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Construction of 2{local nite groups of a type studied by Solomon and Benson

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Abstract

A p{local nite group is an algebraic structure with a classifying space which has many of the properties of p{completed classifying spaces of nite groups. In this paper, we construct a family of 2{local nite groups, which are exotic in the following sense: they are based on certain fusion systems over the Sylow 2{subgroup of $\mathrm{Spin}_7(q)$ (q an odd prime power) shown by Solomon not to occur as the 2{fusion in any actual nite group. Thus, the resulting classifying spaces are not homotopy equivalent to the 2{completed classifying space of any nite group. As predicted by Benson, these classifying spaces are also very closely related to the Dwyer{Wilkerson space $BDI(4)$.

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As one step in the classification of finite simple groups, Ron Solomon [22] considered the problem of classifying all finite simple groups whose Sylow 2–subgroups are isomorphic to those of the Conway group Co_3. The end result of his paper was that Co_3 is the only such group. In the process of proving this, he needed to consider groups G in which all involutions are conjugate, and such that for any involution $x \in G$, there are subgroups $K \lhd H \lhd C_G(x)$ such that K and $C_G(x)/H$ have odd order and $H/K \cong \mathrm{Spin}_7(q)$ for some odd prime power q. Solomon showed that such a group G does not exist. The proof of this statement was also interesting, in the sense that the 2–local structure of the group in question appeared to be internally consistent, and it was only by analyzing its interaction with the p–local structure (where p is the prime of which q is a power) that he found a contradiction.

In a later paper [3], Dave Benson, inspired by Solomon's work, constructed certain spaces which can be thought of as the 2–completed classifying spaces which the groups studied by Solomon would have if they existed. He started with the spaces $BDI(4)$ constructed by Dwyer and Wilkerson having the property that

$$H^*(BDI(4);\mathbb{F}_2) \cong \mathbb{F}_2[x_1,x_2,x_3,x_4]^{GL_4(2)}$$

(the rank four Dickson algebra at the prime 2). Benson then considered, for each odd prime power q, the homotopy fixed point set of the $\mathbb{Z}$–action on $BDI(4)$ generated by an "Adams operation" ψ^q constructed by Dwyer and Wilkerson. This homotopy fixed point set is denoted here $BDI_4(q)$.

In this paper, we construct a family of *2–local finite groups*, in the sense of [6], which have the 2–local structure considered by Solomon, and whose classifying spaces are homotopy equivalent to Benson's spaces $BDI_4(q)$. The results of [6] combined with those here allow us to make much more precise the statement that these spaces have many of the properties which the 2–completed classifying spaces of the groups studied by Solomon would have had if they existed. To explain what this means, we first recall some definitions.

A *fusion system* over a finite p–group S is a category whose objects are the subgroups of S, and whose morphisms are monomorphisms of groups which include all those induced by conjugation by elements of S. A fusion system is *saturated* if it satisfies certain axioms formulated by Puig [19], and also listed in [6, Definition 1.2] as well as at the beginning of Section 1 in this paper. In particular, for any finite group G and any $S \in \mathrm{Syl}_p(G)$, the category $\mathcal{F}_S(G)$ whose objects are the subgroups of S and whose morphisms are those monomorphisms between subgroups induced by conjugation in G is a saturated fusion system over S.

If $\mathcal{F}$ is a saturated fusion system over S, then a subgroup $P \le S$ is called *$\mathcal{F}$–centric* if $C_S(P') = Z(P')$ for all P' isomorphic to P in the category $\mathcal{F}$. A *centric linking system* associated to $\mathcal{F}$ consists of a category $\mathcal{L}$ whose objects are the $\mathcal{F}$–centric subgroups of S, together with a functor $\mathcal{L} \longrightarrow \mathcal{F}$ which is the inclusion on objects, is surjective on all morphism sets and which satisfies certain additional axioms (see [6, Definition 1.7]). These axioms suffice to ensure that the p–completed nerve $|\mathcal{L}|^\wedge_p$ has all of the properties needed to regard it as a "classifying space" of the fusion system $\mathcal{F}$. A *p–local finite group* consists of a triple $(S,\mathcal{F},\mathcal{L})$, where S is a finite p–group, $\mathcal{F}$ is a saturated fusion system over S, and $\mathcal{L}$ is a linking system associated to $\mathcal{F}$. The classifying space of a p–local finite group $(S,\mathcal{F},\mathcal{L})$ is the p–completed nerve $|\mathcal{L}|^\wedge_p$ (which is p–complete since $|\mathcal{L}|$ is always p–good [6, Proposition 1.12]). For example, if G is a finite group and $S \in \mathrm{Syl}_p(G)$, then there is an explicitly defined centric linking system $\mathcal{L}^c_S(G)$ associated to $\mathcal{F}_S(G)$, and the classifying space of the triple $(S,\mathcal{F}_S(G),\mathcal{L}^c_S(G))$ is the space $|\mathcal{L}^c_S(G)|^\wedge_p \simeq BG^\wedge_p$.

Exotic examples of p–local finite groups for odd primes p — ie, examples which do not represent actual groups — have already been constructed in [6], but using ad hoc methods which seemed to work only at odd primes.

In this paper, we first construct a fusion system $\mathcal{F}_{\mathrm{Sol}}(q)$ (for any odd prime power q) over a 2–Sylow subgroup S of $\mathrm{Spin}_7(q)$, with the properties that all elements of order 2 in S are conjugate (ie, the subgroups they generated are all isomorphic in the category), and the "centralizer fusion system" (see the beginning of Section 1) of each such element is isomorphic to the fusion system of $\mathrm{Spin}_7(q)$. We then show that $\mathcal{F}_{\mathrm{Sol}}(q)$ is saturated, and has a unique associated linking system $\mathcal{L}^c_{\mathrm{Sol}}(q)$. We thus obtain a 2–local finite group $(S,\mathcal{F}_{\mathrm{Sol}}(q),\mathcal{L}^c_{\mathrm{Sol}}(q))$ where by Solomon's theorem [22] (as explained in more detail in Proposition 3.4), $\mathcal{F}_{\mathrm{Sol}}(q)$ is not the fusion system of any finite group. Let $B\mathrm{Sol}(q) \stackrel{\text{def}}{=} |\mathcal{L}^c_{\mathrm{Sol}}(q)|^\wedge_2$ denote the classifying space of $(S,\mathcal{F}_{\mathrm{Sol}}(q),\mathcal{L}^c_{\mathrm{Sol}}(q))$. Thus, $B\mathrm{Sol}(q)$ does not have the homotopy type of $BG^\wedge_2$ for any finite group G, but does have many of the nice properties of the 2–completed classifying space of a finite group (as described in [6]).

Relating $B\mathrm{Sol}(q)$ to $BDI_4(q)$ requires taking the "union" of the categories $\mathcal{L}^c_{\mathrm{Sol}}(q^n)$ for all $n \ge 1$. This however is complicated by the fact that an inclusion of fields $\mathbb{F}_{p^m} \subseteq \mathbb{F}_{p^n}$ (ie, $m|n$) does not induce an inclusion of cenric linking systems. Hence we have to replace the centric linking systems $\mathcal{L}^c_{\mathrm{Sol}}(q^n)$ by the full subcategories $\mathcal{L}^{cc}_{\mathrm{Sol}}(q^n)$ whose objects are those 2–subgroups which are centric in $\mathcal{F}^c_{\mathrm{Sol}}(q^\infty) = \bigcup_{n\ge 1}\mathcal{F}^c_{\mathrm{Sol}}(q^n)$, and show that the inclusion induces a homotopy equivalence $B\mathrm{Sol}'(q^n) \stackrel{\text{def}}{=} |\mathcal{L}^{cc}_{\mathrm{Sol}}(q^n)|^\wedge_2 \simeq B\mathrm{Sol}(q^n)$. Inclusions of fields do induce inclusions of these categories, so we can then define $\mathcal{L}^c_{\mathrm{Sol}}(q^\infty) \stackrel{\mathrm{def}}{=} \bigcup_{n\ge 1} \mathcal{L}^{cc}_{\mathrm{Sol}}(q^n)$, and spaces

$$B\mathrm{Sol}(q^\infty) = |\mathcal{L}^c_{\mathrm{Sol}}(q^\infty)|^\wedge_2 \simeq \Big(\bigcup_{n\ge 1} B\mathrm{Sol}^0(q^n)\Big)^\wedge_2 .$$

The category $\mathcal{L}^c_{\mathrm{Sol}}(q^\infty)$ has an "Adams map" ψ^q induced by the Frobenius automorphism $x \mapsto x^q$ of $\bar{\mathbb{F}}_q$. We then show that $B\mathrm{Sol}(q^\infty) \simeq BDI(4)$, the space of Dwyer and Wilkerson mentioned above; and also that $B\mathrm{Sol}(q)$ is equivalent to the homotopy fixed point set of the $\mathbb{Z}$–action on $B\mathrm{Sol}(q^\infty)$ generated by $B\psi^q$. The space $B\mathrm{Sol}(q)$ is thus equivalent to Benson's spaces $BDI_4(q)$ for any odd prime power q.

The paper is organized as follows. Two propositions used for constructing saturated fusion systems, one very general and one more specialized, are proven in Section 1. These are then applied in Section 2 to construct the fusion systems $\mathcal{F}_{\mathrm{Sol}}(q)$, and to prove that they are saturated. In Section 3 we prove the existence and uniqueness of a centric linking systems associated to $\mathcal{F}_{\mathrm{Sol}}(q)$ and study their automorphisms. Also in Section 3 is the proof that $\mathcal{F}_{\mathrm{Sol}}(q)$ is not the fusion system of any finite group. The connections with the space $BDI(4)$ of Dwyer and Wilkerson is shown in Section 4. Some background material on the spinor groups $\mathrm{Spin}(V, \mathfrak{b})$ over fields of characteristic $\neq 2$ is collected in an appendix.

We would like to thank Dave Benson, Ron Solomon, and Carles Broto for their help while working on this paper.

1 Constructing saturated fusion systems

In this section, we first prove a general result which is useful for constructing saturated fusion systems. This is then followed by a more technical result, which is designed to handle the specific construction in Section 2.

We first recall some definitions from [6]. A *fusion system* over a p–group S is a category $\mathcal{F}$ whose objects are the subgroups of $\mathcal{F}$, such that

$$\mathrm{Hom}_S(P,Q) \subseteq \mathrm{Mor}_{\mathcal{F}}(P,Q) \subseteq \mathrm{Inj}(P,Q)$$

for all $P,Q \le S$, and such that each morphism in $\mathcal{F}$ factors as the composite of an $\mathcal{F}$–isomorphism followed by an inclusion. We write $\mathrm{Hom}_{\mathcal{F}}(P,Q) = \mathrm{Mor}_{\mathcal{F}}(P,Q)$ to emphasize that the morphisms are all homomorphisms of groups.

We say that two subgroups $P,Q \le S$ are *$\mathcal{F}$–conjugate* if they are isomorphic in $\mathcal{F}$. A subgroup $P \le S$ is *fully centralized* (*fully normalized*) in $\mathcal{F}$ if $|C_S(P)| \ge |C_S(P')|$ ($|N_S(P)| \ge |N_S(P')|$) for all $P' \le S$ which is $\mathcal{F}$–conjugate to P. A *saturated fusion system* is a fusion system $\mathcal{F}$ over S which satisfies the following two additional conditions:

(I) For each fully normalized subgroup $P \le S$, P is fully centralized and $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$.

(II) For each $P \le S$ and each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$ such that $\varphi(P)$ is fully centralized in $\mathcal{F}$, if we set

$$N_\varphi = \bigl\{ g \in N_S(P) \bigm| \varphi c_g \varphi^{-1} \in \mathrm{Aut}_S(\varphi(P)) \bigr\},$$

then φ extends to a homomorphism $\bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, S)$.

For example, if G is a finite group and $S \in \mathrm{Syl}_p(G)$, then the category $\mathcal{F}_S(G)$ whose objects are the subgroups of S and whose morphisms are the homomorphisms induced by conjugation in G is a saturated fusion system over S. A subgroup $P \le S$ is fully centralized in $\mathcal{F}_S(G)$ if and only if $C_S(P) \in \mathrm{Syl}_p(C_G(P))$, and P is fully normalized in $\mathcal{F}_S(G)$ if and only if $N_S(P) \in \mathrm{Syl}_p(N_G(P))$.

For any fusion system $\mathcal{F}$ over a p–group S, and any subgroup $P \le S$, the "centralizer fusion system" $C_{\mathcal{F}}(P)$ over $C_S(P)$ is defined by setting

$$\mathrm{Hom}_{C_{\mathcal{F}}(P)}(Q,Q') = \bigl\{ (\varphi|_Q) \bigm| \varphi \in \mathrm{Hom}_{\mathcal{F}}(PQ,PQ'),\ \varphi(Q) \le Q',\ \varphi|_P = \mathrm{Id}_P \bigr\}$$

for all $Q,Q' \le C_S(P)$ (see [6, Definition A.3] or [19] for more detail). We also write $C_{\mathcal{F}}(g) = C_{\mathcal{F}}(\langle g \rangle)$ for $g \in S$. If $\mathcal{F}$ is a saturated fusion system and P is fully centralized in $\mathcal{F}$, then $C_{\mathcal{F}}(P)$ is saturated by [6, Proposition A.6] (or [19]).

Proposition 1.1 *Let $\mathcal{F}$ be any fusion system over a p–group S. Then $\mathcal{F}$ is saturated if and only if there is a set $\mathfrak{X}$ of elements of order p in S such that the following conditions hold:*

(a) *Each $x \in S$ of order p is $\mathcal{F}$–conjugate to some element of $\mathfrak{X}$.*

(b) *If x and y are $\mathcal{F}$–conjugate and $y \in \mathfrak{X}$, then there is some morphism $\psi \in \mathrm{Hom}_{\mathcal{F}}(C_S(x), C_S(y))$ such that $\psi(x) = y$.*

(c) *For each $x \in \mathfrak{X}$, $C_{\mathcal{F}}(x)$ is a saturated fusion system over $C_S(x)$.*

Proof Throughout the proof, conditions (I) and (II) always refer to the conditions in the definition of a saturated fusion system, as stated above or in [6, Definition 1.2].

Assume first that $\mathcal{F}$ is saturated, and let $\mathfrak{X}$ be the set of all $x\in S$ of order p such that $\langle x\rangle$ is fully centralized. Then condition (a) holds by definition, (b) follows from condition (II), and (c) holds by [6, Proposition A.6] or [19].

Assume conversely that $\mathfrak{X}$ is chosen such that conditions (a–c) hold for $\mathcal{F}$. Define

$$\mathcal{U}=\Big\{(P,x)\;\Big|\;P\le S,\ |x|=p,\ x\in Z(P)^T,\ \text{some } T\in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P)),\ T\ge \mathrm{Aut}_S(P)\Big\};$$

where $Z(P)^T$ is the subgroup of elements of $Z(P)$ fixed by the action of T. Let $\mathcal{U}_0\subseteq\mathcal{U}$ be the set of pairs (P,x) such that $x\in\mathfrak{X}$. For each $1\ne P\le S$, there is some x such that $(P,x)\in\mathcal{U}$ (since every action of a p–group on $Z(P)$ has nontrivial fixed set); but x need not be unique.

We first check that

$$(P,x)\in\mathcal{U}_0,\ P \text{ fully centralized in } C_{\mathcal{F}}(x)\implies P \text{ fully centralized in } \mathcal{F}. \qquad (1)$$

Assume otherwise: that $(P,x)\in\mathcal{U}_0$ and P is fully centralized in $C_{\mathcal{F}}(x)$, but P is not fully centralized in $\mathcal{F}$. Let $P'\le S$ and $\varphi\in\mathrm{Iso}_{\mathcal{F}}(P,P')$ be such that $|C_S(P)|<|C_S(P')|$. Set $x'=\varphi(x)\in Z(P')$. By (b), there exists $\psi\in\mathrm{Hom}_{\mathcal{F}}(C_S(x'),C_S(x))$ such that $\psi(x')=x$. Set $P''=\psi(P')$. Then $\psi\circ\varphi\in\mathrm{Iso}_{C_{\mathcal{F}}(x)}(P,P'')$, and in particular P'' is $C_{\mathcal{F}}(x)$–conjugate to P. Also, since $C_S(P')\le C_S(x')$, ψ sends $C_S(P')$ injectively into $C_S(P'')$, and $|C_S(P)|<|C_S(P')|\le|C_S(P'')|$. Since $C_S(P)=C_{C_S(x)}(P)$ and $C_S(P'')=C_{C_S(x)}(P'')$, this contradicts the original assumption that P is fully centralized in $C_{\mathcal{F}}(x)$.

By definition, for each $(P,x)\in\mathcal{U}$, $N_S(P)\le C_S(x)$ and hence $\mathrm{Aut}_{C_S(x)}(P)=\mathrm{Aut}_S(P)$. By assumption, there is $T\in\mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$ such that $\alpha(x)=x$ for all $\alpha\in T$; ie, such that $T\le\mathrm{Aut}_{C_{\mathcal{F}}(x)}(P)$. In particular, it follows that

$$\forall (P,x)\in\mathcal{U}:\ \ \mathrm{Aut}_S(P)\in\mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))\iff\mathrm{Aut}_{C_S(x)}(P)\in\mathrm{Syl}_p(\mathrm{Aut}_{C_{\mathcal{F}}(x)}(P)). \qquad (2)$$

We are now ready to prove condition (I) for $\mathcal{F}$; namely, to show for each $P\le S$ fully normalized in $\mathcal{F}$ that P is fully centralized and $\mathrm{Aut}_S(P)\in\mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$. By definition, $|N_S(P)|\ge|N_S(P')|$ for all P' $\mathcal{F}$–conjugate to P. Choose $x\in Z(P)$ such that $(P,x)\in\mathcal{U}$; and let $T\in\mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$ be such that $T\ge\mathrm{Aut}_S(P)$ and $x\in Z(P)^T$. By (a) and (b), there is an element $y\in\mathfrak{X}$ and a homomorphism $\varphi\in\mathrm{Hom}_{\mathcal{F}}(C_S(x),C_S(y))$ such that $\varphi(x)=y$. Set $P'=\varphi P$, and set $T'=\varphi T\varphi^{-1}\in\mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(T'))$. Since $T\ge\mathrm{Aut}_S(P)$ by definition of $\mathcal{U}$, and $\varphi(N_S(P))=N_S(P')$ by the maximality assumption, we see that $T'\ge\mathrm{Aut}_S(P')$. Also, $y\in Z(P')^{T'}$ ($T'y=y$ since $Tx=x$), and this shows that $(P',y)\in\mathcal{U}_0$. The maximality of $|N_S(P')|=|N_{C_S(y)}(P')|$ implies that P' is fully normalized in $C_{\mathcal{F}}(y)$. Hence by condition (I) for the saturated fusion system $C_{\mathcal{F}}(y)$, together with (1) and (2), P fully centralized in $\mathcal{F}$ and $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$.

It remains to prove condition (II) for $\mathcal{F}$. Fix $1 \neq P \subseteq S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$ such that $P' \overset{\mathrm{def}}{=} \varphi P$ is fully centralized in $\mathcal{F}$, and set

$$N_\varphi = \left\{ g \in N_S(P) \mid \varphi c_g \varphi^{-1} \in \mathrm{Aut}_S(P') \right\}.$$

We must show that φ extends to some $\bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, S)$. Choose some $x' \in Z(P')$ of order p which is fixed under the action of $\mathrm{Aut}_S(P')$, and set $x = \varphi^{-1}(x') \in Z(P)$. For all $g \in N_\varphi$, $\varphi c_g \varphi^{-1} \in \mathrm{Aut}_S(P')$ fixes x', and hence $c_g(x) = x$. Thus

$$x \in Z(N_\varphi) \text{ and hence } N_\varphi \subseteq C_S(x); \qquad \text{and} \qquad N_S(P') \subseteq C_S(x'). \tag{3}$$

Fix $y \in \mathfrak{X}$ which is $\mathcal{F}$–conjugate to x and x', and choose

$$\alpha \in \mathrm{Hom}_{\mathcal{F}}(C_S(x), C_S(y)) \qquad \text{and} \qquad \alpha' \in \mathrm{Hom}_{\mathcal{F}}(C_S(x'), C_S(y))$$

such that $\alpha(x) = \alpha'(x') = y$. Set $Q = \alpha(P)$ and $Q' = \alpha'(P')$. Since P' is fully centralized in $\mathcal{F}$, $\alpha'(P') = Q'$, and $C_S(P') \subseteq C_S(x')$, we have

$$\alpha'(C_{C_S(x')}(P')) = \alpha'(C_S(P')) = C_S(Q') = C_{C_S(y)}(Q'). \tag{4}$$

Set $\psi = \alpha' \circ \varphi \circ \alpha^{-1} \in \mathrm{Iso}_{\mathcal{F}}(Q,Q')$. By construction, $\psi(y) = y$, and thus $\psi \in \mathrm{Iso}_{C_{\mathcal{F}}(y)}(Q,Q')$. Since P' is fully centralized in $\mathcal{F}$, (4) implies that Q' is fully centralized in $C_{\mathcal{F}}(y)$. Hence condition (II), when applied to the saturated fusion system $C_{\mathcal{F}}(y)$, shows that ψ extends to a homomorphism $\bar{\psi} \in \mathrm{Hom}_{C_{\mathcal{F}}(y)}(N_\psi, C_S(y))$, where

$$N_\psi = \left\{ g \in N_{C_S(y)}(Q) \mid \psi c_g \psi^{-1} \in \mathrm{Aut}_{C_S(y)}(Q') \right\}.$$

Also, for all $g \in N_\varphi \subseteq C_S(x)$ (see (3)),

$$c_{\psi(\alpha(g))} = \psi c_{\alpha(g)} \psi^{-1} = (\psi\alpha) c_g (\psi\alpha)^{-1} = (\alpha'\varphi) c_g (\alpha'\varphi)^{-1} = c_{\alpha'(h)} \in \mathrm{Aut}_{C_S(y)}(Q')$$

for some $h \in N_S(P')$ such that $\varphi c_g \varphi^{-1} = c_h$. This shows that $\alpha(N_\varphi) \subseteq N_\psi$; and also (since $C_S(Q') = \alpha'(C_S(P'))$ by (4)) that

$$\bar{\psi}(\alpha(N_\varphi)) \subseteq \alpha'(N_{C_S(x')}(P')).$$

We can now define

$$\bar{\varphi} \overset{\mathrm{def}}{=} (\alpha')^{-1} \circ (\bar{\psi} \circ \alpha)|_{N_\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, S);$$

and $\bar{\varphi}|_P = \varphi$. □

Proposition 1.1 will also be applied in a separate paper of Carles Broto and Jesper Møller [7] to give a construction of some "exotic" p–local finite groups at certain odd primes.

Our goal now is to construct certain saturated fusion systems, by starting with the fusion system of $\mathrm{Spin}_7(q)$ for some odd prime power q, and then adding to that the automorphisms of some subgroup of $\mathrm{Spin}_7(q)$. This is a special case of the general problem of studying fusion systems generated by fusion subsystems, and then showing that they are saturated. We first fix some notation. If $\mathcal{F}_1$ and $\mathcal{F}_2$ are two fusion systems over the same p–group S, then $\langle \mathcal{F}_1, \mathcal{F}_2 \rangle$ denotes the fusion system over S generated by $\mathcal{F}_1$ and $\mathcal{F}_2$: the smallest fusion system over S which contains both $\mathcal{F}_1$ and $\mathcal{F}_2$. More generally, if $\mathcal{F}$ is a fusion system over S, and $\mathcal{F}_0$ is a fusion system over a subgroup $S_0 \subseteq S$, then $\langle \mathcal{F}; \mathcal{F}_0 \rangle$ denotes the fusion system over S generated by the morphisms in $\mathcal{F}$ between subgroups of S, together with morphisms in $\mathcal{F}_0$ between subgroups of S_0 only. In other words, a morphism in $\langle \mathcal{F}; \mathcal{F}_0 \rangle$ is a composite

$$P_0 \xrightarrow{\varphi_1} P_1 \xrightarrow{\varphi_2} P_2 \longrightarrow \cdots \longrightarrow P_{k-1} \xrightarrow{\varphi_k} P_k,$$

where for each i, either $\varphi_i \in \mathrm{Hom}_{\mathcal{F}}(P_{i-1}, P_i)$, or $\varphi_i \in \mathrm{Hom}_{\mathcal{F}_0}(P_{i-1}, P_i)$ (and $P_{i-1}, P_i \subseteq S_0$).

As usual, when G is a finite group and $S \in \mathrm{Syl}_p(G)$, then $\mathcal{F}_S(G)$ denotes the fusion system of G over S. If $\Gamma \subseteq \mathrm{Aut}(G)$ is a group of automorphisms which contains $\mathrm{Inn}(G)$, then $\mathcal{F}_S(\Gamma)$ will denote the fusion system over S whose morphisms consist of all restrictions of automorphisms in Γ to monomorphisms between subgroups of S.

The next proposition provides some fairly specialized conditions which imply that the fusion system generated by the fusion system of a group G together with certain automorphisms of a subgroup of G is saturated.

Proposition 1.2 *Fix a finite group G, a prime p dividing $|G|$, and a Sylow p–subgroup $S \in \mathrm{Syl}_p(G)$. Fix a normal subgroup $Z \lhd G$ of order p, an elementary abelian subgroup $U \lhd S$ of rank two containing Z such that $C_S(U) \in \mathrm{Syl}_p(C_G(U))$, and a subgroup $\Gamma \subseteq \mathrm{Aut}(C_G(U))$ containing $\mathrm{Inn}(C_G(U))$ such that $\gamma(U) = U$ for all $\gamma \in \Gamma$. Set*

$$S_0 = C_S(U) \qquad \text{and} \qquad \mathcal{F} \stackrel{\mathrm{def}}{=} \langle \mathcal{F}_S(G); \mathcal{F}_{S_0}(\Gamma) \rangle,$$

and assume the following hold.

(a) *All subgroups of order p in S different from Z are G–conjugate.*

(b) *Γ permutes transitively the subgroups of order p in U.*

(c) $\{\varphi \in \Gamma \mid \varphi(Z) = Z\} = \mathrm{Aut}_{N_G(U)}(C_G(U))$.

(d) *For each $E \le S$ which is elementary abelian of rank three, contains U, and is fully centralized in $\mathcal{F}_S(G)$,*

$$\{\alpha \in \mathrm{Aut}_{\mathcal{F}}(C_S(E)) \mid \alpha(Z) = Z\} = \mathrm{Aut}_G(C_S(E)).$$

(e) *For all $E, E' \le S$ which are elementary abelian of rank three and contain U, if E and E' are Γ–conjugate, then they are G–conjugate.*

Then $\mathcal{F}$ is a saturated fusion system over S. Also, for any $P \le S$ such that $Z \le P$,

$$\{\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S) \mid \varphi(Z) = Z\} = \mathrm{Hom}_G(P, S). \qquad (1)$$

Proposition 1.2 follows from the following three lemmas. Throughout the proofs of these lemmas, references to points (a–e) mean to those points in the hypotheses of the proposition, unless otherwise stated.

Lemma 1.3 *Under the hypotheses of Proposition 1.2, for any $P \le S$ and any central subgroup $Z' \le Z(P)$ of order p,*

$$Z \ne Z' \le U \implies \exists\, \varphi \in \mathrm{Hom}_\Gamma(P, S_0) \text{ such that } \varphi(Z') = Z \qquad (1)$$

and

$$Z' \not\le U \implies \exists\, \alpha \in \mathrm{Hom}_G(P, S_0) \text{ such that } \alpha(Z') \le U. \qquad (2)$$

Proof Note first that $Z \le Z(S)$, since it is a normal subgroup of order p in a p–group.

Assume $Z \ne Z' \le U$. Then $U = ZZ'$, and

$$P \le C_S(Z') = C_S(ZZ') = C_S(U) = S_0$$

since $Z' \le Z(P)$ by assumption. By (b), there is $\gamma \in \Gamma$ such that $\gamma(Z') = Z$. Since $S_0 \in \mathrm{Syl}_p(C_G(U))$, there is $h \in C_G(U)$ such that $h\gamma(P)h^{-1} \le S_0$; and since

$$c_h \in \mathrm{Aut}_{N_G(U)}(C_G(U)) \subseteq \Gamma$$

by (c), $\varphi \overset{\text{def}}{=} c_h \circ \gamma \in \mathrm{Hom}_\Gamma(P, S_0)$ and sends Z' to Z.

If $Z' \not\le U$, then by (a), there is $g \in G$ such that $gZ'g^{-1} \subseteq U \smallsetminus Z$. Since Z is central in S, $gZ'g^{-1}$ is central in gPg^{-1}, and U is generated by Z and $gZ'g^{-1}$, it follows that $gPg^{-1} \le C_G(U)$. Since $S_0 \in \mathrm{Syl}_p(C_G(U))$, there is $h \in C_G(U)$ such that $h(gPg^{-1})h^{-1} \le S_0$; and we can take $\alpha = c_{hg} \in \mathrm{Hom}_G(P, S_0)$. □

We are now ready to prove point (1) in Proposition 1.2.

Lemma 1.4 *Assume the hypotheses of Proposition 1.2, and let*

$$\mathcal{F} = \langle \mathcal{F}_S(G), \mathcal{F}_{S_0}(\Gamma) \rangle$$

be the fusion system generated by G and Γ. Then for all $P, P' \le S$ which contain Z,

$$\{\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,P') \mid \varphi(Z) = Z\} = \mathrm{Hom}_G(P,P').$$

Proof Upon replacing P' by $\varphi(P) \le P'$, we can assume that φ is an isomorphism, and thus that it factors as a composite of isomorphisms

$$P = P_0 \xrightarrow[\cong]{\varphi_1} P_1 \xrightarrow[\cong]{\varphi_2} P_2 \xrightarrow[\cong]{\varphi_3} \cdots \xrightarrow[\cong]{\varphi_{k-1}} P_{k-1} \xrightarrow[\cong]{\varphi_k} P_k = P',$$

where for each i, $\varphi_i \in \mathrm{Hom}_G(P_{i-1},P_i)$ or $\varphi_i \in \mathrm{Hom}_\Gamma(P_{i-1},P_i)$. Let $Z_i \le Z(P_i)$ be the subgroups of order p such that $Z_0 = Z_k = Z$ and $Z_i = \varphi_i(Z_{i-1})$.

To simplify the discussion, we say that a morphism in $\mathcal{F}$ is of type (G) if it is given by conjugation by an element of G, and of type (Γ) if it is the restriction of an automorphism in Γ. More generally, we say that a morphism is of type (G,Γ) if it is the composite of a morphism of type (G) followed by one of type (Γ), etc. We regard Id_P, for all $P \le S$, to be of both types, even if $P \not\le S_0$. By definition, if any nonidentity isomorphism is of type (Γ), then its source and image are both contained in $S_0 = C_S(U)$.

For each i, using Lemma 1.3, choose some $\alpha_i \in \mathrm{Hom}_{\mathcal{F}}(P_iU,S)$ such that $\alpha_i(Z_i) = Z$. More precisely, using points (1) and (2) in Lemma 1.3, we can choose α_i to be of type (Γ) if $Z_i \le U$ (the inclusion if $Z_i = Z$), and to be of type (G,Γ) if $Z \not\le U$. Set $P_i' = \alpha_i(P_i)$. To keep track of the effect of morphisms on the subgroups Z_i, we write them as morphisms between pairs, as shown below. Thus, φ factors as a composite of isomorphisms

$$(P'_{i-1},Z) \xrightarrow{\alpha_{i-1}^{-1}} (P_{i-1},Z_{i-1}) \xrightarrow{\varphi_i} (P_i,Z_i) \xrightarrow{\alpha_i} (P'_i,Z).$$

If φ_i is of type (G), then this composite (after replacing adjacent morphisms of the same type by their composite) is of type (Γ,G,Γ). If φ_i is of type (Γ), then the composite is again of type (Γ,G,Γ) if either $Z_{i-1} \le U$ or $Z_i \le U$, and is of type $(\Gamma,G,\Gamma,G,\Gamma)$ if neither Z_{i-1} nor Z_i is contained in U. So we are reduced to assuming that φ is of one of these two forms.

Case 1 Assume first that φ is of type (Γ,G,Γ); ie, a composite of isomorphisms of the form

$$(P_0,Z) \xrightarrow[(\Gamma)]{\varphi_1} (P_1,Z_1) \xrightarrow[(G)]{\varphi_2} (P_2,Z_2) \xrightarrow[(\Gamma)]{\varphi_3} (P_3,Z).$$

Then $Z_1 = Z$ if and only if $Z_2 = Z$ because φ_2 is of type (G). If $Z_1 = Z_2 = Z$, then φ_1 and φ_3 are of type (G) by (c), and the result follows.

If $Z_1 \neq Z \neq Z_2$, then $U = ZZ_1 = ZZ_2$, and thus $\varphi_2(U) = U$. Neither φ_1 nor φ_3 can be the identity, so $P_i \leq S_0 = C_S(U)$ for all i by definition of $\mathrm{Hom}_\Gamma(-,-)$, and hence φ_2 is of type (Γ) by (c). It follows that $\varphi \in \mathrm{Iso}_\Gamma(P_0,P_3)$ sends Z to itself, and is of type (G) by (c) again.

