduce $\underline{w}(v) \equiv \int w(\tau, v) d\tau$ to. Then (6.2) implies that

$$|(v^{\alpha_0} \underline{w})_v| < \zeta v^{\alpha_0} v^{-1} \int w^2(\tau, v)\, d\tau \,. \tag{6.3}$$

Now, as $Y = 0$, so $\lim_{v \to \infty} v^{\alpha_0} w = 0$ and so integration of (6.3) yields the inequality

$$v^{\alpha_0} |\underline{w}(v)| < \zeta \sup_{v' \geq v} (v'^{\alpha_0} |w(v')|) \left| \int_{v' \geq v} \underline{w}(v') v'^{\alpha_0 - 1} dv' \right| . \tag{6.4}$$

Note that the derivation of (6.4) exploits the lack of a sign change in $\underline{w}$. In any event, recall that $\alpha_0 > 0$, and thus (6.4) implies that

$$v^{\alpha_0} |\underline{w}(v)| < \zeta' v^{-\alpha_0} \sup_{v' \geq v} (v'^{\alpha_0} |\underline{w}(v')|) \tag{6.5}$$

where $\zeta' = 2\zeta\alpha_0^{-1} \sup(v'^{\alpha_0} |w(v')|)$. Of course (6.5) asserts the ridiculous when v is large.

An argument that is almost identical to that just given also establishes the following:

Lemma 6.5 *If there is a single $(m, m') \in I$ with $m > 0$, denote the latter by (p, p'). If there are two $m > 0$ pairs in I and if m' has the same sign for both, denote by (p, p') that for which $|m/m'| \sin^2 \theta_0$ is greatest. Finally, if there are* *two* $m>0$ *pairs and the corresponding* m' *are opposite, denote by* (p,p') *that for which* $|m'|$ *is greatest. Then, the* (p,p') *version of* Y *is never zero and its sign is opposite that of* p'.

Step 3 An embedded copy of N in H is obtained by the choice of a slice across H for the action $\mathbb{R}\times T$. For the purpose of defining such a slice, remember that $\mathbb{R}\times T$ is the subgroup in $\mathrm{Isom}(\mathbb{R}\times(S^1\times S^2))$ where the action of the $\mathbb{R}$ factor is generated by ∂_s while ∂_t and ∂_φ generate the actions of the respective S^1 factors of $T=S^1\times S^1$. This said, the definition of the slice involves three conditions. Before these conditions are stated, remark that at least one of the integers p, q and k must be positive, and with this understood, take (p,p') as in Lemma 6.5. Then, the first condition assert the vanishing of the constant term on the right hand side of (5.1) for (p,p') end of C, and the second asserts the vanishing of the (k,k') version of this same constant. As was the case with the $\theta=\pi/2$ triply punctured spheres, these first two conditions force the vanishing of the (q,q') version of (5.1)'s right hand side constant term. The final condition on the slice involves the (k,k') version of the function Y from Lemma 6.3. According to Lemma 6.4, this function is nowhere zero, and with this understood, a slice of the $\mathbb{R}$ factor in $\mathbb{R}\times T$ is obtained by requiring that $|Y|=1$. Here, it is important to note that $a\in\mathbb{R}\subset\mathbb{R}\times T$ sends Y to $e^{-\sqrt{6}(1+\alpha_0)a}Y$.

Step 4 With $N\subset H$ so identified, define a function, $F\colon N\to\mathbb{R}$ by assigning to each $C\in N$ the value of the (p,p') end version of the function Y. With F understood, then Proposition 6.2 is an immediate consequence of

Lemma 6.6 *So defined, the function* F *has no critical points on* N.

The proof of Lemma 6.6 occupies this step and the next two steps in Proposition 6.2's proof.

Proof of Lemma 6.6 To begin the proof, suppose, for the sake of argument, that $C\in N$ were a critical point of this function F. To study the implications for C, it is important to remember that the tangent space to C in H consists of the 4–dimensional space of bounded sections of C's normal bundle that are annihilated by the C version of the operator in (3.5). In this regard, remember that C is the image of C_0 via a pseudoholomorphic immersion and so the pullback of $T(\mathbb{R}\times(S^1\times S^2))$ via this immersion splits orthogonally as a direct sum of oriented, J–invariant 2–plane bundles. The differential of the immersion canonically identifies one of these subbundles with TC_0; the orthogonal complement of the latter is the 'normal bundle to C' in $\mathbb{R} \times (S^1 \times S^2)$. Thus, with a pseudoholomorphic immersion of C_0 into $\mathbb{R} \times (S^1 \times S^2)$ understood, vector fields pullback to C_0 as sections of the pullback of $T(\mathbb{R} \times (S^1 \times S^2))$. For example, the vectors ∂_s, ∂_t and ∂_φ pullback to sections over C_0 of the pullback of $T(\mathbb{R} \times (S^1 \times S^2))$.

