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Yau–Zaslow formula on K3 surfaces for non-primitive classes

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Abstract

We compute the genus zero family Gromov–Witten invariants for K3 surfaces using the topological recursion formula and the symplectic sum formula for a degeneration of elliptic K3 surfaces. In particular we verify the Yau–Zaslow formula for non-primitive classes of index two.

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1 Introduction

Let $N(d,r)$ be the number of rational curves in K3 surfaces X that represent a homology class $A \in H_2(X,\mathbb{Z})$ of self-intersection $A^2 = 2d-2$ and of index[1] r. Yau and Zaslow [23] give an ingenious heuristic argument to compute the generating function for primitive classes and they also expected that the same formula holds true for classes of arbitrary index. More precisely the Yau–Zaslow conjectural formula says that, for any positive integer r, we have

$$\sum_{d\geq 0} N(d,r)\, t^d = \prod_{d\geq 1} \left(\frac{1}{1-t^d}\right)^{24}. \tag{1.1}$$

Their original approach was pursued by Beauville [1], Chen [4] and Li [16]. In [2, 3], Bryan and the second author showed that $N(d,r)$ can be computed in terms of the twistor family Gromov–Witten invariants of the K3 surfaces. They also proved the Yau–Zaslow formula for primitive classes in K3 surfaces and its higher genera generalization.

In [14], the first author reproved the Yau–Zaslow formula (1.1) for primitive classes and its higher genera generalization using p_g–dimensional family Gromov–Witten invariants defined in [13] — following the approach of [8], he computed those invariants by relating the TRR (topological recursion relation) and the symplectic sum formula of [8] for a suitable degeneration of an elliptic K3 surface. In this article we explain how to use the same approach to compute the p_g–dimensional family Gromov–Witten invariants for non-primitive classes. In particular *we verify the Yau–Zaslow formula for non-primitive classes of index two*. At present, it is not easy to use this approach to handle classes of higher indexes, for example, we do not know how to handle relative invariants with multiplicity greater than 2. An analogous problem for the Seiberg–Witten invariants was studied by Liu [19].

Notice that $N(d,2)$ is different from the family Gromov–Witten invariant $GW^{\mathcal{H}}_{A,0}$ due to the multiple cover contributions, as it was explained by Gathmann in [5]. This is because the family Gromov–Witten invariants count holomorphic maps and each rational curve C representing the primitive class $A/2$ contributes $1/2^3$ to $GW^{\mathcal{H}}_{A,0}$, however, the multiple curve $2C$ contributes zero to $N(d,2)$ because it has negative genus. As a result, the Yau–Zaslow formula for non-primitive classes of index two follows directly from the following theorem.

[1] The index of A is the largest positive integer r such that $r^{-1}A$ is integral. An index one class is called primitive.

Theorem 1.1 *Let X be a K3 surface and $A/2 \in H_2\big(X;\mathbb{Z}\big)$ be a primitive class. Then, the genus $g=0$ family GW invariant of X for the class A is given by*

$$GW^{\mathcal{H}}_{A,0} \,-\, GW^{\mathcal{H}}_{B,0} \,=\, \Big(\frac{1}{2}\Big)^3 GW^{\mathcal{H}}_{A/2,0}$$

where B is any primitive class with $B^2 = A^2$.

In the sequel [15], we apply the same technique to enumerate the number of elliptic curves representing non-primitive classes of index two in K3 surfaces.

The construction of family GW invariants is briefly described in Section 2. We outline the proof of Theorem 1.1 in Section 3. This proof follows the elegant argument used by Ionel and Parker to compute the GW invariants of $E(0)$ [8]. It involves computing the generating functions for the invariants in two different ways, first using the TRR formula, and second using the symplectic sum formula. Section 4 gives the sum formulas of the symplectic sum of $E(2)$ with $E(0)$ along a fixed fiber. The sum formulas yield relations of family invariants of $E(2)$ and relative invariants of $E(0)$. We compute those relative invariants of $E(0)$ in Section 5–7.

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2 Family GW invariants of K3 surfaces

This section briefly describes family GW invariants of K3 surfaces. We first give the definition of family GW invariants of Kähler surfaces with $p_g = \dim H^{2,0} \geq 1$ defined in [13]. Fix a compact Kähler surface (X,J) and choose the $2p_g$–dimensional parameter space

$$\mathcal{H} \,=\, \mathrm{Re}\,\big(H^{0,2}(X) \oplus H^{0,2}(X)\big).$$

Using the Kähler metric, each $\alpha \in \mathcal{H}$ defines an endomorphism K_α of TX by the equation

$$\langle u, K_\alpha v\rangle \,=\, \alpha(u,v).$$

Since $Id + JK_\alpha$ is invertible for each $\alpha \in \mathcal{H}$, the equation

$$J_\alpha \,=\, (Id \,+\, J\,K_\alpha)^{-1} J\,(Id \,+\, J\,K_\alpha)$$

defines a family of almost complex structure on X parameterized by α in the $2p_g$–dimensional linear space $\mathcal{H}$. The family GW invariants are defined, in the same manner as the ordinary GW invariants [22, 17], but using the moduli space of stable (J,α)–holomorphic maps (f,α):

$$\overline{\mathcal{M}}{}^{\mathcal{H}}_{g,k}(X,A,J) = \{\ (f,\alpha) \mid \overline{\partial}_{J_\alpha} = 0\,,\ \alpha \in \mathcal{H}\,,\ [f] = A \in H_2(X;\mathbb{Z})\,\}. \tag{2.1}$$

For each stable (J,α) holomorphic map $f\colon\,(C,j) \to X$ of genus g with $k-$ marked points, collapsing unstable components of the domain determines a point in the Deligne–Mumford space $\overline{\mathcal{M}}_{g,k}$ and evaluation of marked points determines a point in X^k. Thus we have a map

$$st \times ev\ :\ \overline{\mathcal{M}}{}^{\mathcal{H}}_{g,k}(X,A,J)\ \to\ \overline{\mathcal{M}}_{g,k} \times X^k \tag{2.2}$$

where st and ev denote the stabilization map and the evaluation map, respectively. If the space (2.1) is compact, it carries a fundamental homology class

$$[\,\overline{\mathcal{M}}{}^{\mathcal{H}}_{g,k}(X,A,J)\,]$$

which we can push forward by the map (2.2) to obtain a homology class

$$(st \times ev)_*\,\big[\,\overline{\mathcal{M}}{}^{\mathcal{H}}_{g,k}(X,A,J)\,\big]\ \in\ H_{2r}\big(\overline{\mathcal{M}}_{g,k} \times X^k;\mathbb{Q}\big)$$

where $r = -K\cdot A + (g-1) + k + p_g$ and K is the canonical class of X. Then, the family GW invariants are defined by

$$\begin{aligned} &GW^{\mathcal{H}}_{g,k}(X,A,J)\big(\kappa;\beta_1,\cdots,\beta_k\big)\\ &\ = \ (st \times ev)_*\,\big[\,\overline{\mathcal{M}}{}^{\mathcal{H}}_{g,k}(X,A,J)\,\big] \,\cap\, \big(\kappa^* \,\cup\, \beta_1^* \,\cup\, \cdots \,\cup\, \beta_k^*\big)\end{aligned}$$

where κ^* and β_i^* are Poincaré dual of $\kappa \in H_*\big(\overline{\mathcal{M}}_{g,k};\mathbb{Q}\big)$ and $\beta_i \in H_*\big(X^k;\mathbb{Q}\big)$, respectively.

When X is a K3 surface, the family GW invariants reduce to the invariants defined by Bryan and Leung [2] using the twistor family. In particular, (i) they are independent of complex structures and (ii) for any two homology classes A and B of the same index with $A^2 = B^2$, there is an orientation preserving diffeomorphism $h\colon\,X \to X$ such that $h_*A = B$ and

$$GW^{\mathcal{H}}_{g,k}(X,A)(\kappa;\beta_1,\cdots,\beta_k)\ =\ GW^{\mathcal{H}}_{g,k}(X,B)(\kappa;h_*\beta_1,\cdots,h_*\beta_k). \tag{2.3}$$

Below, we will often write the family GW invariants of $K3$ surfaces as simply

$$GW^{\mathcal{H}}_{A,g}(X)\big(\kappa;\beta_1,\cdots,\beta_k\big) \qquad \text{or} \qquad GW^{\mathcal{H}}_{A,g}\big(\kappa;\beta_1,\cdots,\beta_k\big).$$

By dimension count, this invariant vanishes unless

$$\deg(\kappa^*) + \sum \deg(\beta_i^*) = 2(g+k).$$

Let $E(2) \to \mathbb{P}^1$ be an elliptic K3 surface with a section of self intersection number -2. Denote by S and F the section class and the fiber class, respectively. It then follows from (2.3) that for any class A of index 2 with $(A/2)^2 = 2d-2$, we have

$$GW^{\mathcal{H}}_{A,0} = GW^{\mathcal{H}}_{2(S+dF),0}.$$

These are the invariants we aim to compute. We will compute them following a similar approach of [8, 14] — relating the genus 1 TRR Formula and the Symplectic Sum Formula [8].

The Sum Formula yields relations between GW invariants and relative GW invariants of [7]. One can derive a family version of the Sum Formula for the cases of elliptic surfaces (cf [14]). Here, we introduce relative family invariants of $E(2)$ and describe the extension of the Sum Formula in Section 3.

First, we define relative family invariants of $E(2)$ for the classes $2S+dF$, $d \in \mathbb{Z}$. Let $V \cong T^2$ be a smooth fiber of $E(2) \to \mathbb{P}^1$ and choose a smooth bump function μ that vanishes in a small δ–neighborhood of V and is 1 everywhere outside of a 2δ–neighborhood of V. Replacing $\alpha \in \mathcal{H}$ by $\mu\alpha$ and following the construction of relative invariants in [7], one can define the moduli space of 'V–regular' $(J,\mu\alpha)$–holomorphic maps $(f,\mu\alpha)$

$$\mathcal{M}^{\mathcal{H},V}_{g,k,s}(2S+dF) \tag{2.4}$$

where $s=(s_1,\cdots,s_l)$ is a multiplicity vector and $f^{-1}(V)$ consists of marked points p_j, $k+1 \le j \le k+l$, each with the contact order of f with V at p_j being s_j. Since each $s_j \ge 1$ and $(2S+dF)\cdot[V]=2$, the multiplicity vector s is either $(1,1)$ or (2). This moduli space also comes with a map

$$st \times ev \times h : \mathcal{M}^{\mathcal{H},V}_{g,k,s}(S+dF) \to \overline{\mathcal{M}}_{g,k+l} \times E(2)^k \times V^l \tag{2.5}$$

where ev is the evaluation map of first k marked points into $E(2)^k$ and h is the evaluation map of last l marked points into V^l. The moduli space (2.4) is compact (cf Section 6 of [14]) and hence carries a fundamental class

$$[\mathcal{M}^V_{g,k,s}(S+dF)]$$

which we can push forward by the map (2.5) to obtain a homology class

$$(st \times ev \times h)_* \left[\mathcal{M}^{\mathcal{H},V}_{g,k,s}(2S+dF)\right] \in H_{2r}\left(\overline{\mathcal{M}}_{g,k+l} \times E(2)^k \times V^l; \mathbb{Q}\right) \tag{2.6}$$

where $r = g + k + l - 2$. The relative family invariants of $(E(2), V)$ is then defined as

$$\begin{aligned} & GW^V_{2S+dF,g,s}\big(\kappa;\beta_1,\cdots,\beta_k;C_{\gamma_1\cdots\gamma_l}\big) \\ &= \ (st\times ev\times h)_*\left[\mathcal{M}^{\mathcal{H},V}_{g,k,s}\big(2S+dF\big)\right]\cap\big(\kappa^*\cup\beta^*\cup\gamma^*\big) \end{aligned} \tag{2.7}$$

where $\beta^* = \beta_1^* \cup \cdots \cup \beta_k^*$, $\gamma^* = \gamma_1^* \cup \cdots \cup \gamma_l^*$, and γ_j^* is the Poincaré dual of $\gamma_j \in H_*(V;\mathbb{Z})$ in V.

Similarly, we can define relative family invariants of $E(2)$ for the classes $S + (2d-3)F$, $d \in \mathbb{Z}$. In this case, we choose a symplectic submanifold $U \cong T^2$ of $E(2)$ that represents the class $2F$. Repeating the same arguments as above then gives relative family invariants of $(E(2), U)$

$$GW^U_{S+(2d-3)F,g,s}(\kappa;\beta_1,\cdots,\beta_k;C_{\gamma_1\cdots\gamma_l}) \tag{2.8}$$

where s is also either (1,1) or (2) since each $s_j \geq 1$ and $(S+(2d-3)F)\cdot[U] = 2$.

The invariant (2.7) (resp. (2.8)) counts the oriented number of genus g V–regular (resp. U–regular) $(J,\mu\alpha)$–holomorphic maps $(f,\mu\alpha)\colon C \to E(2)$, representing the homology class $2S + dF$ (resp. $S + (2d-3)F$), with $C \in K$ and $f(x_i) \in A_i$ such that these have a contact of order s_j with V (resp. U) along fixed representatives G_j of γ_j in V (resp. U) where K and A_i are representatives of κ and β_i. In particular, both relative invariants (2.7) and (2.8) have the same (formal) dimension and thus vanish unless

$$\deg(\kappa^*) + \sum \deg(\beta_i^*) + \sum \deg(\gamma_j^*) = 2\big(g + k + l - 2\big).$$

3 Outline of computations

Our goal is to compute the $g = 0$ family invariants of $K3$ surfaces for the classes A of index 2. By (2.3), it suffices to compute the invariants of $E(2)$ for the classes $2(S + dF)$. For convenience we assemble them in the generating functions

$$M_g(\,\cdot\,)(t) = \sum GW^{\mathcal{H}}_{2S+dF,g}\big(\,\cdot\,\big)\, t^d\,. \tag{3.1}$$

We further introduce generating functions for invariants of primitive classes, by the formula

$$\begin{aligned} N_g(\,\cdot\,)(t) &= \sum GW^{\mathcal{H}}_{S+dF,g}\big(\,\cdot\,\big)\, t^d\,, \\ P_g(\,\cdot\,)(t) &= \sum GW^{\mathcal{H}}_{S+(2d-3)F,g}\big(\,\cdot\,\big)\, t^d. \end{aligned} \tag{3.2}$$

It then follows from (2.3) that

$$M_0(t) \,-\, P_0(t) \,=\, \sum_{d\geq 0} \big(GW^{\mathcal{H}}_{2(S+dF),0} \,-\, GW^{\mathcal{H}}_{S+(4d-3)F,0} \big)\, t^{2d}$$

since both $2S+dF$ and $S+(2d-3)F$ are primitive with the same square when d is odd. In this and the following four sections we will show:

Proposition 3.1 $\quad M_0(t) - P_0(t) \,=\, \left(\frac{1}{2}\right)^3 N_0(t^2).$

Let A be any class of index 2 and B be any primitive classes such that $A^2 = B^2 = 4(2d-2)$. Proposition 3.1 and (2.3) then imply

$$\begin{aligned} GW^{\mathcal{H}}_{A,0} \,-\, GW^{\mathcal{H}}_{B,0} \;&=\; GW^{\mathcal{H}}_{2(S+dF),0} - GW^{\mathcal{H}}_{S+(4d-3)F,0} \\ &=\; \left(\tfrac{1}{2}\right)^3 GW^{\mathcal{H}}_{S+dF,0} \;=\; \left(\tfrac{1}{2}\right)^3 GW^{\mathcal{H}}_{A/2,0} \end{aligned}$$

and hence prove Theorem 1.1 of the introduction.