Case 2 Assume now that φ is of type $(\Gamma,G,\Gamma,G,\Gamma)$; more precisely, that it is a composite of the form

$$(P_0,Z) \xrightarrow[(\Gamma)]{\varphi_1} (P_1,Z_1) \xrightarrow[(G)]{\varphi_2} (P_2,Z_2) \xrightarrow[(\Gamma)]{\varphi_3} (P_3,Z_3) \xrightarrow[(G)]{\varphi_4} (P_4,Z_4) \xrightarrow[(\Gamma)]{\varphi_5} (P_5,Z),$$

where $Z_2,Z_3 \not\leq U$. Then $Z_1,Z_4 \leq U$ and are distinct from Z, and the groups P_0,P_1,P_4,P_5 all contain U since φ_1 and φ_5 (being of type (Γ)) leave U invariant. In particular, P_2 and P_3 contain Z, since P_1 and P_4 do and φ_2,φ_4 are of type (G). We can also assume that $U \leq P_2,P_3$, since otherwise $P_2 \cap U = Z$ or $P_3 \cap U = Z$, $\varphi_3(Z) = Z$, and hence φ_3 is of type (G) by (c) again. Finally, we assume that $P_2,P_3 \leq S_0 = C_S(U)$, since otherwise $\varphi_3 = \mathrm{Id}$.

Let $E_i \leq P_i$ be the rank three elementary abelian subgroups defined by the requirements that $E_2 = UZ_2$, $E_3 = UZ_3$, and $\varphi_i(E_{i-1}) = E_i$. In particular, $E_i \leq Z(P_i)$ for $i = 2,3$ (since $Z_i \leq Z(P_i)$, and $U \leq Z(P_i)$ by the above remarks); and hence $E_i \leq Z(P_i)$ for all i. Also, $U = ZZ_4 \leq \varphi_4(E_3) = E_4$ since $\varphi_4(Z) = Z$, and thus $U = \varphi_5(U) \leq E_5$. Via similar considerations for E_0 and E_1, we see that $U \leq E_i$ for all i.

Set $H = C_G(U)$ for short. Let $\mathcal{E}_3$ be the set of all elementary abelian subgroups $E \leq S$ of rank three which contain U, and with the property that $C_S(E) \in \mathrm{Syl}_p(C_H(E))$. Since $C_S(E) \leq C_S(U) = S_0 \in \mathrm{Syl}_p(H)$, the last condition implies that E is fully centralized in the fusion system $\mathcal{F}_{S_0}(H)$. If $E \leq S$ is any rank three elementary abelian subgroup which contains U, then there is some $a \in H$ such that $E' = aEa^{-1} \in \mathcal{E}_3$, since $\mathcal{F}_{S_0}(H)$ is saturated and $U \lhd H$. Then $c_a \in \mathrm{Iso}_G(E,E') \cap \mathrm{Iso}_\Gamma(E,E')$ by (c). So upon composing with such isomorphisms, we can assume that $E_i \in \mathcal{E}_3$ for all i, and also that $\varphi_i(C_S(E_{i-1})) = C_S(E_i)$ for each i.

In this way, φ can be assumed to extend to an $\mathcal{F}$–isomorphism $\bar{\varphi}$ from $C_S(E_0)$ to $C_S(E_5)$ which sends Z to itself. By (e), the rank three subgroups E_i are all G–conjugate to each other. Choose $g \in G$ such that $gE_5g^{-1} = E_0$. Then $g\,C_S(E_5)\,g^{-1}$ and $C_S(E_0)$ are both Sylow p–subgroups of $C_G(E_0)$, so there is $h \in C_G(E_0)$ such that $(hg)C_S(E_5)(hg)^{-1} = C_S(E_0)$. By (d), $c_{hg} \circ \bar{\varphi} \in \mathrm{Aut}_{\mathcal{F}}(C_S(E_0))$ is of type (G); and thus $\varphi \in \mathrm{Iso}_G(P_0,P_5)$. □

To finish the proof of Proposition 1.2, it remains only to show:

Lemma 1.5 *Under the hypotheses of Proposition 1.2, the fusion system $\mathcal{F}$ generated by $\mathcal{F}_S(G)$ and $\mathcal{F}_{S_0}(\Gamma)$ is saturated.*

Proof We apply Proposition 1.1, by letting $\mathfrak{X}$ be the set of generators of Z. Condition (a) of the proposition (every $x \in S$ of order p is $\mathcal{F}$–conjugate to an element of $\mathfrak{X}$) holds by Lemma 1.3. Condition (c) holds since $C_{\mathcal{F}}(Z)$ is the fusion system of the group $C_G(Z)$ by Lemma 1.4, and hence is saturated by [6, Proposition 1.3].

It remains to prove condition (b) of Proposition 1.1. We must show that if $y, z \in S$ are $\mathcal{F}$–conjugate and $\langle z \rangle = Z$, then there is $\psi \in \mathrm{Hom}_{\mathcal{F}}(C_S(y), C_S(z))$ such that $\psi(y) = z$. If $y \notin U$, then by Lemma 1.3(2), there is $\varphi \in \mathrm{Hom}_{\mathcal{F}}(C_S(y), S_0)$ such that $\varphi(y) \in U$. If $y \in U \smallsetminus Z$, then by Lemma 1.3(1), there is $\varphi \in \mathrm{Hom}_{\mathcal{F}}(C_S(y), S_0)$ such that $\varphi(y) \in Z$. We are thus reduced to the case where $y, z \in Z$ (and are $\mathcal{F}$–conjugate).

In this case, then by Lemma 1.4, there is $g \in G$ such that $z = gyg^{-1}$. Since $Z \lhd G$, $[G{:}C_G(Z)]$ is prime to p, so S and gSg^{-1} are both Sylow p–subgroups of $C_G(Z)$, and hence are $C_G(Z)$–conjugate. We can thus choose g such that $z = gyg^{-1}$ and $gSg^{-1} = S$. Since $C_S(y) = C_S(z) = S$ ($Z \le Z(S)$ since it is a normal subgroup of order p), this shows that $c_g \in \mathrm{Iso}_G(C_S(y), C_S(z))$, and finishes the proof of (b) in Proposition 1.1. □

2 A fusion system of a type considered by Solomon

The main result of this section and the next is the following theorem:

Theorem 2.1 *Let q be an odd prime power, and fix $S \in \mathrm{Syl}_2(\mathrm{Spin}_7(q))$. Let $z \in Z(\mathrm{Spin}_7(q))$ be the central element of order 2. Then there is a saturated fusion system $\mathcal{F} = \mathcal{F}_{\mathrm{Sol}}(q)$ which satisfies the following conditions:*

(a) $C_{\mathcal{F}}(z) = \mathcal{F}_S(\mathrm{Spin}_7(q))$ *as fusion systems over S.*

(b) *All involutions of S are $\mathcal{F}$–conjugate.*

Furthermore, there is a unique centric linking system $\mathcal{L} = \mathcal{L}^c_{\mathrm{Sol}}(q)$ associated to $\mathcal{F}$.

Theorem 2.1 will be proven in Propositions 2.11 and 3.3. Later, at the end of Section 3, we explain why Solomon's theorem [22] implies that these fusion systems are not the fusion systems of any finite groups, and hence that the spaces $B\mathrm{Sol}(q)$ are not homotopy equivalent to the 2–completed classifying spaces of any finite groups.

Background results needed for computations in $\mathrm{Spin}(V,\mathfrak{b})$ have been collected in Appendix A. We focus attention here on $SO_7(q)$ and $\mathrm{Spin}_7(q)$. In fact, since we want to compare the constructions over $\mathbb{F}_q$ with those over its field extensions, most of the constructions will first be made in the groups $SO_7(\bar{\mathbb{F}}_q)$ and $\mathrm{Spin}_7(\bar{\mathbb{F}}_q)$.

We now fix, for the rest of the section, an odd prime power q. It will be convenient to write $\mathrm{Spin}_7(q^\infty) \overset{\text{def}}{=} \mathrm{Spin}_7(\bar{\mathbb{F}}_q)$, etc. In order to make certain computations more explicit, we set

$$V_\infty = M_2(\bar{\mathbb{F}}_q) \oplus M_2^0(\bar{\mathbb{F}}_q) \cong (\bar{\mathbb{F}}_q)^7 \qquad \text{and} \qquad \mathfrak{b}(A,B) = \det(A) + \det(B)$$

(where $M_2^0(-)$ is the group of (2×2) matrices of trace zero), and for each $n \geq 1$ set $V_n = M_2(\mathbb{F}_{q^n}) \oplus M_2^0(\mathbb{F}_{q^n}) \subseteq V_\infty$. Then $\mathfrak{b}$ is a nonsingular quadratic form on V_∞ and on V_n. Identify $SO_7(q^\infty) = SO(V_\infty,\mathfrak{b})$ and $SO_7(q^n) = SO(V_n,\mathfrak{b})$, and similarly for $\mathrm{Spin}_7(q^n) \subseteq \mathrm{Spin}_7(q^\infty)$. For all $\alpha \in \mathrm{Spin}(M_2(\bar{\mathbb{F}}_q),\det)$ and $\beta \in \mathrm{Spin}(M_2^0(\bar{\mathbb{F}}_q),\det)$, we write $\alpha\oplus\beta$ for their image in $\mathrm{Spin}_7(q^\infty)$ under the natural homomorphism

$$\lambda_{4,3}\colon \mathrm{Spin}_4(q^\infty) \times \mathrm{Spin}_3(q^\infty) \longrightarrow \mathrm{Spin}_7(q^\infty).$$

There are isomorphisms

$$\tilde{e}_4\colon SL_2(q^\infty) \times SL_2(q^\infty) \xrightarrow{\cong} \mathrm{Spin}_4(q^\infty) \quad \text{and} \quad \tilde{e}_3\colon SL_2(q^\infty) \xrightarrow{\cong} \mathrm{Spin}_3(q^\infty)$$

which are defined explicitly in Proposition A.5, and which restrict to isomorphisms

$$SL_2(q^n) \times SL_2(q^n) \cong \mathrm{Spin}_4(q^n) \qquad \text{and} \qquad SL_2(q^n) \cong \mathrm{Spin}_3(q^n)$$

for each n. Let

$$z = \tilde{e}_4(-I,-I) \oplus 1 = 1 \oplus \tilde{e}_3(-I) \in Z(\mathrm{Spin}_7(q))$$

denote the central element of order two, and set

$$z_1 = \tilde{e}_4(-I,I) \oplus 1 \in \mathrm{Spin}_7(q).$$

Here, $1 \in \mathrm{Spin}_k(q)$ $(k = 3,4)$ denotes the identity element. Define $U = \langle z, z_1\rangle$.

Definition 2.2 Define

$$\omega\colon SL_2(q^\infty)^3 \longrightarrow \mathrm{Spin}_7(q^\infty)$$

by setting

$$\omega(A_1,A_2,A_3) = \widetilde{\rho}_4(A_1,A_2)\cdot\widetilde{\rho}_3(A_3)$$

for $A_1,A_2,A_3 \in SL_2(q^\infty)$. Set

$$H(q^\infty) = \omega(SL_2(q^\infty)^3) \qquad \text{and} \qquad [\![A_1,A_2,A_3]\!] = \omega(A_1,A_2,A_3).$$

Since $\widetilde{\rho}_3$ and $\widetilde{\rho}_4$ are isomorphisms, $\mathrm{Ker}(\omega) = \mathrm{Ker}(\rho_{4,3})$, and thus

$$\mathrm{Ker}(\omega) = \langle(-I,-I,-I)\rangle.$$

In particular, $H(q^\infty) = (SL_2(q^\infty)^3)/\{\pm(I,I,I)\}$. Also,

$$z = [\![I,I,-I]\!] \qquad \text{and} \qquad z_1 = [\![-I,I,I]\!],$$

and thus

$$U = \bigl\{[\![\pm I,\pm I,\pm I]\!]\bigr\}$$

(with all combinations of signs).

For each $1 \le n < \infty$, the natural homomorphism

$$\mathrm{Spin}_7(q^n) \longrightarrow SO_7(q^n)$$

has kernel and cokernel both of order 2. The image of this homomorphism is the commutator subgroup $\Omega_7(q^n) \lhd SO_7(q^n)$, which is partly described by Lemma A.4(a). In contrast, since all elements of $\bar{\mathbb{F}}_q$ are squares, the natural homomorphism from $\mathrm{Spin}_7(q^\infty)$ to $SO_7(q^\infty)$ is surjective.

Lemma 2.3 *There is an element $\tau \in N_{\mathrm{Spin}_7(q)}(U)$ of order 2 such that*

$$\tau[\![A_1,A_2,A_3]\!]\tau^{-1} = [\![A_2,A_1,A_3]\!] \tag{1}$$

for all $A_1,A_2,A_3 \in SL_2(q^\infty)$.

Proof Let $\beta \in SO_7(q)$ be the involution defined by setting

$$\beta(X,Y) = (-\psi(X),-Y)$$

for $(X,Y) \in V_\infty = M_2(\bar{\mathbb{F}}_q) \oplus M_2^0(\bar{\mathbb{F}}_q)$, where

$$\psi\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a\end{pmatrix}.$$

Let $\widehat{\beta} \in \mathrm{Spin}_7(q^\infty)$ be a lifting of β. The (-1)–eigenspace of β on V_∞ has orthogonal basis

$$(I,0),\ \left(0,\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\right),\ \left(0,\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\right),\ \left(0,\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\right),$$

and in particular has discriminant 1 with respect to this basis. Hence by Lemma A.4(a), $\bar{\tau} \in \Omega_7(q)$, and so $\tau \in \mathrm{Spin}_7(q)$. Since in addition, the (-1)–eigenspace of $\bar{\tau}$ is 4–dimensional, Lemma A.4(b) applies to show that $\tau^2 = 1$.

By definition of the isomorphisms $\tilde{\omega}_3$ and $\tilde{\omega}_4$, for all $A_i \in SL_2(q^\infty)$ $(i = 1, 2, 3)$ and all $(X, Y) \in V_\infty$,

$$[\![A_1, A_2, A_3]\!](X, Y) = (A_1 X A_2^{-1}, A_3 Y A_3^{-1}).$$

Here, $\mathrm{Spin}_7(q^\infty)$ acts on V_∞ via its projection to $SO_7(q^\infty)$. Also, for all $X, Y \in M_2(\bar{\mathbb{F}}_q)$,

$$\psi(X) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} X^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{-1} \quad \text{and in particular} \quad \psi(XY) = \psi(Y)\psi(X);$$

and $\psi(X) = X^{-1}$ if $\det(X) = 1$. Hence for all $A_1, A_2, A_3 \in SL_2(q^\infty)$ and all $(X, Y) \in V_\infty$,

$$\begin{aligned} \left(\tau [\![A_1, A_2, A_3]\!] \tau^{-1}\right)(X, Y) &= \tau(-A_1 \psi(X) A_2^{-1}, -A_3 Y A_3^{-1}) \\ &= (A_2 X A_1^{-1}, A_3 Y A_3^{-1}) = [\![A_2, A_1, A_3]\!](X, Y). \end{aligned}$$

This shows that (1) holds modulo $\langle z \rangle = Z(\mathrm{Spin}_7(q^\infty))$. We thus have two automorphisms of $H(q^\infty) = (SL_2(q^\infty)^3)/\{\pm(I, I, I)\}$ — conjugation by τ and the permutation automorphism — which are liftings of the same automorphism of $H(q^\infty)/\langle z \rangle$. Since $H(q^\infty)$ is perfect, each automorphism of $H(q^\infty)/\langle z \rangle$ has at most one lifting to an automorphism of $H(q^\infty)$, and thus (1) holds. Also, since U is the subgroup of all elements $[\![\pm I, \pm I, \pm I]\!]$ with all combinations of signs, formula (1) shows that $\tau \in N_{\mathrm{Spin}_7(q)}(U)$. □

Definition 2.4 For each $n \geq 1$, set

$$H(q^n) = H(q^\infty) \cap \mathrm{Spin}_7(q^n) \quad \text{and} \quad H_0(q^n) = \omega(SL_2(q^n)^3) \subseteq H(q^n).$$

Define

$$\Gamma_n = \mathrm{Inn}(H(q^n)) \rtimes \widehat{\Sigma}_3 \subseteq \mathrm{Aut}(H(q^n)),$$

where $\widehat{\Sigma}_3$ denotes the group of permutation automorphisms

$$\widehat{\Sigma}_3 = \left\{ [\![A_1, A_2, A_3]\!] \mapsto [\![A_{\sigma 1}, A_{\sigma 2}, A_{\sigma 3}]\!] \;\middle|\; \sigma \in \Sigma_3 \right\} \subseteq \mathrm{Aut}(H(q^n)).$$

For each n, let ψ^{q^n} be the automorphism of $\mathrm{Spin}_7(q^\infty)$ induced by the field isomorphism $(q \mapsto q^{p^n})$. By Lemma A.3, $\mathrm{Spin}_7(q^n)$ is the fixed subgroup of ψ^{q^n}. Hence each element of $H(q^n)$ is of the form $[\![A_1, A_2, A_3]\!]$, where either $A_i \in SL_2(q^n)$ for each i (and the element lies in $H_0(q^n)$), or $\psi^{q^n}(A_i) = -A_i$ for each i. This shows that $H_0(q^n)$ has index 2 in $H(q^n)$.

The goal is now to choose compatible Sylow subgroups $S(q^n) \in \mathrm{Syl}_2(\mathrm{Spin}_7(q^n))$ (all $n \geq 1$) contained in $N(H(q^n))$, and let $\mathcal{F}_{\mathrm{Sol}}(q^n)$ be the fusion system over $S(q^n)$ generated by conjugation in $\mathrm{Spin}_7(q^n)$ and by restrictions of Γ_n.

Proposition 2.5 *The following hold for each $n \geq 1$.*

(a) $H(q^n) = C_{\mathrm{Spin}_7(q^n)}(U)$.

(b) $N_{\mathrm{Spin}_7(q^n)}(U) = N_{\mathrm{Spin}_7(q^n)}(H(q^n)) = H(q^n)\langle\tau\rangle$, *and contains a Sylow 2–subgroup of* $\mathrm{Spin}_7(q^n)$.

Proof Let $\bar{z}_1 \in SO_7(q)$ be the image of $z_1 \in \mathrm{Spin}_7(q)$. Set $V_- = M_2(\bar{\mathbb{F}}_q)$ and $V_+ = M_2^0(\bar{\mathbb{F}}_q)$: the eigenspaces of $\bar{z}_1$ acting on V. By Lemma A.4(c),

$$C_{\mathrm{Spin}_7(q^\infty)}(U) = C_{\mathrm{Spin}_7(q^\infty)}(z_1)$$

is the group of all elements $\alpha \in \mathrm{Spin}_7(q^\infty)$ whose image $\bar{\alpha} \in SO_7(q^\infty)$ has the form

$$\bar{\alpha} = \alpha_- \oplus \alpha_+ \quad \text{where} \quad \alpha_\pm \in SO(V_\pm).$$

In other words,

$$C_{\mathrm{Spin}_7(q^\infty)}(U) = \rho_{4,3}\big(\mathrm{Spin}_4(q^\infty) \times \mathrm{Spin}_3(q^\infty)\big) = \omega(SL_2(q^\infty)^3) = H(q^\infty).$$

Furthermore, since

$$\tau z_1 \tau^{-1} = \tau [\![-I, I, I]\!] \tau^{-1} = [\![I, -I, I]\!] = z z_1$$

by Lemma 2.3, and since any element of $N_{\mathrm{Spin}_7(q^\infty)}(U)$ centralizes z, conjugation by τ generates $\mathrm{Out}_{\mathrm{Spin}_7(q^\infty)}(U)$. Hence

$$N_{\mathrm{Spin}_7(q^\infty)}(U) = H(q^\infty)\langle\tau\rangle.$$

Point (a), and the first part of point (b), now follow upon taking intersections with $\mathrm{Spin}_7(q^n)$.

If $N_{\mathrm{Spin}_7(q^n)}(U)$ did not contain a Sylow 2–subgroup of $\mathrm{Spin}_7(q^n)$, then since every noncentral involution of $\mathrm{Spin}_7(q^n)$ is conjugate to z_1 (Proposition A.8), the Sylow 2–subgroups of $\mathrm{Spin}_7(q)$ would have no normal subgroup isomorphic to C_2^2. By a theorem of Hall (cf [15, Theorem 5.4.10]), this would imply that they are cyclic, dihedral, quaternion, or semidihedral. This is clearly not the case, so $N_{\mathrm{Spin}_7(q^n)}(U)$ must contain a Sylow 2–subgroup of $\mathrm{Spin}_7(q)$, and this finishes the proof of point (b).

Alternatively, point (b) follows from the standard formulas for the orders of these groups (cf [24, pages 19,140]), which show that

$$\frac{|\mathrm{Spin}_7(q^n)|}{|H(q^n)\langle\tau\rangle|} = \frac{q^{9n}(q^{6n}-1)(q^{4n}-1)(q^{2n}-1)}{2\,[q^n(q^{2n}-1)]^3} = q^{6n}(q^{4n}+q^{2n}+1)\cdot\frac{q^{2n}+1}{2}$$

is odd. □

We next fix, for each n, a Sylow 2–subgroup of $\mathrm{Spin}_7(q^n)$ which is contained in $H(q^n)\langle\tau\rangle = N_{\mathrm{Spin}_7(q^n)}(U)$.

Definition 2.6 Fix elements $A, B \in SL_2(q)$ such that $\langle A, B\rangle = Q_8$ (a quaternion group of order 8), and set $\widehat{A} = [\![A, A, A]\!]$ and $\widehat{B} = [\![B, B, B]\!]$. Let $C(q^\infty) \le C_{SL_2(q^\infty)}(A)$ be the subgroup of elements of 2–power order in the centralizer (which is abelian), and set $Q(q^\infty) = \langle C(q^\infty), B\rangle$. Define

$$S_0(q^\infty) = \omega(Q(q^\infty)^3) \le H_0(q^\infty)$$

and

$$S(q^\infty) = S_0(q^\infty)\langle\tau\rangle \le H(q^\infty) \le \mathrm{Spin}_7(q^\infty).$$

Here, $\tau \in \mathrm{Spin}_7(q)$ is the element of Lemma 2.3. Finally, for each $n \ge 1$, define

$$C(q^n) = C(q^\infty) \cap SL_2(q^n); \qquad Q(q^n) = Q(q^\infty) \cap SL_2(q^n);$$
$$S_0(q^n) = S_0(q^\infty) \cap \mathrm{Spin}_7(q^n); \qquad \text{and} \qquad S(q^n) = S(q^\infty) \cap \mathrm{Spin}_7(q^n).$$

Since the two eigenvalues of A are distinct, its centralizer in $SL_2(q^\infty)$ is conjugate to the subgroup of diagonal matrices, which is abelian. Thus $C(q^\infty)$ is conjugate to the subgroup of diagonal matrices of 2–power order. This shows that each finite subgroup of $C(q^\infty)$ is cyclic, and that each finite subgroup of $Q(q^\infty)$ is cyclic or quaternion.

Lemma 2.7 *For all n, $S(q^n) \in \mathrm{Syl}_2(\mathrm{Spin}_7(q^n))$.*

Proof By [23, 6.23], A is contained in a cyclic subgroup of order $q^n - 1$ or $q^n + 1$ (depending on which of them is divisible by 4). Also, the normalizer of this cyclic subgroup is a quaternion group of order $2(q^n \pm 1)$, and the formula $|SL_2(q^n)| = q^n(q^{2n} - 1)$ shows that this quaternion group has odd index. Thus by construction, $Q(q^n)$ is a Sylow 2–subgroup of $SL_2(q^n)$. Hence $\omega(Q(q^n)^3)$ is a Sylow 2–subgroup of $H_0(q^n)$, so $\omega(Q(q^\infty)^3) \cap \mathrm{Spin}_7(q^n)$ is a Sylow 2–subgroup of $H(q^n)$. It follows that $S(q^n)$ is a Sylow 2–subgroup of $H(q^n)\langle\tau\rangle$, and hence also of $\mathrm{Spin}_7(q^n)$ by Proposition 2.5(b). □

Following the notation of Definition A.7, we say that an elementary abelian 2–subgroup $E \le \mathrm{Spin}_7(q^n)$ has type I if its eigenspaces all have square discriminant, and has type II otherwise. Let $\mathcal{E}_r$ be the set of elementary abelian subgroups of rank r in $\mathrm{Spin}_7(q^n)$ which contain z, and let $\mathcal{E}_r^I$ and $\mathcal{E}_r^{II}$ be the sets of those of type I or II, respectively. In Proposition A.8, we show that there are two conjugacy classes of subgroups in $\mathcal{E}_4^I$ and one conjugacy class of subgroups in $\mathcal{E}_4^{II}$. In Proposition A.9, an invariant $x_C(E) \in E$ is defined, for all $E \in \mathcal{E}_4$ (and where C is one of the conjugacy classes in $\mathcal{E}_4^{I}$) as a tool for determining the conjugacy class of a subgroup. More precisely, E has type I if and only if $x_C(E) \in \langle z \rangle$, and $E \in C$ if and only if $x_C(E) = 1$. The next lemma provides some more detailed information about the rank four subgroups and these invariants.

Recall that we define $\hat{A} = [\![A,A,A]\!]$ and $\hat{B} = [\![B,B,B]\!]$.

Lemma 2.8 *Fix $n \geq 1$, set $E_* = \langle z, z_1, \hat{A}, \hat{B} \rangle \leq S(q^n)$, and let C be the $\mathrm{Spin}_7(q^n)$–conjugacy class of E_*. Let $\mathcal{E}_4^U$ be the set of all elementary abelian subgroups $E \leq S(q^n)$ of rank 4 which contain $U = \langle z, z_1 \rangle$. Fix a generator $X \in C(q^n)$ (the 2–power torsion in $C_{SL_2(q^n)}(A)$), and choose $Y \in C(q^{2n})$ such that $Y^2 = X$. Then the following hold.*

(a) *E_* has type I.*

(b) *$\mathcal{E}_4^U = \{E_{ijk}, E'_{ijk} \mid i,j,k \in \mathbb{Z}\}$ (a finite set), where*

$$E_{ijk} = \langle z, z_1, \hat{A}, [\![X^iB, X^jB, X^kB]\!] \rangle$$

and

$$E'_{ijk} = \langle z, z_1, \hat{A}, [\![X^iYB, X^jYB, X^kYB]\!] \rangle.$$

(c) *$x_C(E_{ijk}) = [\![(-I)^i, (-I)^j, (-I)^k]\!]$ and $x_C(E'_{ijk}) = [\![(-I)^i, (-I)^j, (-I)^k]\!] \cdot \hat{A}$.*

(d) *All of the subgroups E'_{ijk} have type II. The subgroup E_{ijk} has type I if and only if $i \equiv j \pmod 2$, and lies in C (is conjugate to E_*) if and only if $i \equiv j \equiv k \pmod 2$. The subgroups E_{000}, E_{001}, and E_{100} thus represent the three conjugacy classes of rank four elementary abelian subgroups of $\mathrm{Spin}_7(q^n)$ (and $E_* = E_{000}$).*

(e) *For any $\varphi \in \Gamma_n \leq \mathrm{Aut}(H(q^n))$ (see Definition 2.4), if $E', E'' \in \mathcal{E}_4^U$ are such that $\varphi(E') = E''$, then $\varphi(x_C(E')) = x_C(E'')$.*

Proof (a) The set

$$\{(I,0), (A,0), (B,0), (AB,0), (0,A), (0,B), (0,AB)\}$$

is a basis of eigenvectors for the action of E_* on $V_n = M_2(\mathbb{F}_{q^n}) \oplus M_2^0(\mathbb{F}_{q^n})$. (Since the matrices A, B, and AB all have order 4 and determinant one, each has as eigenvalues the two distinct fourth roots of unity, and hence they all have trace zero.) Since all of these have determinant one, E_* has type I by definition.

(b) Consider the subgroups

$$R_0 = \omega(C(q^\infty)^3) \cap S(q^n) = \big\langle [\![X^i, X^j, X^k]\!], [\![X^iY, X^jY, X^kY]\!] \;\big|\; i,j,k \in \mathbb{Z} \big\rangle$$

and

$$R_1 = C_{S(q^n)}(\langle U, \widehat{A}\rangle) = R_0\langle \widehat{B}\rangle.$$

Clearly, each subgroup $E \in \mathcal{E}_4^U$ is contained in

$$C_{S(q^n)}(U) = S_0(q^n) = R_0\langle [\![B^i, B^j, B^k]\!]\rangle.$$

All involutions in this subgroup are contained in $R_1 = R_0\langle [\![B, B, B]\!]\rangle$, and thus $E \subseteq R_1$. Hence $E \cap R_0$ has rank 3, which implies that $E \supseteq \langle z, z_1, \widehat{A}\rangle$ (the 2–torsion in R_0). Since all elements of order two in the coset $R_0\widehat{B}$ have the form

$$[\![X^iB, X^jB, X^kB]\!] \qquad \text{or} \qquad [\![X^iYB, X^jYB, X^kYB]\!]$$

for some i,j,k, this shows that E must be one of the groups E_{ijk} or E'_{ijk}. (Note in particular that $E_* = E_{000}$.)

(c) By Proposition A.9(a), the element $x_C(E) \in \mathcal{E}$ is characterized uniquely by the property that $x_C(E) = g^{-1}\psi^{q^n}(g)$ for some $g \in \mathrm{Spin}_7(q^\infty)$ such that $gEg^{-1} \in \mathcal{C}$. We now apply this explicitly to the subgroups E_{ijk} and E'_{ijk}.

For each i, $Y^{-i}(X^iB)Y^i = Y^{-2i}X^iB = B$. Hence for each i,j,k,

$$[\![Y^i, Y^j, Y^k]\!]^{-1} E_{ijk} [\![Y^i, Y^j, Y^k]\!] = E_*$$

and

$$\psi^{q^n}([\![Y^i, Y^j, Y^k]\!]) = [\![Y^i, Y^j, Y^k]\!]\,[\![(-I)^i, (-I)^j, (-I)^k]\!].$$

Hence

$$x_C(E_{ijk}) = [\![(-I)^i, (-I)^j, (-I)^k]\!].$$

Similarly, if we choose $Z \in C_{SL_2(q^\infty)}(A)$ such that $Z^2 = Y$, then for each i,

$$(Y^iZ)^{-1}(X^iYB)(Y^iZ) = B.$$

Hence for each i,j,k,

$$[\![Y^iZ, Y^jZ, Y^kZ]\!]^{-1} E'_{ijk} [\![Y^iZ, Y^jZ, Y^kZ]\!] = E_*.$$

Since $\psi^{q^n}(Z) = ZA$,

$$\psi^{q^n}([\![Y^iZ, Y^jZ, Y^kZ]\!]) = [\![Y^iZ, Y^jZ, Y^kZ]\!]\,[\![(-I)^iA, (-I)^jA, (-I)^kA]\!],$$

and hence

$$x_C(E'_{ijk}) = [\![(-I)^iA, (-I)^jA, (-I)^kA]\!].$$

(d) This now follows immediately from point (c) and Proposition A.9(b,c).

(e) By Definition 2.4, Γ_n is generated by $\mathrm{Inn}(H(q^n))$ and the permutations of the three factors in $H(q^\infty) = (SL_2(q^\infty))^3/\{\pm(I,I,I)\}$. If $\varphi \in \Gamma_n$ is a permutation automorphism, then it permutes the elements of $\mathcal{E}_4^U$, and preserves the elements $x_C(-)$ by the formulas in (c). If $\varphi \in \mathrm{Inn}(H(q^n))$ and $\varphi(E') = E''$ for $E', E'' \in \mathcal{E}_4^U$, then $\varphi(x_C(E')) = x_C(E'')$ by definition of $x_C(-)$; and so the same property holds for all elements of Γ_n. □

Following the notation introduced in Section 1, $\mathrm{Hom}_{\mathrm{Spin}_7(q^n)}(P,Q)$ (for $P,Q \le S(q^n)$) denotes the set of homomorphisms from P to Q induced by conjugation by some element of $\mathrm{Spin}_7(q^n)$. Also, if $P,Q \le S(q^n) \cap H(q^n)$, $\mathrm{Hom}_{\Gamma_n}(P,Q)$ denotes the set of homomorphisms induced by restriction of an element of Γ_n. Let $\mathcal{F}_n = \mathcal{F}_{\mathrm{Sol}}(q^n)$ be the fusion system over $S(q^n)$ generated by $\mathrm{Spin}_7(q^n)$ and Γ_n. In other words, for each $P,Q \le S(q^n)$, $\mathrm{Hom}_{\mathcal{F}_n}(P,Q)$ is the set of all composites

$$P = P_0 \xrightarrow{\varphi_1} P_1 \xrightarrow{\varphi_2} P_2 \longrightarrow \cdots \longrightarrow P_{k-1} \xrightarrow{\varphi_k} P_k = Q,$$

where $P_i \le S(q^n)$ for all i, and each φ_i lies in $\mathrm{Hom}_{\mathrm{Spin}_7(q^n)}(P_{i-1},P_i)$ or (if $P_{i-1},P_i \le H(q^n)$) $\mathrm{Hom}_{\Gamma_n}(P_{i-1},P_i)$. This clearly defines a fusion system over $S(q^n)$.