With these reminiscences now ended, note that $TH|_C$ has a three-dimensional subspace which is spanned by the projections onto C's normal bundle of the vector fields ∂_s, ∂_t and ∂_φ. Denote the latter by $\ ^s$, $\ ^t$ and $\ ^\varphi$, respectively. The tangent vector, , to N at C provides a fourth basis element for $TH|_C$. In particular, to say that is tangent to N at C places three constraints on the behavior of , whether or not C is a critical point of F. The assumption that C is a critical point places an additional constraint on . As argued subsequently, these constraints are not mutually self-consistent.

Step 5 This step describes the aforementioned four constraints on . In particular, these constraints involve the asymptotics of on both the $(p; p')$ and $(k; k')$ ends of C. In particular, fix attention first on the $(k; k')$ end of C. Reintroduce the coordinates $(\ ; u)$ on this end and then any element of $TH|_C$ at large $|u|$ can be viewed as a pair $(x'; y')$ of functions of and u which obey the linearized version of (2.20) subject to the constraint that $|x'|$ and $|y'/u|$ are bounded as $|u| \to \infty$. As indicated in the proof of Lemma 6.3, both x' and $|u|^{-1} y'$ have limits as $|u| \to \infty$. With this point understood, here are the first and second requirements for : The version of the functions x' and y' obey

$$\lim_{|u| \to \infty} x' = 0$$
$$\lim_{|u| \to \infty} |u|^{-1} y' = 0 \tag{6.6}$$

Indeed, these constraints are required for movement along N in the direction of to preserve the defining slice conditions that come from the $(k; k')$ end.

By the same reasoning, the $(p; p')$ version of the function x' must also satisfy the first point (6.6) if is to be tangent to N. This is the third requirement for . Finally, the $(p; p')$ version of the function y' must satisfy the second point in (6.6) if C is to be a critical point of the function F. This is the fourth constraint on .

Step 6 As is demonstrated momentarily, it is impossible for (6.6) to hold at both the $(k; k')$ and $(p; p')$ ends of C. The proximate cause of this incompatibility stems from the second point in (6.6) which requires both the $(k; k')$ and $(p; p')$ versions of the function y' to have zeros at arbitrarily large values of the coordinate $|u|$. Indeed, the proof that such zeros exist amounts to a linearized version of the argument in the proof of Lemma 6.4 that surrounds (6.2)–(6.5). However, more must be said about these zeros of y'. In particular, both the $(k; k')$ and $(p; p')$ versions of x' and y' have the following property:

> *Given* $R > 0$, *there exists* $\;> 0$ *such that the function* x' *take every value in* $(- \; ; \;)$ *at some* $|u| > R$ *zero of* y'. (6.7)

The validity of (6.7) is justified below, so accept it for now.

This large $|u|$ behavior of the version of y' on the two ends of C is used below to find a constant r and two distinct points in C_0 where $\;' + r\;$ is zero. But $\;' + r\;$ can have at most one zero for the following reason: All of its zeros count with the positive weight to any Euler class computation. Thus, if $\;' + r\;$ has two or more zeros, then a standard perturbation argument provides at least two zeros for $\;' + r\; + \;\;^t$ for any small in absolute value constant . Of course, the latter is also annihilated by the operator D, so all of its zeros also count with positive weight. Thus any Euler number calculation that uses $\;' + r\; + \;\;^t$ gives 2 or more for an answer. On the other hand, for a suitably generic choice of , the Euler count of the zeros of $\;' + r\; + \;\;^t$ can be used to compute the expression $\langle e; [C]\rangle - 2m_C$ that appears in Proposition 3.2. And, as this expression equals 1, so $\;' + r\; + \;\;^t$ has but one zero on C_0.

To find r which makes $\;' + r\;$ vanish at two points, first note that on both $(k; k')$ and $(p; p')$ ends of C, the $\;'$ version of the functions x' and y' has $x' \quad 1$ and $y' \quad 0$. Thus, $\;'$ and are colinear at all $(\; ; u)$ where the version of y' is zero. With the preceding understood, choose $R \quad 1$ and then take r so that $1/r$ is less than the smallest of the two values provided by the $(k; k')$ and $(p; p')$ versions of (6.7). Then (6.7) guarantees a zero for $\;' + r\; = 0$ on the $(k; k')$ end of C and another on the $(p; p')$ end.

Step 7 It remains now only to justify (6.7). For this purpose, note that the pair $(x'; y')$ obey the linearized version of (2.20) and so

$$x'_u = -g^{-2}\sin^{-2}\; y' + \;_1 y' \quad \text{and} \quad x' = \sin^{-2}\; y'_u + \;_2 y'; \tag{6.8}$$

where $\;_{1;2}$ are bounded functions of and u. Note that (6.8) asserts that $(x'; y')$ obeys a Cauchy Riemann equation to order $|y'|$. In particular, arguments akin to those used in Appendix A of [22] can be brought to bear and establish the following: First, the simultaneous zeros of y' and dy' are isolated points. Second, the zero set of y' constitutes an oriented graph, G, whose vertices are these simultaneous zeros. Here, the orientation on the edges is defined by the pullback of dx'. Finally, each vertex has a non-zero and even number of incident edges, with precisely half oriented by dx' to point into the vertex.