Below, we outline how we prove Proposition 3.1.

Let φ_i be the first Chern class of the line bundle $\mathcal{L}^{\mathcal{H}}_i \to \overline{\mathcal{M}}^{\mathcal{H}}_{g,k}\big(X, A\big)$ whose geometric fiber at the point $\big(C; x_1, \cdots, x_k, f, \alpha\big)$ is $T^*_{x_i}C$. Similarly as for the ordinary GW invariants, one can use φ_i to impose descendent constraints on family invariants as follows:

$$\begin{aligned} &GW^{\mathcal{H}}_{g,k}(X,A)(\tau_{m_1}(\beta_1), \cdots, \tau_{m_k}(\beta_k)) \\ &\;=\; (st \times ev)_*\big(\big[\overline{\mathcal{M}}^{\mathcal{H}}_{g,k}(X,A)\big] \cap \varphi_1^{m_1} \cdots \varphi_k^{m_k} \big) \;\cap\; (\beta_1^* \cup \cdots \cup \beta_k^*). \end{aligned}$$

If the constraint $\tau_{m_i}(\beta_i)$ repeats n times and $\deg(\beta_i^*)$ is even, we will use the notation $\tau_{m_i}(\beta_i)^n$.

Recall that $V \cong T^2$ is a fixed smooth fiber of $E(2) \to \mathbb{P}^2$. To save notation, we denote by F the fundamental class of V. Introduce a generating functions for the relative invariants of $(E(2), V)$, by the formula

$$M^V_{1,(2)}(t) \,=\, \sum GW^V_{2S+dF,1,(2)}\big(C_F\big)\, t^d. \tag{3.3}$$

As in Proposition 3.1 of [14], we can combine the $g=1$ TRR formula with the composition law (Proposition 3.7 of [13]) to have

$$M_1\big(\tau(F)\big) \,=\, \tfrac{1}{3}\, t\, M_0' \,-\, \tfrac{2}{3}\, M_0. \tag{3.4}$$

Then, in Proposition 4.4 we apply the Symplectic Sum Formula of [8] to obtain

$$M_1\big(\tau(F)\big) \,=\, M^V_{1,(2)} \,+\, 4\, G_2\, M_0, \tag{3.5}$$

$$M_2\big(\tau(F)^2\big) \,-\, 2\,M_1\big(pt\big) \;=\; 20\,G_2\,M^V_{1,(2)} \,+\, \big(16\,G_2^2 \,+\, 8\,t\,G_2'\big)\,M_0 \tag{3.6}$$

where $G_2(t)$ is the Eisenstein series of weight 2, namely

$$G_2(t) \;=\; \sum_{d\geq 0} \sigma(d)\,t^d \quad \text{where} \quad \sigma(d) \;=\; \sum_{k|d} k,\; d\geq 1 \quad \text{and} \quad \sigma(0) \;=\; -\tfrac{1}{24}.$$

Now, eliminate $M^V_{1,(2)}$ in (3.6) by using (3.4) and (3.5) to obtain

$$M_2\big(\tau(F)^2\big) \,-\, 2\,M_1\big(pt\big) \;=\; \tfrac{20}{3}\,G_2\,t\,M_0' \,-\, \big(64\,G_2^2 \,+\, \tfrac{40}{3}\,G_2 \,-\, 8\,t\,G_2'\big)\,M_0\,. \tag{3.7}$$

Recall that $U \cong T^2$ is a fixed symplectic submanifold of $E(2)$ that represents the class $2F$. Without any further confusion, we will also denote by F the fundamental class of $U \cong T^2$. Introduce a generating function for the relative invariants of $(E(2), U)$, by the formula

$$P^U_{1,(2)}(t) \;=\; \sum GW^U_{S+(2d-3)F,1,(2)}\big(C_F\big)\,t^d\,. \tag{3.8}$$

The genus 1 TRR formula gives a formula like (3.4)

$$P_1\big(\tau(2F)\big) \;=\; \tfrac{1}{3}\,t\,P_0' \,-\, \tfrac{2}{3}\,P_0\,. \tag{3.9}$$

Then, in Proposition 4.5 we apply the sum formula to have formulas like (3.5) and (3.6)

$$P_1\big(\tau(2F)\big) \;=\; P^U_{1,(2)} \,+\, 4\,G_2\,P_0, \tag{3.10}$$

$$P_2\big(\tau(2F)^2\big) \,-\, 2\,P_1\big(pt\big) \;=\; 20\,G_2\,P^U_{1,(2)} \,+\, \big(16\,G_2^2 \,+\, 8\,t\,G_2'\big)\,P_0 \tag{3.11}$$

Similarly, as above, equations (3.9), (3.10) and (3.11) give

$$P_2\big(\tau(2F)^2\big) \,-\, 2\,P_1\big(pt\big) \;=\; \tfrac{20}{3}\,G_2\,t\,P_0' \,-\, \big(64\,G_2^2 \,+\, \tfrac{40}{3}\,G_2 \,-\, 8\,t\,G_2'\big)\,P_0\,. \tag{3.12}$$

Note that the equations (3.7) and (3.12) have the same coefficients. This is true because all coefficients of TRR and sum formula depends only on the topological quantities

$$(2S{+}dF)^2 \;=\; (S{+}(2d{-}3)F)^2,\;\; (2S{+}dF){\cdot}F \;=\; (S{+}(2d{-}3)F){\cdot}2F,\;\; F^2 \;=\; (2F)^2.$$

Hence, (3.7) and (3.12) give

$$\begin{aligned} &3\,\big[\,M_2\big(\tau(F)^2\big) \,-\, P_2\big(\tau(2F)^2\big)\,\big] \,-\, 6\,\big[\,M_1\big(pt\big) \,-\, P_1\big(pt\big)\,\big] \\ &\;=\; 20\,G_2\,t\,\big(\,M_0 \,-\, P_0\,\big)' \,-\, \big(192\,G_2^2 \,+\, 40\,G_2 \,-\, 24\,t\,G_2'\big)\,\big(\,M_0 \,-\, P_0\,\big)\,. \end{aligned} \tag{3.13}$$

Note that by (2.3) both generating functions $M_1\big(pt\big){-}P_1\big(pt\big)$ and $M_0{-}P_0$ have no odd terms. One can also show that the generating function $M_2\big(\tau(F)^2\big) - P_2\big(\tau(2F)^2\big)$ has no odd terms (see [15]). Consequently, comparing odd terms of both sides of (3.13) gives the first order ODE

$$0 = \; 20\,G_o\,t\,\big(\,M_0 - P_0\,\big)' \,-\, \big(384\,G_e\,G_o \,+\, 40\,G_o \,-\, 24\,t\,G_o'\big)\,\big(\,M_0 - P_0\,\big) \tag{3.14}$$

where $G_e(t)$ (resp. $G_o(t)$) is the sum of all even (resp. odd) terms of $G_2(t)$. On the other hand, it follows from the equation (2.7) of [14] that

$$t\,\frac{d}{dt}N_0(t^2) \;=\; 2\,t^2\,N_0'(t^2) \;=\; 48\,G_2(t^2)\,N_0(t^2) \;+\; 2\,N_0(t^2). \tag{3.15}$$

Combining (3.15) with the following relation of certain quasi-modular forms

$$4\,t^2\,G_2'(t^2) \;=\; 32\,G_2^2(t^2) \;-\; 40\,G_2(t)\,G_2(t^2) \;+\; 8\,G_2^2(t) \;-\; t\,G_2'(t) \tag{3.16}$$

(see Lemma 3.2) says that $N_0(t^2)$ also satisfies the same ODE (3.14). Since the initial conditions are $N_0(0)=1$ (cf [14]) and $M_0(0)-P_0(0)=(1/2)^3$ (cf [5]), we can conclude that

$$M_0(t)-P_0(t) \;=\; \left(\tfrac{1}{2}\right)^3 N_0(t^2).$$

This completes the proof of Proposition 3.1 and hence of Theorem 1.1 of the introduction. The main task is, thus, to establish the sum formulas (3.5), (3.6), (3.10), and (3.11).

We end this section with the proof of (3.16).

Lemma 3.2 $\;4\,t^2\,G_2'(t^2) \;=\; 32\,G_2^2(t^2) \;-\; 40\,G_2(t)\,G_2(t^2) \;+\; 8\,G_2^2(t) \;-\; t\,G_2'(t).$

Proof Let $G_2(z)=\sum_{d\geq 0}\sigma(d)\,q^d$ and set

$$E(z) \;=\; -2\,DG_2(2z) \;+\; 32\,G_2^2(2z) \;-\; 40\,G_2(z)\,G_2(2z) \;+\; 8\,G_2^2(z) \;-\; DG_2(z)$$

where $z\in\mathbb{C}$ with $\mathrm{Im(z)}>0$, $q=e^{2\pi iz}$ and $D=q\frac{d}{dq}$ is the logarithmic differential operator. It then suffices to show that $E(z)\equiv 0$.

Since the Eisenstein series $G_2(z)$ of weight 2 satisfies

$$G_2\Big(\frac{az+b}{cz+d}\Big) \;=\; (cz+d)^2\,G_2(z) \;-\; \frac{c(cz+d)}{4\pi i} \quad \text{for any} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

one can show by hand that $E(z)$ is a modular form of weight 4 and of level 2 on the Hecke subgroup $\Gamma_0(2)$. The space of such modular forms is a 2-dimensional vector space with generators

$$G_4(z) \;=\; \frac{1}{24} \;+\; 10\sum_{d\geq 1}\sigma_3(d)\,q^d \;=\; \frac{1}{24} \;+\; 10\,q \;+\; 90\,q^2 \;+\; \cdots$$

$$G_2^{(4)}(z) \;=\; \big[G_2(z) \;-\; 2\,G_2(2z)\big]^2 \;=\; \frac{1}{24^2} \;+\; \frac{1}{12}\,q \;+\; \frac{26}{24}\,q^2 \;+\; \cdots$$

where $\sigma_3(d)=\sum_{k|d}k^3$ (cf [10]). Thus, $E(z)$ can be written as

$$E(z) \;=\; a\,G_4(z) \;+\; b\,G_2^{(4)}(z) \;=\; \Big(\frac{a}{24} \;+\; \frac{b}{24^2}\Big) \;+\; \Big(10\,a \;+\; \frac{b}{12}\Big)\,q \;+\; \cdots$$

for some constants a and b. On the other hand, one can also show by hand that the first two terms in the q–expansion of $E(z)$ vanish. Consequently, $a = b = 0$ and hence $E(z) \equiv 0$. □

4 Symplectic sum formula

Let $E(0) = S^2 \times T^2 \to S^2$ be a rational elliptic surface. To save notation, we also denote by S and F the section class and the fiber class of $E(0)$, respectively. In this section, we apply the sum formula [8] of the symplectic sum of $E(2)$ and $E(0)$ to prove the sum formulas (3.5), (3.6), (3.10), and (3.11).

Recall that $V \cong T^2$ is a fixed fiber of $E(2) \to S^2$. For convenience, we use the same notation V for a fixed fiber of $E(0) \to S^2$. Recall that relative GT (Gromov–Taubes) invariants of $(E(0), V)$ count V–regular maps from possibly disconnected domain [7]. The sum formula of [8] applies to GT invariants to give relations between GT invariants and relative GT invariants. The sum formula also applies to GW invariants. In this case, it gives relations between GW invariants and (partial) relative GT invariants — these invariants are defined to count maps each of whose domain component has contact order at least one with V. We denote such (partial) relative invariants of $(E(0), V)$ for the class $2S + dF$ with the Euler characteristic χ and the multiplicity vector s by

$$G\Phi^V_{2S+dF,\chi,s}\big(C_{\gamma_1\cdots\gamma_l};\kappa;\beta_1,\cdots,\beta_k\big). \tag{4.1}$$

Here, $s = (s_1, \cdots, s_l)$ equals (1,1) or (2), $\gamma_j \in H_*(V;\mathbb{Z})$, $\kappa \in H_*(\overline{\mathcal{M}}_{\chi,k+l};\mathbb{Q})$ and $\beta_i \in H_*(E(0);\mathbb{Z})$; $\overline{\mathcal{M}}_{\chi,k+l}$ is the space of all compact Riemann surface of Euler characteristic χ with $k + l$ marked points. We will also use the notation γ_j^n if the constraint γ_j repeats n times and $\deg(\gamma_j^*)$ is even. By dimension formula of [7], the (partial) GT invariant (4.1) vanishes unless

$$\deg(\kappa^*) + \sum \deg(\beta_i^*) + \sum \deg(\gamma_j^*) = 2\big(4 - \tfrac{1}{2}\chi + k + l - 2\big).$$

Consider the symplectic sum of $E(2)$ and $E(0)$ along V

$$E(2) = E(2) \#_V E(0). \tag{4.2}$$

The Gluing Theorem (Theorem 10.1 of [8]) applies for this sum to give relations between family GW invariants of $E(2)$ for the classes $2S + dF$ and (partial) relative family GT invariants of $(E(2), V)$ for the classes $2S + dF$. The latter count V–regular maps f with possibly disconnected domains such that if f has a disconnected domain of Euler characteristic χ with k marked points f is a pair of V–regular maps (f_1, f_2) satisfying $[f_i] = S + d_iF$ with $d_1 + d_2 = d$

and the domain of f_i lies in $\overline{\mathcal{M}}_{g_i,k_i}$ with $\chi = 4 - 2(g_1 + g_2)$ and $k = k_1 + k_2$. Denote the moduli space of all such pairs (f_1, f_2) by

$$\overline{\mathcal{M}}^{\mathcal{H},V}_{(g_1,g_2),(k_1,k_2),(1,1)}\big(d_1, d_2\big). \tag{4.3}$$

The standard corbodism argument (cf proof of Proposition 3.7 of [13]) then shows that the moduli space (4.3) is corbodant to the product moduli space

$$\overline{\mathcal{M}}^{\mathcal{H},V}_{g_1,k_1,(1)}(S + d_1F) \times \overline{\mathcal{M}}^{V}_{g_2,k_2,(1)}(S + d_2F)$$

where the first factor is a relative family GW moduli space of $(E(2), V)$ and the second is a relative ordinary GW moduli space of $(E(2), V)$. Since $E(2)$ is a K3 surface, the contribution of the second factor to relative ordinary GW invariants of $(E(2), V)$ vanishes. Consequently, the contribution of the moduli space (4.3) to (partial) relative family GT invariants also vanishes. This implies that for the classes $2S + dF$ the (partial) relative family GT invariants of $(E(2), V)$ and the relative family GW invariants GW^V are the same. Therefore, a family version of the sum formula of the sum (4.2) for the classes $2S + dF$ relates $GW^{\mathcal{H}}$ invariants of $E(2)$ and GW^V invariants of $(E(2), V)$.