Proposition 2.9 *Fix $n \ge 1$. Let $E \le S(q^n)$ be an elementary abelian subgroup of rank 3 which contains U, and such that*

$$C_{S(q^n)}(E) \in \mathrm{Syl}_2(C_{\mathrm{Spin}_7(q^n)}(E)).$$

Then

$$\{\varphi \in \mathrm{Aut}_{\mathcal{F}_n}(C_{S(q^n)}(E)) \mid \varphi(z) = z\} = \mathrm{Aut}_{\mathrm{Spin}_7(q^n)}(C_{S(q^n)}(E)). \tag{1}$$

Proof Set

$$\mathrm{Spin} = \mathrm{Spin}_7(q^n); \qquad S = S(q^n); \qquad \Gamma = \Gamma_n; \qquad \text{and} \qquad \mathcal{F} = \mathcal{F}_n$$

for short. Consider the subgroups

$$R_0 = R_0(q^n) \overset{\text{def}}{=} \omega(C(q^\infty)^3) \cap S \quad \text{and} \quad R_1 = R_1(q^n) \overset{\text{def}}{=} C_S(\langle U, \widehat{A}\rangle) = \langle R_0, \widehat{B}\rangle.$$

Here, R_0 is generated by elements of the form $[\![X_1,X_2,X_3]\!]$, where either $X_i \in C(q^n)$, or $X_1 = X_2 = X_3 = X \in C(q^{2n})$ and $\psi^{q^n}(X) = -X$. Also, $C(q^n) \in \mathrm{Syl}_2(C_{SL_2(q^n)}(A))$ is cyclic of order $2^k \ge 4$, where 2^k is the largest power which divides $q^n \pm 1$; and $C(q^{2n})$ is cyclic of order 2^{k+1}. So

$$R_0 \cong (C_{2^k})^3 \qquad \text{and} \qquad R_1 = R_0 \rtimes \langle \widehat{B} \rangle;$$

where $\hat{B} = [\![B,B,B]\!]$ has order 2 and acts on R_0 via $(g \mapsto g^{-1})$. Note that

$$\langle U, \hat{A}\rangle = \langle [\![-I,-I,-I]\!], [\![A,A,A]\!]\rangle \cong C_2^3$$

is the 2–torsion subgroup of R_0.

We claim that

$$R_0 \text{ is the only subgroup of } S \text{ isomorphic to } (C_{2^k})^3. \tag{2}$$

To see this, let $R' \le S$ be any subgroup isomorphic to $(C_{2^k})^3$, and let $E' \cong C_2^3$ be its 2–torsion subgroup. Recall that for any 2–group P, the Frattini subgroup $\mathrm{Fr}(P)$ is the subgroup generated by commutators and squares in P. Thus

$$E' \le \mathrm{Fr}(R') \le \mathrm{Fr}(S) \le \langle R_0, [\![B,B,I]\!]\rangle$$

(note that $[\![B,B,I]\!] = (\;[\![B,I,I]\!])^2$). Any elementary abelian subgroup of rank 4 in $\mathrm{Fr}(S)$ would have to contain $\langle U, \hat{A}\rangle$ (the 2–torsion in $R_0 \cong C_{2^k}^3$), and this is impossible since no element of the coset $R_0\,[\![B,B,I]\!]$ commutes with $\hat{A}$. Thus, $\mathrm{rk}(\mathrm{Fr}(S)) = 3$. Hence $U \le E'$, since otherwise $\langle U, E'\rangle$ would be an elementary abelian subgroup of $\mathrm{Fr}(S)$ of rank ≥ 4. This in turn implies that $R' \le C_S(U)$, and hence that $E' \le \mathrm{Fr}(C_S(U)) \le R_0$. Thus $E' = \langle U, \hat{A}\rangle$ (the 2–torsion in R_0 again). Hence $R' \le C_S(\langle U, \hat{A}\rangle) = \langle R_0, \hat{B}\rangle$, and it follows that $R' = R_0$. This finishes the proof of (2).

Choose generators $x_1, x_2, x_3 \in R_0$ as follows. Fix $X \in C_{SL_2(q^\infty)}(A)$ of order 2^k, and $Y \in C_{SL_2(q^{2n})}(A)$ of order 2^{k+1} such that $Y^2 = X$. Set $x_1 = [\![I,I,X]\!]$, $x_2 = [\![X,I,I]\!]$, and $x_3 = [\![Y,Y,Y]\!]$. Thus, $x_1^{2^{k-1}} = z$, $x_2^{2^{k-1}} = z_1$, and $(x_3)^{2^{k-1}} = \hat{A}$.

Now let $E \le S(q^n)$ be an elementary abelian subgroup of rank 3 which contains U, and such that $C_{S(q^n)}(E) \in \mathrm{Syl}_2(C_{\mathrm{Spin}}(E))$. In particular, $E \le R_1 = C_{S(q^n)}(U)$. There are two cases to consider: that where $E \le R_0$ and that where $E \not\le R_0$.

Case 1: Assume $E \le R_0$. Since R_0 is abelian of rank 3, we must have $E = \langle U, \hat{A}\rangle$, the 2–torsion subgroup of R_0, and $C_S(E) = R_1$. Also, by (2), neither R_0 nor R_1 is isomorphic to any other subgroup of S; and hence

$$\mathrm{Aut}_{\mathcal{F}}(R_i) = \langle \mathrm{Aut}_{\mathrm{Spin}}(R_i), \mathrm{Aut}_{\Gamma}(R_i)\rangle \qquad \text{for } i = 0, 1. \tag{4}$$

By Proposition A.8, $\mathrm{Aut}_{\mathrm{Spin}}(E)$ is the group of all automorphisms of E which send z to itself. In particular, since $H(q^n) = C_{\mathrm{Spin}}(U)$, $\mathrm{Aut}_{H(q^n)}(E)$ is the group of all automorphisms of E which are the identity on U. Also, $\Gamma = \mathrm{Inn}(H(q^n))\;\tau_3$, where τ_3 sends $\hat{A} = [\![A,A,A]\!]$ to itself and permutes the nontrivial elements of $U = \{[\![-I,-I,-I]\!]\}$. Hence $\mathrm{Aut}_{\Gamma}(E)$ is the group of all automorphisms which send U to itself. So if we identify $\mathrm{Aut}(E) = GL_3(\mathbb{Z}/2)$ via the basis $\{z, z_1, \widehat{A}\}$, then

$$\mathrm{Aut}_{\mathrm{Spin}}(E) = T_1 \stackrel{\mathrm{def}}{=} GL_2^1(\mathbb{Z}/2) = \{(a_{ij}) \in GL_3(\mathbb{Z}/2) \mid a_{21} = a_{31} = 0\}$$

and

$$\mathrm{Aut}_{\Gamma}(E) = T_2 \stackrel{\mathrm{def}}{=} GL_1^2(\mathbb{Z}/2) = \{(a_{ij}) \in GL_3(\mathbb{Z}/2) \mid a_{31} = a_{32} = 0\}.$$

By (2) (and since E is the 2–torsion in R_0),

$$N_{\mathrm{Spin}}(E) = N_{\mathrm{Spin}}(R_0) \qquad \text{and} \qquad \{\gamma \in \Gamma \mid \gamma(E) = E\} = \{\gamma \in \Gamma \mid \gamma(R_0) = R_0\}.$$

Since $C_{\mathrm{Spin}}(E) = C_{\mathrm{Spin}}(R_0)\langle\widehat{B}\rangle$, the only nonidentity element of $\mathrm{Aut}_{\mathrm{Spin}}(R_0)$ or of $\mathrm{Aut}_{\Gamma}(R_0)$ which is the identity on E is conjugation by $\widehat{B}$, which is $-I$. Hence restriction from R_0 to E induces isomorphisms

$$\mathrm{Aut}_{\mathrm{Spin}}(R_0)/\{\pm I\} \cong \mathrm{Aut}_{\mathrm{Spin}}(E) \qquad \text{and} \qquad \mathrm{Aut}_{\Gamma}(R_0)/\{\pm I\} \cong \mathrm{Aut}_{\Gamma}(E).$$

Upon identifying $\mathrm{Aut}(R_0) = GL_3(\mathbb{Z}/2^k)$ via the basis $\{x_1, x_2, x_3\}$, these can be regarded as sections

$$\sigma_i \colon T_i \longrightarrow GL_3(\mathbb{Z}/2^k)/\{\pm I\} \cong SL_3(\mathbb{Z}/2^k) \times \{\lambda I \mid \lambda \in (\mathbb{Z}/2^k)^*\}/\{\pm I\}$$

of the natural projection from $GL_3(\mathbb{Z}/2^k)/\{\pm I\}$ to $GL_3(\mathbb{Z}/2)$, which agree on the group $T_0 = T_1 \cap T_2$ of upper triangular matrices.

We claim that σ_1 and σ_2 both map trivially to the second factor. Since this factor is abelian, it suffices to show that T_0 is generated by $[T_1, T_1] \cap T_0$ and $[T_2, T_2] \cap T_0$, and that each T_i is generated by $[T_i, T_i]$ and T_0 — and this is easily checked. (Note that $T_1 \cong T_2 \cong \Sigma_4$.)

By carrying out the above procedure over the field $\mathbb{F}_{q^{2^n}}$, we see that both of these sections σ_i can be lifted further to $SL_3(\mathbb{Z}/2^{k+1})$ (still agreeing on T_0). So by Lemma A.10, there is a section

$$\sigma \colon GL_3(\mathbb{Z}/2) \longrightarrow SL_3(\mathbb{Z}/2^k)$$

which extends both σ_1 and σ_2. By (4), $\mathrm{Aut}_{\mathcal{F}}(R_0) = \mathrm{Im}(\sigma) \times \langle -I \rangle$.

We next identify $\mathrm{Aut}_{\mathcal{F}}(R_1)$. By Lemma 2.8(a), $E' \stackrel{\mathrm{def}}{=} \langle z, z_1, \widehat{A}, \widehat{B}\rangle \subseteq \mathrm{Spin}_7(q^n)$ is a subgroup of rank 4 and type I. So by Proposition A.8, $\mathrm{Aut}_{\mathrm{Spin}}(E')$ contains all automorphisms of $E' \cong C_2^4$ which send $z \in Z(\mathrm{Spin})$ to itself. Hence for any $x \in N_{\mathrm{Spin}}(R_1)$, since $c_x(z) = z$, there is $x_1 \in N_{\mathrm{Spin}}(E')$ such that $c_{x_1}|_{E'} = c_x|_{E'}$ (ie, $xx_1^{-1} \in C_{\mathrm{Spin}}(E')$) and $c_{x_1}(\widehat{B}) = \widehat{B}$ (ie, $[x_1, \widehat{B}] = 1$). Set $x_2 = xx_1^{-1}$.

Since $C_{\mathrm{Spin}}(U) = H(q^n) \subseteq \mathrm{Im}(\omega)$, we see that $C_{\mathrm{Spin}}(E) = K_0\langle\widehat{B}\rangle$, where

$$K_0 = \omega(C_{SL_2(q^\infty)}(A)^3) \cap \mathrm{Spin}$$

is abelian, $R_0 \in \mathrm{Syl}_2(K_0)$, and $\hat{B}$ acts on K_0 by inversion. Upon replacing x_1 by $\hat{B}x_1$ and x_2 by $x_2\hat{B}^{-1}$ if necessary, we can assume that $x_2 \in K_0$. Then

$$[x_2, \hat{B}] = x_2 \cdot (\hat{B}x_2\hat{B}^{-1})^{-1} = x_2^2,$$

while by the original choice of x, x_1 we have

$$[x_2, \hat{B}] = [xx_1^{-1}, \hat{B}] = [x, \hat{B}] \in R_0.$$

Thus $x_2^2 \in R_0 \in \mathrm{Syl}_2(K_0)$, and hence $x_2 \in R_0 \le R_1$. Since $x = x_2x_1$ was an arbitrary element of $N_{\mathrm{Spin}}(R_1)$, this shows that $N_{\mathrm{Spin}}(R_1) \le R_1 \cdot C_{\mathrm{Spin}}(\hat{B})$, and hence that

$$\mathrm{Aut}_{\mathrm{Spin}}(R_1) = \mathrm{Inn}(R_1) \cdot \{\varphi \in \mathrm{Aut}_{\mathrm{Spin}}(R_1) \mid \varphi(\hat{B}) = \hat{B}\}. \tag{5}$$

Since $\mathrm{Aut}_\Gamma(R_1)$ is generated by its intersection with $\mathrm{Aut}_{\mathrm{Spin}}(R_1)$ and the group Σ_3 which permutes the three factors in $H(q^\infty)$ (and since the elements of Σ_3 all fix $\hat{B}$), we also have

$$\mathrm{Aut}_\Gamma(R_1) = \mathrm{Inn}(R_1) \cdot \{\varphi \in \mathrm{Aut}_\Gamma(R_1) \mid \varphi(\hat{B}) = \hat{B}\}.$$

Together with (4) and (5), this shows that $\mathrm{Aut}_{\mathcal{F}}(R_1)$ is generated by $\mathrm{Inn}(R_1)$ together with certain automorphisms of $R_1 = R_0\langle\hat{B}\rangle$ which send $\hat{B}$ to itself. In other words,

$$\begin{aligned}\mathrm{Aut}_{\mathcal{F}}(R_1) &= \Big\langle \mathrm{Inn}(R_1),\ \{\varphi \in \mathrm{Aut}(R_1) \mid \varphi(\hat{B}) = \hat{B},\ \varphi|_{R_0} \in \mathrm{Aut}_{\mathcal{F}}(R_0)\}\Big\rangle \\ &= \Big\langle \mathrm{Inn}(R_1),\ \{\varphi \in \mathrm{Aut}(R_1) \mid \varphi(\hat{B}) = \hat{B},\ \varphi|_{R_0} \in (GL_3(\mathbb{Z}/2))\}\Big\rangle.\end{aligned}$$

Thus

$$\begin{aligned}&\{\varphi \in \mathrm{Aut}_{\mathcal{F}}(R_1) \mid \varphi(z) = z\} \\ &\quad = \Big\langle \mathrm{Inn}(R_1),\ \{\varphi \in \mathrm{Aut}(R_1) \mid \varphi(\hat{B}) = \hat{B},\ \varphi|_{R_0} \in (T_1) = \mathrm{Aut}_{\mathrm{Spin}}(R_0)\}\Big\rangle \\ &\quad = \mathrm{Aut}_{\mathrm{Spin}}(R_1);\end{aligned}$$

the last equality by (5); and (1) now follows.

Case 2: Now assume that $E \not\le R_0$. By assumption, $U \le E$ (hence $E \le C_S(E) \le C_S(U)$), and $C_S(E)$ is a Sylow subgroup of $C_{\mathrm{Spin}}(E)$. Since $C_S(E)$ is not isomorphic to $R_1 = C_S(\langle z, z_1, \hat{A}\rangle)$ (by (2)), this shows that E is not Spin–conjugate to $\langle z, z_1, \hat{A}\rangle$. By Proposition A.8, Spin contains exactly two conjugacy classes of rank 3 subgroups containing z, and thus E must have type II. Hence by Proposition A.8(d), $C_S(E)$ is elementary abelian of rank 4, and also has type II.

Let $\mathcal{C}$ be the $\mathrm{Spin}_7(q^n)$–conjugacy class of the subgroup $E_* = \langle U, \hat{A}, \hat{B}\rangle \cong C_2^4$, which by Lemma 2.8(a) has type I. Let $\mathcal{E}'$ be the set of all subgroups of S which are elementary abelian of rank 4, contain U, and are not in $\mathcal{C}$. By Lemma 2.8(e), for any $\varphi \in \mathrm{Iso}_\Gamma(E', E'')$ and any $E' \in \mathcal{E}'$, $E'' \stackrel{\text{def}}{=} \varphi(E') \in \mathcal{E}'$, and φ sends $x_C(E')$ to $x_C(E'')$. The same holds for $\varphi \in \mathrm{Iso}_{\mathrm{Spin}}(E', E'')$ by definition of the elements $x_C(-)$ (Proposition A.9). Since $C_S(E) \in \mathcal{E}'$, this shows that all elements of $\mathrm{Aut}_{\mathcal{F}}(C_S(E))$ send the element $x_C(C_S(E))$ to itself. By Proposition A.9(c), $\mathrm{Aut}_{\mathrm{Spin}}(C_S(E))$ is the group of automorphisms which are the identity on the rank two subgroup $\langle x_C(C_S(E)), z\rangle$; and (1) now follows. □

One more technical result is needed.

Lemma 2.10 *Fix $n \geq 1$, and let $E, E' \subseteq S(q^n)$ be two elementary abelian subgroups of rank three which contain U, and which are Γ_n–conjugate. Then E and E' are $\mathrm{Spin}_7(q^n)$–conjugate.*

Proof By [23, 3.6.3(ii)], $-I$ is the only element of order 2 in $SL_2(q^\infty)$. Consider the sets

$$J_1 = \{X \in SL_2(q^n) \mid X^2 = -I\}$$

and

$$J_2 = \{X \in SL_2(q^{2n}) \mid \psi^{q^n}(X) = -X,\ X^2 = -I\}.$$

Here, as usual, ψ^{q^n} is induced by the field automorphism $(x \mapsto x^{q^n})$. All elements in J_1 are $SL_2(q)$–conjugate (this follows, for example, from [23, 3.6.23]), and we claim the same is true for elements of J_2.

Let $SL_2^*(q^n)$ be the group of all elements $X \in SL_2(q^{2n})$ such that $\psi^{q^n}(X) = \pm X$. This is a group which contains $SL_2(q^n)$ with index 2. Let k be such that the Sylow 2–subgroups of $SL_2(q^n)$ have order 2^k; then $k \geq 3$ since $|SL_2(q^n)| = q^n(q^{2n}-1)$. Any $S \in \mathrm{Syl}_2(SL_2^*(q^n))$ is quaternion of order $2^{k+1} \geq 16$ (see [15, Theorem 2.8.3]) and its intersection with $SL_2(q^n)$ is quaternion of order 2^k, so all elements in $S \cap J_2$ are S–conjugate. It follows that all elements of J_2 are $SL_2^*(q^n)$–conjugate. If $X, X' \in J_2$ and $X' = gXg^{-1}$ for $g \in SL_2^*(q^n)$, then either $g \in SL_2(q^n)$ or $gX \in SL_2(q^n)$, and in either case X and X' are conjugate by an element of $SL_2(q^n)$.

By Proposition 2.5(a),

$$E, E' \subseteq C_{\mathrm{Spin}_7(q^n)}(U) = H(q^n) \stackrel{\text{def}}{=} \omega(SL_2(q^\infty)^3) \cap \mathrm{Spin}_7(q^n).$$

Thus $E = \langle z, z_1, [\![X_1, X_2, X_3]\!]\rangle$ and $E' = \langle z, z_1, [\![X_1', X_2', X_3']\!]\rangle$, where the X_i are all in J_1 or all in J_2, and similarly for the X_i'. Also, since E and E' are Γ_n–conjugate (and each element of Γ_n leaves $U = \langle z, z_1\rangle$ invariant), the X_i and X_i' must all be in the same set J_1 or J_2. Hence they are all $SL_2(q^n)$–conjugate, and so E and E' are $\mathrm{Spin}_7(q^n)$–conjugate. □

We are now ready to show that the fusion systems $\mathcal{F}_n$ are saturated, and satisfy the conditions listed in Theorem 2.1.

Proposition 2.11 *For a fixed odd prime power q, let $S(q^n) \subseteq S(q^\infty) \subseteq \mathrm{Spin}_7(q^\infty)$ be as defined above. Let $z \in Z(\mathrm{Spin}_7(q^\infty))$ be the central element of order 2. Then for each n, $\mathcal{F}_n = \mathcal{F}_{\mathrm{Sol}}(q^n)$ is saturated as a fusion system over $S(q^n)$, and satisfies the following conditions:*

(a) *For all $P,Q \subseteq S(q^n)$ which contain z, if $\varphi \in \mathrm{Hom}(P,Q)$ is such that $\varphi(z) = z$, then $\varphi \in \mathrm{Hom}_{\mathcal{F}_n}(P,Q)$ if and only if $\varphi \in \mathrm{Hom}_{\mathrm{Spin}_7(q^n)}(P,Q)$.*

(b) *$C_{\mathcal{F}_n}(z) = \mathcal{F}_{S(q^n)}(\mathrm{Spin}_7(q^n))$ as fusion systems over $S(q^n)$.*

(c) *All involutions of $S(q^n)$ are $\mathcal{F}_n$–conjugate.*

Furthermore, $\mathcal{F}_m \subseteq \mathcal{F}_n$ for $m|n$. The union of the $\mathcal{F}_n$ is thus a category $\mathcal{F}_{\mathrm{Sol}}(q^\infty)$ whose objects are the finite subgroups of $S(q^\infty)$.

Proof We apply Proposition 1.2, where $p = 2$, $G = \mathrm{Spin}_7(q^n)$, $S = S(q^n)$, $Z = \langle z \rangle = Z(G)$; and U and $C_G(U) = H(q^n)$ are as defined above. Also, $\Gamma = \Gamma_n \subseteq \mathrm{Aut}(H(q^n))$. Condition (a) in Proposition 1.2 (all noncentral involutions in G are conjugate) holds since all subgroups in $\mathcal{E}_2$ are conjugate (Proposition A.8), and condition (b) holds by definition of Γ. Condition (c) holds since

$$\{\gamma \in \Gamma \mid \gamma(z) = z\} = \mathrm{Inn}(H(q^n))\langle c_\tau \rangle = \mathrm{Aut}_{N_G(U)}(H(q^n))$$

by definition, since $H(q^n) = C_G(U)$, and by Proposition 2.5(b). Condition (d) was shown in Proposition 2.9, and condition (e) in Lemma 2.10. So by Proposition 1.2, $\mathcal{F}_n$ is a saturated fusion system, and $C_{\mathcal{F}_n}(Z) = \mathcal{F}_{S(q^n)}(\mathrm{Spin}_7(q^n))$.

The last statement is clear. □

3 Linking systems and their automorphisms

We next show the existence and uniqueness of centric linking systems associated to the $\mathcal{F}_{\mathrm{Sol}}(q)$, and also construct certain automorphisms of these categories analogous to the automorphisms ψ^q of the group $\mathrm{Spin}_7(q^n)$. One more technical lemma about elementary abelian subgroups, this time about their $\mathcal{F}$–conjugacy classes, is first needed.

Lemma 3.1 *Set $\mathcal{F} = \mathcal{F}_{\mathrm{Sol}}(q)$. For each $r \le 3$, there is a unique $\mathcal{F}$–conjugacy class of elementary abelian subgroups $E \le S(q)$ of rank r. There are two $\mathcal{F}$–conjugacy classes of rank four elementary abelian subgroups $E \le S(q)$: one is the set $\mathcal{C}$ of subgroups* $\mathrm{Spin}_7(q)$*–conjugate to $E_* = \langle z, z_1, \widehat{A}, \widehat{B}\rangle$, while the other contains the other conjugacy class of type I subgroups as well as all type II subgroups. Furthermore,* $\mathrm{Aut}_{\mathcal{F}}(E) = \mathrm{Aut}(E)$ *for all elementary abelian subgroups $E \le S(q)$* except *when E has rank four and is not $\mathcal{F}$–conjugate to E_*, in which case*

$$\mathrm{Aut}_{\mathcal{F}}(E) = \{\alpha \in \mathrm{Aut}(E) \mid \alpha(x_{\mathcal{C}}(E)) = x_{\mathcal{C}}(E)\}.$$

Proof By Lemma 2.8(d), the three subgroups

$$E_* = \langle z, z_1, \widehat{A}, [\![B,B,B]\!]\rangle;\quad E_{001} = \langle z, z_1, \widehat{A}, [\![B,B,XB]\!]\rangle;\quad E_{100} = \langle z, z_1, \widehat{A}, [\![XB,B,B]\!]\rangle$$

(where X is a generator of $C(q)$) represent the three $\mathrm{Spin}_7(q)$–conjugacy classes of rank four subgroups. Clearly, E_{100} and E_{001} are Γ_1–conjugate, hence $\mathcal{F}$–conjugate; and by Lemma 2.8(e), neither is Γ_1–conjugate to E_*. This proves that there are exactly two $\mathcal{F}$–conjugacy classes of such subgroups.

Since E_* and E_{001} both are of type I in $\mathrm{Spin}_7(q)$, their $\mathrm{Spin}_7(q)$–automorphism groups contain all automorphisms which fix z (see Proposition A.8). By Lemma 2.8(e), z is fixed by all Γ–automorphisms of E_{001}, and so $\mathrm{Aut}_{\mathcal{F}}(E_{001})$ is the group of all automorphisms of E_{001} which send $z = x_{\mathcal{C}}(E_{001})$ to itself. On the other hand, E_* contains automorphisms (induced by permuting the three coordinates of H) which permute the three elements z, z_1, zz_1; and these together with $\mathrm{Aut}_{\mathrm{Spin}}(E_*)$ generate $\mathrm{Aut}(E_*)$.

It remains to deal with the subgroups of smaller rank. By Proposition A.8 again, there is just one $\mathrm{Spin}_7(q)$–conjugacy class of elementary abelian subgroups of rank one or two. There are two conjugacy classes of rank three subgroups, those of type I and those of type II. Since E_{100} is of type II and E_{001} of type I, all rank three subgroups of E_{001} have type I, while some of the rank three subgroups of E_{100} have type II. Since E_{001} is $\mathcal{F}$–conjugate to E_{100}, this shows that some subgroup of rank three and type II is $\mathcal{F}$–conjugate to a subgroup of type I, and hence all rank three subgroups are conjugate to each other. Finally, $\mathrm{Aut}_{\mathcal{F}}(E) = \mathrm{Aut}(E)$ whenver $\mathrm{rk}(E) \le 3$ since any such group is $\mathcal{F}$–conjugate to a subgroup of E_* (and we have just seen that $\mathrm{Aut}_{\mathcal{F}}(E_*) = \mathrm{Aut}(E_*)$). □

To simplify the notation, we now define

$$\mathcal{F}_{\mathrm{Spin}}(q^n) \stackrel{\mathrm{def}}{=} \mathcal{F}_{S(q^n)}(\mathrm{Spin}_7(q^n))$$

for all $1 \le n \le \infty$: the fusion system of the group $\mathrm{Spin}_7(q^n)$ at the Sylow subgroup $S(q^n)$. By construction, this is a subcategory of $\mathcal{F}_{\mathrm{Sol}}(q^n)$. We write

$$\mathcal{O}_{\mathrm{Sol}}(q^n) = \mathcal{O}(\mathcal{F}_{\mathrm{Sol}}(q^n)) \qquad \text{and} \qquad \mathcal{O}_{\mathrm{Spin}}(q^n) = \mathcal{O}(\mathcal{F}_{\mathrm{Spin}}(q^n))$$

for the corresponding orbit categories: both of these have as objects the subgroups of $S(q^n)$, and have as morphism sets

$$\mathrm{Mor}_{\mathcal{O}_{\mathrm{Sol}(q^n)}}(P,Q) = \mathrm{Hom}_{\mathcal{F}_{\mathrm{Sol}(q^n)}}(P,Q)/\mathrm{Inn}(Q) \cong \mathrm{Rep}(P,Q)$$

and

$$\mathrm{Mor}_{\mathcal{O}_{\mathrm{Spin}(q^n)}}(P,Q) = \mathrm{Hom}_{\mathcal{F}_{\mathrm{Spin}(q^n)}}(P,Q)/\mathrm{Inn}(Q).$$

Let $\mathcal{O}^c_{\mathrm{Sol}}(q^n) \subseteq \mathcal{O}_{\mathrm{Sol}}(q^n)$ and $\mathcal{O}^c_{\mathrm{Spin}}(q^n) \subseteq \mathcal{O}_{\mathrm{Spin}}(q^n)$ be the centric orbit categories; ie, the full subcategories whose objects are the $\mathcal{F}_{\mathrm{Sol}}(q^n)$– or $\mathcal{F}_{\mathrm{Spin}}(q^n)$–centric subgroups of $S(q^n)$. (We will see shortly that these in fact have the same objects.)

The obstructions to the existence and uniqueness of linking systems associated to the fusion systems $\mathcal{F}_{\mathrm{Sol}}(q^n)$, and to the existence and uniqueness of certain automorphisms of those linking systems, lie in certain groups which were identified in [6] and [5]. It is these groups which are shown to vanish in the next lemma.

Lemma 3.2 *Fix a prime power q, and let*

$$\mathcal{Z}_{\mathrm{Sol}}(q)\colon \mathcal{O}^c_{\mathrm{Sol}}(q) \longrightarrow \mathrm{Ab} \qquad \text{and} \qquad \mathcal{Z}_{\mathrm{Spin}}(q)\colon \mathcal{O}^c_{\mathrm{Spin}}(q) \longrightarrow \mathrm{Ab}$$

be the functors $\mathcal{Z}(P) = Z(P)$. Then for all $i \ge 0$,

$$\varprojlim^i_{\mathcal{O}^c_{\mathrm{Sol}}(q)} (\mathcal{Z}_{\mathrm{Sol}}(q)) = 0 = \varprojlim^i_{\mathcal{O}^c_{\mathrm{Spin}}(q)} (\mathcal{Z}_{\mathrm{Spin}}(q)).$$

Proof Set $\mathcal{F} = \mathcal{F}_{\mathrm{Sol}}(q)$ for short. Let $P_1, \ldots, P_k$ be $\mathcal{F}$–conjugacy class representatives for all $\mathcal{F}$–centric subgroups $P_i \subseteq S(q)$, arranged such that $|P_i| \ge |P_j|$ for $i \le j$. For each i, let $\mathcal{Z}_i \subseteq \mathcal{Z}_{\mathrm{Sol}}(q)$ be the subfunctor defined by setting $\mathcal{Z}_i(P) = \mathcal{Z}_{\mathrm{Sol}}(q)(P)$ if P is conjugate to P_j for some $j \le i$ and $\mathcal{Z}_i(P) = 0$ otherwise. We thus have a filtration

$$0 = \mathcal{Z}_0 \subseteq \mathcal{Z}_1 \subseteq \cdots \subseteq \mathcal{Z}_k = \mathcal{Z}_{\mathrm{Sol}}(q)$$

of $\mathcal{Z}_{\mathrm{Sol}}(q)$ by subfunctors, with the property that for each i, the quotient functor $\mathcal{Z}_i/\mathcal{Z}_{i-1}$ vanishes except on the conjugacy class of P_i (and such that $(\mathcal{Z}_i/\mathcal{Z}_{i-1})(P_i) = \mathcal{Z}_{\mathrm{Sol}}(q)(P_i)$). By [6, Proposition 3.2],

$$\varprojlim^* (\mathcal{Z}_i/\mathcal{Z}_{i-1}) \cong \Lambda^*(\mathrm{Out}_{\mathcal{F}}(P_i); Z(P_i))$$

for each i. Here, $\Lambda^*(\Gamma;M)$ are certain graded groups, defined in [16, section 5] for all finite groups Γ and all finite $\mathbb{Z}_{(p)}[\Gamma]$–modules M. We will show that $\Lambda^*(\mathrm{Out}_{\mathcal{F}}(P_i);Z(P_i)) = 0$ except when $P_i = S(q)$ or $S_0(q)$ (see Definition 2.6).

Fix an $\mathcal{F}$–centric subgroup $P \subseteq S(q)$. For each $j \geq 1$, let $\Omega_j(Z(P)) = \{g \in Z(P) \mid g^{2^j} = 1\}$, and set $E = \Omega_1(Z(P))$ — the 2–torsion in the center of P. For each $j \geq 1$, let $\Omega_j(Z(P)) = \{g \in Z(P) \mid g^{2^j} = 1\}$, and set $E = \Omega_1(Z(P))$ — the 2–torsion in the center of P. We can assume E is fully centralized in $\mathcal{F}$ (otherwise replace P and E by appropriate subgroups in the same $\mathcal{F}$–conjugacy classes).

Assume first that $Q \overset{\text{def}}{=} C_{S(q)}(E) \gneqq P$, and hence that $N_Q(P) \gneqq P$. Then any $x \in N_Q(P) \smallsetminus P$ centralizes $E = \Omega_1(Z(P))$. Hence for each j, x acts trivially on $\Omega_j(Z(P))/\Omega_{j-1}(Z(P))$, since multiplication by p^{j-1} sends this group $N_Q(P)/P$–linearly and monomorphically to E. Since c_x is a nontrivial element of $\mathrm{Out}_{\mathcal{F}}(P)$ of p–power order,

$$\Lambda^*(\mathrm{Out}_{\mathcal{F}}(P);\Omega_j(Z(P))/\Omega_{j-1}(Z(P))) = 0$$

for all $j \geq 1$ by [16, Proposition 5.5], and thus $\Lambda^*(\mathrm{Out}_{\mathcal{F}}(P);Z(P)) = 0$.

Now assume that $P = C_{S(q)}(E) = P$, the centralizer in $S(q)$ of a fully $\mathcal{F}$–centralized elementary abelian subgroup. Since there is a unique conjugacy class of elementary abelian subgroup of any rank ≤ 3, $C_{S(q)}(E)$ always contains a subgroup C_2^4, and hence P contains a subgroup C_2^4 which is self centralizing by Proposition A.8(a). This shows that $Z(P)$ is elementary abelian, and hence that $Z(P) = E$.