The preceding properties of G have (6.7) as a consequence. To see that such is the case, note first that G has no closed, oriented loops since its orientation form, dx', is the differential of a bona fide function. Thus, as there are points in G where $|s|$ is arbitrarily large, so there is an oriented path in G, starting on some large, but constant $|s|$ slice and on which $|s|$ is unbounded. Indeed, such a path is constructed by following an oriented edge until it hits a vertex, and then continuing out from the vertex along another edge with the outward pointing orientation. Only finitely many paths of this sort which start at some finite $|s|$ slice will have $|s|$ bounded since none are closed loops and only finitely many can intersect any given constant $|s|$ locus. Use up the finitely many with both ends hitting a chosen constant $|s|$ locus and then start another at a very large $|s|$ zero of y'. The latter must have one end where $|s|$ is unbounded.

Next, remark that y' is negative on one side of any edge, and positive on the other; and which side has which sign is determined by the orientation because (6.8) asserts that $dx' \wedge dy' > 0$ where $y' = 0$ and $dy' \neq 0$. In particular, as y' is also a bona fide function, there are two distinct paths in G which start on some large, but constant $|s|$ slice on which $|s|$ is unbounded. More to the point, $|s| \to \infty$ in the direction oriented by dx' on one of them, while $|s| \to \infty$ in the direction oriented by dx' on the other. However, since $|x'|$ limits to zero as $|s| \to \infty$, so x' must be negative on the first of these paths, and x' must be positive on the second. Given that $|x'| \to 0$ as $|s| \to \infty$, these last conclusions directly imply (6.7).

(c) The limits on a component of $\mathfrak{M}_I$

Supposing that $\mathfrak{M}_I$ is non-empty, focus attention on a component $H \subset \mathfrak{M}_I$. As was just proved, $H/(\mathbb{R} \times T)$ is an open interval, and so any slice in H of the $\mathbb{R} \times \mathbb{R} \times T$ action is non-compact. This said, the focus here is on the limiting behavior of non-convergent sequences in such a slice. In this regard, the discussion is simplest when the integer q from I is non-zero, for in this case, none of p, q or k is zero and dt has exactly one zero upon pullback to any $C \in H$. In particular, with $q \neq 0$, a slice, H^s, of the $\mathbb{R}$ action on H is obtained by requiring that s equal 0 at the zero of dt. In the case where one of p, q or k is equal to zero, choose some $\alpha \neq 0$ and not equal to one of the finite ratios p'/p, q'/q or k'/k. For such α, the form $dt - \alpha\, d\varphi$ pulls back to any $C \in H$ with but one zero. In this case, a slice, H^s, of the $\mathbb{R}$ action on H is obtained by the condition that $s = 0$ at the zero of $dt - \alpha\, d\varphi$.

The assignment to each $C \in H^s$ of the maximum of s on C defines a continuous function $S\colon H^s \to [0, \infty)$. This function S is proper; as can be shown with an essentially verbatim repetition of arguments from Section 5g. Thus, a sequence $\{C_i\}\in\mathcal{H}^s$ has no convergent subsequences provided that $\{S(C_i)\}$ is unbounded. In particular, fix attention on such a sequence where $S(C_i) > S(C_{i+1})$. For this sequence, minimal modifications of arguments from Section 5g find a subsequence (hence renumbered consecutively from 1) of $\{C_i\}$ which converges as described in Step 4 of Section 5g to one of the $\partial = 2$, triply punctured spheres from Part d of the third point of Proposition 4.2.

Let C denote this limit triply punctured sphere. Up to the action of $\mathcal{T}$, this sphere is described by a set $\hat{I}_C = \{(a,a'),(b,b')\}$ of pairs of integers with $\Delta_C = ab' - ba'$ positive. In this regard, arguments from Section 5g can be employed to deduce that $\hat{I}_C$ is a subset of $\hat{I}$. As is argued below, the pair (m,m') from $\hat{I}$ that is missing from $\hat{I}_C$ obeys

$$\sqrt{3}/\sqrt{2} < |m'/m| < \infty. \qquad (6.9)$$

These last conclusions have the following consequence:

> *Let $\hat{I}$ denote an unordered set of pairs of integers that corresponds to a component of the moduli space of $\partial = 3$, thrice-punctured spheres from Part d of the third point of Proposition 4.2. Then $\hat{I}$ satisfies the three constraints in Part A of Proposition 6.1.*

Before discussing (6.9), agree on the notational convention that orders the integers in $\hat{I}$ so that $\hat{I}_C = \{(p,p'),(q,q')\}$; thus (k,k') is the pair from $\hat{I}$ that is missing from $\hat{I}_C$. Now, to prove (6.9), translate the elements in the convergent sequence $\{C_i\}$ by the $\mathbb{R}$–factor in $\mathbb{R}\times\mathcal{T}$ so that $S(C_i) = 0$. Use $\{\underline{C}_i\}\subset\mathcal{H}$ to denote the resulting sequence. The zero of dt on $\underline{C}_i$ now occurs where $s = -S(C_i)$ and so the corresponding sequence of points has no convergent subsequence in any part of $\mathbb{R}\times(S^1\times S^2)$ where s is bounded from below. Arguments such as found in Section 5g apply now to the sequence $\{\underline{C}i\}$ and find a subsequence which converges as described in Step 4 of Section 5g to a cylinder from Example 6 in Section 4a, parameterized by the pair whose respective components are the quotients of k and k' by their greatest common divisor. This implies that $|k'/k| > \sqrt{3}/\sqrt{2}$ because a cylinder with $|k'/k| < \sqrt{3}/\sqrt{2}$ is ruled out by the arguments from Step 6 of Section 5g which find $\delta_I > 0$ such that $\sin\theta > \delta_I$ on every $C\in\mathfrak{M}_I$.