On the other hand, using Lemma 14.5 of [8] and routine dimension count one can show that there is no 'contribution from the neck' (cf Section 12 of [8]). Moreover, 'rim tori' [7] of $E(2)$–side disappear under the symplectic sum (4.2) — that enables us to work with summed relative invariants. Combined with these observations, the Gluing Theorem then yields a considerably simple sum formulas (4.4) below: Let $\{\gamma_i\}$ be a basis of $H_*(V;\mathbb{Z})$ and $\{\gamma^i\}$ be its dual basis with respect to the intersection form of V. For a vector of nonnegative integers $m = (m_1, \cdots, m_4)$ with $\sum m_i$ is either 2 or 1, we set

$$C_{\gamma_m} = C_{\gamma_1^{m_1}\cdots\gamma_4^{m_4}}, \quad C_{\gamma_{m^*}} = C_{(\gamma^4)^{m_4}\cdots(\gamma^1)^{m_1}}, \quad \text{and} \quad m! = \prod m_i!.$$

For a multiplicity vector $s = (s_1, \cdots, s_l)$, either (1,1) or (2), let $|s| = \prod s_i$. We are now ready to write a sum formula of the symplectic sum (4.2)

$$\begin{aligned} &GW^{\mathcal{H}}_{2S+dF,g}\big(\tau(F)^k, pt^{g-k}\big) \\ &= \sum \frac{|s|}{m!}\, GW^V_{2S+d_1F,g_1,s}\big(C_{\gamma_m}\big)\, G\Phi^V_{2S+d_2F,\chi_2,s}\big(C_{\gamma_{m^*}}; \tau(F)^k, pt^{g-k}\big) \end{aligned} \tag{4.4}$$

where the sum is over all $s = (s_1, \cdots, s_l)$ which is either (1,1) or (2), vectors $m = (m_i)$ as above with $\sum m_i = l(s)$, $d = d_1 + d_2$ and $g = g_1 - \frac{1}{2}\chi_2 + l(s)$.

Similarly, one can also derive a sum formula for the case of family invariants of $E(2)$ for the classes $S + (2d - 3)F$, $d \in \mathbb{Z}$. Recall that $U \cong T^2$ is a fixed symplectic submanifold of $E(2)$ that represents the class $2F$. We also denote by U a fixed fiber of $E(0)\to S^2$ and consider the symplectic sum of $E(2)$ with $E(0)$ along U

$$E(2) \;=\; E(2)\;\#_U\,E(0). \tag{4.5}$$

Since both U and V are fibers of $E(0)$, relative GW invariants of $(E(0),V)$ and $(E(0),U)$ are in fact the same. Thus, the (partial) relative GT invariants of $(E(0),V)$ and $(E(0),U)$ are also the same invariants, ie $G\Phi^V = G\Phi^U$. On the other hand, $U\subset E(2)$ represents $2F$ on $E(2)$, while $U\subset E(0)$ represents F on $E(0)$. With these observations, repeating the same arguments as above for the symplectic sum (4.5) gives a sum formula like (4.4)

$$\begin{aligned} &GW^{\mathcal{H}}_{S+(2d-3)F,g}\big(\tau(2F)^k, pt^{g-k}\big)\\ &\quad= \sum \frac{|s|}{m!}\, GW^{U}_{S+(2d_1-3)F,g_1,s}\big(C_{\gamma_m}\big)\, G\Phi^{V}_{2S+d_2F,\chi_2,s}\big(C_{\gamma_{m^*}};\tau(F)^k, pt^{g-k}\big). \end{aligned} \tag{4.6}$$

Remark 4.1 Once and for all, we fix an (ordered) basis $\{\,pt,\gamma_1,\gamma_2,F\,\}$ of $H_*(V;\mathbb{Z})\cong H_*(U;\mathbb{Z})$ and its (ordered) dual basis $\{\,F,\gamma_2,-\gamma_1,pt\,\}$ with respect to the intersection form of $V\cong U$ where $\{\gamma_1,\gamma_2\}$ is a basis of $H_1\big(V;\mathbb{Z}\big)\cong H_1\big(U;\mathbb{Z}\big)\cong H_1\big(E(0);\mathbb{Z}\big)$ with $\gamma_1\cdot\gamma_2=1$. Then, in the sum formulas (4.4) and (4.6) the splitting of diagonal for contact constraints C_{γ_m} is given as follows:

- if $m=(2,0,0,0)$ then $\gamma_m = pt^2$ and $\gamma_{m^*} = F^2$,
- if $m=(1,0,0,1)$ then $\gamma_m = pt\cdot F$ and $\gamma_{m^*} = pt\cdot F$,
- if $m=(0,1,1,0)$ then $\gamma_m = \gamma_1\cdot\gamma_2$ and $\gamma_{m^*} = (-\gamma_1)\cdot\gamma_2$.

Using the sum formula (4.4), one can derive relations between invariants $GW^{\mathcal{H}}$ and GW^V.

Lemma 4.2 *Let GW^V be the relative family invariants of $(E(2),V)$. Then,*

(a) $\frac{1}{2}\,GW^V_{2S+dF,0,(1,1)}\big(C_{F^2}\big) \;=\; GW^{\mathcal{H}}_{2S+dF,0}$,

(b) $GW^V_{2S+dF,1,(1,1)}\big(C_{\gamma_1\cdot\gamma_2}\big) \;=\; GW^V_{2S+dF,1,(1,1)}\big(C_{pt\cdot F}\big)$,

(c) $GW^V_{2S+dF,1,(1,1)}\big(C_{pt\cdot F}\big) \;=\; GW^{\mathcal{H}}_{2S+dF,1}\big(pt\big) - 2\sum_{d=d_1+d_2} GW^{\mathcal{H}}_{2S+d_1F,0}\, d_2\,\sigma(d_2)$.

Proof (a) By the sum formula (4.4), Remark 4.1 and Lemma 7.1 a, we have

$$\begin{aligned} GW^{\mathcal{H}}_{2S+dF,0} \;&=\; \sum_{d=d_1+d_2} \tfrac{1}{2}\,GW^V_{2S+d_1F,0,(1,1)}\big(C_{F^2}\big)\,G\Phi^V_{2S+d_2F,4,(1,1)}\big(C_{pt^2}\big)\\ &=\; \tfrac{1}{2}\,GW^V_{2S+dF,0,(1,1)}\big(C_{F^2}\big). \end{aligned}$$

(b) Note that the sum formula (4.4) also holds for one dimensional constraints. The sum formula (4.4), Remark 4.1 and Lemma 7.1 b,c,d,e thus give

$$\begin{aligned}
&GW^{\mathcal{H}}_{2S+dF,1}\big(\iota_*(\gamma_1),\iota_*(\gamma_2)\big)\\
&= \sum_{d=d_1+d_2} GW^V_{2S+d_1F,1,(1,1)}\big(C_{pt\cdot F}\big)\, G\Phi^V_{2S+d_2F,4,(1,1)}\big(C_{pt\cdot F};\gamma_1,\gamma_2\big)\\
&+ \sum_{d=d_1+d_2} GW^V_{2S+d_1F,1,(1,1)}\big(C_{\gamma_1\cdot\gamma_2}\big)\, G\Phi^V_{2S+d_2F,4,(1,1)}\big(C_{(-\gamma_1)\cdot\gamma_2};\gamma_1,\gamma_2\big)\\
&+ \sum_{d=d_1+d_2} 2\, GW^V_{2S+d_1F,1,(2)}\big(C_F\big)\, G\Phi^V_{2S+d_2F,2,(2)}\big(C_{pt};\gamma_1,\gamma_2\big)\\
&+ \sum_{d=d_1+d_2} \tfrac{1}{2}\, GW^V_{2S+d_1F,0,(1,1)}\big(C_{F^2}\big)\, G\Phi^V_{2S+d_2F,2,(1,1)}\big(C_{pt^2};\gamma_1,\gamma_2\big)\\
&= \; GW^V_{2S+dF,1,(1,1)}\big(C_{pt\cdot F}\big) \;-\; GW^V_{2S+dF,1,(1,1)}\big(C_{\gamma_1\cdot\gamma_2}\big)
\end{aligned}$$

where $\iota\colon V\hookrightarrow E(2)$ is the inclusion map. Since $E(2)$ is simply connected, the left hand side of the first equality vanishes and hence this shows (b).

(c) We have

$$\begin{aligned}
&GW^{\mathcal{H}}_{2S+dF,1}\big(pt\big)\\
&= \sum_{d=d_1+d_2} GW^V_{2S+d_1F,1,(1,1)}\big(C_{pt\cdot F}\big)\, G\Phi^V_{2S+d_2F,4,(1,1)}\big(C_{pt\cdot F};pt\big)\\
&+ \sum_{d=d_1+d_2} 2\, GW^V_{2S+d_1F,1,(2)}\big(C_F\big)\, G\Phi^V_{2S+d_2F,2,(2)}\big(C_{pt};pt\big)\\
&+ \sum_{d=d_1+d_2} \tfrac{1}{2}\, GW^V_{2S+d_1F,0,(1,1)}\big(C_{F^2}\big)\, G\Phi^V_{2S+d_2F,2,(1,1)}\big(C_{pt^2};pt\big)\\
&= GW^V_{2S+dF,1,(1,1)}\big(C_{pt\cdot F}\big) \;+\; 2\sum_{d=d_1+d_2} GW^{\mathcal{H}}_{2S+d_1F,0}\, d_2\,\sigma(d_2)
\end{aligned}$$

where the first equality follows from the sum formula (4.4) and Remark 4.1, and the second follows from Lemma 7.1 f,g,h. □

The sum formula (4.6) also gives relations between invariants $GW^{\mathcal{H}}$ and GW^U.

Lemma 4.3 *Let GW^U be the relative family invariants of $(E(2),U)$. Then,*

(a) $\frac{1}{2}\, GW^U_{S+(2d-3)F,0,(1,1)}\big(C_{F^2}\big) \;=\; GW^{\mathcal{H}}_{S+(2d-3)F,0}$,

(b) $GW^U_{S+(2d-3)F,1,(1,1)}\big(C_{\gamma_1\cdot\gamma_2}\big) \;=\; GW^U_{S+(2d-3)F,1,(1,1)}\big(C_{pt\cdot F}\big)$,

(c) $GW^U_{S+(2d-3)F,1,(1,1)}\big(C_{pt\cdot F}\big)$

$$= GW^{\mathcal{H}}_{S+(2d-3)F,1}\big(pt\big) \,-\, 2\sum_{d=d_1+d_2} GW^{\mathcal{H}}_{S+(2d_1-3)F,0}\, d_2\, \sigma(d_2)\,.$$

The proof of this lemma is identical to the proof of Lemma 4.2.

Now, we are ready to show the sum formulas (3.5) and (3.6).

Proposition 4.4 *Let M and M^V be the generating functions defined in (3.1) and (3.3), respectively. Then,*

(a) $M_1\big(\tau(F)\big) \,=\, M^V_{1,(2)} \,+\, 4\, M_0\, G_2\,,$

(b) $M_2\big(\tau(F)^2\big) \,=\, 2\, M_1(pt) \,+\, 20\, M^V_{1,(2)}\, G_2 \,+\, M_0\, \big(16\, G_2^2 \,+\, 8\, t\, G_2'\big)\,.$

Proof (a) We have

$$\begin{aligned}
&GW^{\mathcal{H}}_{2S+dF,1}\big(\tau(F)\big)\\
&= \sum_{d=d_1+d_2} GW^V_{2S+d_1F,1,(1,1)}\big(C_{pt\cdot F}\big)\, G\Phi^V_{2S+d_2F,4,(1,1)}\big(C_{pt\cdot F};\tau(F)\big)\\
&+ \sum_{d=d_1+d_2} 2\, GW^V_{2S+d_1F,1,(2)}\big(C_F\big)\, G\Phi^V_{2S+d_2F,0,(2)}\big(C_{pt};\tau(F)\big)\\
&+ \sum_{d=d_1+d_2} \tfrac{1}{2}\, GW^V_{2S+d_1F,0,(1,1)}\big(C_{F^2}\big)\, G\Phi^V_{2S+d_2F,2,(1,1)}\big(C_{pt^2};\tau(F)\big).\\
&= GW^V_{2S+dF,1,(2)}\big(C_F\big) \,+\, \sum_{d=d_1+d_2} 2\, GW^V_{2S+d_1F,0,(1,1)}\big(C_{F^2}\big)\, \sigma(d_2)\\
&= GW^V_{2S+dF,1,(2)}\big(C_F\big) \,+\, \sum_{d=d_1+d_2} 4\, GW^{\mathcal{H}}_{2S+d_1F,0}\, \sigma(d_2) \qquad (4.7)
\end{aligned}$$

where the first equality follows from the sum formula (4.4) and Remark 4.1, the second equality follows from Lemma 7.2 and the third equality follows from Lemma 4.2 a. Then, (a) follows from (4.7) and definition of generating functions.