We can assume P is fully normalized in $\mathcal{F}$, so

$$\mathrm{Aut}_{S(q)}(P) \in \mathrm{Syl}_2(\mathrm{Aut}_{\mathcal{F}}(P))$$

by condition (I) in the definition of a saturated fusion system. Since $P = C_{S(q)}(E)$ (and $E = Z(P)$), this shows that

$$\mathrm{Ker}\big[\mathrm{Out}_{\mathcal{F}}(P) \longrightarrow \mathrm{Aut}_{\mathcal{F}}(E)\big]$$

has odd order. Also, since E is fully centralized, any $\mathcal{F}$–automorphism of E extends to an $\mathcal{F}$–automorphism of $P = C_{S(q)}(E)$, and thus this restriction map between automorphism groups is onto. By [16, Proposition 6.1(i,iii)], it now follows that

$$\Lambda^i(\mathrm{Out}_{\mathcal{F}}(P);Z(P)) \cong \Lambda^i(\mathrm{Aut}_{\mathcal{F}}(E);E). \tag{1}$$

By Lemma 3.1, $\mathrm{Aut}_{\mathcal{F}}(E) = \mathrm{Aut}(E)$, except when E lies in one certain $\mathcal{F}$–conjugacy class of subgroups $E \cong C_2^4$; and in this case $P = E$ and $\mathrm{Aut}_{\mathcal{F}}(E)$ is the group of automorphisms fixing the element $x_C(E)$. In this last (exceptional) case, $O_2(\mathrm{Aut}_{\mathcal{F}}(E)) \neq 1$ (the subgroup of elements which are the identity on $E/\langle x_C(E)\rangle$), so

$$\Lambda^*(\mathrm{Out}_{\mathcal{F}}(P); Z(P)) = \Lambda^*(\mathrm{Aut}_{\mathcal{F}}(E); E) = 0 \qquad (2)$$

by [16, Proposition 6.1(ii)]. Otherwise, when $\mathrm{Aut}_{\mathcal{F}}(E) = \mathrm{Aut}(E)$, by [16, Proposition 6.3] we have

$$\Lambda^i(\mathrm{Aut}_{\mathcal{F}}(E); E) = \begin{cases} \mathbb{Z}/2 & \text{if } \mathrm{rk}(E) = 2,\ i = 1 \\ \mathbb{Z}/2 & \text{if } \mathrm{rk}(E) = 1,\ i = 0 \\ 0 & \text{otherwise.} \end{cases} \qquad (3)$$

By points (1), (2), and (3), the groups $\Lambda^*(\mathrm{Out}_{\mathcal{F}}(P); Z(P))$ vanish except in the two cases $E = \langle z\rangle$ or $E = U$, and these correspond to $P = S(q)$ or $P = N_{S(q)}(U) = S_0(q)$.

We can assume that $P_k = S(q)$ and $P_{k-1} = S_0(q)$. We have now shown that $\varprojlim(Z_{k-2}) = 0$, and thus that $Z_{\mathrm{Sol}}(q)$ has the same higher limits as Z_k/Z_{k-2}. Hence $\varprojlim^j(Z_{\mathrm{Sol}}(q)) = 0$ for all $j \geq 2$, and there is an exact sequence

$$0 \longrightarrow \varprojlim^0(Z_{\mathrm{Sol}}(q)) \longrightarrow \underbrace{\varprojlim^0(Z_k/Z_{k-1})}_{\cong\mathbb{Z}/2} \longrightarrow \underbrace{\varprojlim^1(Z_{k-1}/Z_{k-2})}_{\cong\mathbb{Z}/2}$$
$$\longrightarrow \varprojlim^1(Z_{\mathrm{Sol}}(q)) \longrightarrow 0.$$

One easily checks that $\varprojlim^0(Z_{\mathrm{Sol}}(q)) = 0$, and hence we also get $\varprojlim^1(Z_{\mathrm{Sol}}(q)) = 0$.

The proof that $\varprojlim^i(Z_{\mathrm{Spin}}(q)) = 0$ for all $i \geq 1$ is similar, but simpler. If $\mathcal{F} = \mathcal{F}_{\mathrm{Spin}}(q)$, then for any $\mathcal{F}$–centric subgroup $P \lneqq S(q)$, there is an element $x \in N_S(P) \setminus P$ such that $[x, P] = \langle z\rangle$, and c_x is a nontrivial element of $O_2(\mathrm{Out}_{\mathcal{F}}(P))$. Thus

$$\Lambda^*(\mathrm{Out}_{\mathcal{F}}(P); Z(P)) = 0$$

for all such P by [16, Proposition 6.1(ii)] again. □

We are now ready to construct classifying spaces $B\mathrm{Sol}(q)$ for these fusion systems $\mathcal{F}_{\mathrm{Sol}}(q)$. The following proposition finishes the proof of Theorem 2.1, and also contains additional information about the spaces $B\mathrm{Sol}(q)$.

To simplify notation, we write $\mathcal{L}^c_{\mathrm{Spin}}(q^n) = \mathcal{L}^c_{S(q^n)}(\mathrm{Spin}_7(q^n))$ $(n \geq 1)$ to denote the centric linking system for the group $\mathrm{Spin}_7(q^n)$. The field automorphism $(x \mapsto x^q)$ induces an automorphism of $\mathrm{Spin}_7(q^n)$ which sends $S(q^n)$ to itself; and this in turn induces automorphisms $\psi^q_{\mathcal{F}} = \psi^q_{\mathcal{F}}(\mathrm{Sol})$, $\psi^q_{\mathcal{F}}(\mathrm{Spin})$, and $\psi^q_{\mathcal{L}}(\mathrm{Spin})$ of the fusion systems $\mathcal{F}_{\mathrm{Sol}}(q^n) \supseteq \mathcal{F}_{\mathrm{Spin}}(q^n)$ and of the linking system $\mathcal{L}^c_{\mathrm{Spin}}(q^n)$.

Proposition 3.3 *Fix an odd prime q, and $n \geq 1$. Let $S = S(q^n) \in \mathrm{Syl}_2(\mathrm{Spin}_7(q^n))$ be as defined above. Let $z \in Z(\mathrm{Spin}_7(q^n))$ be the central element of order 2. Then there is a centric linking system*

$$\mathcal{L} = \mathcal{L}^c_{\mathrm{Sol}}(q^n) \xrightarrow{\quad\pi\quad} \mathcal{F}_{\mathrm{Sol}}(q^n)$$

associated to the saturated fusion system $\mathcal{F} \overset{\text{def}}{=} \mathcal{F}_{\mathrm{Sol}}(q^n)$ over S, which has the following additional properties.

(a) *A subgroup $P \leq S$ is $\mathcal{F}$–centric if and only if it is $\mathcal{F}_{\mathrm{Spin}}(q^n)$–centric.*

(b) *$\mathcal{L}^c_{\mathrm{Sol}}(q^n)$ contains $\mathcal{L}^c_{\mathrm{Spin}}(q^n)$ as a subcategory, in such a way that $\pi|_{\mathcal{L}^c_{\mathrm{Spin}}(q^n)}$ is the usual projection to $\mathcal{F}^c_{\mathrm{Spin}}(q^n)$, and that the distinguished monomorphisms*

$$P \xrightarrow{\ \delta_P\ } \mathrm{Aut}_{\mathcal{L}}(P)$$

for $\mathcal{L} = \mathcal{L}^c_{\mathrm{Sol}}(q^n)$ are the same as those for $\mathcal{L}^c_{\mathrm{Spin}}(q^n)$.

(c) *Each automorphism of $\mathcal{L}^c_{\mathrm{Spin}}(q^n)$ which covers the identity on $\mathcal{F}^c_{\mathrm{Spin}}(q^n)$ extends to an automorphism of $\mathcal{L}^c_{\mathrm{Sol}}(q^n)$ which covers the identity on $\mathcal{F}^c_{\mathrm{Sol}}(q^n)$. Furthermore, such an extension is unique up to composition with the functor*

$$C_z\colon \mathcal{L}^c_{\mathrm{Sol}}(q^n) \xrightarrow{\quad\quad} \mathcal{L}^c_{\mathrm{Sol}}(q^n)$$

which is the identity on objects and sends $\varphi \in \mathrm{Mor}_{\mathcal{L}^c_{\mathrm{Sol}}(q^n)}(P,Q)$ to $\hat{z} \circ \varphi \circ \hat{z}^{-1}$ ("conjugation by z").

(d) *There is a unique automorphism $\psi^q_{\mathcal{L}} \in \mathrm{Aut}(\mathcal{L}^c_{\mathrm{Sol}}(q^n))$ which covers the automorphism of $\mathcal{F}_{\mathrm{Sol}}(q^n)$ induced by the field automorphism $(x \mapsto x^q)$, which extends the automorphism of $\mathcal{L}^c_{\mathrm{Spin}}(q^n)$ induced by the field automorphism, and which is the identity on $\pi^{-1}(\mathcal{F}_{\mathrm{Sol}}(q))$.*

Proof By Proposition 2.11, $\mathcal{F} = \mathcal{F}_{\mathrm{Sol}}(q^n)$ is a saturated fusion system over $S = S(q^n) \in \mathrm{Syl}_2(\mathrm{Spin}_7(q^n))$, with the property that $C_{\mathcal{F}}(z) = \mathcal{F}_{\mathrm{Spin}}(q^n)$. Point (a) follows as a special case of [6, Proposition 2.5(a)].

Since $\varprojlim^i_{\mathcal{O}^c_{\mathrm{Sol}}(q^n)} (\mathcal{Z}_{\mathrm{Sol}}(q^n)) = 0$ for $i = 2,3$ by Lemma 3.2, there is by [6, Proposition 3.1] a centric linking system $\mathcal{L} = \mathcal{L}^c_{\mathrm{Sol}}(q^n)$ associated to $\mathcal{F}$, which is unique up to isomorphism (an isomorphism which commutes with the projection to $\mathcal{F}_{\mathrm{Sol}}(q^n)$ and with the distinguished monomorphisms). Furthermore, $\pi^{-1}(\mathcal{F}_{\mathrm{Spin}}(q^n))$ is a linking system associated to $\mathcal{F}_{\mathrm{Spin}}(q^n)$, such a linking system is unique up to isomorphism since $\varprojlim^2(\mathcal{Z}_{\mathrm{Spin}}(q^n)) = 0$ (Lemma 3.2 again), and this proves (b).

(c) By [5, Theorem 6.2] (more precisely, by the same proof as that used in [5]), the vanishing of $\varprojlim^i(\mathcal{Z}_{\mathrm{Sol}}(q^n))$ for $i=1,2$ (Lemma 3.2) shows that each automorphism of $\mathcal{F}=\mathcal{F}_{\mathrm{Sol}}(q^n)$ lifts to an automorphism of $\mathcal{L}$, which is unique up to a natural isomorphism of functors; and any such natural isomorphism sends each object $P\le S$ to a isomorphism c_g for some $g\in Z(P)$. Similarly, the vanishing of $\varprojlim^i(\mathcal{Z}_{\mathrm{Spin}}(q^n))$ for $i=1,2$ shows that each automorphism of $\mathcal{F}_{\mathrm{Spin}}(q^n)$ lifts to an automorphism of $\mathcal{L}^c_{\mathrm{Spin}}(q^n)$, also unique up to a natural isomorphism of functors. Since $\mathcal{L}^c_{\mathrm{Sol}}(q^n)$ and $\mathcal{L}^c_{\mathrm{Spin}}(q^n)$ have the same objects by (a), this shows that each automorphism of $\mathcal{L}^c_{\mathrm{Spin}}(q^n)$ which covers the identity on $\mathcal{F}^c_{\mathrm{Spin}}(q^n)$ extends to a unique automorphism of $\mathcal{L}^c_{\mathrm{Sol}}(q^n)$ which covers the identity on $\mathcal{F}_{\mathrm{Sol}}(q^n)$.

It remains to show, for any $\alpha\in\mathrm{Aut}(\mathcal{L}^c_{\mathrm{Sol}}(q^n))$ which covers the identity on $\mathcal{F}^c_{\mathrm{Sol}}(q^n)$ and such that $\alpha|_{\mathcal{L}^c_{\mathrm{Spin}}(q^n)}=\mathrm{Id}$, that α is the identity or conjugation by z. We have already noted that α must be naturally isomorphic to the identity; ie, that there are elements $\gamma(P)\in Z(P)$, for all P in $\mathcal{L}^c_{\mathrm{Sol}}(q^n)$, such that

$$\alpha(\varphi)=\gamma(Q)\circ\varphi\circ\gamma(P)^{-1}\quad\text{for all }\varphi\in\mathrm{Mor}_{\mathcal{L}^c_{\mathrm{Sol}}(q^n)}(P,Q),\text{ all }P,Q.$$

Since α is the identity on $\mathcal{L}^c_{\mathrm{Spin}}(q^n)$, the only possibilities are $\gamma(P)=1$ for all P (hence $\alpha=\mathrm{Id}$), or $\gamma(P)=z$ for all P (hence α is conjugation by z).

(d) Now consider the automorphism $\psi^q_{\mathcal{F}}\in\mathrm{Aut}(\mathcal{F}_{\mathrm{Sol}}(q^n))$ induced by the field automorphism $(x\mapsto x^q)$ of $\mathbb{F}_{q^n}$. We have just seen that this lifts to an automorphism $\psi^q_{\mathcal{L}}$ of $\mathcal{L}^c_{\mathrm{Sol}}(q^n)$, which is unique up to natural isomorphism of functors. The restriction of $\psi^q_{\mathcal{L}}$ to $\mathcal{L}^c_{\mathrm{Spin}}(q^n)$, and the automorphism $\psi^q_{\mathcal{L}}(\mathrm{Spin})$ of $\mathcal{L}^c_{\mathrm{Spin}}(q^n)$ induced directly by the field automorphism, are two liftings of $\psi^q_{\mathcal{F}}|_{\mathcal{F}_{\mathrm{Spin}}(q^n)}$, and hence differ by a natural isomorphism of functors which extends to a natural isomorphism of functors on $\mathcal{L}^c_{\mathrm{Sol}}(q^n)$. Upon composing with this natural isomorphism, we can thus assume that $\psi^q_{\mathcal{L}}$ does restrict to the automorphism of $\mathcal{L}^c_{\mathrm{Spin}}(q^n)$ induced by the field automorphism.

Now consider the action of $\psi^q_{\mathcal{L}}$ on $\mathrm{Aut}_{\mathcal{L}}(S_0(q))$, which by assumption is the identity on $\mathrm{Aut}_{\mathcal{L}^c_{\mathrm{Spin}}(q)}(S_0(q))$, and in particular on $\delta(S_0(q))$ itself. Thus, with respect to the extension

$$1\longrightarrow S_0(q)\longrightarrow\mathrm{Aut}_{\mathcal{L}}(S_0(q))\longrightarrow\Sigma_3\longrightarrow 1;$$

$\psi^q_{\mathcal{L}}$ is the identity on the kernel and on the quotient, and hence is described by a cocycle

$$\eta\in Z^1(\Sigma_3;Z(S_0(q)))=Z^1(\Sigma_3;(\mathbb{Z}/2)^2).$$

Since $H^1(\Sigma_3;(\mathbb{Z}/2)^2)=0$, η must be a coboundary, and thus the action of $\psi^q_{\mathcal{L}}$ on $\mathrm{Aut}_{\mathcal{L}}(S_0(q))$ is conjugation by an element of $Z(S_0(q))$. Since it is the identity on $\mathrm{Aut}_{L^c_{\mathrm{Spin}}(q)}(S_0(q))$, it must be conjugation by 1 or z. If it is conjugation by z, then we can replace ^q_L (on the whole category L) by its composite with z; ie, by its composite with the functor which is the identity on objects and sends $2\ \mathrm{Mor}_L(P;Q)$ to b b.

In this way, we can assume that ^q_L is the identity on $\mathrm{Aut}_L(S_0(q))$. By construction, every morphism in $F_{\mathrm{Sol}}(q)$ is a composite of morphisms in $F_{\mathrm{Spin}}(q)$ and restrictions of automorphisms in $F_{\mathrm{Sol}}(q)$ of $S_0(q)$. Since ^q_L is the identity on $^{-1}(F_{\mathrm{Spin}}(q))$, this shows that it is the identity on $^{-1}(F_{\mathrm{Sol}}(q))$.

It remains to check the uniqueness of ^q_L. If ^{0} is another functor with the same properties, then by (e), $(\ ^0)^{-1}\ ^q_L$ is either the identity or conjugation by z; and the latter is not possible since conjugation by z is not the identity on $^{-1}(F_{\mathrm{Sol}}(q))$. □

This nishes the construction of the classifying spaces $B\mathrm{Sol}(q) = |L_{\mathrm{Sol}}(q)|^\wedge_2$ for the fusion systems constructed in Section 2. We end the section with an explanation of why these are not the fusion systems of nite groups.

Proposition 3.4 *For any odd prime power q, there is no nite group G whose fusion system is isomorphic to that of $F_{\mathrm{Sol}}(q)$.*

Proof Let G be a nite group, x $S \in \mathrm{Syl}_2(G)$, and assume that $S = S(q) \in \mathrm{Syl}_2(\mathrm{Spin}_7(q))$, and that the fusion system $F_S(G)$ satis es conditions (a) and (b) in Theorem 2.1. In particular, all involutions in G are conjugate, and the centralizer of any involution $z \in G$ has the fusion system of $\mathrm{Spin}_7(q)$. When q 3 (mod 8), Solomon showed [22, Theorem 3.2] that there is no nite group whose fusion system has these properties. When q 1 (mod 8), he showed (in the same theorem) that there is no such G such that $\hat{H} \stackrel{\mathrm{def}}{=} C_G(z)/O_{2'}(C_G(z))$ is isomorphic to a subgroup of $\mathrm{Aut}(\mathrm{Spin}_7(q))$ which contains $\mathrm{Spin}_7(q)$ with odd index. (Here, $O_{2'}(-)$ means largest odd order normal subgroup.)

Let G be a nite group whose fusion system is isomorphic to $F_{\mathrm{Sol}}(q)$, and again set $\hat{H} \stackrel{\mathrm{def}}{=} C_G(z)/O_{2'}(C_G(z))$ for some involution $z \in G$. Set $H = O^{2'}(\hat{H}/\langle z\rangle)$: the smallest normal subgroup of $\hat{H}/\langle z\rangle$ of odd index. Then H has the fusion system of $\Omega_7(q) = \mathrm{Spin}_7(q)/Z(\mathrm{Spin}_7(q))$. We will show that $H = \Omega_7(q')$ for some odd prime power q'. It then follows that $O^{2'}(\hat{H}) = \mathrm{Spin}_7(q')$, thus contradicting Solomon's theorem and proving our claim.

The following \classi cation free" argument for proving that $H = \Omega_7(q')$ for some q' was explained to us by Solomon. We refer to the appendix for general results about the groups $\mathrm{Spin}_n(q)$ and $\Omega_n(q)$. Fix $S \in \mathrm{Syl}_2(H)$. Thus S is isomorphic to a Sylow 2–subgroup of $\Omega_7(q)$, and has the same fusion.

We first claim that H must be simple. By definition ($H = O^{2'}(\widehat{H}/\langle z\rangle)$), H has no proper normal subgroup of odd index, and H has no proper normal subgroup of odd order since any such subgroup would lift to an odd order normal subgroup of $\widehat{H} = C_G(z)/O_{2'}(C_G(z))$. Hence for any proper normal subgroup $N \lhd H$, $Q \overset{\text{def}}{=} N \cap S$ is a proper normal subgroup of S, which is strongly closed in S with respect to H in the sense that no element of Q can be H–conjugate to an element of $S\smallsetminus Q$. Using Lemma A.4(a), one checks that the group $\Omega_7(q)$ contains three conjugacy classes of involutions, classified by the dimension of their (-1)–eigenspace. It is not hard to see (by taking products) that any subgroup of S which contains all involutions in one of these conjugacy classes contains all involutions in the other two classes as well. Furthermore, S is generated by the set of all of its involutions, and this shows that there are no proper subgroups which are strongly closed in S with respect to H. Since we have already seen that the intersection with S of any proper normal subgroup of H would have to be such a subgroup, this shows that H is simple.

Fix an isomorphism

$$S \xrightarrow[\cong]{\varphi} S' \in \mathrm{Syl}_2(\Omega_7(q))$$

which preserves fusion. Choose $x' \in S'$ whose (-1)–eigenspace is 4–dimensional, and such that $\langle x'\rangle$ is fully centralized in $\mathcal{F}_{S'}(\Omega_7(q))$. Then

$$C_{O_7(q)}(x') = O_4^+(q) \times O_3(q)$$

by Lemma A.4(c). Since $\Omega_4^+(q) \subseteq O_4^+(q)$ and $\Omega_3(q) \subseteq O_3(q)$ both have index 4, $C_{\Omega_7(q)}(x')$ is isomorphic to a subgroup of $O_4^+(q) \times O_3(q)$ of index 4, and contains a normal subgroup $K' = \Omega_4^+(q) \times \Omega_3(q)$ of index 4. Since $\langle x'\rangle$ is fully centralized, $C_{S'}(x')$ is a Sylow 2–subgroup of $C_{\Omega_7(q)}(x')$, and hence $S'_0 \overset{\text{def}}{=} S' \cap K'$ is a Sylow 2–subgroup of K'.

Set $x = \varphi^{-1}(x') \in S$. Since $S \cong S'$ have the same fusion in H and $\Omega_7(q)$, $C_S(x) \cong C_{S'}(x')$ have the same fusion in $C_H(x)$ and $C_{\Omega_7(q)}(x')$. Hence

$$H_1(C_H(x);\mathbb{Z}_{(2)}) \cong H_1(C_{\Omega_7(q)}(x');\mathbb{Z}_{(2)})$$

(homology is determined by fusion), both have order 4, and thus $C_H(x)$ also has a unique normal subgroup $K \lhd H$ of index 4. Set $S_0 = K \cap S$. Thus $\varphi(S_0) = S'_0$, and using Alperin's fusion theorem one can show that this isomorphism is fusion preserving with respect to the inclusions of Sylow subgroups $S_0 \subseteq K$ and $S'_0 \subseteq K'$.

Using the isomorphisms of Proposition A.5:

$$\Omega_4^+(q) = SL_2(q) \times_{\langle x \rangle} SL_2(q) \qquad \text{and} \qquad \Omega_3(q) = PSL_2(q),$$

we can write $K' = K'_1 \times_{\langle x' \rangle} K'_2$, where $K'_1 = SL_2(q)$ and $K'_2 = SL_2(q) \;\; PSL_2(q)$. Set $S'_i = S' \cap K'_i \in \mathrm{Syl}_2(K'_i)$; thus $S'_0 = S'_1 \times_{\langle x' \rangle} S'_2$. Set $S_i = \varphi^{-1}(S'_i)$, so that $S_0 = S_1 \times_{\langle x \rangle} S_2$ is normal of index 4 in $C_S(x)$. The fusion system of K thus splits as a central product of fusion systems, one of which is isomorphic to the fusion system of $SL_2(q)$.

We now apply a theorem of Goldschmidt, which says very roughly that under these conditions, the group K also splits as a central product. To make this more precise, let K_i be the normal closure of S_i in $K \lhd C_H(x)$. By [14, Corollary A2], since S_1 and S_2 are strongly closed in S_0 with respect to K,

$$[K_1, K_2] \le \langle x \rangle O_{2'}(K).$$

Using this, it is not hard to check that $S_i \in \mathrm{Syl}_2(K_i)$. Thus K_1 has same fusion as $SL_2(q)$ and is subnormal in $C_H(x)$ ($K_1 \lhd K \lhd C_H(x)$), and an argument similar to that used above to prove the simplicity of H shows that $K_1/(\langle x \rangle O_{2'}(K_1))$ is simple. Hence K_1 is a 2–component of $C_H(x)$ in the sense described by Aschbacher in [1]. By [1, Corollary III], this implies that H must be isomorphic to a Chevalley group of odd characteristic, or to M_{11}. It is now straightforward to check that among these groups, the only possibility is that $H = \Omega_7(q')$ for some odd prime power q'. □

4 Relation with the Dwyer-Wilkerson space

We now want to examine the relation between the spaces $B\mathrm{Sol}(q)$ which we have just constructed, and the space $BDI(4)$ constructed by Dwyer and Wilkerson in [9]. Recall that this is a 2–complete space characterized by the property that its cohomology is the Dickson algebra in four variables over $\mathbb{F}_2$; ie, the ring of invariants $\mathbb{F}_2[x_1, x_2, x_3, x_4]^{GL_4(2)}$. We show, for any odd prime power q, that $BDI(4)$ is homotopy equivalent to the 2–completion of the union of the spaces $B\mathrm{Sol}(q^n)$, and that $B\mathrm{Sol}(q)$ is homotopy equivalent to the homotopy fixed point set of an Adams map from $BDI(4)$ to itself.

We would like to define an infinite "linking system" $\mathcal{L}^c_{\mathrm{Sol}}(q^\infty)$ as the union of the finite categories $\mathcal{L}^c_{\mathrm{Sol}}(q^n)$, and then set $B\mathrm{Sol}(q^\infty) = |\mathcal{L}^c_{\mathrm{Sol}}(q^\infty)|^\wedge_2$. The difficulty with this approach is that a subgroup which is centric in the fusion system $\mathcal{F}_{\mathrm{Sol}}(q^m)$ need not be centric in a larger fusion system $\mathcal{F}_{\mathrm{Sol}}(q^n)$ (for $m|n$). To get around this problem, we define $\mathcal{L}^{cc}_{\mathrm{Sol}}(q^n) \subseteq \mathcal{L}^c_{\mathrm{Sol}}(q^n)$ to be the full subcategory whose objects are those subgroups of $S(q^n)$ which are $\mathcal{F}_{\mathrm{Sol}}(q^\infty)$-centric; or equivalently $\mathcal{F}_{\mathrm{Sol}}(q^k)$-centric for all $k \in n\mathbb{Z}$. Similarly, we define $\mathcal{L}^{cc}_{\mathrm{Spin}}(q^n)$ to be the full subcategory of $\mathcal{L}^{c}_{\mathrm{Spin}}(q^n)$ whose objects are those subgroups of $S(q^n)$ which are $\mathcal{F}_{\mathrm{Spin}}(q^\infty)$-centric. We can then define $\mathcal{L}^{c}_{\mathrm{Sol}}(q^\infty)$ and $\mathcal{L}^{c}_{\mathrm{Spin}}(q^\infty)$ to be the unions of these categories.

For these definitions to be useful, we must first show that $|\mathcal{L}^{cc}_{\mathrm{Sol}}(q^n)|^\wedge_2$ has the same homotopy type as $|\mathcal{L}^{c}_{\mathrm{Sol}}(q^n)|^\wedge_2$. This is done in the following lemma.

Lemma 4.1 *For any odd prime power q and any $n \geq 1$, the inclusions*

$$|\mathcal{L}^{cc}_{\mathrm{Sol}}(q^n)|^\wedge_2 \subseteq |\mathcal{L}^{c}_{\mathrm{Sol}}(q^n)|^\wedge_2 \qquad \text{and} \qquad |\mathcal{L}^{cc}_{\mathrm{Spin}}(q^n)|^\wedge_2 \subseteq |\mathcal{L}^{c}_{\mathrm{Spin}}(q^n)|^\wedge_2$$

are homotopy equivalences.

Proof It clearly suffices to show this when $n = 1$.

Recall, for a fusion system $\mathcal{F}$ over a p-group S, that a subgroup $P \leq S$ is *$\mathcal{F}$-radical* if $\mathrm{Out}_{\mathcal{F}}(P)$ is p-reduced; ie, if $O_p(\mathrm{Out}_{\mathcal{F}}(P)) = 1$. We will show that

$$\text{all } \mathcal{F}_{\mathrm{Sol}}(q)\text{-centric } \mathcal{F}_{\mathrm{Sol}}(q)\text{-radical subgroups of } S(q) \text{ are } \mathcal{F}_{\mathrm{Sol}}(q^\infty)\text{-centric} \tag{1}$$

and similarly

$$\text{all } \mathcal{F}_{\mathrm{Spin}}(q)\text{-centric } \mathcal{F}_{\mathrm{Spin}}(q)\text{-radical subgroups of } S(q) \text{ are } \mathcal{F}_{\mathrm{Spin}}(q^\infty)\text{-centric.} \tag{2}$$

In other words, (1) says that for each $P \leq S(q)$ which is an object of $\mathcal{L}^{c}_{\mathrm{Sol}}(q)$ but not of $\mathcal{L}^{cc}_{\mathrm{Sol}}(q)$, $O_2\big(\mathrm{Out}_{\mathcal{F}_{\mathrm{Sol}}(q)}(P)\big) \neq 1$. By [16, Proposition 6.1(ii)], this implies that

$$\Lambda^*(\mathrm{Out}_{\mathcal{F}_{\mathrm{Sol}}(q)}(P); H^*(BP;\mathbb{F}_2)) = 0.$$

Hence by [6, Propositions 3.2 and 2.2] (and the spectral sequence for a homotopy colimit), the inclusion $\mathcal{L}^{cc}_{\mathrm{Sol}}(q) \subseteq \mathcal{L}^{c}_{\mathrm{Sol}}(q)$ induces an isomorphism

$$H^*\big(|\mathcal{L}^{c}_{\mathrm{Sol}}(q)|;\mathbb{F}_2\big) \xrightarrow{\ \cong\ } H^*\big(|\mathcal{L}^{cc}_{\mathrm{Sol}}(q)|;\mathbb{F}_2\big);$$

and thus $|\mathcal{L}^{cc}_{\mathrm{Sol}}(q)|^\wedge_2 \simeq |\mathcal{L}^{c}_{\mathrm{Sol}}(q)|^\wedge_2$. The proof that $|\mathcal{L}^{cc}_{\mathrm{Spin}}(q)|^\wedge_2 \simeq |\mathcal{L}^{c}_{\mathrm{Spin}}(q)|^\wedge_2$ is similar, using (2).

Point (2) is shown in Proposition A.12, so it remains only to prove (1). Set $\mathcal{F} = \mathcal{F}_{\mathrm{Sol}}(q)$, and set $\mathcal{F}_k = \mathcal{F}_{\mathrm{Sol}}(q^k)$ for all $1 \leq k \leq \infty$. Let $E \leq Z(P)$ be the 2-torsion in the center of P, so that $P \leq C_{S(q)}(E)$. Set

$$E' = \begin{cases} \langle z \rangle & \text{if } \mathrm{rk}(E) = 1 \\ \langle z, z_1 \rangle & \text{if } \mathrm{rk}(E) = 2 \\ \langle z, z_1, \widehat{A} \rangle & \text{if } \mathrm{rk}(E) = 3 \\ E & \text{if } \mathrm{rk}(E) = 4 \end{cases}$$

in the notation of Definition 2.6. In all cases, E is $\mathcal{F}$–conjugate to E' by Lemma 3.1. We claim that E' is fully centralized in $\mathcal{F}_k$ for all $k \leq \infty$. This is clear when $\mathrm{rk}(E') = 1$ ($E' = Z(S(q^k))$), follows from Proposition 2.5(a) when $\mathrm{rk}(E') = 2$, and from Proposition A.8(a) (all rank 4 subgroups are self centralizing) when $\mathrm{rk}(E') = 4$. If $\mathrm{rk}(E') = 3$, then by Proposition A.8(d), the centralizer in $\mathrm{Spin}_7(q^k)$ (hence in $S(q^k)$) of any rank 3 subgroup has an abelian subgroup of index 2; and using this (together with the construction of $S(q^k)$ in Definition 2.6), one sees that E' is fully centralized in $\mathcal{F}_k$.

If $E' \neq E$, choose $\varphi \in \mathrm{Hom}_{\mathcal{F}}(E, S(q))$ such that $\varphi(E) = E'$; then φ extends to $\bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(C_{S(q)}(E), S(q))$ by condition (II) in the definition of a saturated fusion system, and we can replace P by $\bar{\varphi}(P)$ and E by $\varphi(E)$. We can thus assume that E is fully centralized in $\mathcal{F}_k$ for each $k \leq \infty$. So by [6, Proposition 2.5(a)], P is $\mathcal{F}_k$–centric if and only if it is $C_{\mathcal{F}_k}(E)$–centric; and this also holds when $k = \infty$. Furthermore, since $\mathrm{Out}_{C_{\mathcal{F}}(E)}(P) \lhd \mathrm{Out}_{\mathcal{F}}(P)$, $O_2(\mathrm{Out}_{C_{\mathcal{F}}(E)}(P))$ is a normal 2–subgroup of $\mathrm{Out}_{\mathcal{F}}(P)$, and thus

$$O_2\big(\mathrm{Out}_{C_{\mathcal{F}}(E)}(P)\big) \leq O_2(\mathrm{Out}_{\mathcal{F}}(P)).$$

Hence P is $C_{\mathcal{F}}(E)$–radical if it is $\mathcal{F}$–radical. So it remains to show that

all $C_{\mathcal{F}}(E)$–centric $C_{\mathcal{F}}(E)$–radical subgroups of $S(q)$ are also $C_{\mathcal{F}_\infty}(E)$–centric. (3)

If $\mathrm{rk}(E) = 1$, then $C_{\mathcal{F}}(E) = \mathcal{F}_{\mathrm{Spin}}(q)$ and $C_{\mathcal{F}_\infty}(E) = \mathcal{F}_{\mathrm{Spin}}(q^\infty)$, and (3) follows from (2). If $\mathrm{rk}(E) = 4$, then $P = E = C_{S(q^\infty)}(E)$ by Proposition A.8(a), so P is $\mathcal{F}_\infty$–centric, and the result is clear.