Being an open interval, $\mathcal{H}/(\mathbb{R}\times\mathcal{T})$ has two ends and so two distinct, triply punctured, $\partial = 2$ spheres can be expected as limits of non-convergent sequences in $\mathcal{H}^s$. As argued in the next subsection, this is indeed the case; and the next lemma describes the relation between resulting two versions of Proposition 5.1's data set $\hat{I}_C$:

Lemma 6.7 *Suppose that $I_C = \{(p, p'), (q, q')\}$ satisfies Proposition 5.1's constraints with the additional requirement that $k \equiv p + q$ and $k' \equiv p' + q'$ obey $|k'/k| > \sqrt{3}/\sqrt{2}$. Then I_C contains a unique pair (m, m') such that the following is true:*

- $|m'/m| > \sqrt{3}/2$.
- *When $(m, m') = (p, p')$, define $I_{C'} \equiv \{(q, q'), (-k, -k')\}$; and when $(m, m') = (q, q')$, define $I_{C'} \equiv \{(-k, -k'), (p, p')\}$. Then $I_{C'}$ also satisfies Proposition 5.1's constraints.*

Note here that I_C is never the same as $I_{C'}$.

Proof of Lemma 6.7 The discussion here considers various cases in turn, with the sign(p) = sign(q) case treated first. Consider first the existence of (m, m') in this case. To start the story, note that $\Delta \equiv pq' - qp' > 0$ is both $k'p - kp'$ and $kq' - qk'$, and as the signs of p, q and k agree, so

$$q'/q > k'/k > p'/p \,. \tag{6.10}$$

Thus, at least one of q'/q or p'/p has absolute value greater than $\sqrt{3}/\sqrt{2}$. Note that the first and third constraints in Proposition 5.1 hold automatically for $I_{C'}$ no matter which of the pair (p, p') or (q, q') ends up as (m, m'). Thus, only the second constraint is open to debate.

For the sake of argument, suppose that $k'/k > 0$ so that q'/q is guaranteed larger than $\sqrt{3}/\sqrt{2}$. Either the obvious choice, $\{(-k, -k'), (p, p')\}$, for $I_{C'}$ obeys the constraint of the second point of Proposition 5.1 or not. If not, then $|p'/p|$ had better be greater than $\{\sqrt{3}/\sqrt{2}\}$, and it is. Indeed, if the obvious $\{(-k, -k'), (p, p')\}$ does not obey the constraint in Proposition 5.1's second point, then $p' + k'$ must be negative and both p' and k' must also have the same sign unless $p' = 0$. Now, $p' \neq 0$ is precluded by the second point of Proposition 5.1; the latter requires $q' > 0$ when $p' = 0$, so $k' = q' + p'$ makes $k' > 0$ when $p' = 0$. Thus, both k' and p' are negative. Since $k' < 0$, so $k < 0$ and then the positivity of $\Delta = pk' - p'k$ requires $p < 0$. But then $|p'/p| > \sqrt{3}/\sqrt{2}$ by virtue of the third constraint in Proposition 5.1.

Thus, $(m, m') = (p, p')$ satisfies the first constraint of Lemma 6.7 and so the set $\{(q, q'), (-k, -k')\}$ is also a candidate for $I_{C'}$. To verify the second constraint of Proposition 5.1, consider the consequences of its violation. This occurs when $q' > 0$ and $k' > 0$. But, as previously argued, $k' < 0$ or else the set $\{(-k, -k'), (p, p')\}$ would serve for $I_{C'}$.

Now consider the uniqueness question for this first case when $k'/k > 0$. In particular, in the light of the preceding discussion, it is enough to verify that

$\{(q, q'), (-k, -k')\}$ cannot satisfy the requirements of Lemma 6.7 when $\{(-k, -k'), (p, p')\}$ does. For this purpose, note that when $-k'$ and p' have opposite signs then the second point of Proposition 5.1 applies to $\{(-k, -k'), (p, p')\}$ to force $k' > 0$ and $p' > 0$. If the second point is to apply to $\{(q, q'), (-k, -k')\}$ as well, then q' must be negative. However, if k' is positive, then so is k and thus so is q, and then (6.10) is violated. On the other hand, if k' and p' have the same sign, then k' and p' have opposite signs. This requires q' and p' to have opposite signs so $q' > 0$ and $p' < 0$ by an application of the second point in Proposition 5.1 to $\{(p, p'), (q, q')\}$. But now the fact that $q' > 0$ and $k' < 0$ violates this same point when applied to $\{(q, q'), (-k, -k')\}$.

The argument for the case where the signs of p and q agree and $k'/k < 0$ is obtained from the preceding argument by changing various signs.