(b) The sum formula (4.4) and Remark 4.1 give

$$\begin{aligned}
&GW^{\mathcal{H}}_{2S+dF,2}\big(\tau(F)^2\big)\\
&= \sum_{d=d_1+d_2} \tfrac{1}{2}\, GW^V_{2S+d_1F,2,(1,1)}\big(C_{pt^2}\big)\, G\Phi^V_{2S+d_2F,4,(1,1)}\big(C_{F^2};\tau(F)^2\big)\\
&+ \sum_{d=d_1+d_2} 2\, GW^V_{2S+d_1F,2,(2)}\big(C_{pt}\big)\, G\Phi^V_{2S+d_2F,0,(2)}\big(C_F;\tau(F)^2\big)\\
&+ \sum_{d=d_1+d_2} GW^V_{2S+d_1F,1,(1,1)}\big(C_{pt\cdot F}\big)\, G\Phi^V_{2S+d_2F,2,(1,1)}\big(C_{pt\cdot F};\tau(F)^2\big)
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{d=d_1+d_2} GW^V_{2S+d_1F,1,(1,1)}\big(C_{\gamma_1\cdot\gamma_2}\big)\, G\Phi^V_{2S+d_2F,2,(1,1)}\big(C_{(-\gamma_1)\cdot\gamma_2};\tau(F)^2\big)\\
&+ \sum_{d=d_1+d_2} 2\, GW^V_{2S+d_1F,1,(2)}\big(C_F\big)\, G\Phi^V_{2S+d_2F,1,(2)}\big(C_{pt};\tau(F)^2\big)\\
&+ \sum_{d=d_1+d_2} \tfrac{1}{2}\, GW^V_{2S+d_1F,0,(1,1)}\big(C_{F^2}\big)\, G\Phi^V_{2S+d_2F,0,(1,1)}\big(C_{pt^2};\tau(F)^2\big) \qquad (4.8)
\end{aligned}
$$

By Lemma 7.3, the right hand side of (4.8) becomes

$$
\begin{aligned}
&GW^V_{2S+dF,1,(1,1)}\big(C_{pt\cdot F}\big) + GW^V_{2S+dF,1,(1,1)}\big(C_{\gamma_1\cdot\gamma_2}\big)\\
&+ \sum_{d=d_1+d_2} 20\, GW^V_{2S+d_1F,1,(2)}\big(C_F\big)\,\sigma(d_2)\\
&+ \sum_{d=d_1+d_2} \tfrac{1}{2}\, GW^V_{2S+d_1F,0,(1,1)}\big(C_{F^2}\big)\Big(\sum_{k_1+k_2=d_2} 16\,\sigma(k_1)\,\sigma(k_2) + 12\, d_2\,\sigma(d_2)\Big).
\end{aligned}
$$

This can be further simplified by using Lemma 4.2 to give

$$
\begin{aligned}
&2\, GW^{\mathcal{H}}_{2S+dF,1}\big(pt\big) + \sum_{d=d_1+d_2} 20\, GW^V_{2S+d_1F,1,(2)}\big(C_F\big)\cdot\sigma(d_2)\\
&+ \sum_{d=d_1+d_2} GW^{\mathcal{H}}_{2S+d_1F,0}\Big(\sum_{k_1+k_2=d_2} 16\,\sigma(k_1)\,\sigma(k_2) + 8\, d_2\,\sigma(d_2)\Big). \qquad (4.9)
\end{aligned}
$$

Thus, (b) follows from (4.8), (4.9) and definition of generating functions. □

The same computation shows (3.10) and (3.11):

Proposition 4.5 *Let P and P^U be the generating functions defined in (3.2) and (3.8), respectively. Then,*

(a) $P_1\big(\tau(2F)\big) = P^U_{1,(2)} + 4\, P_0\, G_2,$

(b) $P_2\big(\tau(2F)^2\big) = 2\, P_1(pt) + 20\, P^U_{1,(2)}\, G_2 + P_0\,\big(16\, G_2^2 + 8\, t\, G_2'\big).$

5 GW invariants of $E(0)$

In order to complete the proof of the sum formulas (3.5), (3.6), (3.10), and (3.11), we need to compute the (partial) Gromov–Taubes invariants of $E(0)$ that appeared in Section 4. Those invariants are expressed in terms of the relative invariants of $E(0)$. The aim of this section is to compute various GW invariants of $E(0)$ which we use in later sections to compute the required relative invariants.

Recall that S and F denote the section class and the fiber class of $E(0)$, respectively. We always denote the genus g GW invariants of $E(0)$ for the class A by

$$\Phi_{A,g}\big(\kappa;\beta_1,\cdots,\beta_k\big)$$

where $\kappa\in H_*(\overline{\mathcal{M}}_{g,k};\mathbb{Q})$ and $\beta_i\in H_*(E(0);\mathbb{Z})$. Note that by dimension count this invariant vanishes unless

$$\deg(\kappa^*)\,+\,\sum\deg(\beta_i^*)\;=\;2\,(\,A\cdot(2F)+g-1+k\,)$$

where κ^* and β_i^* are Poincaré dual of κ and β_i, respectively.

We start with the genus 0 GW invariants for the trivial homology class. The lemma below directly follows from Proposition 1.2 of [11].

Lemma 5.1 *Let Φ denote the GW invariants of $E(0)$. Then,*

$$\Phi_{0,0}\big(\kappa;\beta_1,\cdots,\beta_n\big)=0 \text{ unless } n=3 \quad\text{and}\quad \Phi_{0,0}\big(\beta_1,\beta_2,\beta_3\big)=\int_{E(0)}\beta_1^*\cup\beta_2^*\cup\beta_3^*$$

where β_i^* denote the Poincaré dual of $\beta_i\in H_*(E(0);\mathbb{Z})$.

Recall that $V\cong T^2$ is a fixed fiber of $E(0)\to S^2$. We always denote the genus g relative GW invariants of $(E(0),V)$ for the class A with the multiplicity vector s by

$$\Phi^V_{A,g,s}\big(\kappa;\beta_1,\cdots,\beta_k;C_{\gamma_1\cdots\gamma_l}\big)$$

where $s=(s_1,\cdots,s_l)$ with $\sum s_j=A\cdot[V]=A\cdot F$ and $\gamma_j\in H_*(V;\mathbb{Z})$. By dimension formula of [7], this relative invariant vanishes unless

$$\deg(\kappa^*)\,+\,\sum\deg(\beta_i^*)\,+\,\sum\deg(\gamma_j^*)\;=\;2\,\big(\,A\cdot(2F)+g-1+k+l-A\cdot F\,\big)$$

where γ_i^* is the Poincaré dual of γ_i.

Recall that $\{\gamma_1,\gamma_2\}$ is a basis of $H_1(V;\mathbb{Z})\cong H_1(E(0);\mathbb{Z})$ with $\gamma_1\cdot\gamma_2=1$ and F also denotes the fundamental class of V.

Lemma 5.2 [8, 18, 14] *Let Φ and Φ^V denote the GW invariants of $E(0)$ and the relative GW invariants of $(E(0),V)$, respectively. Then,*

(a) $\Phi_{S,0}\big(pt\big)\;=\;\Phi^V_{S,0,(1)}\big(C_{pt}\big)\;=\;\Phi^V_{S,0,(1)}\big(pt;C_F\big)\;=\;1,$

(b) $\Phi_{S,0}\big(\gamma_1,\gamma_2\big)\;=\;\Phi^V_{S,0,(1)}\big(\gamma_1;C_{\gamma_2}\big)\;=\;\Phi^V_{S,0,(1)}\big(\gamma_1,\gamma_2;C_F\big)\;=\;1,$

(c) $\Phi_{S+dF,1}\big(\tau(F),pt\big)\;=\;\Phi^V_{S+dF,1,(1)}\big(\tau(F);C_{pt}\big)$
$\;=\;\Phi^V_{S+dF,1,(1)}\big(\tau(F),pt;C_F\big)\;=\;2\,\sigma(d),$

(d) $\Phi_{S+dF,1}\big(pt^2\big) = 2\,d\,\sigma(d)$; $\Phi^V_{S+dF,1}\big(pt;C_{pt}\big) = d\,\sigma(d)$,

(e) $\Phi_{S+dF,1}\big(pt,\gamma_1,\gamma_2\big) = \Phi^V_{S+dF,1,(1)}\big(pt,\gamma_1;C_{\gamma_2}\big) = d\,\sigma(d)$;
$\Phi^V_{S+dF,1,(1)}\big(\gamma_1,\gamma_2;C_{pt}\big) = 0$,

(f) $\Phi_{dF,1}\big(S\big) = 2\,\sigma(d)$.

We will often use the following simple observations:

Remark 5.3 Consider $E(0) = S^2 \times T^2$ with a product complex structure. Since there is no nontrivial holomorphic map from S^2 to T^2, any nontrivial holomorphic map $f\colon S^2 \to E(0)$ should represent a class aS, $a \geq 1$, with its image a section. Thus, the image of such maps can't pass through generic two points, a generic geometric representative of γ_1 or γ_2 and a generic point, or a generic section and a generic point. Combined with the Gromov Convergence Theorem [21, 20, 9], this observation gives vanishing results of certain genus 0 invariants. For example, $\Phi_{aS+dF,0}\big(\,\cdot\,\big) = \delta_{d0}\,\Phi_{aS,0}\big(\,\cdot\,\big)$ and

$$\Phi_{aS,0}\big(pt,\gamma_1,\,\cdot\,\big) = \Phi_{aS,0}\big(S,pt,\,\cdot\,\big) = \Phi_{aS,0}\big(S,S,\,\cdot\,\big) = 0\,.$$

Fix a product complex structure on $E(0)$ and let $f\colon S^2 \to E(0)$ be a holomorphic map representing a class aS, $a \geq 1$. Then f is a branched covering of some section S_0 of $E(0)$ and since the normal bundle of S_0 is trivial $H^1(f^*TE(0)) = H^1(:\ \oplus f^*TS_0) = 0$. This shows that the linearization L_f of the holomorphic map equation at f has a trivial cokernal. With this observation, we will use the product complex structure of $E(0)$ for computation of both absolute and relative GW invariants for the following cases:

Lemma 5.4 *Let Φ and Φ^V denote the GW invariants of $E(0)$ and the relative GW invariants of $(E(0),V)$, respectively. Then, we have*

(a) $2\,\Phi^V_{2S,0,(2)}\big(\tau(F);C_{pt}\big) = \Phi^V_{2S,0,(2)}\big(pt,\tau(F);C_F\big) = 1$,

(b) $\Phi^V_{S,0,(1)}\big(\tau(F);C_F\big) = \Phi_{S,0}\big(\tau(F)\big) = \Phi^V_{2S,0,(2)}\big(\tau(F)^2;C_F\big) = 0$.

Proof (a) Fix a product complex structure on $E(0)$ and let V_1, V_2, V be distinct fibers and p be a point in V. Denote by

$$\mathcal{M}^V_{0,2,(2)}\big(E(0),2S\big)(V_1,V_2,p) \;\subset\; \overline{\mathcal{M}}_{0,3}\big(E(0),2S\big) \tag{5.1}$$

the cut-down moduli space that consists of all maps $(f,C;x_1,x_2,x_3)$ satisfying (i) the contact order of f with V at x_3 is 2 and (ii) $f(x_i) \in V_i$, $i = 1,2$, and $f(x_3) = p$. This space is smooth of expected complex dimension 1 with no boundary stratum; by stability each map in the space (5.1) has a smooth domain. Moreover, each map f in the space (5.1) has no non-trivial automorphism and the linearization L_f has a trivial cokernal. Therefore, the invariant $\Phi^V_{2S,0,(2)}\big(\tau(F),F;C_{pt}\big)$ is equal to the (homology) Euler class of the relative cotangent bundle $\mathcal{L}_1$ over the cut-down moduli space (5.1) whose fiber over $(f,C;x_1,x_2,x_3)$ is $T^*_{x_1}C$.

Note that each map f in (5.1) is a 2-fold branched covering of the fixed section $S_p\cong\mathbb{P}^1$ containing the point p. Let p_1,p_2, and p_3 be distinct points of $\mathbb{P}^1$ and $P=\{p_3\}$. Then the space (5.1) can be identified with

$$\mathcal{M}^P_{0,2,(2)}\big(\mathbb{P}^1,2\big)(p_1,p_2)\ \subset\ \overline{\mathcal{M}}_{0,3}\big(\mathbb{P}^1,2\big) \tag{5.2}$$

the space of degree 2 stable maps $f\colon\big(\mathbb{P}^1,x_1,x_2,x_2\big)\ \to\ \big(\mathbb{P}^1,P\big)$ satisfying (i) the contact order of f with p_3 at x_3 is 2, and (ii) $f(x_i)=p_i$ for $i=1,2$. Under this identification, the relative cotangent bundle $\mathcal{L}_1$ over the space (5.1) becomes the relative cotangent bundle, still denoted by $\mathcal{L}_1$, over the space (5.2); this bundle $\mathcal{L}_1$ has a fiber $T^*_{x_1}\mathbb{P}^1$ at f.

For each map f in the space (5.2), choose local holomorphic coordinates z centered at x_1 and w centered at p_1. Then, there is a local expansion $f(z)=\sum_{k\geq1}a_k\,z^k$. The leading coefficient a_1 is the 1-jet of f at x_1 modulo higher order terms. Thus, we have a global section

$$a_1\ \in\ \mathcal{L}_1 \tag{5.3}$$

over the space (5.2). The zero set of this section consists of degree two branched coverings $\big(\mathbb{P}^1,x_1,x_2,x_3\big)\to\big(\mathbb{P}^1,p_1,p_2,p_3\big)$ with the ramification indexes (2,1,2) at marked points (x_1,x_2,x_3). Since there is only one such map, the (homology) Euler class of the bundle $\mathcal{L}_1$ over the space (5.2) is one. Consequently, we have

$$2\,\Phi^V_{2S,0,(2)}\big(\tau(F);C_{pt}\big)\ =\ \Phi^V_{2S,0,(2)}\big(\tau(F),F;C_{pt}\big)\ =\ 1.$$

By the same arguments as above, the invariant $\Phi^V_{2S,0,(2)}\big(pt,\tau(F);C_F\big)$ is the number of degree two branched coverings $\big(\mathbb{P}^1,x_1,x_2,x_3\big)\to\big(\mathbb{P}^1,p_1,p_2,p_3\big)$ with the ramification indexes (1,2,2). Since the number of such maps is 1, we have

$$\Phi^V_{2S,0,(2)}\big(pt,\tau(F);C_F\big)\ =\ 1.$$

(b) Similarly, as above, one can show that the invariant $\Phi^V_{S,0,(1)}\big(\tau(F),F;C_F\big)$ is the Euler class of the relative cotangent bundle

$$\mathcal{L}_1\ \to\ \Big(\,\mathcal{M}^P_{0,2,(1)}\big(\mathbb{P}^1,1\big)(p_1,p_2)\Big)\ \times V$$

with a section defined similarly as in (5.3). The zero set of this section is empty since there is no degree 1 map $\mathbb{P}^1 \to \mathbb{P}^1$ with ramification indexes $(2,1,1)$. Thus, we have

$$\Phi^V_{S,0,(1)}\big(\tau(F); C_F\big) \;=\; \Phi^V_{S,0,(1)}\big(\tau(F), F; C_F\big) \;=\; 0.$$

Repeating the same argument, we also have

$$\Phi_{S,0}\big(\tau(F)\big) \;=\; \Phi_{S,0}\big(\tau(F), F^2\big) \;=\; 0.$$

Similarly, the invariant $\Phi^V_{2S,0,(2)}\big(\tau(F)^2; C_F\big)$ is the Euler class of the bundle

$$\mathcal{L}_1 \oplus \mathcal{L}_2 \;\to\; \Big(\mathcal{M}^P_{0,2,(2)}(\mathbb{P}^1,2)(p_1,p_2)\Big) \times V$$

with a section defined similarly as in (5.3). The zero set of this section is empty since there is no degree 2 map $\mathbb{P}^1 \to \mathbb{P}^1$ with ramification indexes $(2,2,2)$. Thus, the invariant $\Phi^V_{2S,0,(2)}\big(\tau(F)^2; C_F\big)$ is trivial. □

6 GW invariants of $E(0)$ with $\tau(F)$ constraints

The aim of this section is to prove:

Lemma 6.1 *Let Φ denote the GW invariants of $E(0)$. Then,*

(a) $\Phi_{2S,0}\big(\tau(F)^2, pt\big) \;=\; 1,$

(b) $\Phi_{2S,0}\big(\tau(F)^2, \gamma_1, \gamma_2\big) \;=\; 2,$

(c) $\Phi_{2S+dF,1}\big(\tau(F)^3, pt\big) \;=\; 24\,\sigma(d),$

(d) $\Phi_{2S+dF,1}\big(\tau(F)^2, pt^2\big) \;=\; 16\,d\,\sigma(d).$

One can prove Lemma 6.1 applying the genus 0 and 1 TRR formulas for the descendent constraint $\tau(F)$. In fact, these TRR formulas consist of:

(1) the relation between the tautological class ψ (see below) and some boundary strata of the Deliegn-Mumford space $\overline{\mathcal{M}}_{g,k}$,

(2) the relation between $\tau(F)$ constraint and $\psi(F)$ constraint.