If $\mathrm{rk}(E) = 3$, then by Proposition A.8(d), $C_{\mathcal{F}}(E) \subseteq C_{\mathcal{F}_\infty}(E)$ are the fusion systems of a pair of semidirect products $A \rtimes C_2 \subseteq A_\infty \rtimes C_2$, where $A \subseteq A_\infty$ are abelian and C_2 acts on A_∞ by inversion. Also, E is the full 2–torsion subgroup of A_∞, since otherwise $\mathrm{rk}(A_\infty) > 3$ would imply $A_\infty \rtimes C_2 \subseteq \mathrm{Spin}_7(q^\infty)$ contains a subgroup C_2^5 (contradicting Proposition A.8). If $P \leq A$, then either $\mathrm{Out}_{C_{\mathcal{F}}(E)}(P)$ has order 2, which contradicts the assumption that P is radical; or P is elementary abelian and $\mathrm{Out}_{C_{\mathcal{F}}(E)}(P) = 1$, in which case $P \leq Z(A \rtimes C_2)$ is not centric. Thus $P \not\leq A$; $P \cap A \supseteq E$ contains all 2–torsion in A_∞, and hence P is centric in $A_\infty \rtimes C_2$.

If $\mathrm{rk}(E) = 2$, then by Proposition 2.5(a), $C_{\mathcal{F}_\infty}(E)$ and $C_{\mathcal{F}}(E)$ are the fusion systems of the groups

$$H(q^\infty) = SL_2(q^\infty)^3/\{\pm(I, I, I)\} \tag{4}$$

and

$$H(q) = H(q^\infty) \cap \mathrm{Spin}_7(q) \supseteq H_0(q) \stackrel{\mathrm{def}}{=} SL_2(q)^3/\{\pm(I, I, I)\}.$$

If $P \subseteq S(q)$ is centric and radical in the fusion system of $H(q)$, then by Lemma A.11(c), its intersection with $H_0(q) = SL_2(q)^3/\{\pm(I,I,I)\}$ is centric and radical in the fusion system of that group. So by Lemma A.11(a,f),

$$P \cap H_0(q) = (P_1 \times P_2 \times P_3)/\{\pm(I,I,I)\} \tag{5}$$

for some P_i which are centric and radical in the fusion system of $SL_2(q)$. Since the Sylow 2–subgroups of $SL_2(q)$ are quaternion [15, Theorem 2.8.3], the P_i must be nonabelian and quaternion, so each $P_i/\{\pm I\}$ is centric in $PSL_2(q^\infty)$. Hence $P \cap H_0(q)$ is centric in $SL_2(q)^3/\{\pm(I,I,I)\}$ by (5), and so P is centric in $H(q^\infty)$ by (4). □

We would like to be able to regard $B\mathrm{Spin}_7(q)$ as a subcomplex of $B\mathrm{Sol}(q)$, but there is no simple natural way to do so. Instead, we set

$$B\mathrm{Spin}_7'(q) = |\mathcal{L}^{cc}_{\mathrm{Spin}}(q)|^\wedge_2 \subseteq |\mathcal{L}^{cc}_{\mathrm{Sol}}(q)|^\wedge_2 \subseteq B\mathrm{Sol}(q);$$

then $B\mathrm{Spin}_7'(q) \simeq B\mathrm{Spin}_7(q)^\wedge_2$ by [5, Proposition 1.1] and Lemma 4.1. Also, we write

$$B\mathrm{Sol}'(q) = |\mathcal{L}^{cc}_{\mathrm{Sol}}(q)|^\wedge_2 \subseteq B\mathrm{Sol}(q) \overset{\mathrm{def}}{=} |\mathcal{L}^{c}_{\mathrm{Sol}}(q)|^\wedge_2$$

to denote the subcomplex shown in Lemma 4.1 to be equivalent to $B\mathrm{Sol}(q)$; and set

$$B\mathrm{Spin}_7'(q^\infty) = |\mathcal{L}^{c}_{\mathrm{Spin}}(q^\infty)|^\wedge_2.$$

From now on, when we talk about the inclusion of $B\mathrm{Spin}_7(q)$ into $B\mathrm{Sol}(q)$, as long as it need only be well defined up to homotopy, we mean the composite

$$B\mathrm{Spin}_7(q) \simeq B\mathrm{Spin}_7'(q) \subseteq B\mathrm{Sol}'(q)$$

(for some choice of homotopy equivalence). Similarly, if we talk about the inclusion of $B\mathrm{Sol}(q^m)$ into $B\mathrm{Sol}(q^n)$ (for $m|n$) where it need only be defined up to homotopy, we mean these spaces identified with their equivalent subcomplexes $B\mathrm{Sol}'(q^m) \subseteq B\mathrm{Sol}'(q^n)$.

Lemma 4.2 *Let q be any odd prime. Then for all $n \geq 1$,*

$$\begin{array}{ccc} H^*(B\mathrm{Sol}(q^n);\mathbb{F}_2) & \longrightarrow & H^*(BH(q^n);\mathbb{F}_2)^{C_3} \\ \big\downarrow & & \big\downarrow \\ H^*(B\mathrm{Spin}_7(q^n);\mathbb{F}_2) & \longrightarrow & H^*(BH(q^n);\mathbb{F}_2) \end{array} \tag{1}$$

(with all maps induced by inclusions of groups or spaces) is a pullback square.

Proof By [6, Theorem B], $H^*(B\mathrm{Sol}(q^n);\mathbb{F}_2)$ is the ring of elements in the cohomology of $S(q^n)$ which are stable relative to the fusion. By the construction in Section 2, the fusion in $\mathrm{Sol}(q^n)$ is generated by that in $\mathrm{Spin}_7(q^n)$, together with the permutation action of C_3 on the subgroup $H(q^n) \subseteq \mathrm{Spin}_7(q^n)$, and hence (1) is a pullback square. □

Proposition 4.3 *For each odd prime q, there is a category $\mathcal{L}^c_{\mathrm{Sol}}(q^\infty)$, together with a functor*

$$\psi: \mathcal{L}^c_{\mathrm{Sol}}(q^\infty) \longrightarrow \mathcal{F}_{\mathrm{Sol}}(q^\infty);$$

such that the following hold:

(a) *For each $n \geq 1$, $\psi^{-1}(\mathcal{F}_{\mathrm{Sol}}(q^n)) = \mathcal{L}^{cc}_{\mathrm{Sol}}(q^n)$.*

(b) *There is a homotopy equivalence*

$$B\mathrm{Sol}(q^\infty) \stackrel{\mathrm{def}}{=} |\mathcal{L}^c_{\mathrm{Sol}}(q^\infty)|^\wedge_2 \xrightarrow[\simeq]{\quad\omega\quad} BDI(4)$$

such that the following square commutes up to homotopy

$$\begin{array}{ccc} B\mathrm{Spin}^0_7(q^\infty)^\wedge_2 & \xrightarrow{\ \lambda(q^\infty)\ } & B\mathrm{Sol}(q^\infty) \\ \omega_0 \Big\downarrow \simeq & & \Big\downarrow \omega\ \simeq \\ B\mathrm{Spin}(7)^\wedge_2 & \xrightarrow{\ \widehat{\ }\ } & BDI(4). \end{array} \qquad (1)$$

Here, ω_0 is the homotopy equivalence of [13], induced by some fixed choice of embedding of the Witt vectors for $\bar{\mathbb{F}}_q$ into $\mathbb{C}$, while $\lambda(q^\infty)$ is the union of the inclusions $|\mathcal{L}^{cc}_{\mathrm{Spin}}(q^n)|^\wedge_2 \subseteq |\mathcal{L}^{cc}_{\mathrm{Sol}}(q^n)|^\wedge_2$, and $\widehat{\ }$ is the inclusion arising from the construction of $BDI(4)$ in [9].

Furthermore, there is an automorphism $\psi^q_{\mathcal{L}} \in \mathrm{Aut}(\mathcal{L}^c_{\mathrm{Sol}}(q^\infty))$ of categories which satisfies the conditions:

(c) *the restriction of $\psi^q_{\mathcal{L}}$ to each subcategory $\mathcal{L}^{cc}_{\mathrm{Sol}}(q^n)$ is equal to the restriction of $\psi^q_{\mathcal{L}} \in \mathrm{Aut}(\mathcal{L}^c_{\mathrm{Sol}}(q^n))$ as defined in Proposition 3.3(d);*

(d) *$\psi^q_{\mathcal{L}}$ covers the automorphism $\psi^q_{\mathcal{F}}$ of $\mathcal{F}_{\mathrm{Sol}}(q^\infty)$ induced by the field automorphism $(x \mapsto x^q)$; and*

(e) *for each n, $(\psi^q_{\mathcal{L}})^n$ fixes $\mathcal{L}^{cc}_{\mathrm{Sol}}(q^n)$.*

Proof By Proposition 2.11, the inclusions $\mathrm{Spin}_7(q^m) \subseteq \mathrm{Spin}_7(q^n)$ for all $m|n$ induce inclusions of fusion systems $\mathcal{F}_{\mathrm{Sol}}(q^m) \subseteq \mathcal{F}_{\mathrm{Sol}}(q^n)$. Since the restriction of a linking system over $\mathcal{F}^{cc}_{\mathrm{Sol}}(q^n)$ is a linking system over $\mathcal{F}^{cc}_{\mathrm{Sol}}(q^m)$, the uniqueness of linking systems (Proposition 3.3) implies that we get inclusions $\mathcal{L}^{cc}_{\mathrm{Sol}}(q^m) \subseteq \mathcal{L}^{cc}_{\mathrm{Sol}}(q^n)$. We define $\mathcal{L}^{c}_{\mathrm{Sol}}(q^\infty)$ to be the union of the finite categories $\mathcal{L}^{cc}_{\mathrm{Sol}}(q^n)$. (More precisely, fix a sequence of positive integers $n_1|n_2|n_3|\cdots$ such that every positive integer divides some n_i, and set

$$\mathcal{L}^{c}_{\mathrm{Sol}}(q^\infty) = \bigcup_{i=1}^{\infty} \mathcal{L}^{cc}_{\mathrm{Sol}}(q^{n_i}).$$

Then by uniqueness again, we can identify $\mathcal{L}^{cc}_{\mathrm{Sol}}(q^n)$ for each n with the appropriate subcategory.)

Let $\pi\colon \mathcal{L}^{c}_{\mathrm{Sol}}(q^\infty) \longrightarrow \mathcal{F}_{\mathrm{Sol}}(q^\infty)$ be the union of the projections from $\mathcal{L}^{cc}_{\mathrm{Sol}}(q^{n_i})$ to $\mathcal{F}_{\mathrm{Sol}}(q^{n_i}) \subseteq \mathcal{F}_{\mathrm{Sol}}(q^\infty)$. Condition (a) is clearly satisfied. Also, using Proposition 3.3(d) and Lemma 4.1, we see that there is an automorphism $\psi^q_{\mathcal{L}}$ of $\mathcal{L}^{c}_{\mathrm{Sol}}(q^\infty)$ which satisfies conditions (c,d,e) above. (Note that by the fusion theorem as shown in [6, Theorem A.10], morphisms in $\mathcal{L}^{c}_{\mathrm{Sol}}(q^n)$ are generated by those between radical subgroups, and hence by those in $\mathcal{L}^{cc}_{\mathrm{Sol}}(q^n)$.)

It remains only to show that $|\mathcal{L}^{c}_{\mathrm{Sol}}(q^\infty)|^\wedge_2 \simeq BDI(4)$, and to show that square (1) commutes. The space $BDI(4)$ is 2–complete by its construction in [9]. By Lemma 4.1,

$$H^*(B\mathrm{Sol}(q^\infty);\mathbb{F}_2) = \varprojlim_n H^*\big(|\mathcal{L}^{c}_{\mathrm{Sol}}(q^n)|;\mathbb{F}_2\big) = \varprojlim_n H^*\big(B\mathrm{Sol}(q^n);\mathbb{F}_2\big).$$

Hence by Lemma 4.2 (and since the inclusions $B\mathrm{Spin}_7(q^n) \longrightarrow B\mathrm{Sol}(q^n)$ commute with the maps induced by inclusions of fields $\mathbb{F}_{q^m} \subseteq \mathbb{F}_{q^n}$), there is a pullback square

$$\begin{array}{ccc} H^*(B\mathrm{Sol}(q^\infty);\mathbb{F}_2) & \longrightarrow & H^*(BH(q^\infty);\mathbb{F}_2)^{C_3} \\ \big\downarrow & & \big\downarrow \\ H^*(B\mathrm{Spin}_7(q^\infty);\mathbb{F}_2) & \longrightarrow & H^*(BH(q^\infty);\mathbb{F}_2). \end{array} \qquad (2)$$

Also, by [13, Theorem 1.4], there are maps

$$B\mathrm{Spin}_7(q^\infty) \longrightarrow B\mathrm{Spin}(7) \quad \text{and} \quad BH(q^\infty) \longrightarrow B\big(SU(2)^3/\{\pm(I,I,I)\}\big)$$

which induce isomorphisms of $\mathbb{F}_2$–cohomology, and hence homotopy equivalences after 2–completion. So by Propositions 4.7 and 4.9 (or more directly by the computations in [9, section 3]), the pullback of the above square is the ring of Dickson invariants in the polynomial algebra $H^*(BC_2^4;\mathbb{F}_2)$, and thus isomorphic to $H^*(BDI(4);\mathbb{F}_2)$.

Point (b), including the commutativity of (1), now follows from the following lemma. □

Lemma 4.4 *Let X be a 2–complete space such that $H^*(X;\mathbb{F}_2)$ is the Dickson algebra in 4 variables. Assume further that there is a map $B\mathrm{Spin}(7) \xrightarrow{f} X$ such that $H^*(f|_{BC_2^4};\mathbb{F}_2)$ is the inclusion of the Dickson invariants in the polynomial algebra $H^*(BC_2^4;\mathbb{F}_2)$. Then $X \simeq BDI(4)$. More precisely, there is a homotopy equivalence between these spaces such that the composite*

$$B\mathrm{Spin}(7) \xrightarrow{\quad f \quad} X \simeq BDI(4)$$

is the inclusion arising from the construction in [9].

Proof In fact, Notbohm [18, Theorem 1.2] has proven that the lemma holds even without the assumption about $B\mathrm{Spin}(7)$ (but with the more precise assumption that $H^*(X;\mathbb{F}_2)$ is isomorphic as an algebra over the Steenrod algebra to the Dickson algebra). The result as stated above is much more elementary (and also implicit in [9]), so we sketch the proof here.

Since $H^*(X;\mathbb{F}_2)$ is a polynomial algebra, $H^*(\Omega X;\mathbb{F}_2)$ is isomorphic as a graded vector space to an exterior algebra on the same number of variables, and in particular is finite. Hence X is a 2–compact group. By [11, Theorem 8.1] (the centralizer decomposition for a p–compact group), there is an $\mathbb{F}_2$–homology equivalence

$$\underset{\mathbf{A}}{\underrightarrow{\mathrm{hocolim}}}(\Phi) \xrightarrow{\quad \varphi \quad} X.$$

Here, $\mathbf{A}$ is the category of pairs (V,φ), where V is a nontrivial elementary abelian 2–group, and $\varphi\colon BV \longrightarrow X$ makes $H^*(BV;\mathbb{F}_2)$ into a finitely generated module over $H^*(X;\mathbb{F}_2)$ (see [10, Proposition 9.11]). Morphisms in $\mathbf{A}$ are defined by letting $\mathrm{Mor}_{\mathbf{A}}((V,\varphi),(V',\varphi'))$ be the set of monomorphisms $V \longrightarrow V'$ of groups which make the obvious triangle commute up to homotopy. Also,

$$\Phi\colon \mathbf{A}^{\mathrm{op}} \longrightarrow \mathrm{Top} \quad \text{is the functor} \quad \Phi(V,\varphi) = \mathrm{Map}(BV,X)_\varphi.$$

By [9, Lemma 1.6(1)] and [17, Theoreme 0.4], $\mathbf{A}$ is equivalent to the category of elementary abelian 2–groups E with $1 \le \mathrm{rk}(E) \le 4$, whose morphisms consist of all group monomorphisms. Also, if $BC_2 \xrightarrow{\varphi} X$ is the restriction of f to any subgroup $C_2 \subseteq \mathrm{Spin}(7)$, then in the notation of Lannes,

$$T_{C_2}(H^*(X;\mathbb{F}_2);\varphi^*) \cong H^*(B\mathrm{Spin}(7);\mathbb{F}_2)$$

by [9, Lemmas 16.(3), 3.10 and 3.11], and hence

$$H^*(\mathrm{Map}(BC_2,X)_\varphi;\mathbb{F}_2) \cong H^*(B\mathrm{Spin}(7);\mathbb{F}_2)$$

by Lannes [17, Theoreme 3.2.1]. This shows that

$$\mathrm{Map}(BC_2, X)_{\iota} \simeq \ {}^{\wedge}_{2} \simeq B\mathrm{Spin}(7)^{\wedge}_{2};$$

and thus that the centralizers of other elementary abelian 2-groups are the same as their centralizers in $B\mathrm{Spin}(7)^{\wedge}_{2}$. In other words, is equivalent in the homotopy category to the diagram used in [9] to define $BDI(4)$. By [9, Proposition 7.7] (and the remarks in its proof), this homotopy functor has a unique homotopy lifting to spaces. So by definition of $BDI(4)$,

$$X \simeq \underset{\mathbf{A}}{\mathrm{hocolim}}(\ )^{\wedge}_{2} \simeq BDI(4). \qquad \square$$

Set $B\psi^q \stackrel{\mathrm{def}}{=} |\psi^q_L|$, a self homotopy equivalence of $B\mathrm{Sol}(q^\infty) \simeq BDI(4)$. By construction, the restriction of $B\psi^q$ to the maximal torus of $B\mathrm{Sol}(q^\infty)$ is the map induced by $x \mapsto x^q$, and hence this is an "Adams map" as defined by Notbohm [18]. In fact, by [18, Theorem 3.5], there is an Adams map from BDI(4) to itself, unique up to homotopy, of degree any 2-adic unit.

Following Benson [3], we define $BDI_4(q)$ for any odd prime power q to be the homotopy fixed point set of the $\mathbb{Z}$-action on $B\mathrm{Sol}(q^\infty) \simeq BDI(4)$ induced by the Adams map $B\psi^q$. By "homotopy fixed point set" in this situation, we mean that the following square is a homotopy pullback:

$$\begin{array}{ccc}
BDI_4(q) & \longrightarrow & B\mathrm{Sol}(q^\infty) \\
\big\downarrow & & \big\downarrow \\
B\mathrm{Sol}(q^\infty) & \xrightarrow{(\mathrm{Id}, B\psi^q)} & B\mathrm{Sol}(q^\infty) \times B\mathrm{Sol}(q^\infty).
\end{array}$$

The actual pullback of this square is the subspace $B\mathrm{Sol}(q)$ of elements fixed by $B\psi^q$, and we thus have a natural map $B\mathrm{Sol}(q) \xrightarrow{\ '\ } BDI_4(q)$.

Theorem 4.5 *For any odd prime power q, the natural map*

$$B\mathrm{Sol}(q) \xrightarrow[\ ,\ ]{\ '\ } BDI_4(q)$$

is a homotopy equivalence.

Proof Since $BDI(4)$ is simply connected, the square used to define $BDI_4(q)$ remains a homotopy pullback square after 2-completion by [4, II.5.3]. Thus $BDI_4(q)$ is 2-complete. Also, $B\mathrm{Sol}(q) \stackrel{\mathrm{def}}{=} |\mathcal{L}^c_{\mathrm{Sol}}(q)|^{\wedge}_{2}$ is 2-complete since $|\mathcal{L}^c_{\mathrm{Sol}}(q)|$ is 2-good [6, Proposition 1.12], and hence it suffices to prove that the map between these spaces is an $\mathbb{F}_2$–cohomology equivalence. By Lemma 4.2, this means showing that the following commutative square is a pullback square:

$$\begin{array}{ccc} H^*(BDI_4(q);\mathbb{F}_2) & \longrightarrow & H^*(BH(q);\mathbb{F}_2)^{C_3} \\ \big\downarrow & & \big\downarrow \\ H^*(B\mathrm{Spin}_7(q);\mathbb{F}_2) & \longrightarrow & H^*(BH(q);\mathbb{F}_2). \end{array} \tag{1}$$

Here, the maps are induced by the composite

$$B\mathrm{Spin}_7(q) \simeq B\mathrm{Spin}'_7(q)^\wedge_2 \quad B\mathrm{Sol}(q) \longrightarrow BDI_4(q)$$

and its restriction to $BH(q)$. Also, by Proposition 4.3(b), the following diagram commutes up to homotopy:

$$\begin{array}{ccccc} B\mathrm{Spin}_7(q) & \xrightarrow{\text{incl}} & B\mathrm{Spin}_7(q^\infty) & \xrightarrow{\lambda'} & B\mathrm{Spin}(7) \\ \big\downarrow{\scriptstyle \lambda(q)} & & \big\downarrow{\scriptstyle \lambda(q^\infty)} & & \big\downarrow{\scriptstyle \hat{\lambda}} \\ B\mathrm{Sol}(q) & \xrightarrow{\text{incl}} & B\mathrm{Sol}(q^\infty) & \longrightarrow & BDI(4) \end{array} \tag{2}$$

By [12, Theorem 12.2], together with [13, section 1], for any connected reductive Lie group $\mathbb{G}$ and any algebraic epimorphism ψ on $\mathbb{G}(\bar{\mathbb{F}}_q)$ with finite fixed subgroup, there is a homotopy pullback square

$$\begin{array}{ccc} B(\mathbb{G}(\bar{\mathbb{F}}_q)^\psi)^\wedge_2 & \xrightarrow{\text{incl}} & B\mathbb{G}(\bar{\mathbb{F}}_q)^\wedge_2 \\ \big\downarrow{\scriptstyle \text{incl}} & & \big\downarrow \\ B\mathbb{G}(\bar{\mathbb{F}}_q)^\wedge_2 & \xrightarrow{(\mathrm{Id},B\psi)} & B\mathbb{G}(\bar{\mathbb{F}}_q)^\wedge_2 \times B\mathbb{G}(\bar{\mathbb{F}}_q)^\wedge_2 . \end{array} \tag{3}$$

We need to apply this when $\mathbb{G} = \mathrm{Spin}_7$ or $\mathbb{G} = H = (SL_2)^3/\{\pm(I,I,I)\}$. In particular, if $\psi = \psi^q$ is the automorphism induced by the field automorphism $(x \mapsto x^q)$, then $\mathrm{Spin}_7(\bar{\mathbb{F}}_q)^\psi = \mathrm{Spin}_7(q)$ by Lemma A.3, and $H(\bar{\mathbb{F}}_q)^\psi = H(q) \overset{\text{def}}{=} H(\bar{\mathbb{F}}_q) \cap \mathrm{Spin}_7(q)$. We thus get a description of $B\mathrm{Spin}_7(q)$ and $BH(q)$ as homotopy pullbacks.

By [13, Theorem 1.4], $B\mathbb{G}(\bar{\mathbb{F}}_q)^\wedge_2 \simeq B\mathbb{G}(\mathbb{C})^\wedge_2$. Also, we can replace the complex Lie groups $\mathrm{Spin}_7(\mathbb{C})$ and $H(\mathbb{C})$ by maximal compact subgroups $\mathrm{Spin}(7)$ and $\overline{H} \overset{\text{def}}{=} SU(2)^3/\{\pm(I,I,I)\}$, since these have the same homotopy type.

If we set $\mathfrak{R} = H^*(B\mathbb{G}(\bar{\mathbb{F}}_q);\mathbb{F}_2) = H^*(B\mathbb{G}(\mathbb{C});\mathbb{F}_2)$, then there are Eilenberg-Moore spectral sequences

$$E_2 = \mathrm{Tor}_{\mathfrak{R}\otimes\mathfrak{R}^{\mathrm{op}}}(\mathfrak{R},\mathfrak{R}) \Longrightarrow H^*(B(\mathbb{G}(\bar{\mathbb{F}}_q)^\psi);\mathbb{F}_2);$$

where the $(\mathfrak{R} \otimes \mathfrak{R}^{\rm op})$–module structure on $\mathfrak{R}$ is defined by setting $(a \otimes b)x = a\,x\,B(b)$. When $\mathbb{G} = \mathrm{Spin}_7$ or H, then $\mathfrak{R}$ is a polynomial algebra by Proposition 4.7 and the above remarks, and B acts on $\mathfrak{R}$ via the identity. The above spectral sequence thus satisfies the hypotheses of [20, Theorem II.3.1], and hence collapses. (Alternatively, note that in this case, E_2 is generated multiplicatively by $E_2^{0,*}$ and $E_2^{-1,*}$ by (5) below.) Similarly, when $\mathfrak{R} = H^*(BDI(4);\mathbb{F}_2)$, there is an analogous spectral sequence which converges to $H^*(BDI_4(q);\mathbb{F}_2)$, and which collapses for the same reason. By the above remarks, these spectral sequences are natural with respect to the inclusions $BH(-) \subseteq B\mathrm{Spin}_7(-)$, and (using the naturality of ψ^q shown in Proposition 3.3(d)) of $B\mathrm{Spin}_7(-)$ into $B\mathrm{Sol}(-)$ or $BDI(4)$.

To simplify the notation, we now write

$$\mathfrak{A} \stackrel{\rm def}{=} H^*(BDI(4);\mathbb{F}_2), \quad \mathfrak{B} \stackrel{\rm def}{=} H^*(B\mathrm{Spin}(7);\mathbb{F}_2), \quad \text{and} \quad \mathfrak{C} \stackrel{\rm def}{=} H^*(\overline{H};\mathbb{F}_2)$$

to denote these cohomology rings. The Frobenius automorphism ψ^q acts via the identity on each of them. We claim that the square

$$\begin{array}{ccc} \mathrm{Tor}_{\mathfrak{A}\otimes\mathfrak{A}^{\rm op}}(\mathfrak{A},\mathfrak{A}) & \longrightarrow & \mathrm{Tor}_{\mathfrak{C}\otimes\mathfrak{C}^{\rm op}}(\mathfrak{C},\mathfrak{C})^{C_3} \\ \downarrow & & \downarrow \\ \mathrm{Tor}_{\mathfrak{B}\otimes\mathfrak{B}^{\rm op}}(\mathfrak{B},\mathfrak{B}) & \longrightarrow & \mathrm{Tor}_{\mathfrak{C}\otimes\mathfrak{C}^{\rm op}}(\mathfrak{C},\mathfrak{C}) \end{array} \tag{4}$$

is a pullback square. Once this has been shown, it then follows that in each degree, square (1) has a finite filtration under which each quotient is a pullback square. Hence (1) itself is a pullback.

For any commutative $\mathbb{F}_2$–algebra $\mathfrak{R}$, let $\Omega_{\mathfrak{R}/\mathbb{F}_2}$ denote the $\mathfrak{R}$–module generated by elements dr for $r \in \mathfrak{R}$ with the relations $dr = 0$ if $r \in \mathbb{F}_2$,

$$d(r+s) = dr + ds \qquad \text{and} \qquad d(rs) = r\,ds + s\,dr.$$

Let $\Omega^*_{\mathfrak{R}/\mathbb{F}_2}$ denote the *ring of Kähler differentials*: the exterior algebra (over $\mathfrak{R}$) of $\Omega_{\mathfrak{R}/\mathbb{F}_2} = \Omega^1_{\mathfrak{R}/\mathbb{F}_2}$. When $\mathfrak{R}$ is a polynomial algebra, there are natural identifications

$$\mathrm{Tor}_{\mathfrak{R}\otimes\mathfrak{R}^{\rm op}}(\mathfrak{R},\mathfrak{R}) \cong HH_*(\mathfrak{R};\mathfrak{R}) \cong \Omega^*_{\mathfrak{R}/\mathbb{F}_2}. \tag{5}$$

The first isomorphism holds for arbitrary algebras, and is shown, e.g., in [25, Lemma 9.1.3]. The second holds for smooth algebras over a field [25, Theorem 9.4.7] (and polynomial algebras are smooth as shown in [25, section 9.3.1]). In particular, the isomorphisms (5) hold for $\mathfrak{R} = \mathfrak{A}, \mathfrak{B}, \mathfrak{C}$, which are shown to be polynomial algebras in Proposition 4.7 below. Thus, square (4) is isomorphic to the square

$$\begin{array}{ccc} \Omega_{\mathfrak{A}/\mathbb{F}_2} & \longrightarrow & (\Omega_{\mathfrak{C}/\mathbb{F}_2})^{C_3} \\ \downarrow & & \downarrow \\ \Omega_{\mathfrak{B}/\mathbb{F}_2} & \longrightarrow & \Omega_{\mathfrak{C}/\mathbb{F}_2}\, , \end{array} \tag{6}$$

which is shown to be a pullback square in Propositions 4.7 and 4.9 below. □

It remains to prove that square (6) in the above proof is a pullback square. In what follows, we let $D_i(x_1,\dots,x_n)$ denote the i-th Dickson invariant in variables $x_1,\dots,x_n$. This is the (2^n-2^{n-i})-th symmetric polynomial in the elements (equivalently in the nonzero elements) of the vector space $\langle x_1,\dots,x_n\rangle_{\mathbb{F}_2}$. We refer to [26] for more detail. Note that what he denotes $c_{n,i}$ is what we call $D_{n-i}(x_1,\dots,x_n)$.

Lemma 4.6 *For any n,*

$$\begin{aligned} D_1(x_1,\dots,x_{n+1}) &= \prod_{x\in\langle x_1,\dots,x_n\rangle_{\mathbb{F}_2}} (x_{n+1}+x) + D_1(x_1,\dots,x_n)^2 \\ &= x_{n+1}^{2^n} + \sum_{i=1}^{n} x_{n+1}^{2^{n-i}} D_i(x_1,\dots,x_n) + D_1(x_1,\dots,x_n)^2. \end{aligned}$$

Proof The first equality is shown in [26, Proposition 1.3(b)]; here we prove them both simultaneously. Set $V_n=\langle x_1,\dots,x_n\rangle_{\mathbb{F}_2}$. Since $\sigma_i(V_n)=0$ whenever 2^n-i is not a power of 2 (cf [26, Proposition 1.1]),

$$\begin{aligned} D_1(x_1,\dots,x_{n+1}) &= \sum_{i=0}^{2^n} \sigma_i(V_n)\,\sigma_{2^n-i}(x_{n+1}+V_n) \\ &= \prod_{x\in V_n}(x_{n+1}+x) + \sum_{i=1}^{n} D_i(x_1,\dots,x_n)\,\sigma_{2^{n-i}}(x_{n+1}+V_n). \end{aligned}$$

Also, since $\sigma_i(V_n)=0$ for $0<i<2^{n-1}$ as noted above,

$$\sigma_k(x_{n+1}+V_n) = \sum_{i=0}^{k} x_{n+1}^{k-i}\binom{2^n-i}{k-i}\sigma_i(V_n) = \begin{cases} 0 & \text{if } 0<k<2^{n-1} \\ D_1(x_1,\dots,x_n) & \text{if } k=2^{n-1}. \end{cases}$$

This proves the first equality, and the second follows since

$$\prod_{x\in V_n}(x_{n+1}+x) = x_{n+1}^{2^n} + \sum_{i=1}^{2^n} x_{n+1}^{2^n-i}\sigma_i(V_n) = x_{n+1}^{2^n} + \sum_{i=1}^{n} x_{n+1}^{2^{n-i}} D_i(x_1,\dots,x_n). \qquad □$$

In the following proposition (and throughout the rest of the section), we work with the polynomial ring $\mathbb{F}_2[x,y,z,w]$, with the natural action of $GL_4(\mathbb{F}_2)$. Let

$$GL_2^2(\mathbb{F}_2),\ GL_1^3(\mathbb{F}_2) \subseteq GL_4(\mathbb{F}_2)$$

be the subgroups of automorphisms of $V \stackrel{\text{def}}{=} \langle x,y,z,w\rangle_{\mathbb{F}_2}$ which leave invariant the subspaces $\langle x,y\rangle$ and $\langle x,y,z\rangle$, respectively. Also, let $GL_{2'}^2(\mathbb{F}_2) \subseteq GL_2^2(\mathbb{F}_2)$ be the subgroup of automorphisms which are the identity modulo $\langle x,y\rangle$. Thus, when described in terms of block matrices (with respect to the given basis $\{x,y,z,w\}$),

$$GL_1^3(\mathbb{F}_2) = \left\{\begin{pmatrix} A & X \\ 0 & 1\end{pmatrix}\right\}, \quad GL_2^2(\mathbb{F}_2) = \left\{\begin{pmatrix} B & Y \\ 0 & C\end{pmatrix}\right\}, \quad \text{and} \quad GL_{2'}^2(\mathbb{F}_2) = \left\{\begin{pmatrix} B & Y \\ 0 & I\end{pmatrix}\right\},$$

for $A \in GL_3(\mathbb{F}_2)$, X a column vector, $B,C \in GL_2(\mathbb{F}_2)$, and $Y \in M_2(\mathbb{F}_2)$.