Consider next the case for Lemma 6.7 when p and q have different signs. Again, the existence question is treated first. To start, suppose that p is negative. Thus (p, p') obeys the first constraint of Lemma 6.7 because of the third constraint in Proposition 5.1. In this case, $I_{C'} = \{(q, q'), (-k, -k')\}$ does not also satisfy the constraints from Proposition 5.1 only if both $k' > 0$ and $q' \le 0$. In this regard, consider first the case that $k > 0$ too. As k and q have the same sign, the positivity of Δ demands $q'/q > k'/k > \sqrt{3}/\sqrt{2}$, so the (q, q') pair all obeys the first constraint in Lemma 6.7. Moreover, now $I_{C'} = \{(-k, -k'), (p, p')\}$ has the same sign for both primed pair and so obeys the second constraint in Proposition 5.1.

Suppose next that $p < 0$ and $k < 0$. As noted, the second constraint of Proposition 5.1 fails for the corresponding $I_{C'}$ only when both $k' > 0$ and $q' \le 0$. However, this last possibility is precluded by virtue of the fact that $\Delta = kq' - qk'$ is positive.

Now suppose that $p > 0$ and $q < 0$. Here, the (q, q') pair obeys the first constraint in Lemma 6.7. The corresponding $I_{C'}$ does not obey the second constraint of Proposition 5.1 only when $k' < 0$ and $p' > 0$. However, this requires $q' < 0$ and so is precluded by the $\{(p, p'), (q, q')\}$ version of the second constraint of Proposition 5.1.

The argument for the uniqueness assertion in Lemma 6.7 for the case where p and q have different signs is straightforward and left to the reader.

Finally, consider the case where one of p or q vanishes. The existence argument where $p = 0$ is given below; the $q = 0$ existence argument and the uniqueness arguments are left to the reader as both are reasonably straightforward.

In the case where $p = 0$, then $p' \neq 0$ and so the (p, p') pair obeys the first condition in Lemma 6.7. The second condition of Proposition 5.1 is violated for the corresponding $I_{C'}$ only if $q' > 0$ and $k' > 0$. Since $p = 0$, the positivity of requires that k and q have opposite sign to p'. If $p' < 0$, then both have positive sign and $q'/q > k'/k > \sqrt{3}/\sqrt{2}$; thus (q, q') obeys the first constraint in Lemma 6.7. Moreover, the corresponding $I_{C'}$ obeys the second constraint as its primed entries have the same sign. Meanwhile, if $p' > 0$, then k and q are negative and now (q, q') obeys the first constraint in Lemma 6.7 by virtue of the fact that $\{(p, p'), (q, q')\}$ obeys the third constraint in Proposition 5.1. In this case, the corresponding $I_{C'} = \{(-k, -k'), (p, p')\}$ obeys the second constraint in Proposition 5.1 because its first primed entry is negative and the second is positive.

(d) The existence of $\partial = 3$, thrice-punctured sphere moduli spaces

This subsection gives a construction for points in the moduli spaces that are described by Proposition 6.1. For this purpose, fix a set I as described in Part A of the proposition, and fix one of the two ways to order I as $\{(p, p'), (q, q'), (k, k')\}$ so that the three points of this same Part A hold. The subset $\{(p, p'), (q, q')\} \subset I$ then labels a component of the moduli space of $\partial = 2$, thrice-punctured spheres from Proposition 5.1. There is a unique point, C', in this same moduli space where the following two constraints hold: The first constraint requires the vanishing of the constant term on the right side of (5.1) for each end of C'. The statement of the second constraint depends on whether or not one of p, q or $p + q$ is zero. If none vanish, then C' is constrained so that the sole point where dt is zero on TC' occurs where $s = 0$. In the case where one of these integers equals zero, fix $\neq 0$ nor equal to any m'/m for $(m, m') \in I$. With fixed, the second constraint requires that $s = 0$ at the sole point in C where $dt + \, d\varphi$ is zero on TC'.

Now consider that the pair (k', k) labels a component of the moduli space of pseudoholomorphic cylinders from Example 6 of Section 4a. To be precise here, introduce m to denote the greatest, common (positive) divisor of k and k', and then introduce $\underline{k} \equiv k/m$ and $\underline{k}' \equiv k'/m$. It is the pair $(\underline{k}, \underline{k}')$ that determines the moduli space. In any event, with $S \in \mathbb{R}$ fixed, the latter moduli space contains a unique point, C_S, where the following two conditions hold: First, the constant term on the right-hand side of (5.1) is zero for each end of C_S. Second, the maximum value of s is C_S is equal to S.

When $S \gg 1$, these subvarieties C' and C_S are used below to construct an element, C, in the I–labeled component, $\mathfrak{M}_I$, of Proposition 6.1's moduli space of thrice-punctured. Of course, the construction of C verifies the existence assertion of Proposition 6.1. Moreover, certain properties of the construction are used to prove some of the other assertions (4.15) of Proposition 6.1. In particular, the latter are described in the following result.