The relation (1) is also called TRR formula and the relation (2) follows from *relations between generalized correlators* (Theorem 1.2 of [11]). In our case, the computation using TRR formulas for $\tau(F)$ is quite complicated, so we will separate the computation into two steps: we first use (1) to compute relevant GW invariants of $E(0)$ with $\psi(F)$ constraints and then apply (2) to those invariants with $\psi(F)$ constraints to compute invariants with $\tau(F)$ constraints shown in Lemma 6.1.

In the next section, we apply the Symplectic sum formula of [8] to the invariants in Lemma 6.1 to compute the (partial) Gromov–Taubes invariants that appeared in the proof of Proposition 4.4. After various preliminary lemmas, we give the proof of Lemma 6.1 at the end of this section.

Let ψ_i be the first Chern class of the relative line bundle L_i over $\overline{\mathcal{M}}_{g,k}$ whose geometric fiber at the point $(C; x_1, \cdots, x_k)$ is $T^*_{x_i}$. When $g = 0, 1$, there are relations between the class ψ and some boundary strata of $\overline{\mathcal{M}}_{g,k}$ (cf section 4 of [6]). Combining with the composition law of GW invariants [22], those relations give the TRR formulas for GW invariants of $E(0)$ with $\psi(F)$ constraint: Let $\{H_\alpha\}$ and $\{H^\alpha\}$ be bases of $E(0)$ dual by the intersection form. For $\beta = \beta_1 \otimes \cdots \otimes \beta_n$ in $[H_*(E(0);\mathbb{Z})]^{\otimes n}$ and an unordered partition of $\pi = (\pi_1, \pi_2)$ of $\{1, \cdots, n\}$ with $\pi_1 \neq \emptyset$, we set $\beta_{\pi_i} = \beta_{l_1} \otimes \cdots \otimes \beta_{l_k}$ where $\pi_i = \{l_1, \cdots, l_k\}$ and $l_1 < \cdots < l_k$. We then have

$$\begin{aligned} &\Phi_{A,0}\big(\psi(F), \beta_1, \cdots, \beta_{n+2}\big) \\ &= \pm \sum_\alpha \sum \Phi_{A_1,0}\big(F, \beta_{\pi_1}, H_\alpha\big)\, \Phi_{A_2,0}\big(H^\alpha, \beta_{\pi_2}, \beta_{n+1}, \beta_{n+2}\big) \end{aligned} \tag{6.1}$$

$$\begin{aligned} &\Phi_{A,1}\big(\psi(F), \beta_1, \cdots, \beta_n\big) \\ &= \sum_\alpha \tfrac{1}{24}\, \Phi_{A,0}\big(F, \beta_1, \cdots, \beta_n, H_\alpha, H^\alpha\big) \\ &\pm \sum_\alpha \sum \Phi_{A_1,0}\big(F, \beta_{\pi_1}, H_\alpha\big)\, \Phi_{A_2,1}\big(H^\alpha, \beta_{\pi_2}\big) \end{aligned} \tag{6.2}$$

where the sum is over $A = A_1 + A_2$ and partitions π as above, and the sign depends on the permutation (π_1, π_2) and the degree of β_i. In particular, if $\deg(\beta_i)$ are all even for $1 \le i \le n$ the sign is positive.

From now on, we always denote the fundamental class of $E(0)$ by 1.

Lemma 6.2 *Let Φ denote the GW invariants of $E(0)$. Then,*

(a) $\Phi_{S+dF,1}\big(\psi(F), pt\big) = 2\,\sigma(d)$,

(b) $\Phi_{S,0}\big(\psi(F), S, F^2\big) = 1$,

(c) $\Phi_{S,0}\big(\psi(F), pt, F, 1\big) = 1$,

(d) $\Phi_{S+dF,1}\big(\psi(F)^2, S\big) = 4\,\sigma(d)$,

(e) $\Phi_{2S,0}\big(\psi(F)^2, pt, F^2\big) = 2$,

(f) $\Phi_{dF,1}\big(\psi(F), S, 1\big) = 0$.

Proof (a) It follows from the genus 1 TRR formula (6.1) and Remark 5.3 that

$$\Phi_{S+dF,1}\big(\psi(F),pt\big) \;=\; \sum_\alpha \tfrac{1}{24}\,\delta_{d,0}\,\Phi_{S,0}\big(F,pt,H_\alpha,H^\alpha\big) \;+\; \Phi_{S,0}\big(F,pt\big)\,\Phi_{dF,1}\big(1,pt\big) \;+\; \Phi_{S,0}\big(F^2,pt\big)\,\Phi_{dF,1}\big(S\big). \tag{6.3}$$

The first term in the right hand side vanishes by Remark 5.3. The second term also vanishes since $\Phi_{dF,1}(pt,\cdot\,) = 0$. The last term equals $2\,\sigma(d)$ by Lemma 5.2 a,f. Thus, (a) follows from (6.3).

(b) We have

$$\Phi_{S,0}\big(\psi(F),S,F^2\big) \;=\; \Phi_{0,0}\big(F,S,1\big)\,\Phi_{S,0}\big(pt,F^2\big) \;=\; 1$$

where the first equality follows from the genus $g=0$ TRR formula (6.1) and the second equality follows from Lemma 5.1 and Lemma 5.2 a.

(c) We have

$$\Phi_{S,0}\big(\psi(F),pt,F,1\big) \;=\; \Phi_{S,0}\big(F^2,pt\big)\,\Phi_{0,0}\big(S,F,1\big) \;=\; 1$$

where the first equality follows from the genus $g=0$ TRR formula (6.1) and the second equality follows from Lemma 5.1 and Lemma 5.2 a.

(d) It follows from the genus $g=1$ TRR formula (6.2) that

$$\begin{aligned}\Phi_{S+dF,1}\big(\psi(F)^2,S\big) \;=\;& \sum_\alpha \tfrac{1}{24}\,\Phi_{S+dF,0}\big(F,\psi(F),S,H_\alpha,H^\alpha\big)\\ &+\; \sum_\alpha\sum \Phi_{A_1,0}\big(F,S,H_\alpha\big)\,\Phi_{A_2,1}\big(H^\alpha,\psi(F)\big)\\ &+\; \sum_\alpha\sum \Phi_{A_1,0}\big(F,\psi(F),H_\alpha\big)\,\Phi_{A_2,1}\big(H^\alpha,S\big)\\ &+\; \sum_\alpha\sum \Phi_{A_1,0}\big(F,\psi(F),S,H_\alpha\big)\,\Phi_{A_2,1}\big(H^\alpha\big)\end{aligned}$$

where the sum is over all decompositions $A_1+A_2=S+dF$. The first term in the right hand side vanishes by Remark 5.3. The second term becomes

$$\Phi_{0,0}\big(F,S,1\big)\,\Phi_{S+dF,1}\big(pt,\psi(F)\big).$$

This equals $2\,\sigma(d)$ by Lemma 5.1 and (a). The third term vanishes since $\overline{\mathcal{M}}_{0,3} = \{pt\}$ and the last term becomes

$$\Phi_{S,0}\big(F^2,\psi(F),S\big)\,\Phi_{dF,1}\big(S\big).$$

This equals $2\,\sigma(d)$ by (b) and Lemma 5.2 f. Thus, we have (d).

(e) The genus 0 TRR formula (6.1) and Lemma 5.1 give

$$\begin{aligned}\Phi_{2S,0}\big(\psi(F)^2, pt, F^2\big) &= \sum_\alpha \Phi_{S,0}\big(F,\psi(F),pt,H_a\big)\,\Phi_{S,0}\big(H^a,F^2\big)\\ &+ \sum_\alpha \Phi_{S,0}\big(F,pt,H_a\big)\,\Phi_{S,0}\big(H^a,\psi(F),F^2\big)\\ &+ \sum_\alpha \Phi_{S,0}\big(F,\psi(F),H_a\big)\,\Phi_{S,0}\big(H^a,pt,F^2\big).\end{aligned}$$

The first term in the right hand side becomes

$$\Phi_{S,0}\big(F,\psi(F),pt,1\big)\,\Phi_{S,0}\big(pt,F^2\big).$$

This equals 1 by (c) and Lemma 5.2 a. The second term becomes

$$\Phi_{S,0}\big(F^2,pt\big)\,\Phi_{S,0}\big(S,\psi(F),F^2\big).$$

This equals 1 by Lemma 5.2 a and (b). Since $\overline{\mathcal{M}}_{0,3} = \{pt\}$, the last term vanishes. Thus, we have (e).

(f) The genus $g=1$ TRR formula (6.2) and Remark 5.3 give

$$\begin{aligned}\Phi_{dF,1}\big(\psi(F),S,1\big) &= \sum_\alpha \tfrac{1}{24}\,\delta_{d0}\,\Phi_{0,0}\big(F,S,1,H_\alpha,H^\alpha\big)\\ &+ \sum_\alpha \Phi_{0,0}\big(F,S,H_a\big)\,\Phi_{dF,1}\big(H^\alpha,1\big)\\ &+ \sum_\alpha \Phi_{0,0}\big(F,1,H_\alpha\big)\,\Phi_{dF,1}\big(H^\alpha,S\big)\\ &+ \sum_\alpha \Phi_{0,0}\big(F,1,S,H_a\big)\,\Phi_{dF,1}\big(H^\alpha\big).\end{aligned}$$

The first term in the right hand side vanishes by Lemma 5.1. The second term becomes

$$\Phi_{0,0}\big(S,F,1\big)\,\Phi_{dF,1}\big(pt,1\big).$$

This vanishes by the fact $\Phi_{dF,1}\big(pt,\cdot\,\big)=0$. The third term becomes

$$\Phi_{0,0}\big(F,1,S\big)\,\Phi_{dF,1}\big(F,S\big).$$

This also vanishes by the fact $\Phi_{dF,1}\big(pt,\cdot\,\big)=0$. The last term vanishes as well by Lemma 5.1. Thus, we have (f). □

Observe that by Lemma 5.1, Remark 5.3 and the dimension count the invariant $\Phi_{A,0}(F,B)$ vanishes unless $A=S$ and $B=pt$ and that by Lemma 5.2 a and the Divisor Axiom we have $\Phi_{S,0}(F,pt)=1$. This observation together with Theorem 1.2 of [11] gives

$$\Phi_{A,g}\big(\tau(F),\cdot\big) \;=\; \Phi_{A,g}\big(\psi(F),\cdot\big) \;+\; \Phi_{A-S,g}\big(1,\cdot\big). \tag{6.4}$$

Lemma 6.3 *Let Φ denote the GW invariants of $E(0)$. Then,*

(a) $\Phi_{S,0}\big(\tau(F),F,S\big) \;=\; 1$,

(b) $\Phi_{S,0}\big(\tau(F),pt,F,1\big)=1$.

Proof (a) It follows from (6.4), the fact $\overline{\mathcal{M}}_{0,3}=\{pt\}$ and Lemma 5.1 that

$$\Phi_{S,0}\big(\tau(F),F,S\big) \;=\; \Phi_{S,0}\big(\psi(F),F,S\big) \;+\; \Phi_{0,0}\big(1,F,S\big) \;=\; 1.$$

(b) We have

$$\Phi_{S,0}\big(\tau(F),pt,F,1\big) \;=\; \Phi_{S,0}\big(\psi(F),pt,F,1\big) \;+\; \Phi_{0,0}\big(1,pt,F,1\big) \;=1.$$

where the first equality follows from (6.4) and the second equality follows from Lemma 6.2 c and Lemma 5.1. □

Note that for $B\in H_2(E(0);\mathbb{Z})$ the dot product $B\cdot F\in H_0(E(0);\mathbb{Z})$ corresponds under Poincaré duality to the cup product in cohomology. The generalized Divisor Axiom (Lemma 1.4 of [11]) thus yields

$$\Phi_{A,g}\big(\tau(F),B,\cdot\big) \;=\; (B\cdot A)\,\Phi_{A,g}\big(\tau(F),\cdot\big) \;+\; (B\cdot F)\,\Phi_{A,g}\big(pt,\cdot\big). \tag{6.5}$$

Combining this relation with (6.4) then gives

$$\begin{aligned}&\Phi_{A,g}\big(\psi(F),B,\cdot\big)\\ &\;=\; (B\cdot A)\,\Phi_{A,g}\big(\psi(F),\cdot\big) \;+\; (B\cdot F)\,\Phi_{A,g}\big(pt,\cdot\big) \;+\; (B\cdot S)\,\Phi_{A-S,g}\big(1,\cdot\big).\end{aligned} \tag{6.6}$$

Lemma 6.4 *Let Φ denote the GW invariants of $E(0)$. Then,*

(a) $\Phi_{S,0}\big(\psi(F),1,\gamma_1,\gamma_2\big)=1$,

(b) $\Phi_{2S+dF,1}\big(\psi(F)^3,pt\big) \;=\; 12\,\sigma(d)$,

(c) $\Phi_{S+dF,1}\big(\psi(F)^2,1,pt\big) \;=\; 4\,\sigma(d)$,

(d) $\Phi_{2S+dF,1}\big(\psi(F),\tau(F),pt^2\big) \;=\; 14\,d\,\sigma(d)$,

(e) $\Phi_{S+dF,1}\big(\psi(F),1,pt^2\big) \;=\; 2\,d\,\sigma(d)$.