We need to make more precise the relation between V (or the polynomial ring $\mathbb{F}_2[x,y,z,w]$) and the cohomology of Spin(7). To do this, let $W \subseteq \text{Spin}(7)$ be the inverse image of the elementary abelian subgroup

$$\big\langle \text{diag}(-1,-1,-1,-1,1,1,1),\, \text{diag}(-1,-1,1,1,-1,-1,1),\\ \text{diag}(-1,1,-1,1,-1,1,-1)\big\rangle \subseteq SO(7).$$

Thus, $W \cong C_2^4$. Fix a basis $\{z,\varepsilon,\varepsilon',\varepsilon''\}$ for W, where $z \in Z(\text{Spin}(7))$ is the nontrivial element. Identify $V = W^*$ in such a way that $\{x,y,z,w\} \subseteq V$ is the dual basis to $\{z,\varepsilon,\varepsilon',\varepsilon''\}$. This gives an identification

$$H^*(BW;\mathbb{F}_2) = \mathbb{F}_2[x,y,z,w],$$

arranged such that the action of $N_{\text{Spin}(7)}(W)/W$ on $V = \langle x,y,z,w\rangle$ consists of all automorphisms which leave $\langle x,y,z\rangle$ invariant, and thus can be identified with the action of $GL_1^3(\mathbb{F}_2)$. Finally, set

$$\overline{H} = C_{\text{Spin}(7)}(z) = \text{Spin}(4) \times_{C_2} \text{Spin}(3) \cong SU(2)^3/\{\pm(I,I,I)\}$$

(the central product). Then in the same way, the action of $N_{\overline{H}}(W)/W$ on $H^*(BW;\mathbb{F}_2)$ can be identified with that of $GL_{2'}^2(\mathbb{F}_2)$.

Proposition 4.7 *The inclusions*

$$BW \longrightarrow B\overline{H} \longrightarrow B\text{Spin}(7) \longrightarrow BDI(4)$$

as defined above, together with the identification $H^*(BW;\mathbb{F}_2) = \mathbb{F}_2[x,y,z,w]$, *induce isomorphisms*

$$\mathfrak{A} \stackrel{\text{def}}{=} H^*(BDI(4);\mathbb{F}_2) \cong \mathbb{F}_2[x,y,z,w]^{GL_4(\mathbb{F}_2)} = \mathbb{F}_2[a_8,a_{12},a_{14},a_{15}]$$

$$\mathfrak{B} \stackrel{\text{def}}{=} H^*(B\text{Spin}(7);\mathbb{F}_2) \cong \mathbb{F}_2[x,y,z,w]^{GL_1^3(\mathbb{F}_2)} = \mathbb{F}_2[b_4,b_6,b_7,b_8] \qquad (*)$$

$$\mathfrak{C} \stackrel{\text{def}}{=} H^*(B\overline{H};\mathbb{F}_2) \cong \mathbb{F}_2[x,y,z,w]^{GL_{2'}^2(\mathbb{F}_2)} = \mathbb{F}_2[c_2,c_3,c_4',c_4''];$$

where

$$a_8 = D_1(x,y,z,w),\ a_{12} = D_2(x,y,z,w),$$
$$a_{14} = D_3(x,y,z,w),\ a_{15} = D_4(x,y,z,w),$$
$$b_4 = D_1(x,y,z),\quad b_6 = D_2(x,y,z),\quad b_7 = D_3(x,y,z),\quad b_8 = \prod_{\lambda\in\langle x,y,z\rangle}(w+\lambda);$$

and

$$c_2 = D_1(x,y),\quad c_3 = D_2(x,y),\quad c_4' = \prod_{\lambda\in\langle x,y\rangle}(z+\lambda),\quad c_4'' = \prod_{\lambda\in\langle x,y\rangle}(w+\lambda).$$

Furthermore,

(a) *the natural action of Σ_3 on $\overline{H} = SU(2)^3/\{\pm(I,I,I)\}$ induces the action on $\mathfrak{C}$ which fixes c_2, c_3 and permutes $\{c_4', c_4'', c_4' + c_4''\}$; and*

(b) *the above variables satisfy the relations*

$$a_8 = b_8 + b_4^2 \quad a_{12} = b_8 b_4 + b_6^2 \quad a_{14} = b_8 b_6 + b_7^2 \quad a_{15} = b_8 b_7$$
$$b_4 = c_4' + c_2^2 \quad b_6 = c_2 c_4' + c_3^2 \quad b_7 = c_3 c_4' \quad b_8 = c_4''(c_4' + c_4'').$$

Proof The formulas for $\mathfrak{A} = H^*(BDI(4);\mathbb{F}_2)$ are shown in [9]. From [9, Lemmas 3.10 and 3.11], we see there are (some) identifications

$$H^*(B\mathrm{Spin}(7);\mathbb{F}_2) \cong \mathbb{F}_2[x,y,z,w]^{GL_1^3(\mathbb{F}_2)} \quad\text{and}\quad H^*(B\overline{H};\mathbb{F}_2) \cong \mathbb{F}_2[x,y,z,w]^{GL_{2,0}^2(\mathbb{F}_2)}.$$

From the explicit choices of subgroups $W \subseteq \overline{H} \subseteq \mathrm{Spin}(7)$ as described above (and by the descriptions in Proposition A.8 of the automorphism groups), the images of $H^*(B\mathrm{Spin}(7);\mathbb{F}_2)$ and $H^*(B\overline{H};\mathbb{F}_2)$ in $\mathbb{F}_2[x,y,z,w]$ are seen to be contained in the rings of invariants, and hence these isomorphisms actually are equalities as claimed.

We next prove the equalities in (∗) between the given rings of invariants and polynomial algebras. The following argument was shown to us by Larry Smith. If k is a field and V is an n–dimensional vector space over k, then a *system of parameters* in the polynomial algebra $k[V]$ is a set of n homogeneous elements $f_1,\dots,f_n$ such that $k[V]/(f_1,\dots,f_n)$ is finite dimensional over k. By [21, Proposition 5.5.5], if V is an n–dimensional $k[G]$–representation, and $f_1,\dots,f_n \in k[V]^G$ is a system of parameters the product of whose degrees is equal to $|G|$, then $k[V]^G$ is a polynomial algebra with $f_1,\dots,f_n$ as generators. By [21, Proposition 8.1.7], $\mathbb{F}_2[x,y,z,w]$ is a free finitely generated module over the ring generated by its Dickson invariants (this holds for polynomial algebras over any $\mathbb{F}_p$), and thus $\mathbb{F}_2[x,y,z,w]/(a_8,a_{12},a_{14},a_{15})$ is finite. (This can also be shown directly using the relation in Lemma 4.6.) So assuming the relations in point (b), the quotients $\mathbb{F}_2[x,y,z,w]/(b_4,b_6,b_7,b_8)$ and $\mathbb{F}_2[x,y,z,w]/(c_2,c_3,c_4',c_4'')$ are also finite. In each case, the product of the degrees of the generators is clearly equal to the order of the group in question, and this finishes the proof of the last equality in the second and third lines of ( ).

It remains to prove points (a) and (b). Using Lemma 4.6, the c_i are expressed as polynomials in x,y,z,w as follows:

$$\begin{aligned} c_2 &= D_1(x,y) = x^2 + xy + y^2 \\ c_3 &= D_2(x,y) = xy(x+y) \\ c_4' &= D_1(x,y,z) + D_1(x,y)^2 = z^4 + z^2D_1(x,y) + zD_2(x,y) = z^4 + z^2c_2 + zc_3 \\ c_4'' &= D_1(x,y,w) + D_1(x,y)^2 = w^4 + w^2D_1(x,y) + wD_2(x,y) = w^4 + w^2c_2 + wc_3 . \end{aligned} \tag{1}$$

In particular,

$$c_4' + c_4'' = (z+w)^4 + (z+w)^2D_1(x,y) + (z+w)D_2(x,y) = \prod_{\in\langle x,y\rangle}(z+w+\ ). \tag{2}$$

Furthermore, by (1), we get

$$\begin{aligned} Sq^1(c_2) &= c_3 \\ Sq^1(c_3) &= Sq^1(c_4') = Sq^1(c_4'') = 0 \\ Sq^2(c_3) &= x^2y^2(x+y) + xy(x+y)^3 = c_2c_3 \\ Sq^2(c_4') &= z^4c_2 + z^2c_2^2 + zc_2c_3 = c_2c_4' \\ Sq^3(c_4') &= Sq^1(c_2c_4') = c_3c_4' \\ Sq^2(c_4'') &= c_2c_4'' \\ Sq^3(c_4'') &= c_3c_4''. \end{aligned} \tag{3}$$

The permutation action of Σ_3 on $\overline{H} = SU(2)^3/\{\pm(I,I,I)\}$ permutes the three elements $\ ,\ ,\ +\ $ of $Z(\overline{H}) \cong W$, and thus (via the identification $V \cong W$ described above) induces the identity on $x,y \in V$ and permutes the elements $\{z,w,z+w\}$ modulo $\langle x,y\rangle$. Hence the induced action of Σ_3 on $\mathfrak{C} = \mathbb{F}_2[V]^{GL_2^2{}_0(\mathbb{F}_2)}$ is the restriction of the action on $\mathbb{F}_2[V] = \mathbb{F}_2[x,y,z,w]$ which fixes x,y and permutes $\{z,w,z+w\}$. So by (1) and (2), we see that this action fixes c_2,c_3 and permutes the set $\{c_4',c_4'',c_4'+c_4''\}$. This proves (a).

It remains to prove the formulas in (b). From (1) and (3) we get

$$\begin{aligned} b_4 &= D_1(x,y,z) = c_4' + c_2^2, \\ b_6 &= D_2(x,y,z) = Sq^2(b_4) = c_2c_4' + c_3^2, \\ b_7 &= D_3(x,y,z) = Sq^1(b_6) = c_3c_4'. \end{aligned}$$

Also, by (1) and (2),

$$b_8 = \prod_{\in\langle x,y,z\rangle} (w + \ ) = \prod_{\in\langle x,y\rangle} (w + \ ) \prod_{\in\langle x,y\rangle} (w + z + \ ) = c_4''(c_4' + c_4'').$$

This proves the formulas for the b_i in terms of c_i. Finally, we have

$$\begin{aligned} a_8 &= D_1(x,y,z,w) = b_8 + b_4^2, \\ a_{12} &= D_2(x,y,z,w) = Sq^4(b_8 + b_4^2) = Sq^4(c_4''(c_4' + c_4'') + (c_4' + c_2^2)^2) \\ &= c_4'c_4''(c_4' + c_4'') + c_2^2c_4''(c_4' + c_4'') + c_2^2c_4'^2 + c_3^4 = b_8b_4 + b_6^2 \\ a_{14} &= D_3(x,y,z,w) = Sq^2(a_{12}) = c_2c_4'c_4''(c_4' + c_4'') + c_3^2c_4''(c_4' + c_4'') + c_3^2c_4'^2 \\ &= b_8b_6 + b_7^2 \\ a_{15} &= D_4(x,y,z,w) = Sq^1(a_{14}) = c_3c_4'c_4''(c_4' + c_4'') = b_8b_7\,; \end{aligned}$$

and this finishes the proof of the proposition. □

Lemma 4.8 *Let* $\in \mathrm{Aut}(\mathfrak{C})$ *be the algebra involution which exchanges* c_4' *and* c_4'' *and leaves* c_2 *and* c_3 *fixed. An element of* $\mathfrak{C}$ *will be called "–invariant" if it is fixed by this involution. Then the following hold:*

(a) *If* $\in \mathfrak{B}$ *is –invariant, then* $\in \mathfrak{A}$.

(b) *If* $\in \mathfrak{B}$ *is such that* $c_4'^{\,i}$ *is –invariant, then* $= {}' b_8^i$ *for some* $' \in \mathfrak{A}$.

Proof Point (a) follows from Proposition 4.7 upon regarding $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{C}$ as the fixed subrings of the groups $GL_4(\mathbb{F}_2)$, $GL_1^3(\mathbb{F}_2)$ and $GL_{2'}^2(\mathbb{F}_2)$ acting on $\mathbb{F}_2[x,y,z,w]$, but also follows from the following direct argument. Let m be the degree of as a polynomial in b_8; we argue by induction on m. Write $= {}_0 + b_8^m {}_1$, where ${}_1 \in \mathbb{F}_2[b_4,b_6,b_7]$, and where ${}_0$ has degree $< m$ (as a polynomial in b_8). If $m = 0$, then $= {}_1 \in \mathbb{F}_2[b_4,b_6,b_7] \subseteq \mathbb{F}_2[c_2,c_3,c_4']$, and hence $\in \mathbb{F}_2[c_2,c_3]$ since it is –invariant. But from the formulas in Proposition 4.7(b), we see that $\mathbb{F}_2[b_4,b_6,b_7] \cap \mathbb{F}_2[c_2,c_3]$ contains only constant polynomials (hence it is contained in $\mathfrak{A}$).

Now assume $m \geq 1$. Then, expressed as a polynomial in c_2,c_3,c_4',c_4'', the largest power of c_4'' which occurs in is $c_4''^{2m}$. Since is –invariant, the highest power of c_4' which occurs is $c_4'^{2m}$; and hence by Proposition 4.7(b), the total degree of each term in $_1$ (its degree as a polynomial in b_4, b_6, b_7) is at most m. So for each term $b_4^r b_6^s b_7^t$ in $_1$,

$$b_4^r b_6^s b_7^t b_8^m - a_8^{m-r-s-t} a_{12}^r a_{14}^s a_{15}^t$$

is a sum of terms which have degree $< m$ in b_8, and thus lies in $\mathfrak{A}$ by the induction hypothesis.

To prove (b), note first that since $c_4'^{\,i}$ is -invariant and divisible by $c_4'^{\,i}$, it must also be divisible by $c_4''^{\,i}$, and hence $c_4''^{\,i} |$. Furthermore, by Proposition 4.7, all elements of $\mathfrak{B}$ as well as c_4' are invariant under the involution which fixes c_4' and sends $c_4'' \mapsto c_4' + c_4''$. Thus $(c_4' + c_4'')^i |$. Since $b_8 = c_4''(c_4' + c_4'')$, we can now write $= {}' b_8^i$ for some $\in \mathfrak{B}$. Finally, since

$$c_4'^{\,i} = {}' c_4'^{\,i} c_4''^{\,i} (c_4' + c_4'')^i$$

is -invariant, $'$ is also -invariant, and hence $' \in \mathfrak{A}$ by (a). □

Note that C_3 $_3 = GL_2(\mathbb{F}_2)$ act on $\mathfrak{C} = \mathbb{F}_2[x, y, z, w]^{GL^2_{2'}(\mathbb{F}_2)}$: via the action of the group $GL^2_2(\mathbb{F}_2)/GL^2_{2'}(\mathbb{F}_2)$, or equivalently by permuting c_4', c_4'', and $c_4' + c_4''$ (and fixing c_2, c_3). Thus $\mathfrak{A} = \mathfrak{B} \cap \mathfrak{C}^{C_3}$, since $GL_4(\mathbb{F}_2)$ is generated by the subgroups $GL^3_1(\mathbb{F}_2)$ and $GL^2_2(\mathbb{F}_2)$. This is also shown directly in the following lemma.

Proposition 4.9 *The following square is a pullback square, where all maps are induced by inclusions between the subrings of* $\mathbb{F}_2[x, y, z, w]$*:*

$$\begin{array}{ccc} \Omega_{\mathfrak{A}/\mathbb{F}_2} & \longrightarrow & \left(\Omega_{\mathfrak{C}/\mathbb{F}_2}\right)^{C_3} \\ \big\downarrow & & \big\downarrow \\ \Omega_{\mathfrak{B}/\mathbb{F}_2} & \longrightarrow & \Omega_{\mathfrak{C}/\mathbb{F}_2}\,. \end{array}$$

Proof Let be the involution of Lemma 4.8: the algebra involution of $\mathfrak{C}$ which exchanges c_4' and c_4'' and leaves c_2 and c_3 fixed. By construction, all elements in the image of $\Omega_{\mathfrak{B}/\mathbb{F}_2}$ are invariant under the involution which fixes c_4' (and c_2, c_3), and sends c_4'' to $c_4' + c_4''$. Hence elements in the image of $\Omega_{\mathfrak{B}/\mathbb{F}_2}$ are fixed by C_3 if and only if they are fixed by $_3$, if and only if they are -invariant. So it will suffice to show that all of the above maps are injective, and that all -invariant elements in the image of $\Omega_{\mathfrak{B}/\mathbb{F}_2}$ lie in the image of $\Omega_{\mathfrak{A}/\mathbb{F}_2}$. The injectivity is clear, and the square is a pullback for $\Omega^0_{-/\mathbb{F}_2}$ by Lemma 4.8.

Fix a σ–invariant element

$$\begin{aligned}\omega &= P_1\,db_4 + P_2\,db_6 + P_3\,db_7 + P_4\,db_8 \\ &= P_2c_4'\,dc_2 + P_3c_4'\,dc_3 + P_4c_4'\,dc_4'' + (P_1 + P_2c_2 + P_3c_3 + P_4c_4'')\,dc_4' \in \Omega^1_{\mathfrak{B}/\mathbb{F}_2},\end{aligned} \tag{1}$$

where $P_i \in \mathfrak{B}$ for each i. By applying σ to (1) and comparing the coefficients of dc_2 and dc_3, we see that P_2c_4' and P_3c_4' are σ–invariant. Also, upon comparing the coefficients of dc_4', we get the equation

$$P_1 + P_2c_2 + P_3c_3 + P_4c_4'' = \sigma(P_4)c_4''. \tag{2}$$

So by Lemma 4.8, $P_2 = P_2'b_8$ and $P_3 = P_3'b_8$ for some $P_2', P_3' \in \mathfrak{A}$. Upon subtracting

$$P_2'\,da_{14} + P_3'\,da_{15} = P_2\,db_6 + P_3\,db_7 + (P_2'b_6 + P_3'b_7)\,db_8$$

from ω and introducing an appropriate modification to P_4, we can now assume that $P_2 = P_3 = 0$. With this assumption and (2), we have

$$P_1 + P_4c_4'' = \sigma(P_4c_4') = \sigma(P_4)\,c_4'';$$

so that

$$P_1c_4' = (P_4 + \sigma(P_4))c_4'c_4'' \tag{3}$$

is σ–invariant. This now shows that $P_1 = P_1'b_8$ for some $P_1' \in \mathfrak{A}$, and upon subtracting $P_1'\,da_{12}$ from ω we can assume that $P_1 = 0$. This leaves $\omega = P_4\,db_8 = P_4\,da_8$. By (3) again, P_4 is σ–invariant, so $P_4 \in \mathfrak{A}$ by Lemma 4.8 again, and thus $\omega \in \Omega^1_{\mathfrak{A}/\mathbb{F}_2}$.

The remaining cases are proved using the same techniques, and so we sketch them more briefly. To prove the result in degree two, fix a σ–invariant element

$$\begin{aligned}\omega &= P_1\,db_4\,db_6 + P_2\,db_4\,db_7 + P_3\,db_4\,db_8 + P_4\,db_6\,db_7 + P_5\,db_6\,db_8 + P_6\,db_7\,db_8 \\ &= P_4{c_4'}^2\,dc_2\,dc_3 + (P_1c_4' + P_4c_3c_4' + P_5c_4'c_4'')\,dc_2\,dc_4' + P_5{c_4'}^2\,dc_2\,dc_4'' \\ &\quad + (P_2c_4' + P_4c_2c_4' + P_6c_4'c_4'')\,dc_3\,dc_4' + P_6{c_4'}^2\,dc_3\,dc_4'' \\ &\quad + (P_3c_4' + P_5c_2c_4' + P_6c_3c_4')\,dc_4'\,dc_4'' \in \Omega^2_{\mathfrak{B}/\mathbb{F}_2}.\end{aligned}$$

Using Lemma 4.8, we see that $P_4 = P_4'b_8^2$, and hence can assume that $P_4 = 0$. One then eliminates P_1 and P_2, then P_5 and P_6, and finally P_3.

If

$$\begin{aligned}\omega &= P_1\,db_4\,db_6\,db_7 + P_2\,db_4\,db_6\,db_8 + P_3\,db_4\,db_7\,db_8 + P_4\,db_6\,db_7\,db_8 \\ &= (P_1{c_4'}^2 + P_4{c_4'}^2c_4'')\,dc_2\,dc_3\,dc_4' + (P_2{c_4'}^2 + P_4c_3{c_4'}^2)\,dc_2\,dc_4'\,dc_4'' \\ &\quad + (P_3{c_4'}^2 + P_4c_2{c_4'}^2)\,dc_3\,dc_4'\,dc_4'' + P_4{c_4'}^3\,dc_2\,dc_3\,dc_4'' \in \Omega^3_{\mathfrak{B}/\mathbb{F}_2}\end{aligned}$$

is –invariant, then we eliminate successively P_1, then P_4, then P_2 and P_3. Finally, if

$$\omega = P\, db_4\, db_6\, db_7\, db_8 = P c_4'^{\,3}\, dc_2\, dc_3\, dc_4'\, dc_4'' \in \Omega^4_{\mathfrak{B}/\mathbb{F}_2}$$

is –invariant, then $P = P' b_8^3$ for some $P' \in \mathfrak{A}$ by Lemma 4.8 again, and so

$$\omega = P'\, da_8\, da_{12}\, da_{14}\, da_{15} \in \Omega^4_{\mathfrak{A}/\mathbb{F}_2}. \qquad \square$$

A Appendix: Spinor groups over finite fields

Let F be any field of characteristic $\neq 2$. Let V be a vector space over F, and let $\mathfrak{b}\colon V \longrightarrow F$ be a nonsingular quadratic form. As usual, $O(V,\mathfrak{b})$ denotes the group of isometries of $(V,\mathfrak{b})$, and $SO(V,\mathfrak{b})$ the subgroup of isometries of determinant 1. We will be particularly interested in elementary abelian 2–subgroups of such orthogonal groups.

Lemma A.1 *Fix an elementary abelian 2–subgroup $E \subseteq O(V,\mathfrak{b})$. For each irreducible character $\chi \in \mathrm{Hom}(E,\{\pm 1\})$, let $V_\chi \subseteq V$ denote the corresponding eigenspace: the subspace of elements $v \in V$ such that $g(v) = \chi(g)\cdot v$ for all $g \in E$. Then the restriction of $\mathfrak{b}$ to each subspace V_χ is nonsingular, and V is the orthogonal direct sum of the V_χ.*

Proof Elementary. $\square$

We give a very brief sketch of the definition of spinor groups via Clifford algebras; for more details we refer to [8, section II.7] or [2, section 22]. Let $T(V)$ denote the tensor algebra of V, and set

$$C(V,\mathfrak{b}) = T(V)/\langle (v \otimes v) - \mathfrak{b}(v) \rangle,$$

the Clifford algebra of $(V,\mathfrak{b})$. To simplify the notation, we regard F as a subring of $C(V,\mathfrak{b})$, and V as a subgroup of its additive group; thus the class of $v_1 \otimes \cdots \otimes v_k$ will be written $v_1 \cdots v_k$. Note that $vw + wv = 0$ if $v,w \in V$ and $v \perp w$. Hence if $\dim_F(V) = n$, and $\{v_1,\ldots,v_n\}$ is an orthogonal basis, then the set of 1 and all $v_{i_1}\cdots v_{i_k}$ for $i_1 < \cdots < i_k$ $(1 \leq k \leq n)$ is an F–basis for $C(V,\mathfrak{b})$.

Write $C(V,\mathfrak{b}) = C_0 \oplus C_1$, where C_0 and C_1 consist of classes of elements of even or odd degree, respectively. Let $G \subseteq C(V,\mathfrak{b})^*$ denote the group of invertible elements u such that $uVu^{-1} = V$, and let $\rho\colon G \longrightarrow O(V,\mathfrak{b})$ be the homomorphism

$$\rho(u) = \begin{cases} (v \mapsto -uvu^{-1}) & \text{if } u \in C_1 \\ (v \mapsto uvu^{-1}) & \text{if } u \in C_0\,. \end{cases}$$

In particular, for any nonisotropic element $v \in V$ (ie, $\mathfrak{b}(v) \neq 0$), $v \in G$ and $\rho(v)$ is the reflection in the hyperplane $v^{\perp}$. By [8, section II.7], ρ is surjective and $\mathrm{Ker}(\rho) = F^*$.

Let J be the antiautomorphism of $C(V,\mathfrak{b})$ induced by the antiautomorphism $v_1 \otimes \cdots \otimes v_k \mapsto v_k \otimes \cdots \otimes v_1$ of $T(V)$. Since $O(V,\mathfrak{b})$ is generated by hyperplane reflections, G is generated by F^* and nonisotropic elements $v \in V$. In particular, for any $u = \lambda v_1 \cdots v_k \in G$,

$$J(u)\cdot u = \lambda^2 \cdot v_k \cdots v_1 \cdot v_1 \cdots v_k = \lambda^2 \cdot \mathfrak{b}(v_1) \cdots \mathfrak{b}(v_k) \in F^* = \mathrm{Ker}(\rho);$$

implying that $\rho(J(u)) = \rho(u)^{-1}$ for all $u \in G$. There is thus a homomorphism

$$\widetilde{\sigma}\colon G \xrightarrow{\quad} F^* \qquad \text{defined by} \qquad \widetilde{\sigma}(u) = u \cdot J(u).$$

In particular, $\widetilde{\sigma}(\lambda) = \lambda^2$ for $\lambda \in F^* \subseteq G$, while for any set of nonisotropic elements $v_1, \ldots, v_k$ of V,

$$\widetilde{\sigma}(v_1 \cdots v_k) = (v_1 \cdots v_k)(v_k \cdots v_1) = \mathfrak{b}(v_1) \cdots \mathfrak{b}(v_k).$$

Hence $\widetilde{\sigma}$ factors through a homomorphism

$$\sigma_{V,\mathfrak{b}}\colon O(V,\mathfrak{b}) \xrightarrow{\quad} F^*/F^{*2} = F^*/\{u^2 \mid u \in F^*\},$$

called the *spinor norm*.

Set $G^+ = \rho^{-1}(SO(V,\mathfrak{b})) = G \cap C_0$, and define

$$\mathrm{Spin}(V,\mathfrak{b}) = \mathrm{Ker}(\widetilde{\sigma}|_{G^+}) \qquad \text{and} \qquad \Omega(V,\mathfrak{b}) = \mathrm{Ker}(\sigma_{V,\mathfrak{b}}|_{SO(V,\mathfrak{b})}).$$

In particular, $\Omega(V,\mathfrak{b})$ has index 2 in $SO(V,\mathfrak{b})$ if F is a finite field, and $\Omega(V,\mathfrak{b}) = SO(V,\mathfrak{b})$ if F is algebraically closed (all units are squares). We thus get a commutative diagram

$$
\text{(A.2)}\qquad
\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \{\pm 1\} & \longrightarrow & F^* & \xrightarrow{\lambda\mapsto\lambda^2} & F^{*2} & \longrightarrow & 1\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \operatorname{Spin}(V,\mathfrak{b}) & \longrightarrow & G^+ & \xrightarrow{\ \tilde{}\ } & F^* & \longrightarrow & 1\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Omega(V,\mathfrak{b}) & \longrightarrow & SO(V,\mathfrak{b}) & \xrightarrow{\theta_{V,\mathfrak{b}}} & F^*/F^{*2} & \longrightarrow & 1\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}
$$

where all rows and columns are short exact, and where all columns are central extensions of groups. If $\dim(V) \geq 3$ (or if $\dim(V) = 2$ and the form $\mathfrak{b}$ is hyperbolic), then $\Omega(V,\mathfrak{b})$ is the commutator subgroup of $SO(V,\mathfrak{b})$ [8, section II.8].

The following lemma follows immediately from this description of $\operatorname{Spin}(V,\mathfrak{b})$, together with the analogous description of the corresponding spinor group over the algebraic closure of F.

Lemma A.3 *Let $\overline{F}$ be the algebraic closure of F, and set $\overline{V} = \overline{F} \otimes_F V$ and $\bar{\mathfrak{b}} = \mathrm{Id}_{\overline{F}} \otimes \mathfrak{b}$. Then $\operatorname{Spin}(V,\mathfrak{b})$ is the subgroup of $\operatorname{Spin}(\overline{V},\bar{\mathfrak{b}})$ consisting of those elements fixed by all Galois automorphisms $\sigma \in \operatorname{Gal}(\overline{F}/F)$.* □

For any nonsingular quadratic form $\mathfrak{b}$ on a vector space V, the *discriminant* of $\mathfrak{b}$ (or of V) is the determinant of the corresponding symmetric bilinear form B, related to $\mathfrak{b}$ by the formulas

$$\mathfrak{b}(v) = B(v,v) \qquad \text{and} \qquad B(v,w) = \tfrac{1}{2}\big(\mathfrak{b}(v+w) - \mathfrak{b}(v) - \mathfrak{b}(w)\big) .$$

Note that the discriminant is well defined only modulo squares in F^*. When $W \subseteq V$ is a subspace, then we define the discriminant of W to mean the discriminant of $\mathfrak{b}|_W$. In what follows, we say that the discriminant of a quadratic form is a square or a nonsquare to mean that it is the identity or not in the quotient group F^*/F^{*2}.

Lemma A.4 *Fix an involution $x \in SO(V,\mathfrak{b})$, and let $V = V_+ \oplus V_-$ be its eigenspace decomposition. Then the following hold.*

(a) *$x \in \Omega(V,\mathfrak{b})$ if and only if the discriminant of V_- is a square.*

(b) *If $x \in \Omega(V,\mathfrak{b})$, then it lifts to an element of order 2 in* $\mathrm{Spin}(V,\mathfrak{b})$ *if and only if* $\dim(V_-) \in 4\mathbb{Z}$.

(c) *If $x \in \Omega(V,\mathfrak{b})$, and if $\alpha \in \Omega(V,\mathfrak{b})$ is such that $[x,\alpha]=1$, then $\alpha = \alpha_+ \oplus \alpha_-$, where $\alpha_\pm \in O(V_\pm,\mathfrak{b})$. Also, the liftings of x and α commute in* $\mathrm{Spin}(V,\mathfrak{b})$ *if and only if* $\det(\alpha_-)=1$.

Proof Let $\{v_1,\dots,v_k\}$ be an orthogonal basis for V_- (k is even). Then $x = \rho(v_1 \cdots v_k)$ in the above notation, since $\rho(v_i)$ is the reflection in the hyperplane $v_i^\perp$. Hence by the commutativity of Diagram (A.2),

$$\theta_{V,\mathfrak{b}}(x) \equiv \mathfrak{b}(v_1)\cdots\mathfrak{b}(v_k) = \det(\mathfrak{b}|_{V_-}) \pmod{F^{\times 2}}.$$

Thus $x \in \Omega(V,\mathfrak{b}) = \mathrm{Ker}(\theta_{V,\mathfrak{b}})$ if and only if V_- has square discriminant.

In particular, if $x \in \Omega(V,\mathfrak{b})$, then the product of the $\mathfrak{b}(v_i)$ is a square, and hence (upon replacing v_1 by a scalar multiple) we can assume that $\mathfrak{b}(v_1)\cdots\mathfrak{b}(v_k)=1$. Then $\widetilde{x} \overset{\text{def}}{=} v_1 \cdots v_k \in \mathrm{Spin}(V,\mathfrak{b}) = \mathrm{Ker}(\widetilde{\theta})$. Since $vw=-wv$ in the Clifford algebra whenever $v \perp w$, and since $(v_i)^2 = \mathfrak{b}(v_i)$ for each i,

$$\widetilde{x}^2 = (-1)^{k(k-1)/2}(v_1)^2\cdots(v_k)^2 = (-1)^{k(k-1)/2} = \begin{cases} 1 & \text{if } k \equiv 0 \pmod 4 \\ -1 & \text{if } k \equiv 2 \pmod 4. \end{cases}$$

This proves (b).

It remains to prove (c). The first statement ($\alpha = \alpha_+ \oplus \alpha_-$) is clear. Fix liftings $\widetilde{\alpha}_\pm \in C(V_\pm,\mathfrak{b})^*$. Rather than defining the direct sum of an element of $C(V_+,\mathfrak{b})$ with an element of $C(V_-,\mathfrak{b})$, we regard the groups $C(V_\pm,\mathfrak{b})^*$ as (commuting) subgroups of $C(V,\mathfrak{b})^*$, and set

$$\widetilde{\alpha} = \widetilde{\alpha}_+ \cdot \widetilde{\alpha}_- = \widetilde{\alpha}_- \cdot \widetilde{\alpha}_+ \in \mathrm{Spin}(V,\mathfrak{b}).$$

Let $\widetilde{x} = v_1 \cdots v_k$ be as above. Clearly, $\widetilde{x}$ commutes with all elements of $C(V_+,\mathfrak{b})$. Since

$$(v_1 \cdots v_k)\, v_i = (-1)^{k-1}\, v_i\, (v_1 \cdots v_k) = -v_i\,(v_1 \cdots v_k)$$

for $i = 1,\dots,k$, we have $\widetilde{x}\xi = (-1)^i \xi\widetilde{x}$ for all $\xi \in C_i(V_-,\mathfrak{b})$ ($i=0,1$). In particular, since $[\widetilde{\alpha}_+,\widetilde{\alpha}_-]=1$, $[\widetilde{x},\widetilde{\alpha}] = [\widetilde{x},\widetilde{\alpha}_-] = \det(\alpha_-)$, and this finishes the proof. □

We will need explicit isomorphisms which describe $\mathrm{Spin}_3(F)$ and $\mathrm{Spin}_4(F)$ in terms of $SL_2(F)$. These are constructed in the following proposition, where $M_2^0(F)$ denotes the vector space of matrices of trace zero. Note that the determinant is a nonsingular quadratic form on $M_2(F)$ and on $M_2^0(F)$, in both cases with square discriminant.