Proposition 6.8 *Fix a set, I, of three pair of integers that satisfy the constraints in Part A of Proposition 6.1, and fix one of the two orderings I as $\{(p, p'), (q, q'), (k, k')\}$ so as to satisfy these constraints. Given $\varepsilon > 0$, there exists $S_0 > 1$ and an embedding, ψ, from $[S_0, \infty)$ into H_I with the following properties:*

- *If $C = \psi(S)$, then $C = C_- \cup C_+$ where $C_\pm$ are open sets, the (k, k') end of C is in C_-, the (p, p') and (q, q') ends are in C_+, and*

$$\sup_{z \in C_-} \operatorname{dist}(z, C_S) + \sup_{z \in C_-} \operatorname{dist}(z, C') < \varepsilon \tag{6.11}$$

- *The projection of the image of ψ to $\mathfrak{M}_I/(\mathbb{R} \times T)$ defines a proper embedding of the closed half line $[S_0, \infty)$ into $\mathfrak{M}_I/(\mathbb{R} \times T)$.*

Moreover, if $C \in \mathfrak{M}_I$ and $S(C) > S_0 + 1$, then there exists $\gamma \in \mathbb{R} \times T$ such that $\gamma(C)$ is in the image of the version of the map ψ as defined by one or the other of the two orderings of I that satisfy the constraints in Part A of Proposition 6.1.

This proof of this proposition is given momentarily. Accept it for now, and here is the completion of the following proof.

Proof of Proposition 6.1 Consider first the assertions of Part A: The fact that $\mathfrak{M}_I = \emptyset$ unless I obeys the constraints in Part A was established in Subsection 6c, above, while the sufficiency of these constraints follows directly from the asserted existence of Proposition 6.8's map ψ. (Remember that Lemma 6.9 justifies the claimed existence of two orderings for I that obey Part A's conditions.)

As for the assertions in Part B, the claimed structure of a component of $\mathfrak{M}_I$ as the product $(0, 1) \times (\mathbb{R} \times T/\Gamma_I)$ is proved as Proposition 6.2. The fact that $\mathfrak{M}_I$ has but a single component follows from the third point of Proposition 6.8. Finally, the asserted properties of the two point compactification of $\mathfrak{M}_I/(\mathbb{R} \times T)$ are direct consequences of the first and second points of Proposition 6.8.

The remainder of this subsection contains the following proof.

Proof of Proposition 6.8 The construction of C from C' and C_S constitutes a by now standard 'gluing construction' for pseudoholomorphic subvarieties.

Such gluing constructions were first introduced by Floer [4] as analogies to similar constructions that are used to construct self-dual connections on 4–manifolds. In any event, the details of the construction are left to the reader save for the outline that follows.

To begin the outline, remark that all of the gluing construction can be construed in the following way: The subvariety C is obtained from a small normed section, , of certain complex line bundle over C_0. Here, this line bundle is the normal bundle to a symplectic immersion of C_0 whose image is very close to C on the $s < 3S/4$ portion of the latter and whose image is very close to C_S on the complement of the $s < -S/4$ portion of this cylinder's $(-\underline{k}, -\underline{k}')$ end. To be more explicit, the image of this immersion agrees with C' on the $s < -S/4$ portion of the latter and it sits very close to C_S on the complement of the $s < 3S/4$ portion of this cylinder's $(-\underline{k}, -\underline{k}')$ end. In particular, near this part of C_S, the immersion is an embedding obtained by composing the exponential map from (3.12) with a suitably chosen, m-multivalued section of the cylinder's normal bundle. Then, where $s \in [-S/4, 3S/4]$, these two portions of the immersion are extended and sutured together using cut-off functions.

In any event, the resulting immersion of C_0 is everywhere symplectic and pseudoholomorphic for an almost complex structure that is pointwise close to J, even where $s \in [S/4, 3S/4]$. (Proposition 2.3 is needed for this last conclusion.) With the immersion of C_0 constructed, the subvariety C is the image of the composition of a section, , of C_0's normal bundle with an exponential map such as that used in (3.12).

Defined as it is by , the subvariety C is pseudoholomorphic provided solves an inhomogeneous version of the C_0 version of the equation in the third point of (3.12). In this regard, x n 0 and then the relevant inhomogeneous term is $O(e^{S/})$ as measured with the $L^2_{n;}$ norm in (3.14). Here, 1 is independent of S when the latter is large. (Proposition 2.3 is required to obtain these bounds.) The implicit function theorem can be employed as in Step 1 of the proof of Proposition 3.2 to find a small pointwise norm solution to this inhomogeneous equation which is L^2 orthogonal to the kernel of D. The existence of such a solution verifies the first point of Proposition 6.8.

Of course, the application here of the implicit function theorem to the inhomogeneous version of (3.12) requires some sort of S independent bound on the norm of the inverse of the operator D that appears in this version of (3.12). In the case at hand, such a bound exists when S is large; its existence follows from the fact that the analogous D has trivial cokernel on both C' and on the m–fold covering space of C_S. The fact that the C' version of D has trivial cokernel is proved as Proposition 4.6. Analogous arguments, or a separation of variables analysis can be used to establish this same conclusion for the operator on the m-fold cover of C_S. (Remember that C_S is invariant under an S^1 subgroup of T.)