Proof (a) It follows from the genus 0 TRR formula (6.1), Lemma 5.1 and Lemma 5.2 b that

$$2\,\Phi_{S,0}\big(\psi(F),1,\gamma_1,\gamma_2\big) \;=\; 2\,\Phi_{0,0}\big(F,1,S\big)\,\Phi_{S,0}\big(F,\gamma_1,\gamma_2\big) \;=\; 2.$$

(b) The genus $g=1$ TRR formula (6.2) and Remark 5.3 give

$$\begin{aligned}
&\Phi_{2S+dF,1}\big(\psi(F)^3,pt\big)\\
&= \sum_\alpha \tfrac{1}{24}\,\delta_{d0}\,\Phi_{2S,0}\big(F,\psi(F)^2,pt,H_\alpha,H^\alpha\big)\\
&+ \sum_\alpha\sum \Phi_{A_1,0}\big(F,pt,H_\alpha\big)\,\Phi_{A_2,1}\big(H^\alpha,\psi(F)^2\big)\\
&+ \sum_\alpha\sum 2\,\Phi_{A_1,0}\big(F,\psi(F),H_\alpha\big)\,\Phi_{A_2,1}\big(H^\alpha,\psi(F),pt\big)\\
&+ \sum_\alpha\sum 2\,\Phi_{A_1,0}\big(F,\psi(F),pt,H_\alpha\big)\,\Phi_{A_2,1}\big(H^\alpha,\psi(F)\big)\\
&+ \sum_\alpha\sum \Phi_{A_1,0}\big(F,\psi(F)^2,H_\alpha\big)\,\Phi_{A_2,1}\big(H^\alpha,pt\big)\\
&+ \sum_\alpha\sum \Phi_{A_1,0}\big(F,\psi(F)^2,pt,H_\alpha\big)\,\Phi_{A_2,1}\big(H^\alpha\big)
\end{aligned}$$

where the sum is over all decompositions $A_1+A_2=2S+dF$. The first term in the right hand side vanishes by Remark 5.3. The second term becomes

$$\Phi_{S,0}\big(F^2,pt\big)\,\Phi_{S+dF,1}\big(S,\psi(F)^2\big).$$

This equals $4\,\sigma(d)$ by Lemma 5.2 a and Lemma 6.2 d. The third term vanishes since $\overline{\mathcal{M}}_{0,3}=\{pt\}$. The fourth term becomes

$$2\,\Phi_{S,0}\big(F,\psi(F),pt,1\big)\,\Phi_{S+dF,1}\big(pt,\psi(F)\big).$$

This equals $4\,\sigma(d)$ by Lemma 6.2 c,a. Since $\dim_{\mathbb{C}}\overline{\mathcal{M}}_{0,4}=1$, the fifth term vanishes. The last term becomes

$$\Phi_{2S,0}\big(F^2,\psi(F)^2,pt\big)\,\Phi_{dF,1}\big(S\big).$$

by dimension count and Remark 5.3. This equals $4\,\sigma(d)$ by Lemma 6.2 e and Lemma 5.2 f. Thus, we have (b).

(c) The genus $g=1$ TRR formula (6.2) gives

$$\begin{aligned}
&\Phi_{S+dF,1}\big(\psi(F)^2,1,pt\big)\\
&= \sum_\alpha \tfrac{1}{24}\,\Phi_{S+dF,0}\big(F,\psi(F),1,pt,H_\alpha,H^\alpha\big)
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{\alpha} \sum \Phi_{A_1,0}\big(F, 1, H_\alpha\big)\, \Phi_{A_2,1}\big(H^\alpha, \psi(F), pt\big) \\
&+ \sum_{\alpha} \sum \Phi_{A_1,0}\big(F, pt, H_\alpha\big)\, \Phi_{A_2,1}\big(H^\alpha, \psi(F), 1, \big) \\
&+ \sum_{\alpha} \sum \Phi_{A_1,0}\big(F, \psi(F), H_\alpha\big)\, \Phi_{A_2,1}\big(H^\alpha, 1, pt\big) \\
&+ \sum_{\alpha} \sum \Phi_{A_1,0}\big(F, 1, pt, H_\alpha\big)\, \Phi_{A_2,1}\big(H^\alpha, \psi(F)\big) \\
&+ \sum_{\alpha} \sum \Phi_{A_1,0}\big(F, \psi(F), 1, H_\alpha\big)\, \Phi_{A_2,1}\big(H^\alpha, pt\big) \\
&+ \sum_{\alpha} \sum \Phi_{A_1,0}\big(F, \psi(F), pt, H_\alpha\big)\, \Phi_{A_2,1}\big(H^\alpha, 1\big) \\
&+ \sum_{\alpha} \sum \Phi_{A_1,0}\big(F, \psi(F), 1, pt, H_\alpha\big)\, \Phi_{A_2,1}\big(H^\alpha\big) \qquad (6.7)
\end{aligned}
$$

where the sum is over all decompositions $A_1 + A_2 = 2S + dF$. The first term in the right hand side of (6.7) vanishes by Remark 5.3. The second term becomes

$$\Phi_{0,0}\big(F, 1, S\big)\, \Phi_{S+dF,1}\big(F, \psi(F), pt\big) \qquad (6.8)$$

since $\Phi_{dF,1}\big(pt, \cdot\big) = 0$. The first factor of (6.8) is 1 by Lemma 5.1 and the second factor is $2\,\sigma(d)$ by the Divisor Axiom (6.6) and Lemma 6.2 a. Thus the second term in the right hand side of (6.7) is $2\,\sigma(d)$. The third term vanishes by Lemma 5.1 and Lemma 6.2 f;

$$
\begin{aligned}
&\sum_{\alpha} \sum \Phi_{A_1,0}\big(F, pt, H_\alpha\big)\, \Phi_{A_2,1}\big(H^\alpha, \psi(F), 1\big) \\
&= \Phi_{S,0}\big(F^2, pt\big)\, \Phi_{dF,1}\big(S, \psi(F), 1\big) \;=\; 0.
\end{aligned}
$$

The fourth term vanishes since $\overline{\mathcal{M}}_{0,3} = \{pt\}$. The fifth term vanishes by routine dimension count and Remark 5.3. The sixth term and the seventh term vanish by routine dimension count and the fact $\Phi_{dF,1}\big(pt, \cdot\big) = 0$. The last term in the right hand side of (6.7) becomes

$$\Phi_{S,0}\big(F^2, \psi(F), 1, pt\big)\, \Phi_{dF,1}\big(S\big).$$

This equals $2\,\sigma(d)$ by Lemma 6.2 c, the Divisor Axiom (6.6) and Lemma 5.2 f. Therefore, we have (c).

(d) The genus $g = 1$ TRR formula (6.2) gives

$$\Phi_{2S+dF,1}\big(\psi(F), \tau(F), pt^2\big)$$

$$
\begin{aligned}
&= \sum_\alpha \tfrac{1}{24}\,\Phi_{2S+dF,0}\big(F,\tau(F),pt^2,H_\alpha,H^\alpha\big)\\
&= \sum_\alpha \sum 2\,\Phi_{A_1,0}\big(F,pt,H_\alpha\big)\,\Phi_{A_2,1}\big(H^\alpha,\tau(F),pt\big)\\
&= \sum_\alpha \sum \Phi_{A_1,0}\big(F,\tau(F),H_\alpha\big)\,\Phi_{A_2,1}\big(H^\alpha,pt^2\big)\\
&= \sum_\alpha \sum 2\,\Phi_{A_1,0}\big(F,\tau(F),pt,H_\alpha\big)\,\Phi_{A_2,1}\big(H^\alpha,pt\big)\\
&= \sum_\alpha \sum \Phi_{A_1,0}\big(F,pt^2,H_\alpha\big)\,\Phi_{A_2,1}\big(H^\alpha,\tau(F)\big)\\
&= \sum_\alpha \sum \Phi_{A_1,0}\big(F^2,\tau(F),pt^2,H_\alpha\big)\,\Phi_{A_2,1}\big(H^\alpha\big)
\end{aligned}
$$

where the sum is over all decompositions $A_1 + A_2 = 2S + dF$. The first term of the right hand side vanishes by Remark 5.3. The second term becomes

$$2\,\Phi_{S,0}\big(F^2,pt\big)\,\Phi_{S+dF,1}\big(S,\tau(F),pt\big).$$

This equals $8\,d\,\sigma(d)$ by Lemma 5.2 a, the Divisor Axiom (6.5) and Lemma 5.2 c,d;

$$\Phi_{S+dF,1}\big(S,\tau(F),pt\big) \;=\; d\,\Phi_{S+dF,1}\big(\tau(F),pt\big) \;+\; \Phi_{S+dF,1}\big(pt^2\big) \;=\; 4\,d\,\sigma(d).$$

The third term becomes

$$\Phi_{S,0}\big(F,\tau(F),S\big)\,\Phi_{S+dF,1}\big(F,pt^2\big) \;=\; 2\,d\,\sigma(d).$$

This equals $2\,d\,\sigma(d)$ by Lemma 6.3 a and Lemma 5.2 d. The fourth term becomes

$$2\,\Phi_{S,0}\big(F,\tau(F),pt,1\big)\,\Phi_{S+dF,1}\big(pt^2\big).$$

This equals $4\,d\,\sigma(d)$ by Lemma 6.3 b and Lemma 5.2 d. The last two terms vanish by Remark 5.3. Thus, we have (d).

(e) It follows from (6.4) and $\Phi_{dF,1}(pt,\cdot) = 0$ that

$$\Phi_{S+dF,1}\big(\tau(F),1,pt^2\big) \;=\; \Phi_{S+dF,1}\big(\psi(F),1,pt^2\big).$$

By the genus $g = 1$ TRR formula (6.2) and Remark 5.3, this becomes

$$
\begin{aligned}
&\sum_\alpha \tfrac{1}{24}\,\Phi_{S+dF,1}\big(F,1,pt^2,H_\alpha,H^\alpha\big) \;+\; \sum_\alpha \Phi_{0,0}\big(F,1,H_\alpha\big)\,\Phi_{S+dF,1}\big(H^\alpha,pt^2\big)\\
&+ \sum_\alpha 2\,\Phi_{S,0}\big(F,pt,H_\alpha\big)\,\Phi_{dF,1}\big(H^\alpha,1,pt\big) \;+\; \sum_\alpha 2\,\Phi_{S,0}\big(F,1,pt,H_\alpha\big)\,\Phi_{dF,1}\big(H^\alpha,pt\big)\\
&+ \sum_\alpha \Phi_{S,0}\big(F,pt^2,H_\alpha\big)\,\Phi_{dF,1}\big(H^\alpha,1\big) \;+\; \sum_\alpha \Phi_{S,0}\big(F,1,pt^2,H_\alpha\big)\,\Phi_{dF,1}\big(H^\alpha\big).
\end{aligned}
$$

The first term vanishes by Remark 5.3. The second term becomes $2\,d\,\sigma(d)$ by Lemma 5.1 and Lemma 5.2 d. The third term and the fourth term vanishes since $\Phi_{dF,1}\big(pt,\cdot\big)=0$. The last two terms vanish by Remark 5.3. Thus, we have (e). □

Now, we are ready to prove Lemma 6.1.

Proof of Lemma 6.1 (a) The relation (6.4) gives

$$\Phi_{2S,0}\big(\tau(F)^2,pt\big) \;=\; \Phi_{2S,0}\big(\psi(F),\tau(F),pt\big) \;+\; \Phi_{S,0}\big(1,\psi(F),pt\big) \;+\; \Phi_{0,0}\big(1^2,pt\big).$$

This equals 1 by the fact $\overline{\mathcal{M}}_{0,3}=\{pt\}$ and Lemma 5.1.

(b) By (6.4) we have

$$\begin{aligned}&\Phi_{2S,0}\big(\tau(F)^2,\gamma_1,\gamma_2\big)\\ &\quad=\;\Phi_{2S,0}\big(\psi(F)^2,\gamma_1,\gamma_2\big)\;+\;2\,\Phi_{S,0}\big(1,\psi(F),\gamma_1,\gamma_2\big)\;+\;\Phi_{0,0}\big(1^2,\gamma_1,\gamma_2\big).\end{aligned}$$

Then, (b) follows from the fact $\dim_{\mathbb{C}}\overline{\mathcal{M}}_{0,4}=1$, Lemma 6.4 a and Lemma 5.1.

(c) It follows from (6.4) that

$$\begin{aligned}&\Phi_{2S+dF,1}\big(\tau(F)^3,pt\big)\\ &\quad=\;\Phi_{2S+dF,1}\big(\psi(F)^3,pt\big)\;+\;3\,\Phi_{S+dF,1}\big(1,\psi(F)^2,pt\big)\;+\;3\,\Phi_{dF,1}\big(1^2,\psi(F),pt\big).\end{aligned}$$

Now, Lemma 6.4 b,c and the fact $\Phi_{dF,1}(pt,\cdot\,)=0$ show (c).

(d) Using (6.4) yields

$$\begin{aligned}&\Phi_{2S+dF,1}\big(\tau(F)^2,pt^2\big)\\ &\quad=\;\Phi_{2S+dF,1}\big(\psi(F),\tau(F),pt^2\big)\;+\;\Phi_{S+dF,1}\big(1,\psi(F),pt^2\big)\;+\;\Phi_{dF,1}\big(1^2,pt^2\big).\end{aligned}$$

This equals $16\,d\,\sigma(d)$ by Lemma 6.4 d,e and the fact $\Phi_{dF,1}(pt,\cdot\,)=0$. □

7 Relative Gromov–Taubes invariants of $E(0)$

The aim of this section is to compute the (partial) relative Gromov–Taubes invariants $G\Phi^V$ of $(E(0),V)$ that appeared in Section 4, thereby completing the proof of the sum formulas (3.5), (3.6), (3.10), and (3.11). Applying the Symplectic Sum Formula of [8], we will compute those invariants.