Proposition A.5 *Define*

$$\rho_3\colon SL_2(F) \longrightarrow \Omega(M_2^0(F),\det)$$

and

$$\rho_4\colon SL_2(F)\times SL_2(F) \longrightarrow \Omega(M_2(F),\det)$$

by setting

$$\rho_3(A)(X) = AXA^{-1} \qquad \text{and} \qquad \rho_4(A,B)(X) = AXB^{-1}.$$

Then ρ_3 *and* ρ_4 *are both epimorphisms, and lift to unique isomorphisms*

$$SL_2(F) \xrightarrow[\cong]{\widetilde{\rho}_3} \mathrm{Spin}(M_2^0(F),\det)$$

and

$$SL_2(F)\times SL_2(F) \xrightarrow[\cong]{\widetilde{\rho}_4} \mathrm{Spin}(M_2(F),\det).$$

Proof See [24, pages 142, 199] for other ways of defining these isomorphisms. By Lemma A.3, it suffices to prove this (except for the uniqueness of the lifting) when F is algebraically closed. In particular,

$$\Omega(M_2^0(F),\det) = SO(M_2^0(F),\det) \quad \text{and} \quad \Omega(M_2(F),\det) = SO(M_2(F),\det)$$

in this case.

For general V and $\mathfrak{b}$, the group $SO(V,\mathfrak{b})$ is generated by reflections fixing nonisotropic subspaces (ie, of nonvanishing discriminant) of codimension 2 (cf [8, section II.6(1)]). Hence to see that ρ_3 and ρ_4 are surjective, it suffices to show that such elements lie in their images. A codimension 2 reflection in $SO(M_2^0(F),\det)$ is of the form R_X (the reflection fixing the line generated by X) for some $X\in M_2^0(F)$ which is nonisotropic (ie, $\det(X)\neq 0$). Since F is algebraically closed, we can assume $X\in SL_2(F)$. Then $X^2=-I$ (since $\mathrm{Tr}(X)=0$ and $\det(X)=1$), and $R_X=\rho_3(X)$ since it has order 2 and fixes X. Thus ρ_3 is onto.

Similarly, any 2–dimensional nonisotropic subspace $W\subseteq V$ has an orthonormal basis $\{Y,Z\}$, and ZY^{-1} and $Y^{-1}Z$ have trace zero (since they are orthogonal to the identity matrix) and determinant one. Hence their square is $-I$, and one repeats the above argument to show that $R_W=\rho_4(ZY^{-1},Y^{-1}Z)$. So ρ_4 is onto.

The liftings $\widetilde{\rho}_m$ exist and are unique since $SL_2(F)$ is the universal central extension of $PSL_2(F)$ (or universal among central extensions by 2–groups if $F=\mathbb{F}_3$). □

We now restrict to the case $F = \mathbb{F}_q$ where q is an odd prime power. We refer to [2, section 21] for a description of quadratic forms in this situation, and the notation for the associated orthogonal groups. If n is odd and $\mathfrak{b}$ is any nonsingular quadratic form on $\mathbb{F}_q^n$, then every nonsingular quadratic form is isomorphic to $u\mathfrak{b}$ for some $u \in \mathbb{F}_q$, and hence one can write $SO_n(q) = SO(\mathbb{F}_q^n; \mathfrak{b}) = SO(\mathbb{F}_q^n; u\mathfrak{b})$ without ambiguity. If n is even, then there are exactly two isomorphism classes of quadratic forms on $\mathbb{F}_q^n$; and one writes $SO_n^+(q) = SO(\mathbb{F}_q^n; \mathfrak{b})$ when $\mathfrak{b}$ is the hyperbolic form (equivalently, has discriminant $(-1)^{n/2}$ modulo squares), and $SO_n^-(q) = SO(\mathbb{F}_q^n; \mathfrak{b})$ when $\mathfrak{b}$ is not hyperbolic (equivalently, has discriminant $(-1)^{n/2} u$ for $u \in \mathbb{F}_q$ not a square). This notation extends in the obvious way to $\Omega_n(q)$ and $\mathrm{Spin}_n(q)$.

The following lemma does, in fact, hold for for orthogonal representations over arbitrary fields of characteristic $\neq 2$. But to simplify the proof (and since we were unable to find a reference), we state it only in the case of finite fields.

Lemma A.6 *Assume $F = \mathbb{F}_q$, where q is a power of an odd prime. Let V be an F–vector space, and let $\mathfrak{b}$ be a nonsingular quadratic form on V. Let $P \subseteq O(V; \mathfrak{b})$ be a 2–subgroup which is orthogonally irreducible; ie, such that V has no splitting as an orthogonal direct sum of nonzero P–invariant subspaces. Then $\dim_F(V)$ is a power of 2; and if $\dim(V) > 1$ then $\mathfrak{b}$ has square discriminant.*

Proof This means showing that each orthogonal group $O(\mathbb{F}_q{}^n; \mathfrak{b})$, such that either n is not a power of 2, or $n = 2^k \geq 2$ and the quadratic form $\mathfrak{b}$ has nonsquare discriminant, contains some subgroup $O_m(q) \times O_{n-m}(q)$ (for $0 < m < n$) of odd index. We refer to the standard formulas for the orders of these groups (see [24, p.165]): if $\varepsilon = \pm 1$ then

$$|O_{2n}(q)| = 2q^{n(n-1)}(q^n - \varepsilon)\prod_{i=1}^{n-1}(q^{2i}-1) \quad \text{and} \quad |O_{2n+1}(q)| = 2q^{n^2}\prod_{i=1}^{n}(q^{2i}-1).$$

We will also use repeatedly the fact that for all $0 < i < 2^k$ ($k \geq 1$), the largest powers of 2 dividing $(q^{2^k+i}-1)$ and (q^i-1) are the same. In other words, $(q^{2^k+i}-1)/(q^i-1)$ is invertible in $\mathbb{Z}_{(2)}$.

For any $n \geq 1$,

$$\frac{|O_{2n+1}(q)|}{|O_{2n}(q)|\,|O_1(q)|} = q^n\,\frac{q^n+\varepsilon}{2}$$

is odd for an appropriate choice of ε. Thus, there are no irreducibles of odd dimension.

Assume n is not a power of 2, and write $n = 2^k + m$ where $0 < m < 2^k$ and $k \geq 1$. Then

$$\frac{|O_{2n}(q)|}{|O^+_{2^{k+1}}(q)|\,|O_{2m}(q)|} = q^{m2^{k+1}} \left(\prod_{i=1}^{m-1} \frac{q^{2(2^k+i)}-1}{q^{2i}-1} \right) \left(\frac{q^{2^k+m}-\varepsilon}{q^m-\varepsilon} \right) \left(\frac{q^{2^k}+1}{2} \right);$$

and each factor is invertible in $\mathbb{Z}_{(2)}$. When $n = 2m = 2^k$ and $k \geq 1$, then $O^-_{2n}(q)$ is the orthogonal group for the quadratic form with nonsquare discriminant, and

$$\frac{|O^-_{2n}(q)|}{|O^+_{2m}(q)|\,|O^-_{2m}(q)|} = q^{2m^2} \left(\prod_{i=1}^{m-1} \frac{q^{2(m+i)}-1}{q^{2i}-1} \right) \frac{q^{2m}+1}{2};$$

and again each factor is invertible in $\mathbb{Z}_{(2)}$. Finally,

$$\frac{|O_2(q)|}{|O_1(q)|\,|O_1(q)|} = \frac{q-\varepsilon}{2}$$

is odd whenever $q \equiv 1 \pmod 4$ and $\varepsilon = -1$, or $q \equiv 3 \pmod 4$ and $\varepsilon = +1$; and these are precisely the cases where the quadratic form on $\mathbb{F}_q{}^2$ has nonsquare discriminant. □

We must classify the conjugacy classes of those elementary abelian 2–subgroups of $\mathrm{Spin}_7(q)$ which contain its center. The following definition will be useful when doing this.

Definition A.7 Fix an odd prime power q. Identify $SO_7(q) = SO(\mathbb{F}_q^7, \mathfrak{b})$ and $\mathrm{Spin}_7(q) = \mathrm{Spin}(\mathbb{F}_q^7, \mathfrak{b})$, where $\mathfrak{b}$ is a nonsingular quadratic form with square discriminant. An elementary abelian 2–subgroup of $SO_7(q)$ or of $\mathrm{Spin}_7(q)$ will be called of *type I* if its eigenspaces all have square discriminant (with respect to $\mathfrak{b}$), and of *type II* otherwise. Let $\mathcal{E}_n$ be the set of elementary abelian 2–subgroups in $\mathrm{Spin}_7(q)$ which contain $Z(\mathrm{Spin}_7(q)) = C_2$ and have rank n. Let $\mathcal{E}_n^I$ and $\mathcal{E}_n^{II}$ be the subsets of $\mathcal{E}_n$ consisting of those subgroups of types I and II, respectively.

In the following two propositions, we collect together the information which will be needed about elementary abelian 2–subgroups of $\mathrm{Spin}_7(q)$. We fix $\mathrm{Spin}_7(q) = \mathrm{Spin}(V, \mathfrak{b})$, where $V = \mathbb{F}_q^7$, and $\mathfrak{b}$ is a nonsingular quadratic form *with square discriminant*. Let $z \in Z(\mathrm{Spin}_7(q))$ be the generator. For any subgroup $H \subseteq \mathrm{Spin}_7(q)$ or any element $g \in \mathrm{Spin}_7(q)$, let $\overline{H}$ and $\bar{g}$ denote their images in $\Omega_7(q) \subseteq SO_7(q)$. For each elementary abelian 2–subgroup $E \subseteq \mathrm{Spin}_7(q)$, and each character $\chi \in \mathrm{Hom}(\overline{E}, \{\pm 1\})$, $V_\chi \subseteq V$ denotes the eigenspace of (and V_1 denotes the eigenspace of the trivial character). Also (when $z \in E$), $\mathrm{Aut}(E;z)$ denotes the group of all automorphisms of E which send z to itself.

Proposition A.8 *For any odd prime power q, the following table describes the numbers of $\mathrm{Spin}_7(q)$-conjugacy classes in each of the sets $\mathcal{E}_n^I$ and $\mathcal{E}_n^{II}$, the dimensions and discriminants of the eigenspaces of subgroups in these sets, and indicates in which cases $\mathrm{Aut}_{\mathrm{Spin}_7(q)}(E)$ contains all automorphisms which x z.*

Set of subgroups	$\mathcal{E}_2^I$	$\mathcal{E}_3^I$	$\mathcal{E}_3^{II}$	$\mathcal{E}_4^I$	$\mathcal{E}_4^{II}$
Nr. conj. classes	1	1	1	2	1
$\dim(V_1)$	3	1		0	
$\dim(V)$, $\ne 1$	4	2		1	
$\mathrm{discr}(V_1;\mathfrak{b})$	square	square	nonsq.	—	—
$\mathrm{discr}(V;\mathfrak{b})$, $\ne 1$	square	square	nonsq.	square	both
$\mathrm{Aut}_{\mathrm{Spin}_7(q)}(E) = \mathrm{Aut}(E;z)$	yes	yes	yes	yes	no

There are no subgroups in $\mathcal{E}_2$ of type II, and no subgroups of rank 5. Furthermore, we have:

(a) *For all $E \in \mathcal{E}_4$, $C_{\mathrm{Spin}_7(q)}(E) = E$.*

(b) *If $E, E' \in \mathcal{E}_4^I$, then $\overline{E'} = g\overline{E}g^{-1}$ for some $g \in SO_7(q)$, and E and E' are $\mathrm{Spin}_7(q)$-conjugate if and only if $g \in \Omega_7(q)$.*

(c) *If $E \in \mathcal{E}_4^{II}$, then there is a unique element $1 \ne \bar{x} = \bar{x}(E) \in \overline{E}$ with the property that for $1 \ne \in \mathrm{Hom}(\overline{E}, \{\pm 1\})$, V has square discriminant if $(\bar{x}) = 1$ and nonsquare discriminant if $(\bar{x}) = -1$. Also, the image of $\mathrm{Aut}_{\mathrm{Spin}_7(q)}(E)$ in $\mathrm{Aut}(\overline{E})$ is the group of all automorphisms which send $\bar{x}$ to itself; and if X E denotes the inverse image of $\langle\bar{x}\rangle$ $\overline{E}$, then $\mathrm{Aut}_{\mathrm{Spin}_7(q)}(E)$ contains all automorphisms of $\overline{E}$ which are the identity on X and the identity modulo $\langle z\rangle$.*

(d) *If $E \in \mathcal{E}_3$, then $C_{\mathrm{Spin}_7(q)}(E) = A \rtimes C_2$, where A is abelian and C_2 acts on A by inversion. If $E \in \mathcal{E}_3^{II}$, then the Sylow 2-subgroups of $C_{\mathrm{Spin}_7(q)}(E)$ are elementary abelian of rank 4 (and type II).*

Proof Write $\mathrm{Spin} = \mathrm{Spin}_7(q)$ for short. Fix an elementary abelian subgroup E Spin such that $z \in E$.

Step 1 We first show that $\mathrm{rk}(E) \le 4$, and that the dimensions of the eigenspaces V_χ for $\chi \in \mathrm{Hom}(\bar{E}, \{\pm 1\})$ are as described in the table.

By Lemma A.4, every involution in $\bar{E}$ has a 4–dimensional (-1)–eigenspace. In particular, if $\mathrm{rk}(E) = 2$, ($\bar{E} \cong C_2$), then $\dim(V_\chi) = 4$ for $1 \ne \chi \in \mathrm{Hom}(\bar{E}, \{\pm 1\})$, while $\dim(V_1) = 3$.

Now assume $\mathrm{rk}(E) = n$ for some $n > 2$. Assume we have shown, for all $E' \in \mathcal{E}_{n-1}$, that the eigenspace of the trivial character of $\bar{E}'$ is r–dimensional. For each $1 \ne \chi \in \mathrm{Hom}(\bar{E}, \{\pm 1\})$, let $E_\chi \in \mathcal{E}_{n-1}$ be the subgroup such that $\bar{E}_\chi = \mathrm{Ker}(\chi)$; then $V_1 \oplus V_\chi$ is the eigenspace of the trivial character of $\bar{E}_\chi = \mathrm{Ker}(\chi)$, and thus $\dim(V_1) + \dim(V_\chi) = r$. Hence all nontrivial characters of $\bar{E}$ have eigenspaces of the same dimension. Since there are $2^{n-1} - 1$ nontrivial characters of $\bar{E}$, we have $\dim(V_1) + (2^{n-1} - 1)\dim(V_\chi) = 7$, and these two equations completely determine $\dim(V_1)$ and $\dim(V_\chi)$. Using this procedure, the dimensions of the eigenspaces are shown inductively to be equal to those given by the table. Also, this shows that $\mathrm{rk}(E) \le 4$, since otherwise $\mathrm{rk}(\bar{E}) \ge 4$, so the V_χ for $\chi \ne 1$ must be trivial (they cannot all have dimension ≥ 1), so E acts on V via the identity, which contradicts the assumption that $E \le \mathrm{Spin}_7(q)$.

Step 2 We next show that $\mathcal{E}_2'' = \emptyset$, describe the discriminants of the eigenspaces of characters of $\bar{E}$ for $E \in \mathcal{E}_n$ (for all n), and show that subgroups of rank 4 are self centralizing. In particular, this proves (a) together with the first statement of (c).

If $E \in \mathcal{E}_2$, then $E = \langle z, g \rangle$ for some noncentral involution $g \in \mathrm{Spin}_7(q)$, and the eigenspaces of $\bar{E} = \langle \bar{g} \rangle$ have square discriminant by Lemma A.4(a) (and since the ambient space V has square discriminant by assumption). Thus $\mathcal{E}_2'' = \emptyset$.

If $E \in \mathcal{E}_3$, then the sum of any two eigenspaces of $\bar{E}$ is an eigenspace of $\bar{g}$ for some $g \in E \smallsetminus \langle z \rangle$. Hence the sum of any two eigenspaces of $\bar{E}$ has square discriminant, so either all of the eigenspaces have square discriminant ($E \in \mathcal{E}_3'$), or all of the eigenspaces have nonsquare discriminant ($E \in \mathcal{E}_3''$).

Assume $\mathrm{rk}(E) = 4$. We have seen that all eigenspaces of $\bar{E}$ are 1–dimensional. By Lemma A.4(c), for each $a \in C_{\mathrm{Spin}_7(q)}(E)$, $\bar{a}(V_\chi) = V_\chi$ for each $\chi \ne 1$, and since $\dim(V_\chi) = 1$ it must act on each V_χ via $\pm\mathrm{Id}$. Thus $\bar{a} \in \Omega_7(q)$ has order 2; let $V_\pm$ be its eigenspaces. Then $\dim(V_-)$ is even since $\det(\bar{a}) = 1$, and V_- has square discriminant by Lemma A.4(a). If $\dim(V_-) = 4$, then $|a| = 2$ (Lemma A.4(b)), and hence $a \in E$ since otherwise $\langle E, a \rangle$ would have rank 5. If $\dim(V_-) = 2$, then V_- is the sum of the eigenspaces of two distinct characters χ_1, χ_2 of $\bar{E}$, there is some $g \in E$ such that $\chi_1(\bar{g}) \ne \chi_2(\bar{g})$, hence $\det(\bar{g}|_{V_-}) =$

$_1(\bar g)\ _2(\bar g) = -1$, so $[g, a] = z$ by Lemma A.4(c), and this contradicts the assumption that $[a, E] = 1$. If $\dim(V_-) = 6$, then V_- is the sum of the eigenspaces of all but one of the nontrivial characters of $\bar E$, and this gives a similar contradiction to the assumption $[a, E] = 1$. Thus, $C_{\mathrm{Spin}_7(q)}(E) = E$.

Now assume that $E \in E_4^{II}$, and let $\bar x \in O_7(q)$ be the element which acts via $-\mathrm{Id}$ on eigenspaces with nonsquare discriminant, and via the identity on those with square discriminant. Since b has square discriminant on V, the number of eigenspaces of $\bar E$ on which the discriminant is nonsquare is even, so $\bar x \in \Omega_7(q)$ by Lemma A.4(a), and lifts to an element $x \in \mathrm{Spin}_7(q)$. Also, for each $g \in E$, the (-1)–eigenspace of $\bar g$ has square discriminant (Lemma A.4(a) again), hence contains an even number of eigenspaces of $\bar E$ of nonsquare discriminant, and by Lemma A.4(c) this shows that $[g, x] = 1$. Thus $x \in C_{\mathrm{Spin}_7(q)}(E) = E$, and this proves the first statement in (c).

Step 3 We next check the numbers of conjugacy classes of subgroups in each of the sets E_n^I and E_n^{II}, and describe $\mathrm{Aut}_{\mathrm{Spin}}(E)$ in each case. This finishes the proof of (b) and (c), and of all points in the above table.

From the above description, we see immediately that if E and E' have the same rank and type, then any isomorphism $\in \mathrm{Iso}(\bar E, \bar E')$, such that $(\bar x(E)) = \bar x(E')$ if $E, E' \in E_4^{II}$, has the property that for all $\in \mathrm{Hom}(\bar E', \{\pm 1\})$, V and V have the same dimension and the same discriminant (modulo squares). Hence for any such , there is an element $g \in O_7(q)$ such that $g(V\ ) = V$ for all ; and $= c_g \in \mathrm{Iso}(\bar E, \bar E')$ for such g. Upon replacing g by $-g$ if necessary, we can assume that $g \in SO_7(q)$. This shows that

$$E, E' \text{ have the same rank and type} \implies \bar E \text{ and } \bar E' \text{ are } SO_7(q)\text{–conjugate} \tag{1}$$

and also that

$$\mathrm{Aut}_{SO_7(q)}(\bar E) = \begin{cases} \mathrm{Aut}(\bar E) & \text{if } E \notin E_4^{II} \\ \mathrm{Aut}(\bar E, \bar x(E)) & \text{if } E \in E_4^{II}. \end{cases} \tag{2}$$

We next claim that

$$E \notin E_4^{I} \implies \exists \gamma \in SO_7(q) \setminus \Omega_7(q) \text{ such that } [\gamma, \bar E] = 1. \tag{3}$$

To prove this, choose 1–dimensional nonisotropic summands $W \quad V$ and $W' \quad V$, where $\ ,\ $ are two distinct characters of $\bar E$, and where W has square discriminant and W' has nonsquare discriminant. Let $\gamma \in SO_7(q)$ be the involution with (-1)–eigenspace $W \quad W'$. Then $[\gamma, \bar E] = 1$, since γ sends each eigenspace of $\bar{E}$ to itself, and $\gamma \notin \Omega_7(q)$ since its (-1)–eigenspace has nonsquare discriminant (Lemma A.4(a)).

If E has rank 4 and type I, then $\mathrm{Aut}(\bar{E}) \cong GL_3(\mathbb{F}_2)$ is simple, and in particular has no subgroup of index 2. Hence by (2), each element of $\mathrm{Aut}(\bar{E})$ is induced by conjugation by an element of $\Omega_7(q)$. Also, if $g \in SO_7(q)$ centralizes $\bar{E}$, then $g(V_\rho) = V_\rho$ for all $\rho \in \mathrm{Hom}(\bar{E},\{\pm 1\})$, g acts via $-\mathrm{Id}$ on an even number of eigenspaces (since it has determinant $+1$), and hence $g \in \Omega_7(q)$ by Lemma A.4(a). Thus

$$E \in \mathcal{E}_4^{I} \implies N_{SO_7(q)}(\bar{E}) \subseteq \Omega_7(q) \tag{4}$$

If $E \notin \mathcal{E}_4^{I}$, then by (3), for any $g \in SO_7(q)$, there is $\gamma \in SO_7(q) \smallsetminus \Omega_7(q)$ such that $c_g|_E = c_{g\gamma}|_E$, and either g or $g\gamma$ lies in $\Omega_7(q)$. Thus $\mathrm{Iso}_{SO_7(q)}(\bar{E},\bar{E}') = \mathrm{Iso}_{\Omega_7(q)}(\bar{E},\bar{E}')$ for any E'. Together with (1), this shows that E is Spin–conjugate to all other subgroups of the same rank and type, and together with (2) it shows that

$$\mathrm{Im}\left[\mathrm{Aut}_{\mathrm{Spin}}(E) \longrightarrow \mathrm{Aut}(\bar{E})\right] = \begin{cases} \mathrm{Aut}(\bar{E}) & \text{if } E \notin \mathcal{E}_4^{II} \\ \mathrm{Aut}(\bar{E},\bar{x}(E)) & \text{if } E \in \mathcal{E}_4^{II}. \end{cases} \tag{5}$$

If $E \in \mathcal{E}_4^{I}$, then by (4) and (2), $\mathrm{Aut}_{\Omega_7(q)}(\bar{E}) = \mathrm{Aut}_{SO_7(q)}(\bar{E}) = \mathrm{Aut}(\bar{E})$, and so (5) also holds in this case. Furthermore, for any $g \in SO_7(q) \smallsetminus \Omega_7(q)$, $\bar{E}$ and $g\bar{E}g^{-1}$ are representatives for two distinct $\Omega_7(q)$–conjugacy classes — since by (4), no element of the coset $g\,\Omega_7(q)$ normalizes $\bar{E}$.

We have now determined in all cases the number of conjugacy classes of subgroups of a given rank and type, and the image of $\mathrm{Aut}_{\mathrm{Spin}}(E)$ in $\mathrm{Aut}(\bar{E})$. We next claim that if $\mathrm{rk}(E) < 4$ or $E \in \mathcal{E}_4^{I}$, then

$$E \notin \mathcal{E}_4^{II} \implies \mathrm{Aut}_{\mathrm{Spin}}(E) \supseteq \left\{\alpha \in \mathrm{Aut}(E) \,\middle|\, \alpha(z) = z,\ \alpha \equiv \mathrm{Id} \ (\mathrm{mod}\ \langle z\rangle)\right\}. \tag{6}$$

Together with (5), this will finish the proof that $\mathrm{Aut}_{\mathrm{Spin}}(E)$ is the group of all automorphisms of E which send z to itself. We also claim that

$$E \in \mathcal{E}_4^{II} \implies \mathrm{Aut}_{\mathrm{Spin}}(E) \supseteq \left\{\alpha \in \mathrm{Aut}(E) \,\middle|\, \alpha|_X = \mathrm{Id}_X,\ \alpha \equiv \mathrm{Id} \ (\mathrm{mod}\ \langle z\rangle)\right\}, \tag{7}$$

where $X \subseteq E$ denotes the inverse image of $\langle \bar{x}(E)\rangle \subseteq \bar{E}$, and this will finish the proof of (c).

We prove (6) and (7) together. Fix $\alpha \in \mathrm{Aut}(E)$ ($\alpha \neq \mathrm{Id}$) which sends z to itself, induces the identity on $\bar{E}$, and such that $\alpha|_X = \mathrm{Id}_X$ if $E \in \mathcal{E}_4^{II}$. Then there is $1 \neq \rho \in \mathrm{Hom}(\bar{E},\{\pm 1\})$ such that $\alpha(g) = g$ when $\rho(\bar{g}) = 1$ and $\alpha(g) = zg$

when $(\bar g) = -1$. Choose any character such that $V \neq 0$ and $V \neq 0$, and let $W \quad V$ and $W' \quad V$ be 1–dimensional nonisotropic subspaces with the same discriminant (this is possible when $E \in E_4^{II}$ since $\bar x(E) \in \mathrm{Ker}(\ )$). Let $\bar g \in O_7(q)$ be the involution whose (-1)–eigenspace is $W \quad W'$. Then $\bar g \in \Omega_7(q)$ by Lemma A.4(a), so $\bar g$ lifts to $g \in \mathrm{Spin}_7(q)$, and using Lemma A.4(c) one sees that $c_g =$.

Step 4 It remains to prove (d). Assume $E \in E_3$. Let $1 = {}_1; {}_2; {}_3; {}_4$ be the four characters of $\bar E$, and set $V_i = V_{\ i}$. Then $\dim(V_1) = 1$, $\dim(V_i) = 2$ for $i = 2;3;4$, and the V_i either all have square discriminant or all have nonsquare discriminant. For each $g \in C_{\mathrm{Spin}}(E)$, we can write $\bar g = \bigoplus_{i=1}^4 g_i$, where $g_i \in O(V_i; \mathfrak{b}_i)$. For each pair of distinct indices $i; j \in \{2;3;4\}$, there is some $g \in E$ whose (-1)–eigenspace is $V_i \quad V_j$, and hence $\det(g_i \quad g_j) = 1$ by Lemma A.4(c). This shows that the g_i all have the same determinant. Let $A \quad C_{\mathrm{Spin}}(E)$ be the subgroup of index 2 consisting of those g such that $\det(g_i) = 1$ for all i.

Now, $SO_1(\bar{\mathbb{F}}_q) = 1$, while $SO_2(\bar{\mathbb{F}}_q) = \bar{\mathbb{F}}_q$ is the group of diagonal matrices of the form $\mathrm{diag}(u; u^{-1})$ with respect to a hyperbolic basis of $\bar{\mathbb{F}}_q{}^2$. Thus A is contained in a central extension of C_2 by $(\bar{\mathbb{F}}_q)^3$, and any such extension is abelian since $H_2((\bar{\mathbb{F}}_q)^3) = 0$. Hence A is abelian. The groups $O_2(q)$ are all dihedral (see [24, Theorem 11.4]). Hence for any $g \in C_{\mathrm{Spin}}(E) \setminus A$, $\bar g$ has order 2 and (-1)–eigenspace of dimension 4 (its intersection with each V_i is 1–dimensional), and hence $|g| = 2$ by Lemma A.4(b). Thus all elements of $C_{\mathrm{Spin}}(E) \setminus A$ have order 2, so the centralizer must be a semidirect product of A with a group of order 2 which acts on it by inversion.

Now assume that $E \in E_3^{II}$; ie, that the V_i all have nonsquare discriminant. Then for $i = 2;3;4$, $SO(V_i; \mathfrak{b}_i)$ has order $q \quad 1$, whichever is not a multiple of 4 (see [24, Theorem 11.4] again). Thus if $g \in A \quad C_{\mathrm{Spin}}(E)$ has 2–power order, then $g_i = \quad I$ for each i, the number of i for which $g_i = \mathrm{Id}$ is even (since the (-1)–eigenspace of $\bar g$ has square discriminant), and hence $g \in E$. In other words, $E \in \mathrm{Syl}_2(A)$. A Sylow 2–subgroup of $C_{\mathrm{Spin}}(E)$ is thus generated by E together with an element of order 2 which acts on E by inversion; this is an elementary abelian subgroup of rank 4, and is necessarily of type II. □

We also need some more precise information about the subgroups of $\mathrm{Spin}_7(q)$ of rank 4 and type II. Let ${}^q \in \mathrm{Aut}(\mathrm{Spin}_7(\bar{\mathbb{F}}_q))$ denote the automorphism induced by the field automorphism $(x \mapsto x^q)$. By Lemma A.3, $\mathrm{Spin}_7(q)$ is precisely the subgroup of elements fixed by ^{q}.

Proposition A.9 *Fix an odd prime power q, and let $z \in Z(\mathrm{Spin}_7(q))$ be the central involution. Let $\mathcal{C}$ and $\mathcal{C}'$ denote the two conjugacy classes of subgroups $E \subseteq \mathrm{Spin}_7(q)$ of rank 4 and type I. Then the following hold.*

(a) *For each $E \in \mathcal{E}_4$, there is an element $a \in \mathrm{Spin}_7(\bar{\mathbb{F}}_q)$ such that $aEa^{-1} \in \mathcal{C}$. For any such a, if we set*

$$x_{\mathcal{C}}(E) \stackrel{\text{def}}{=} a^{-1}\,\psi^q(a);$$

then $x_{\mathcal{C}}(E) \in E$ and is independent of the choice of a.

(b) *$E \in \mathcal{C}$ if and only if $x_{\mathcal{C}}(E) = 1$, and $E \in \mathcal{C}'$ if and only if $x_{\mathcal{C}}(E) = z$.*

(c) *Assume $E \in \mathcal{E}_4^{II}$, and set $\ (E) = \langle z, x_{\mathcal{C}}(E)\rangle$. Then $\mathrm{rk}(\ (E)) = 2$, and*

$$\mathrm{Aut}_{\mathrm{Spin}_7(q)}(E) = \{\alpha \in \mathrm{Aut}(E) \mid \alpha|_{\ (E)} = \mathrm{Id}\}.$$

The four eigenspaces of $\bar{E}$ contained in the (-1)–eigenspace of $\overline{x_{\mathcal{C}}(E)}$ all have nonsquare discriminant, and the other three eigenspaces all have square discriminant.

Proof (a) For all $E \in \mathcal{E}_4$, E has type I as a subgroup of $\mathrm{Spin}_7(q^2)$ since all elements of $\mathbb{F}_q$ are squares in $\mathbb{F}_{q^2}$. Hence by Proposition A.8(b), for all $E' \in \mathcal{C}$, there is $\bar{a} \in SO_7(q^2) \subseteq \Omega_7(q^4)$ such that $\bar{a}\bar{E}\bar{a}^{-1} = \bar{E}'$. Upon lifting $\bar{a}$ to $a \in \mathrm{Spin}_7(q^4)$, this proves that there is $a \in \mathrm{Spin}_7(\bar{\mathbb{F}}_q)$ such that $aEa^{-1} \in \mathcal{C}$.

Fix any such a, and set

$$x = x_{\mathcal{C}}(E) = a^{-1}\psi^q(a).$$

For all $g \in E$, $\psi^q(g) = g$ and $\psi^q(aga^{-1}) = aga^{-1}$ since $E, aEa^{-1} \subseteq \mathrm{Spin}_7(q)$, and hence

$$aga^{-1} = \psi^q(a)\, g\, \psi^q(a^{-1}) = a(xgx^{-1})a^{-1}.$$

Thus, $x \in C_{\mathrm{Spin}_7(\bar{\mathbb{F}}_q)}(E)$, and so $x \in E$ since it is self centralizing in each $\mathrm{Spin}_7(q^k)$ (Proposition A.8(a)).

We next check that $x_{\mathcal{C}}(E)$ is independent of the choice of a. Assume $a, b \in \mathrm{Spin}_7(\bar{\mathbb{F}}_q)$ are such that $aEa^{-1} \in \mathcal{C}$ and $bEb^{-1} \in \mathcal{C}$. Then by Proposition A.8(b), there is $g \in \mathrm{Spin}_7(q)$ such that $gbE(gb)^{-1} = aEa^{-1}$. Set $E' = aEa^{-1} \in \mathcal{C}$, then $gba^{-1} \in N_{\mathrm{Spin}_7(\bar{\mathbb{F}}_q)}(E')$. Furthermore, since $\mathrm{Aut}_{\mathrm{Spin}_7(q)}(E')$ contains all automorphisms which send z to itself, and since E' is self centralizing in each of the groups $\mathrm{Spin}_7(q^k)$ (both by Proposition A.8 again), we see that $N_{\mathrm{Spin}_7(\bar{\mathbb{F}}_q)}(E')$ is contained in $\mathrm{Spin}_7(q)$. Thus, $ba^{-1} \in \mathrm{Spin}_7(q)$, so $\psi^q(ba^{-1}) =$

ba^{-1}; and this proves that $x_C(E) = a^{-1}\,{}^q(a) = b^{-1}\,{}^q(b)$ is independent of the choice of a.