The use made here of the implicit function theorem has the second point of Proposition 6.8 as a straightforward consequence.

Implicit function theorem conclusions typically have uniqueness as well as existence ramifications, and the former play a key role in the proof of the final assertion of Proposition 6.8. In particular, as applied here, the implicit function theorem can be used to prove that is the only small pointwise norm solution to this inhomogeneous equation which is L^2 orthogonal to the kernel of D. This last conclusion has the following consequence:

> *There exists* > 0 *such that if* S *is large and if a given* $C \in \mathfrak{M}_I$ *has all points distance or less from* $C_S \cup C'$*, then there exists* $\gamma \in \mathbb{R}\ T$ *such that* $\gamma(C)$ *is in the image of* . (6.12)

With (6.12) understood, the proof of the final assertion of Proposition 6.8 begins by noting that the two orderings of I that satisfy the constraints of Part A of Proposition 6.1 each define, for S 1, a pair (C_S, C'). Then, the arguments from Section 6c lead to the following conclusion: Given > 0, a good portion of the points in C have distance or less from one of these versions of $C_S \cup C'$ when $S(C)$ is large. To be more precise, fix $R > 1$ and there exists S_0 such that when $S(C) > S_0$, then the portion of C with distance less than from $C_S \cup C'$ contains all points except possibly those in the set, U, of points that both lie in the bounded component of the locus in C's (k, k') end where $s \quad S(C) - R$, and lie where $s \quad R$.

This said, turn attention to the subset U. For this purpose, reintroduce the cylinder C where both $\underline{k}\,' - \underline{k}'t = 0$ and $h/f = (\underline{k}'/\underline{k})\sin^2 {}_0$. According to Proposition 1.2, when R is large, then all of the $s \quad S(C) - R$ half cylinder in C_S whose boundary is nearest to U has distance less than $e^{-R/}$ from $\underline{C}$. The same is true for the $s \quad R$ half cylinder in C'. Here, 1 can be taken to be independent of R. (Remember that $|\underline{k}'/\underline{k}| > \sqrt{3}/\sqrt{2}$ and that $\sin^2 {}_0$ is bounded away from zero.) With these points taken, the manner of convergence described in Section 6c implies the following: When is fixed, R is large, and then $S(C)$ is very large, the points in U where $s = R$ or $s = S(C)$R obey

$$|\underline{k}\,' - \underline{k}'t| < {}^2 + 2e^{-R/} \quad \text{and} \quad |(h/f) - (\underline{k}'/\underline{k})\sin^2 {}_0| \quad {}^2 + 2e^{-R/}\ . \qquad (6.13)$$

Now, as remarked previously on many occasions, both $\underline{k}' - \underline{k}'t$ and h/f are subject to the mercy of the maximum principle. This implies that $|\underline{k}' - \underline{k}'t|$ is bounded by $^2 + 2e^{-R/}$ on the whole of U, as is $|(h/f) - (\underline{k}'/\underline{k})\sin^2 {}_0|$ provided that $f \neq 0$ on U. In this regard, the convergence described in Section 6c implies that the sign of f is that of $\underline{k}$ on both boundaries of U, and as f is also subject to the maximum priniciple, so f is, indeed, nowhere zero on U.

Next, remark that as U lies in the (k, k') end of C and is far from a critical point of f (where $s = 0$), so it can be parameterized as in (2.19) by variables $(, u)$ using the functions (x, y) of $(, u)$. The conclusions of the previous paragraph now translate as the assertion that both $|x|$ and $|u|^{-1}|y|$ are bounded by $^2 + 2e^{-R/}$ on U.

The preceding point is relevant because the metric distance of a point in U from the cylinder $\underline{C}$ is bounded by $(|x| + |u|^{-1}|y|)$ where is independent of R and when $S(C)$ is large. Thus, small , large R and very large $S(C)$ implies that C obeys the assumptions in (6.12) for one or the other of the two possible versions of $C_S \cup C'$ that are provided by I.

(e) The number of double points

The next proposition describes the double point number m_C of a subvariety C that comes from any of the moduli space components that appear in Proposition 6.1. In particular, this proposition directly implies all assertions of Theorem A.4 that concern subvarieties from Theorem A.3.

Proposition 6.9 *Let $I = \{(p, p'), (q, q'), (k, k')\}$ satisfy the requirements in Proposition 6.1 so as to describe a component of the moduli space of $\partial = 3$, thrice-punctured spheres that appear in Part d of the third point of Proposition 4.2. Let C denote some subvariety in this component of the moduli space. Then, m_C is equal to one half of the number of ordered pairs $(, ') \in S^1 \times S^1$ with $\neq '$, neither equal to 1 and such that $^p\, '^q = ^{p'}\, '^{q'} = 1$.*

Note that the count for m_C is the same as that given in Proposition 5.9 for the case when $\{(p, p'), (q, q')\}$ determine the latters moduli space component. This said, it follows that $m_C = 0$ here if and only if either $|pq' - p'q|$ is one or two, or if both integers in at least one of the pair (p, p'), (q, q') and (k, k') is evenly divisible by $pq' - qp'$.