The (partial) Gromov–Taubes invariants $G\Phi^V$ of $(E(0),V)$ defined in (4.1) can be expressed in terms of the relative invariants Φ^V of $(E(0),V)$. When the multiplicity vector $s=(2)$, the invariants $G\Phi^V_s$ for the classes $2S+dF$, $d\in\mathbb{Z}$, count V–regular maps from a connected domain and hence

$$G\Phi^V_{2S+dF,\chi,(2)} \;=\; \Phi^V_{2S+dF,g,(2)} \quad \text{where} \quad g=1-\tfrac{1}{2}\chi. \tag{7.1}$$

When $s=(1,1)$, the invariants $G\Phi^V_s$ for the classes $2S+dF$ decompose as a sum of two invariants; one is the invariant that counts V–regular maps from a connected domain, and the other is the invariant that counts pairs of V–regular maps (f_1,f_2), each from a connected domain and having contact order 1 with V. Denote the latter invariants by $T\Phi^V$. Then, we have

$$G\Phi^V_{2S+dF,\chi,(1,1)} \;=\; T\Phi^V_{2S+dF,\chi,(1,1)} \;+\; \Phi^V_{2S+dF,g,(1,1)} \quad \text{where} \quad g=1-\tfrac{1}{2}\chi. \tag{7.2}$$

For $\beta=\beta_1\otimes\cdots\otimes\beta_n\in\big[H_*\big(E(0)\big)\big]^{\otimes n}$ and an ordered partition $\pi=(\pi_1,\pi_2)$ of $\{1,\cdots,n\}$, we set $\beta_{\pi_i}=\beta_{l_1}\otimes\cdots\otimes\beta_{l_k}$ where $\pi_i=\{l_1,\cdots,l_k\}$. It then follows that

$$\begin{aligned} &T\Phi^V_{2S+dF,\chi,(1,1)}\big(C_{\gamma_i\cdot\gamma_j};\beta\big)\\ &= \pm\sum \Phi^V_{S+d_1F,g_1,(1)}\big(C_{\gamma_i};\beta_{\pi_1}\big)\,\Phi^V_{S+d_2F,g_2,(1)}\big(C_{\gamma_j};\beta_{\pi_2}\big) \end{aligned} \tag{7.3}$$

where the sum is over all $\chi=(2-2g_1)+(2-2g_2)$, $d=d_1+d_2$, and partitions $\pi=(\pi_1,\pi_2)$ as above, and the sign depends on the permutation (π_1,π_2) and the degrees of β_i. In particular, if $\deg(\beta_i)$ are all even the sign is positive.

The following lemma computes the invariants $G\Phi^V$ appeared in the proof of Lemma 4.2. The proof of this lemma easily follows from (7.2), (7.3), Remark 5.3, and Lemma 5.2.

Lemma 7.1 *Let $G\Phi^V$ be the invariants of $(E(0),V)$ defined in (4.1). Then,*

(a) $G\Phi^V_{2S+dF,4,(1,1)}\big(C_{pt^2}\big) \;=\; \delta_{d0}$,

(b) $G\Phi^V_{2S+dF,4,(1,1)}\big(C_{pt\cdot F};\gamma_1,\gamma_2\big) \;=\; \delta_{do}$,

(c) $G\Phi^V_{2S+dF,4,(1,1)}\big(C_{(-\gamma_1)\cdot\gamma_2};\gamma_1,\gamma_2\big) \;=\; -\delta_{d0}$,

(d) $G\Phi^V_{2S+dF,2,(2)}\big(C_{pt};\gamma_1,\gamma_2\big) \;=\; 0$,

(e) $G\Phi^V_{2S+dF,2,(1,1)}\big(C_{pt^2};\gamma_1,\gamma_2\big) \;=\; 0$,

(f) $G\Phi^V_{2S+dF,4,(1,1)}\big(C_{pt\cdot F};pt\big) \;=\; \delta_{d0}$,

(g) $G\Phi^V_{2S+dF,2,(2)}\big(C_{pt};pt\big) \;=\; 0$,

(h) $G\Phi^V_{2S+dF,2,(1,1)}\big(C_{pt^2};pt\big) \;=\; 2\,d\,\sigma(d)$.

The following lemma lists the invariants $G\Phi^V$ that entered in the proof of Proposition 4.4 a and is an immediate consequence of (7.2), (7.3), Remark 5.3, Lemma 5.2, and Lemma 5.4.

Lemma 7.2 *Let $G\Phi^V$ be the invariants of $(E(0), V)$ defined in (4.1). Then,*

(a) $G\Phi^V_{2S+dF,4,(1,1)}\big(C_{pt\cdot F};\tau(F)\big) \;=\; 0,$

(b) $G\Phi^V_{2S+dF,2,(2)}\big(C_{pt};\tau(F)\big) \;=\; \frac{1}{2}\,\delta_{d0}\,,$

(c) $G\Phi^V_{2S+dF,2,(1,1)}\big(C_{pt^2};\tau(F)\big) \;=\; 4\,\sigma(d).$

Lastly, Lemma 7.3 below computes the invariants $G\Phi^V$ that appeared in the proof of Proposition 4.4 b. In order to prove this lemma, we will apply the Symplectic Sum Formula of [8] to the GW invariants of $E(0)$ in shown Lemma 6.1 by writing $E(0)$ as a symplectic sum

$$E(0) \;=\; E(0)\,\#_V\, E(0) \tag{7.4}$$

and by splitting constraints in various ways. In this case, there is also no contribution from the neck (cf Lemma 16.1 of [8]) and hence we have the following sum formulas: Let $\{\gamma_i\}$ be a basis of $H_*(V;\mathbb{Z})$ and $\{\gamma^i\}$ be its dual basis with respect to the intersection form of V. For $\beta = \beta_1\otimes\cdots\otimes\beta_n$ in $[H_*(E(0);\mathbb{Z})]^{\otimes n}$ and $n = n_1+n_2$, we set

$$\beta' = \beta_1\otimes\cdots\otimes\beta_{n_1} \qquad\text{and}\qquad \beta'' = \beta_{n_1+1}\otimes\cdots\otimes\beta_n.$$

For $\tau(F)^m$ and $m = m_1+m_2$, we set

$$\tau' \;=\; \tau(F)^{m_1} \qquad\text{and}\qquad \tau'' \;=\; \tau(F)^{m_2}.$$

Then, for such splitting of constraints $\beta = \beta'\otimes\beta''$ and $\tau(F)^m = \tau'\cdot\tau''$ the sum formula of the symplectic sum (7.4) for the class $S + dF$ becomes

$$\begin{aligned}&\Phi_{S+dF,g}\big(\beta,\tau(F)^m\big)\\ &\;=\; \sum\sum_i \Phi^V_{S+d_1F,g_1,(1)}\big(\beta',\tau';C_{\gamma_i}\big)\,\Phi^V_{S+d_2F,g_2,(1)}\big(C_{\gamma^i};\beta'',\tau''\big)\end{aligned} \tag{7.5}$$

where the sum is over all $d = d_1+d_2$ and $g = g_1+g_2$. Similarly, the sum formula of the sum (7.4) applied to the class $2S + dF$ with the splitting of constraints $\beta = \beta'\otimes\beta''$ and $\tau(F)^m = \tau'\cdot\tau''$ gives

$$\begin{aligned}&\Phi_{2S+dF,g}\big(\beta,\tau(F)^m\big)\\ &\;=\; \sum\sum_{i,j} \tfrac{1}{2}\,\Phi^V_{2S+d_1F,g_1,(1,1)}\big(\beta',\tau';C_{\gamma_i\cdot\gamma_j}\big)\,T\Phi^V_{2S+d_2F,\chi_2,(1,1)}\big(C_{\gamma^j\cdot\gamma^i};\beta'',\tau''\big)\end{aligned}$$

$$+ \sum \sum_{i,j} \tfrac{1}{2}\, \Phi^V_{2S+d_1F,g_1,(1,1)}\big(\beta', \tau'; C_{\gamma_i\cdot\gamma_j}\big)\, \Phi^V_{2S+d_2F,g_2,(1,1)}\big(C_{\gamma^j\cdot\gamma^i}; \beta'', \tau''\big)$$
$$+ \sum \sum_{i} 2\, \Phi^V_{2S+d_1F,g_1,(2)}\big(\beta', \tau'; C_{\gamma_i}\big)\, \Phi^V_{2S+d_2F,g_2,(2)}\big(C_{\gamma^i}; \beta', \tau''\big)$$
$$+ \sum \sum_{i,j} \tfrac{1}{2}\, T\Phi^V_{2S+d_1F,\chi_1,(1,1)}\big(\beta', \tau'; C_{\gamma_i\cdot\gamma_j}\big)\, \Phi^V_{2S+d_2F,g_2,(1,1)}\big(C_{\gamma^j\cdot\gamma^i}; \beta'', \tau'\big) \tag{7.6}$$

where the sum is over all $d = d_1 + d_2$, and $g = g_1 + 2 - \frac{1}{2}\chi_2$ for the first term, $g = g_1 + g_2 + 1$ for the second term, $g = g_1 + g_2$ for the third term and $g = 2 - \frac{1}{2}\chi_1 + g_2$ for the last term.

Lemma 7.3 *Let $G\Phi^V$ be the invariants of $(E(0), V)$ defined in (4.1). Then*

(a) $G\Phi^V_{2S+dF,4,(1,1)}\big(C_{F^2}; \tau(F)^2\big) = 0$,

(b) $G\Phi^V_{2S+dF,2,(2)}\big(C_F; \tau(F)^2\big) = 0$,

(c) $G\Phi^V_{2S+dF,2,(1,1)}\big(C_{pt\cdot F}; \tau(F)^2\big) = \delta_{d0}$,

(d) $G\Phi V_{2S+dF,2,(1,1)}\big(C_{(-\gamma_1)\cdot\gamma_2}; \tau(F)^2\big) = \delta_{d0}$,

(e) $G\Phi^V_{2S+dF,0,(2)}\big(C_{pt}; \tau(F)^2\big) = 10\,\sigma(d)$,

(f) $G\Phi^V_{2S+dF,0,(1,1)}\big(C_{pt^2}; \tau(F)^2\big) = \sum_{d_1+d_2=d} 16\,\sigma(d_1)\,\sigma(d_2) + 12\,d\,\sigma(d)$.

Proof (a) Using (7.2), (7.3), Remark 5.3, and Lemma 5.4 b gives

$$G\Phi^V_{2S+dF,4,(1,1)}\big(C_{F^2}; \tau(F)^2\big) = 2\,\delta_{d0}\, \Phi^V_{S,0}\big(C_F; \tau(F)\big)\, \Phi^V_{S,0}\big(C_F; \tau(F)\big) = 0.$$

(b) By (7.1), Remark 5.3 and Lemma 5.4 b we have

$$G\Phi^V_{2S+dF,2,(2)}\big(C_F; \tau(F)^2\big) = \delta_{d0}\, \Phi^V_{2S,0,(2)}\big(C_F; \tau(F)^2\big) = 0.$$

(c) It follows from the sum formula Remark 4.1 that

$$\Phi_{S+dF,1}\big(\tau(F)^2\big) = \sum_{d_1+d_2=d} \Phi^V_{S+d_1F,0,(1)}\big(C_{pt}\big)\, \Phi^V_{S+d_2F,1,(1)}\big(C_F; \tau(F)^2\big).$$

By Remark 5.3 and Lemma 5.2 a, the right hand side of this becomes

$$\Phi^V_{S+dF,1,(1)}\big(C_F; \tau(F)^2\big).$$

Similarly, the sum formula (7.5), Remark 5.3 and Lemma 5.4 b yields

$$\Phi_{S+dF,1}\big(\tau(F)^2\big) = \sum_{d_1+d_2=d} \Phi^V_{S+d_1F,1,(1)}\big(\tau(F); C_{pt}\big)\, \Phi^V_{S+d_2F,0,(1)}\big(C_F; \tau(F)\big)$$

$$+ \sum_{d_1+d_2=d} \Phi^V_{S+d_1F,0,(1)}\big(\tau(F); C_F\big)\, \Phi^V_{S+d_2F,1,(1)}\big(C_{pt}; \tau(F)\big)$$
$$= 0.$$

Thus, we have

$$\Phi^V_{S+dF,1,(1)}\big(C_F; \tau(F)^2\big) \;=\; \Phi_{S+dF,1}\big(\tau(F)^2\big) \;=\; 0. \tag{7.7}$$

This together with (7.3) and Lemma 5.4 b then implies that

$$\begin{aligned} T\Phi^V_{2S+dF,2,(1,1)}\big(C_{F\cdot pt}; \tau(F)^2\big) \;&=\; \Phi^V_{S+dF,1,(1)}\big(C_F; \tau(F)^2\big)\, \Phi^V_{S,0,(1)}\big(C_{pt}\big) \\ &\quad + 2\,\Phi^V_{S,0,(1)}\big(C_F; \tau(F)\big)\, \Phi^V_{S+dF,1,(1)}\big(C_{pt}; \tau(F)\big) \\ &= 0. \end{aligned} \tag{7.8}$$

On the other hand, applying the sum formula (7.6) we obtain

$$\begin{aligned} \Phi_{2S,0}\big(pt, \tau(F)^2\big) \;&=\; \tfrac{1}{2}\,\Phi^V_{2S,0,(1,1)}\big(pt; C_{pt^2}\big)\, T\Phi^V_{2S,4,(1,1)}\big(C_{F^2}; \tau(F)^2\big) \\ &\quad + 2\,\Phi^V_{2S,0,(2)}\big(pt; C_{pt}\big)\, \Phi^V_{2S,0,(2)}\big(C_F; \tau(F)^2\big) \\ &\quad + T\Phi^V_{2S,4,(1,1)}\big(pt; C_{pt\cdot F}\big)\, \Phi^V_{2S,0,(1,1)}\big(C_{F\cdot pt}; \tau(F)^2\big). \end{aligned}$$

By Remark 5.3, (7.2) and Lemma 7.1 f, the right hand side of this can be simplified as

$$\Phi^V_{2S,0,(1,1)}\big(C_{F\cdot pt}; \tau(F)^2\big).$$

Consequently, we have

$$\Phi^V_{2S,0,(1,1)}\big(C_{F\cdot pt}; \tau(F)^2\big) \;=\; \Phi_{2S,0}\big(pt, \tau(F)^2\big) \;=\; 1 \tag{7.9}$$

where the second equality follows from Lemma 6.1 a. Now, (c) follows from (7.2), Remark 5.3, (7.8) and (7.9):

$$\begin{aligned} &G\Phi^V_{2S+dF,2,(1,1)}\big(C_{pt\cdot F}; \tau(F)^2\big) \\ &= T\Phi^V_{2S+dF,2,(1,1)}\big(C_{pt\cdot F}; \tau(F)^2\big) + \delta_{d0}\,\Phi^V_{2S,0,(1,1)}\big(C_{pt\cdot F}; \tau(F)^2\big) \\ &= \delta_{d0}. \end{aligned}$$