(b) If $E \in C$, then we can choose $a = 1$, and so $x_C(E) = 1$.

If $E \in C'$, then by Proposition A.8(b), there is $a \in \mathrm{Spin}_7(q^2)$ such that $\bar{a} \in SO_7(q) \setminus \Omega_7(q)$ and $aEa^{-1} \in C$. Then ${}^q(a) \neq a$ since $a \notin \mathrm{Spin}_7(q)$ (Proposition A.3), but ${}^q(\bar{a}) = \bar{a}$ since $\bar{a} \in SO_7(q)$. Thus, $x_C(E) = a^{-1}\,{}^q(a) = z$ in this case.

We have now shown that $x_C(E) \in \langle z \rangle$ if E has type I, and it remains to prove the converse. Fix $a \in \mathrm{Spin}_7(\bar{\mathbb{F}}_q)$ such that $aEa^{-1} \in C$. If $x_C(E) \in \langle z \rangle$, then ${}^q(a) \in \{a, za\}$, so ${}^q(\bar{a}) = \bar{a}$, and hence $\bar{a} \in SO_7(q)$. Conjugation by an element of $SO_7(q)$ sends eigenspaces with square discriminant to eigenspaces with square discriminant, so all eigenspaces of E must have square discriminant since all eigenspaces of aEa^{-1} do. Hence E has type I.

(c) Now write Spin = $\mathrm{Spin}_7(q)$ for short. Assume $E \in E_4^{II}$, and set $x = x_C(E)$ and $(E) = \langle z, x \rangle$. Then $x \notin \langle z \rangle$ by (b), and thus (E) has rank 2.

By (a) (the uniqueness of x having the given properties), each element of $\mathrm{Aut}_{\mathrm{Spin}}(E)$ restricts to the identity on (E). We have already seen (Proposition A.8(c)) that there is an element $\bar{x}(E) \in \bar{E}$ such that the image in $\mathrm{Aut}(\bar{E})$ of $\mathrm{Aut}_{\mathrm{Spin}}(E)$ is the group of automorphisms which fix $\bar{x}(E)$, and this shows that $\bar{x}(E) = \bar{x}$: the image in $\bar{E}$ of x. Since we already showed (Proposition A.8(c) again) that $\mathrm{Aut}_{\mathrm{Spin}}(E)$ contains all automorphisms which are the identity on (E) and the identity modulo $\langle z \rangle$, this finishes the proof that $\mathrm{Aut}_{\mathrm{Spin}}(E)$ is the group of all automorphisms which are the identity on (E). The last statement (about the discriminants of the eigenspaces) follows directly from the first statement of Proposition A.8(c). □

Throughout the rest of the section, we collect some more technical results which will be needed in Sections 2 and 4.

Lemma A.10 *Fix $k \geq 2$. Let $A = e_{13}(2^{k-1}) \in GL_3(\mathbb{Z}/2^k)$ be the elementary matrix which has off diagonal entry 2^{k-1} in position $(1,3)$. Let T_1 and T_2 be the two maximal parabolic subgroups of $GL_3(2)$:*

$$T_1 = GL_2^1(\mathbb{Z}/2) = \{(a_{ij}) \in GL_3(2) \mid a_{21} = a_{31} = 0\}$$

and

$$T_2 = GL_1^2(\mathbb{Z}/2) = \{(a_{ij}) \in GL_3(2) \mid a_{31} = a_{32} = 0\}.$$

Set $T_0 = T_1 \cap T_2$: the group of upper triangular matrices in $GL_3(2)$. Assume that

$$ _i\colon T_i \longrightarrow SL_3(\mathbb{Z}/2^k)$$

are lifts of the inclusions (for $i = 1, 2$) such that $_1|_{T_0} = {}_2|_{T_0}$. Then there is a homomorphism

$$ \colon GL_3(2) \longrightarrow SL_3(\mathbb{Z}/2^k)$$

such that $|_{T_1} = {}_1$, and either $|_{T_2} = {}_2$, or $|_{T_2} = c_A \circ {}_2$.

Proof We first claim that any two liftings $, {}'\colon T_2 \longrightarrow SL_3(\mathbb{Z}/2^k)$ are conjugate by an element of $SL_3(\mathbb{Z}/2^k)$. This clearly holds when $k = 1$, and so we can assume inductively that $\equiv {}'$ (mod 2^{k-1}). Let $M_3^0(\mathbb{F}_2)$ be the group of 3×3 matrices of trace zero, and define $\colon T_2 \longrightarrow M_3^0(\mathbb{F}_2)$ via the formula

$$ {}'(B) = (I + 2^{k-1} (B)) (B)$$

for $B \in T_2$. Then is a 1–cocycle. Also, $H^1(T_2; M_3^0(\mathbb{F}_2)) = 0$ by [9, Lemma 4.3] (the module is $\mathbb{F}_2[T_2]$–projective), so is the coboundary of some $X \in M_3^0(\mathbb{F}_2)$, and and ${}'$ differ by conjugation by $I + 2^{k-1}X$.

By [9, Theorem 4.1], there exists a section defined on $GL_3(2)$ such that $|_{T_1} = {}_1$. Let $B \in SL_3(\mathbb{Z}/2^k)$ be such that $|_{T_2} = c_B \circ {}_2$. Since $|_{T_0} = {}_2|_{T_0}$, B must commute with all elements in (T_0), and one easily checks that the only such elements are $A = e_{13}(2^{k-1})$ and the identity. □

Recall that a p–subgroup P of a finite group G is *p–radical* if $N_G(P)/P$ is p–reduced; ie, if $O_p(N_G(P)/P) = 1$. (Here, $O_p(-)$ denotes the largest normal p–subgroup.) We say here that P is *$F_p(G)$–radical* if $\mathrm{Out}_G(P)$ ($= \mathrm{Out}_{F_p(G)}(P)$) is p–reduced. In Section 4, some information will be needed involving the $F_2(\mathrm{Spin}_7(q))$–radical subgroups of $\mathrm{Spin}_7(q)$ which are also 2–centric. We first note the following general result.

Lemma A.11 *Fix a finite group G and a prime p. Then the following hold for any p–subgroup $P \subseteq G$ which is p–centric and $F_p(G)$–radical.*

(a) *If $G = G_1 \times G_2$, then $P = P_1 \times P_2$, where P_i is p–centric in G_i and $F_p(G_i)$–radical.*

(b) *If $P \subseteq H \lhd G$, then P is p–centric in H and $F_p(H)$–radical.*

(c) *If $H \lhd G$ has p–power index, then $P \cap H$ is p–centric in H and $F_p(H)$–radical.*

(d) *If $G \lhd \bar{G}$ has p–power index, then $P = G \cap \bar{P}$ for some $\bar{P} \le \bar{G}$ which is p–centric in $\bar{G}$ and $\mathcal{F}_p(\bar{G})$–radical.*

(e) *If $Q \lhd G$ is a central p–subgroup, then $Q \le P$, and P/Q is p–centric in G/Q and $\mathcal{F}_p(G/Q)$–radical.*

(f) *If $\alpha\colon \hat{G} \twoheadrightarrow G$ is an epimorphism such that* $\operatorname{Ker}(\alpha) \le Z(\hat{G})$, *then $\alpha^{-1}(P)$ is p–centric in $\hat{G}$ and $\mathcal{F}_p(\hat{G})$–radical.*

Proof Point (a) follows from [16, Proposition 1.6(ii)]: $P = P_1 \times P_2$ for $P_i \le G_i$ since P is p–radical, and P_i must be p–centric in G_i and $\mathcal{F}_p(G_i)$–radical since

$$C_G(P) = C_{G_1}(P_1) \times C_{G_2}(P_2) \quad \text{and} \quad \operatorname{Out}_G(P) = \operatorname{Out}_{P_1}(G_1) \times \operatorname{Out}_{P_2}(G_2).$$

Point (b) holds since $C_H(P) \le C_G(P)$ and $O_p(\operatorname{Out}_H(P)) \le O_p(\operatorname{Out}_G(P))$.

It remains to prove the other four points.

(e) Fix a central p–subgroup $Q \le Z(G)$. Then $P \ge Q$, since otherwise $1 \ne N_{QP}(P)/P \le O_p(N_G(P)/P)$. Also, P/Q is p–centric in G/Q, since otherwise there would be $x \in G{\smallsetminus}P$ of p–power order such that

$$1 \ne [c_x] \in \operatorname{Ker}\bigl[\operatorname{Out}_G(P) \longrightarrow \operatorname{Out}_{G/Q}(P/Q) \times \operatorname{Out}_G(Q)\bigr] \le O_p(\operatorname{Out}_G(P)).$$

It remains only to prove that P/Q is $\mathcal{F}_p(G/Q)$–radical, and to do this it suffices to show that

$$\operatorname{Out}_{G/Q}(P/Q) \cong \operatorname{Out}_G(P).$$

Equivalently, since P/Q and P are p–centric, we must show that

$$\frac{N_{G/Q}(P/Q)}{C'_{G/Q}(P/Q) \times P/Q} \cong \frac{N_G(P)}{C'_G(P) \times P};$$

and this is clear once we have shown that

$$C'_{G/Q}(P/Q) \cong C'_G(P).$$

Any $\bar{x} \in C'_{G/Q}(P/Q)$ lifts to an element $x \in G$ of order prime to p, whose conjugation action on P induces the identity on Q and on P/Q. By [15, Corollary 5.3.3], all such automorphisms of P have p–power order, and thus x centralizes P. Since Q is a p–group and $C'_{G/Q}(P/Q)$ has order prime to p, this shows that the projection modulo Q sends $C'_{G/Q}(P/Q)$ isomorphically to $C'_G(P)$.

(f) Let $\widetilde{G} \twoheadrightarrow G$ be an epimorphism whose kernel is central. Clearly, $\psi^{-1}P$ is p-centric in $\widetilde{G}$. It remains only to prove that $\psi^{-1}P$ is $\mathcal{F}_p(\widetilde{G})$-radical, and to do this it suffices to show that

$$\mathrm{Out}_{\widetilde{G}}(\psi^{-1}P) = \mathrm{Out}_G(P).$$

Equivalently, since P and $\psi^{-1}(P)$ are p-centric, we must show that

$$\frac{N_{\widetilde{G}}(\psi^{-1}P)}{C'_{\widetilde{G}}(\psi^{-1}P)\cdot\psi^{-1}P} \cong \frac{N_G(P)}{C'_G(P)\cdot P};$$

and this is clear once we have shown that

$$C'_{\widetilde{G}}(\psi^{-1}P) = C'_G(P).$$

This follows by exactly the same argument as in the proof of (e).

(c) Set $P' = P \cap H$ for short. Let

$$N_H(P') \xrightarrow{\;\;\psi\;\;} \mathrm{Out}_H(P') = N_H(P')/(C_H(P')\cdot P')$$

be the natural projection, and set

$$K = \psi^{-1}(O_p(\mathrm{Out}_H(P'))) \le N_H(P').$$

Then $K \cong O_p(N_H(P'))$ is an extension of $C_H(P')\cdot P'$ by $O_p(\mathrm{Out}_H(P'))$. It suffices to show that $p \nmid [K{:}P']$, since this implies that $O_p(\mathrm{Out}_H(P')) = 1$ (ie, P' is $\mathcal{F}_p(H)$-radical), and that any Sylow p-subgroup of $C_H(P')$ is contained in P' (hence P' is p-centric in H).

Assume otherwise: that $p \mid [K{:}P']$. Note first that $P' \lhd N_G(P)$, and that $N_G(P) \le N_G(K)$; ie, $N_G(P)$ normalizes P' and K. The first statement is obvious, and the second is verified by observing directly that $N_G(P)$ normalizes $N_H(P')$ and $C_H(P')$. Thus the action of $N_G(P)$ on K induces an action of $N_G(P)$, and in particular of P, on K/P'. Let K_0/P' denote the fixed subgroup of this action of P. Since $p \mid [K{:}P']$ by assumption, and since P is a p-group, $p \mid |K_0/P'|$. A straightforward check also shows that $K_0 \lhd N_G(P)$, and therefore that $PK_0 \lhd N_G(P)$. Also, since $P' \le K_0 \le H$,

$$PK_0/P \cong K_0/(P \cap K_0) = K_0/P'$$

is a normal subgroup of $N_G(P)/P$ of order a multiple of p. Since P is p-centric in G by assumption,

$$\mathrm{Out}_G(P) = N_G(P)/(C_G(P)\cdot P) = N_G(P)/(C'_G(P)\times P);$$

and hence the image of PK_0/P in $\mathrm{Out}_G(P)$ is a normal subgroup which also has order a multiple of p.

By definition of K as an extension of $C_H(P')P'$ by a p-group, if $x\in K$ has order prime to p, then $x\in C_H(P')$. Hence if $x\in K_0$ has order prime to p, then for every $z\in P$, $[x,z]\in P'$, so x acts trivially on P/P'. Since x also centralizes P', it follows that x centralizes P. This shows that the image of PK_0/P in $\mathrm{Out}_G(P)$ is a p-group, thus a nontrivial normal p-subgroup of $\mathrm{Out}_G(P)$, and this contradicts the original assumption that P is $\mathcal{F}_p(G)$-radical.

(d) Let $G\lhd \bar G$ be a normal subgroup of p-power index and let $P\le G$ be a p-centric and $\mathcal{F}_p(G)$-radical subgroup. Let

$$N_{\bar G}(P) \xrightarrow{\ \ \pi\ \ } \mathrm{Out}_{\bar G}(P) = N_{\bar G}(P)\big/ C_{\bar G}(P)P$$

be the natural surjection, and set

$$K = \pi^{-1}\big(O_p(\mathrm{Out}_{\bar G}(P))\big) \le N_{\bar G}(P).$$

Then K is an extension of $C_{\bar G}(P)P$ by $O_p(\mathrm{Out}_{\bar G}(P))$. Fix any $\bar P\in \mathrm{Syl}_p(K)$. We will show that $\bar P\cap G = P$, and that $\bar P$ is p-centric in $\bar G$ and $\mathcal{F}_p(\bar G)$-radical.

For each $x\in K\cap G\le N_G(P)$,

$$\pi(x)\in O_p(\mathrm{Out}_{\bar G}(P))\cap \mathrm{Out}_G(P) \le O_p(\mathrm{Out}_G(P)) = 1.$$

Hence

$$x\in \mathrm{Ker}\big[N_G(P)\xrightarrow{\ \pi\ }\mathrm{Out}_{\bar G}(P)\big] = (C_{\bar G}(P)P)\cap G = C_G(P)P = C'_G(P)\times P,$$

where $C'_G(P)\le C_G(P)$ is of order prime to p. Since the opposite inclusion is obvious, this shows that $K\cap G = C'_G(P)\times P$, and hence (since $\bar P\in \mathrm{Syl}_p(K)$) that $\bar P\cap G = P$.

Next, note that $(K\cap G)\lhd K$ and $K/(K\cap G)\le \bar G/G$, and hence $K/C'_G(P)$ has p-power order. Since $\bar P\in \mathrm{Syl}_p(K)$, $\bar P$ is an extension of P by $K/(K\cap G)$, and $N_K(\bar P)$ is an extension of a subgroup of $(K\cap G)=(C'_G(P)\times P)$ by $K/(K\cap G)$. Also, an element $x\in C'_G(P)$ normalizes $\bar P$ if and only if $[x,\bar P]\in \bar P\cap C'_G(P) = 1$. Hence

$$N_K(\bar P) = C_K(\bar P)\bar P = C'_G(\bar P)\times \bar P, \qquad (1)$$

where $C'_G(\bar P) = C'_G(P)\cap C_G(\bar P)$ has order prime to p and is normal in $N_K(\bar P)$. Since $C_{\bar G}(\bar P)\le C_{\bar G}(P)\le K$, (1) shows that $C_{\bar G}(\bar P)\le C'_G(\bar P)\times \bar P$, and hence that $\bar P$ is p-centric in $\bar G$.

It remains to show that $\bar P$ is $\mathcal{F}_p(\bar G)$-radical. Note first that $K\lhd N_{\bar G}(P)$ by construction, so for any $x\in N_{\bar G}(P)$, $x\bar P x^{-1}\in \mathrm{Syl}_p(K)$. Since K is an extension of $C'_G(P) \times P$ by the p–group $K/(K \cap G)$, and since $C'_G(P) \lhd K$, it follows that K is a split extension of $C'_G(P)$ by $\bar{P}$. Hence for any $x \in N_{\bar{G}}(P)$, $x\bar{P}x^{-1} = y\bar{P}y^{-1}$ for some $y \in C'_G(P)$. Consequently, the restriction map

$$N_{\bar{G}}(\bar{P})/C_{\bar{G}}(\bar{P}) = \mathrm{Aut}_{\bar{G}}(\bar{P}) \longrightarrow \mathrm{Aut}_{\bar{G}}(P) = N_{\bar{G}}(P)/C_{\bar{G}}(P) \qquad (2)$$

is surjective. Also, if $x \in C_{\bar{G}}(P) \subseteq K$ normalizes $\bar{P}$, then $x \in N_K(\bar{P}) = \bar{P} \times C'_G(\bar{P})$ by (1), and so $c_x \in \mathrm{Inn}(\bar{P})$. Thus the kernel of the map in (2) is contained in $\mathrm{Inn}(\bar{P})$. Consequently,

$$\mathrm{Out}_{\bar{G}}(\bar{P}) = \mathrm{Aut}_{\bar{G}}(\bar{P})/\mathrm{Inn}(\bar{P}) \cong \mathrm{Aut}_{\bar{G}}(P)/\mathrm{Aut}_{\bar{P}}(P) = \mathrm{Out}_{\bar{G}}(P)/O_p(\mathrm{Out}_{\bar{G}}(P)),$$

and it follows that $\bar{P}$ is $F_p(\bar{G})$–radical. □

This is now applied to show the following:

Proposition A.12 *Fix an odd prime power q, and let $P \subseteq \mathrm{Spin}_7(q)$ be any subgroup which is 2–centric and $F_2(\mathrm{Spin}_7(q))$–radical. Then P is centric in $\mathrm{Spin}_7(\bar{\mathbb{F}}_q)$; ie, $C_{\mathrm{Spin}_7(\bar{\mathbb{F}}_q)}(P) = Z(P)$.*

Proof Let z be the central involution in $\mathrm{Spin}_7(q)$. By Lemma A.11(e), $z \in P$, and $\bar{P} \stackrel{\mathrm{def}}{=} P/\langle z\rangle$ is 2–centric in $\Omega_7(q)$ and is $F_2(\Omega_7(q))$–radical. So by Lemma A.11(d), there is a 2–subgroup $\hat{P} \subseteq O_7(q)$ such that $\hat{P} \cap \Omega_7(q) = \bar{P}$, and such that $\hat{P}$ is 2–centric in $O_7(q)$ and is $F_2(O_7(q))$–radical.

Let $V = \bigoplus_{i=1}^m V_i$ be a maximal decomposition of V as an orthogonal direct sum of $\hat{P}$–representations, and set $\mathfrak{b}_i = \mathfrak{b}|_{V_i}$. We assume these are arranged so that for some k, $\dim(V_i) > 1$ when $i \le k$ and $\dim(V_i) = 1$ when $i > k$. Let V_+ be the sum of those 1–dimensional components V_i with square discriminant, and let V_- be the sum of those 1–dimensional components V_i with nonsquare discriminant. We will be referring to the two decompositions

$$(V,\mathfrak{b}) = \bigoplus_{i=1}^{m}(V_i,\mathfrak{b}_i) = \bigoplus_{i=1}^{k}(V_i,\mathfrak{b}_i) \oplus (V_+,\mathfrak{b}_+) \oplus (V_-,\mathfrak{b}_-),$$

both of which are orthogonal direct sums. We also write

$$V^{(\infty)} = \bar{\mathbb{F}}_q \otimes_{\mathbb{F}_q} V \qquad \text{and} \qquad V_i^{(\infty)} = \bar{\mathbb{F}}_q \otimes_{\mathbb{F}_q} V_i,$$

and let $\mathfrak{b}^{(\infty)}$ and $\mathfrak{b}_i^{(\infty)}$ be the induced quadratic forms.

Step 1 For each i, set

$$D_i = \{\pm \mathrm{Id}_{V_i}\} \subseteq O(V_i,\mathfrak{b}_i),$$

a subgroup of order 2; and write

$$D = \prod_{i=1}^{m} D_i \subseteq O(V,\mathfrak{b}); \qquad \text{and} \qquad D^* = \prod_{V_i \subseteq V^*} D_i \subseteq O(V^*,\mathfrak{b}^*).$$

Thus D and D^* are elementary abelian 2–groups of rank m and $\dim(V^*)$, respectively. We first claim that

$$\widehat{P} \supseteq D, \tag{1}$$

and that

$$\widehat{P} = \prod_{i=1}^{m} P_i \quad \text{where } \forall i,\ P_i \text{ is 2–centric in } O(V_i,\mathfrak{b}_i) \text{ and } \mathcal{F}_2(O(V_i,\mathfrak{b}_i))\text{–radical.} \tag{2}$$

Clearly, $[D,\widehat{P}] = 1$ (and D is a 2–group), so $D \subseteq \widehat{P}$ since $\widehat{P}$ is 2–centric. This proves (1). The V_i are thus distinct (pairwise nonisomorphic) as $\widehat{P}$–representations, since they are pairwise nonisomorphic as D–representations. The decomposition as a sum of V_i's is thus unique (not only up to isomorphism), since $\mathrm{Hom}_{\widehat{P}}(V_i,V_j) = 0$ for $i \neq j$.

Let $\widehat{C}$ be the group of elements of $O(V,\mathfrak{b})$ which send each V_i to itself, and let $\widehat{N}$ be the group of elements which permute the V_i. By the uniqueness of the decomposition of V,

$$\widehat{P}\, C_{O(V,\mathfrak{b})}(\widehat{P}) \subseteq \widehat{C} = \prod_{i=1}^{m} O(V_i,\mathfrak{b}_i) \qquad \text{and} \qquad N_{O(V,\mathfrak{b})}(\widehat{P}) \subseteq \widehat{N}.$$

Since $\widehat{P}$ is 2–centric in $O(V,\mathfrak{b})$ and $\mathcal{F}_2(O(V,\mathfrak{b}))$–radical, it is also 2–centric in $\widehat{N}$ and $\mathcal{F}_2(\widehat{N})$–radical (this holds for any subgroup which contains $N_{O(V,\mathfrak{b})}(\widehat{P})$). So by Lemma A.11(b) (and since $\widehat{C} \lhd \widehat{N}$), $\widehat{P}$ is 2–centric in $\widehat{C}$ and $\mathcal{F}_2(\widehat{C})$–radical. Point (2) now follows from Lemma A.11(a).

Step 2 Whenever $\dim(V_i) > 1$ (ie, $1 \le i \le k$), then by Lemma A.6, $\dim(V_i)$ is even, and $\mathfrak{b}_i$ has square discriminant. So by Lemma A.4(a), $-\mathrm{Id}_{V_i} \in \Omega(V_i,\mathfrak{b}_i)$ for such i. Together with (1), this shows that

$$\overline{P} = \widehat{P} \cap \Omega_7(q) \supseteq \prod_{i=1}^{k} D_i \times \bigl(\Omega(V_+,\mathfrak{b}_+) \cap D_+\bigr) \times \bigl(\Omega(V_-,\mathfrak{b}_-) \cap D_-\bigr). \tag{3}$$

Also, by Lemma A.4(a) again,

$$\begin{aligned}\Omega(V_*;\mathfrak{b}_*)\cap D^* &= SO(V_*;\mathfrak{b}_*)\cap D^*\\ &= \big\langle -\mathrm{Id}_{V_i\oplus V_j} \;\big|\; k+1\le i<j\le m;\ V_i,V_j\subseteq V_*\big\rangle.\end{aligned}\tag{4}$$

Step 3 By (3) and (4), the V_i are distinct as $\bar{P}$–representations (not only as $\hat{P}$–representations), except possibly when $\dim(V_*)=2$. We first check that this exceptional case cannot occur. If $\dim(V_+)=2$ and its two irreducible summands are isomorphic as $\bar{P}$–representations, then the image of $\bar{P}$ under projection to $O(V_+;\mathfrak{b}_+)$ is just $\{\pm\mathrm{Id}_{V_+}\}$. Hence we can write $V_+=W\oplus W'$, where $W\perp W'$ are 1–dimensional, $\bar{P}$–invariant, and have nonsquare discriminant. Also, $\dim(V_-)$ is odd, since V_+ and the V_i for $i\le k$ are all even dimensional. So $-\mathrm{Id}_{V_-\oplus W}$ lies in $C_{\Omega_7(q)}(\bar{P})$ but not in $\bar{P}$. But this is impossible, since $\bar{P}$ is 2–centric in $\Omega_7(q)$. The argument when $\dim(V_-)=2$ is similar.

The V_i are thus distinct as $\bar{P}$–representations. So for all $i\neq j$, $\mathrm{Hom}_P(V_i,V_j)=0$, and hence

$$\mathrm{Hom}_{\bar{\mathbb{F}}_q[P]}(V_i^{(\infty)},V_j^{(\infty)})=\bar{\mathbb{F}}_q\otimes_{\mathbb{F}_q}\mathrm{Hom}_{\mathbb{F}_q[P]}(V_i,V_j)=0.$$

Thus any element of $O(V^{(\infty)},\mathfrak{b}^{(\infty)})$ which centralizes $\bar{P}$ sends each $V_i^{(\infty)}$ to itself. In other words,

$$C_{\mathrm{Spin}_7(\bar{\mathbb{F}}_q)}(P)/\langle z\rangle\cong C_{\Omega_7(\bar{\mathbb{F}}_q)}(\bar{P})\subseteq\prod_{i=1}^{m}O(V_i^{(\infty)},\mathfrak{b}_i^{(\infty)}).$$

If $\dim(V_*)\ge 2$, then since $\bar{P}$ contains all involutions in $O(V_*;\mathfrak{b}_*)$ which are $\hat{P}$–invariant and have even dimensional (-1)–eigenspace (see (3)), Lemma A.4(c) shows that each element of $\mathrm{Spin}_7(\bar{\mathbb{F}}_q)$ which commutes with P must act on V_* via $\pm\mathrm{Id}$. Also, for $1\le i\le k$, since $-\mathrm{Id}_{V_i}\in\bar{P}$ by (3), each element in the centralizer of P acts on V_i with determinant 1 (Lemma A.4(c) again). We thus conclude that

$$C_{\mathrm{Spin}_7(\bar{\mathbb{F}}_q)}(P)/\langle z\rangle\subseteq\prod_{i=1}^{k}SO(V_i^{(\infty)},\mathfrak{b}_i^{(\infty)})\times\{\pm\mathrm{Id}_{V_+}\}\times\{\pm\mathrm{Id}_{V_-}\}.\tag{5}$$

Step 4 We next show that

$$C_{\mathrm{Spin}_7(\bar{\mathbb{F}}_q)}(P)/\langle z\rangle\subseteq\prod_{i=1}^{k}\{\pm\mathrm{Id}_{V_i}\}\times\{\pm\mathrm{Id}_{V_+}\}\times\{\pm\mathrm{Id}_{V_-}\}.\tag{6}$$

Using (5), this means showing, for each $1 \le i \le k$, that

$$\mathrm{pr}_i\big(C_{\mathrm{Spin}_7(\bar{\mathbb{F}}_q)}(P)/\langle z\rangle\big) \subseteq \{\pm \mathrm{Id}_{V_i}\}; \qquad (7)$$

where pr_i denotes the projection of $O_7(\bar{\mathbb{F}}_q) = O(V^{(\infty)}, \mathfrak{b}^{(\infty)})$ to $O(V_i^{(\infty)}, \mathfrak{b}_i^{(\infty)})$. By Lemma A.6, $\dim(V_i) = 2$ or 4. We consider these two cases separately.

Case 4A If $\dim(V_i) = 4$, then by (2) and Lemma A.11(c), $P_i' \stackrel{\mathrm{def}}{=} P_i \cap \Omega(V_i, \mathfrak{b}_i)$ is 2-centric in $\Omega(V_i, \mathfrak{b}_i)$ and is $\mathcal{F}_2(\Omega(V_i, \mathfrak{b}_i))$-radical. Also, by Proposition A.5,

$$\Omega(V_i, \mathfrak{b}_i) = \Omega_4^+(q) = SL_2(q) \times_{C_2} SL_2(q).$$

By Lemma A.11(a,f), under this identification, we have $P_i' = Q \times_{C_2} Q'$, where Q and Q' are 2-centric in $SL_2(q)$ and $\mathcal{F}_2(SL_2(q))$-radical. The Sylow 2-subgroups of $SL_2(q)$ are quaternion groups of order ≥ 8, all subgroups of a quaternion 2-group are quaternion or cyclic, and cyclic 2-subgroups of $SL_2(q)$ cannot be both 2-centric and $\mathcal{F}_2(SL_2(q))$-radical. So Q and Q' must be quaternion of order ≥ 8. By [23, 3.6.3], any cyclic 2-subgroup of $SL_2(\bar{\mathbb{F}}_q)$ of order ≥ 4 is conjugate to a subgroup of diagonal matrices, whose centralizer is the group of all diagonal matrices in $SL_2(\bar{\mathbb{F}}_q)$. Knowing this, one easily checks that all nonabelian quaternion 2-subgroups of $SL_2(\bar{\mathbb{F}}_q)$ are centric in $SL_2(\bar{\mathbb{F}}_q)$. It follows that P_i' is centric in

$$SO(V_i^{(\infty)}, \mathfrak{b}_i^{(\infty)}) = SL_2(\bar{\mathbb{F}}_q) \times_{C_2} SL_2(\bar{\mathbb{F}}_q);$$

and hence that

$$\mathrm{pr}_i\big(C_{\mathrm{Spin}_7(\bar{\mathbb{F}}_q)}(P)/\langle z\rangle\big) \subseteq C_{SO(V_i^{(\infty)}, \mathfrak{b}_i^{(\infty)})}(P_i') = Z(P_i') = \{\pm \mathrm{Id}_{V_i}\}.$$

Thus (7) holds in this case.

Case 4B If $\dim(V_i) = 2$, then $O(V_i, \mathfrak{b}_i) = O_2^{\pm}(q)$ is a dihedral group of order $2(q \mp 1)$ [24, Theorem 11.4]. Hence $P_i \in \mathrm{Syl}_2(O(V_i, \mathfrak{b}_i))$, since the Sylow subgroups are the only radical 2-subgroups of a dihedral group. Fix V_j for any $k < j \le m$, and choose $\gamma \in O(V_i, \mathfrak{b}_i)$ of determinant (-1) whose (-1)-eigenspace has the same discriminant as V_j. Since $P_i \in \mathrm{Syl}_2(O(V_i, \mathfrak{b}_i))$, we can assume (after conjugating if necessary) that $\gamma \in P_i$. Then $(-\mathrm{Id}_{V_j}) \oplus \gamma$ lies in $\bar{P} = \hat{P} \cap \Omega_7(q)$. Hence for any $g \in C_{\mathrm{Spin}_7(\bar{\mathbb{F}}_q)}(P)/\langle z\rangle$, $\mathrm{pr}_i(g) \in O(V_i^{(\infty)}, \mathfrak{b}_i^{(\infty)})$ leaves both eigenspaces of γ invariant, and has determinant 1 by (5). Thus $\mathrm{pr}_i(g) = \pm \mathrm{Id}_{V_i}$; and so (7) holds in this case.

Step 5 Clearly, $-\mathrm{Id}_{V_\pm}$ lies in $SO(V_\pm, \mathfrak{b}_\pm)$ if and only if $\dim(V_\pm)$ is even (which is the case for exactly one of the two spaces $V_\pm$), and this holds if and only if $-\mathrm{Id}_V \in \Omega(V, \mathfrak{b})$. Also, since each V_i for $1 \le i \le k$ has square discriminant (Lemma A.6 again), $-\mathrm{Id}_{V_i} \in \Omega(V_i, \mathfrak{b}_i)$ for all such i. Thus (6) and (1) imply that

$$C_{\mathrm{Spin}_7(\bar{\mathbb{F}}_q)}(P)/\langle z\rangle \le \widehat{P} \cap \Omega_7(q) = \bar{P};$$

and hence that P is centric in $\mathrm{Spin}_7(\bar{\mathbb{F}}_q)$. □

Proposition A.12 does *not* hold in general if $\mathrm{Spin}_7(-)$ is replaced by an arbitrary algebraic group. For example, assume q is an odd prime power, and let $P \le SL_5(q)$ be the group of diagonal matrices of 2–power order. Then P is 2–centric in $SL_5(q)$ and $F_2(SL_5(q))$–radical, but is definitely not 2–centric in $SL_5(\bar{\mathbb{F}}_q)$.

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