By the way, in spite of the appearance to the contrary, the preceding and Proposition 6.9's count of m_C is insensitive to the ordering of the set I.

The remainder of this subsection contains the following proof.

Proof of Proposition 6.9 Let C be as described in the proposition, and, with small and su ciently generic and > 0 but small, let C denote the translate of C via the element in T that moves t to $t +$ and $'$ to $' +$. Here, is chosen so that the Reeb orbits that are determined by C are pairwise distinct from those deterimined by C . This understood, then it follows from the gluing construction of C's moduli space in Section 6d that the intersection number between C and C is given by the formula in (5.64) where m is the number that is described in Step 1 of the proof of Proposition 5.9. In this regard, note that m is equal to Proposition 6.9's asserted value for m_C.

The preceding understood, then the relationship between the afore-mentioned $C - C$ intersection number and the number m_C is as described in Step 4 of the proof of Proposition 5.9; the same argument gives the justi cation. Moreover, the claim in (5.67) is valid here as well, and with an identical proof. Of course, this implies that m_C has the value asserted by Proposition 6.9.

Acknowledgements

The author thanks Jerrel Mast for pointing out some errors in a preprint version of this article.

The author is supported in part by the National Science Foundation.

References

[1] **M F Atiyah, V K Patodi, I M Singer**, *Spectral asymmetry in Riemannian geometry, I*, Math. Proc. Cambridge Philos. Soc. 77 (1975a) 43{69

[2] **Y Eliashberg**, *Invariants in contact topology*, from: \Proceedings of the International Congress of Mathematicians, Berlin (1998)", Vol II, Documenta Mathematica, Extra Volume ICM (1998) 327{338

[3] **Y Eliashberg, E Hofer**, in preparation

[4] **A Floer**, *Morse theory for Lagrangian intersections*, J. Di . Geom. 28 (1988) 513{547

[5] **P Gri ths, J Harris**, *Principles of Algebraic Geometry*, J Wiley and Sons, New York (1978)

[6] **M Gromov**, *Pseudoholomorphic spheres in symplectic manifolds*, Invent. Math 82 (1985) 307{347

[7] **R Gompf**, private communication.

[8] H Hofer, *Pseudoholmorphic curves in symplectizations with applications to the Weinstein conjecture in dimension 3*, Invent Math 114 (1993) 515{563

[9] H Hofer, *Dynamics, topology and holomorphic curves*, from: \Proceedings of the International Congress of Mathematicians, Berlin 1998", Vol I, Documenta Mathematica, Extra Volume ICM (1998) 255{280

[10] H Hofer, *Holomorphic curves and dynamics in dimension 3*, from: \Symplectic Geometry and Topology", (Eliashberg and Traynor, editors) IAS/Park City Mathematics Series 7, American Mathematical Society, Providence RI (1999) 37{101

[11] H Hofer, K Wysocki, E Zehnder, *Properties of pseudoholomorphic curves in symplectizations I: Asymptotics*, Ann. Inst. Henri Poincare 13 (1996) 337{379

[12] H Hofer, K Wysocki, E Zehnder, *Properties of pseudoholomorphic curves in symplectizations II: Embedding controls and algebraic invariants*, Geom. and Funct. Anal. 5 (1995) 270{328

[13] H Hofer, K Wysocki, E Zehnder, *Properties of pseudoholomorphic curves in symplectizations III: Fredholm theory*, preprint.

[14] R B Lockhart, R C McOwen, *Elliptic di erential operators on non-compact manifolds*, Ann. Sci. Norm. Sup. Pisa IV-12 (1985) 409{446

[15] C Luttinger, unpublished

[16] D McDu , *The local behavior of J{holomorphic curves in almost complex manifolds*, J. Di . Geom. 34 (1990) 679{712

[17] C B Morrey, *Multiple Integrals in the Calculus of Variations*, Springer{Verlag, Berlin (1966)

[18] D McDu , D Salamon, *J{Holomorphic Curves and Quantum Cohomology*, University Lecture Series 6, American Mathematical Society, Providence RI (1994)

[19] C H Taubes, *Seiberg{Witten invariants and pseudoholomorphic subvarieties for self-dual, harmonic 2{forms*, Geom. Topol. 3 (1999) 167{210

[20] C H Taubes, *Seiberg{Witten invariants, self-dual harmonic 2{forms and the Hofer{Wysocki{Zehnder formalism*, from: \Surveys in Di erential Geometry, VII", International Press (2000) 625{672

[21] C H Taubes, *The geometry of the Seiberg{Witten invariants*, from: \Proceedings of the International Congress of Mathematicians, Berlin 1998", Vol II, Documenta Mathematica, Extra Volume ICM (1998) 493{504

[22] C H Taubes, *The structure of pseudoholomorphic subvarieties for a degenerate almost complex structure and symplectic form on* S^1 B^3, Geom. Topol. 2 (1998) 221{332

[23] C H Taubes, L^2 *Moduli Spaces on 4{Manifolds with Cylindrical End*, International Press, Cambridge MA (1993)

[24] C H Taubes, *Gr ! SW; from pseudoholomorphic curves to Seiberg{Witten solutions*, J. Di . Geom. 51 (1999) 203{334.