(d) We have

$$\begin{aligned} &\Phi_{2S,0}\big(\gamma_1, \gamma_2, \tau(F)^2\big) \\ &= \tfrac{1}{2}\,\Phi^V_{2S,0,(1,1)}\big(\gamma_1, \gamma_2; C_{pt^2}\big)\, T\Phi^V_{2S,4,(1,1)}\big(C_{F^2}; \tau(F)^2\big) \\ &+ 2\,\Phi^V_{2S,0,(2)}\big(\gamma_1, \gamma_2; C_{pt}\big)\, \Phi^V_{2S,0,(2)}\big(C_F; \tau(F)^2\big) \\ &+ T\Phi^V_{2S,4,(1,1)}\big(\gamma_1, \gamma_2; C_{pt\cdot F}\big)\, \Phi^V_{2S,0,(1,1)}\big(C_{F\cdot pt}; \tau(F)^2\big) \\ &+ T\Phi^V_{2S,4,(1,1)}\big(\gamma_1, \gamma_2; C_{\gamma_1\cdot\gamma_2}\big)\, \Phi^V_{2S,0,(1,1)}\big(C_{(-\gamma_1)\cdot\gamma_2}; \tau(F)^2\big) \end{aligned}$$

$$= \Phi^V_{2S,0,(1,1)}\big(C_{F\cdot pt};\tau(F)^2\big) + \Phi^V_{2S,0,(1,1)}\big(C_{(-\gamma_1)\cdot\gamma_2};\tau(F)^2\big).$$

where the first equality follows from the sum formula (7.6) and the second follows from Remark 5.3, (7.2) and Lemma 7.1 b,c. This together with Lemma 6.1 b and (7.9) shows

$$\Phi^V_{2S,0,(1,1)}\big(C_{(-\gamma_1)\cdot\gamma_2};\tau(F)^2\big) = 1.$$

On the other hand, (7.3) and dimension count give

$$T\Phi^V_{2S+dF,2,(1,1)}\big(C_{(-\gamma_1)\cdot\gamma_2};\tau(F)^2\big) = 0.$$

Consequently, we have

$$\begin{aligned}
&G\Phi^V_{2S+dF,2,(1,1)}\big(C_{(-\gamma_1)\cdot\gamma_2};\tau(F)^2\big)\\
&= T\Phi^V_{2S+dF,2,(1,1)}\big(C_{(-\gamma_1)\cdot\gamma_2};\tau(F)^2\big) + \delta_{d0}\,\Phi^V_{2S,0,(1,1)}\big(C_{(-\gamma_1)\cdot\gamma_2};\tau(F)^2\big)\\
&= \delta_{d0}.
\end{aligned}$$

(e) Using the sum formula (7.6), we have

$$\begin{aligned}
&\Phi_{2S+dF,1}\big(pt,\tau(F)^3\big)\\
&= \sum_{d_1+d_2=d} \tfrac{1}{2}\,\Phi^V_{2S+d_1F,1,(1,1)}\big(pt,\tau(F);C_{pt^2}\big)\,T\Phi^V_{2S+d_2F,4,(1,1)}\big(C_{F^2};\tau(F)^2\big)\\
&+ \sum_{d_1+d_2=d} 2\,\Phi^V_{2S+d_1F,1,(2)}\big(pt,\tau(F);C_{pt}\big)\,\Phi^V_{2S+d_2F,0,(2)}\big(C_F;\tau(F)^2\big)\\
&+ \sum_{d_1+d_2=d} \Phi^V_{2S+d_1F,0,(1,1)}\big(pt,\tau(F);C_{pt\cdot F}\big)\,T\Phi^V_{2S+d_2F,2,(1,1)}\big(C_{pt\cdot F};\tau(F)^2\big)\\
&+ \sum_{d_1+d_2=d} \Phi^V_{2S+d_1F,0,(1,1)}\big(pt,\tau(F);C_{pt\cdot F}\big)\,\Phi^V_{2S+d_2F,0,(1,1)}\big(C_{pt\cdot F};\tau(F)^2\big)\\
&+ \sum_{d_1+d_2=d} 2\,\Phi^V_{2S+d_1F,0,(2)}\big(pt,\tau(F);C_F\big)\,\Phi^V_{2S+d_2F,1,(2)}\big(C_{pt};\tau(F)^2\big)\\
&+ \sum_{d_1+d_2=d} T\Phi^V_{2S+d_1F,2,(1,1)}\big(pt,\tau(F);C_{pt\cdot F}\big)\,\Phi^V_{2S+d_2F,0,(1,1)}\big(C_{pt\cdot F};\tau(F)^2\big)\\
&+ \sum_{d_1+d_2=d} \tfrac{1}{2}\,T\Phi^V_{2S+d_1F,4,(1,1)}\big(pt,\tau(F);C_{F^2}\big)\,\Phi^V_{2S+d_2F,1,(1,1)}\big(C_{pt^2};\tau(F)^2\big).
\end{aligned} \tag{7.10}$$

The first term in the right hand side vanishes by (7.3) and Lemma 5.4 b. The second term vanishes by Lemma 5.4 b. The third and the fourth terms vanish by Remark 5.3. The fifth term becomes

$$2\,\Phi^V_{2S+dF,1,(2)}\big(C_{pt};\tau(F)^2\big)$$

by Remark 5.3 and Lemma 5.4 a. The sixth term becomes

$$T\Phi^V_{2S+dF,2,(1,1)}\big(pt,\tau(F);C_{pt\cdot F}\big)$$

by Remark 5.3 and (7.9). The last term vanishes by (7.3) and Lemma 5.4 b. Thus, together with Lemma 6.1 c, we have

$$24\,\sigma(d) \;=\; 2\,\Phi^V_{2S+dF,1,(2)}\big(C_{pt};\tau(F)^2\big) + T\Phi^V_{2S+dF,2,(1,1)}\big(pt,\tau(F);C_{pt\cdot F}\big). \tag{7.11}$$

On the other hand, it follows from (7.3), Remark 5.3, Lemma 5.2 a,c and Lemma 5.4 b that

$$\begin{aligned} T\Phi^V_{2S+dF,2,(1,1)}\big(pt,\tau(F);C_{pt\cdot F}\big) \;=\;& \Phi^V_{S,0,(1)}\big(C_{pt}\big)\,\Phi^V_{S+dF,1,(1)}\big(pt,\tau(F);C_F\big)\\ &+\Phi^V_{S+dF,1,(1)}\big(\tau(F);C_{pt}\big)\Phi^V_{S,0,(1)}\big(pt;C_F\big)\\ &+\Phi^V_{S+dF,1,(1)}\big(pt;C_{pt}\big)\,\Phi^V_{S,0,(1)}\big(\tau(F);C_F\big)\\ =\;& 4\,\sigma(d). \end{aligned} \tag{7.12}$$

Then, (e) follows from (7.1), (7.11) and (7.12);

$$G\Phi^V_{2S+dF,0,(2)}\big(C_{pt};\tau(F)^2\big) \;=\; \Phi^V_{2S+dF,1,(2)}\big(C_{pt};\tau(F)^2\big) \;=\; 10\,\sigma(d).$$

(f) The sum formula (7.6) gives

$$\begin{aligned} &\Phi_{2S+dF,1}\big(pt^2,\tau(F)^2\big)\\ &= \sum_{d_1+d_2=d}\tfrac12\,\Phi^V_{2S+d_1F,1,(1,1)}\big(pt^2;C_{pt^2}\big)\,T\Phi^V_{2S+d_2,4,(1,1)}\big(C_{F^2};\tau(F)^2\big)\\ &+ \sum_{d_1+d_2=d} 2\,\Phi^V_{2S+d_1F,1,(2)}\big(pt^2;C_{pt}\big)\,\Phi^V_{2S+d_2F,0,(2)}\big(C_F;\tau(F)^2\big)\\ &+ \sum_{d_1+d_2=d} \Phi^V_{2S+d_1F,0,(1,1)}\big(pt^2;C_{pt\cdot F}\big)\,T\Phi^V_{2S+d_2F,2,(1,1)}\big(C_{pt\cdot F};\tau(F)^2\big)\\ &+ \sum_{d_1+d_2=d} \Phi^V_{2S+d_1F,0,(1,1)}\big(pt^2;C_{pt\cdot F}\big)\,\Phi^V_{2S+d_2F,0,(1,1)}\big(C_{pt\cdot F};\tau(F)^2\big)\\ &+ \sum_{d_1+d_2=d} 2\,\Phi^V_{2S+d_1F,0,(2)}\big(pt^2;C_F\big)\,\Phi^V_{2S+d_2F,1,(2)}\big(C_{pt};\tau(F)^2\big)\\ &+ \sum_{d_1+d_2=d} T\Phi^V_{2S+d_1F,2,(1,1)}\big(pt^2;C_{pt\cdot F}\big)\,\Phi^V_{2S+d_2F,0,(1,1)}\big(C_{pt\cdot F};\tau(F)^2\big)\\ &+ \sum_{d_1+d_2=d} \tfrac12\,T\Phi^V_{2S+d_1F,4,(1,1)}\big(pt^2;C_{F^2}\big)\,\Phi^V_{2S+d_2F,1,(1,1)}\big(C_{pt^2};\tau(F)^2\big). \end{aligned} \tag{7.13}$$

The first two terms in the right hand side vanish by Lemma 5.4 b, while the next three terms vanish by Remark 5.3. Thus, by Lemma 6.1 d and Remark 5.3, the equation (7.13) becomes

$$16\,d\,\sigma(d) \;=\; T\Phi^V_{2S+dF,2,(1,1)}\big(pt^2;C_{pt\cdot F}\big)\,\Phi^V_{2S,0,(1,1)}\big(C_{pt\cdot F};\tau(F)^2\big)$$

$$+\ \tfrac{1}{2}\,T\Phi^V_{2S,4,(1,1)}\big(pt^2;C_{F^2}\big)\,\Phi^V_{2S+dF,1,(1,1)}\big(C_{pt^2};\tau(F)^2\big). \tag{7.14}$$

By (7.3) and Lemma 5.2 a,d, the first $T\Phi^V$ invariant in the right hand side becomes

$$\begin{aligned} &T\Phi^V_{2S+dF,2,(1,1)}\big(pt^2;C_{pt\cdot F}\big)\\ &=\ \Phi^V_{S,0,(1)}\big(C_{pt}\big)\,\Phi^V_{S+dF,1,(1)}\big(pt^2;C_F\big)\ +\ 2\,\Phi^V_{S+dF,1,(1)}\big(pt;C_{pt}\big)\,\Phi^V_{S,0,(1)}\big(pt;C_F\big)\\ &=\ 4\,d\,\sigma(d). \end{aligned} \tag{7.15}$$

By (7.3) and Lemma 5.2 a, the second $T\Phi^V$ invariant becomes

$$T\Phi^V_{2S,4,(1,1)}\big(pt^2;C_{F^2}\big)\ =\ 2\,\Phi^V_{S,0,(1)}\big(pt;C_F\big)\,\Phi^V_{S,0,(1)}\big(pt;C_F\big)\ =\ 2. \tag{7.16}$$

Then, by (7.14), (7.15), (7.9) and (7.16) we have

$$\Phi^V_{2S+dF,1,(1,1)}\big(C_{pt^2};\tau(F)^2\big)\ =\ 12\,d\,\sigma(d). \tag{7.17}$$

On the other hand, the sum formula (7.5) and Remark 5.3 give

$$\begin{aligned} &\Phi_{S+dF,2}\big(\tau(F)^2\big)\\ &=\sum_{d_1+d_2=d}\Phi^V_{S+d_1F,1,(1)}\big(pt;C_{pt}\big)\,\Phi^V_{S+d_2F,1,(1)}\big(C_F;\tau(F)^2\big)\\ &+\ \Phi^V_{S,0,(1)}\big(pt;C_F\big)\,\Phi^V_{S+F,2,(1)}\big(C_{pt};\tau(F)^2\big). \end{aligned} \tag{7.18}$$

The first term of the right hand side vanishes by (7.7) and hence by Lemma 5.2 a, the equation (7.18) becomes

$$\Phi_{S+dF,2}\big(\tau(F)^2\big)\ =\ \Phi^V_{S+F,2,(1)}\big(C_{pt};\tau(F)^2\big). \tag{7.19}$$

Splitting constraints in a different way, the sum formula (7.5) and Remark 5.3 give

$$\begin{aligned} &\Phi_{S+dF,2}\big(\tau(F)^2\big)\\ &=\ \Phi^V_{S+dF,2,(1)}\big(pt,\tau(F);C_{pt}\big)\,\Phi^V_{S,0,(1)}\big(C_F;\tau(F)\big)\\ &+\sum_{d_1+d_2=d}\Phi^V_{S+d_1F,1,(1)}\big(pt,\tau(F);C_F\big)\,\Phi^V_{S+d_2F,(1)}\big(C_{pt};\tau(F)\big). \end{aligned} \tag{7.20}$$

The first term of the right hand side vanishes by Lemma 5.4 b. Thus, by Lemma 5.2 c the equation (7.20) becomes

$$\Phi_{S+dF,2}\big(\tau(F)^2\big)\ =\sum_{d_1+d_2=d}4\,\sigma(d_1)\,\sigma(d_2). \tag{7.21}$$

Relating (7.19) and (7.21), we have

$$\Phi^V_{S+F,2,(1)}\big(C_{pt};\tau(F)^2\big)\ =\sum_{d_1+d_2=d}4\,\sigma(d_1)\,\sigma(d_2). \tag{7.22}$$

Consequently, (7.3), Remark 5.3, (7.22) and Lemma 5.2 a,c show that

$$\begin{aligned} &T\Phi^V_{2S+dF,0,(1,1)}\big(C_{pt^2};\tau(F)^2\big) \\ &= 2\,\Phi^V_{S+dF,2,(1)}\big(C_{pt};\tau(F)^2\big)\,\Phi^V_{S,0,(1)}\big(C_{pt}\big) \\ &+ \sum_{d_1+d_2=d} 2\,\Phi^V_{S+d_1F,1,(1)}\big(C_{pt};\tau(F)\big)\,\Phi^V_{S+d_2F,1,(1)}\big(C_{pt};\tau(F)\big) \\ &= \sum_{d_1+d_2=d} 16\,\sigma(d_1)\,\sigma(d_2). \end{aligned} \tag{7.23}$$

Thus, we have

$$\begin{aligned} &G\Phi^V_{2S+dF,0,(1,1)}\big(C_{pt^2};\tau(F)^2\big) \\ &= T\Phi^V_{2S+dF,0,(1,1)}\big(C_{pt^2};\tau(F)^2\big) + \Phi^V_{2S+dF,1,(1,1)}\big(C_{pt^2};\tau(F)^2\big) \\ &= \sum_{d_1+d_2=d} 16\,\sigma(d_1)\,\sigma(d_2) + 12\,d\,\sigma(d) \end{aligned}$$

where the first equality follows from (7.2) and the second follows from (7.17) and (7.23). □

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