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Characterisation of a class of equations with solutions over torsion-free groups

Roger Fenn
Colin Rourke

Abstract We study equations over torsion-free groups in terms of their "t–shape" (the occurences of the variable t in the equation). A t–shape is *good* if any equation with that shape has a solution. It is an outstanding conjecture [5] that all t–shapes are good. In [2] we proved the conjecture for a large class of t–shapes called *amenable*. In [1] Clifford and Goldstein characterised a class of good t–shapes using a transformation on t–shapes called the *Magnus derivative*. In this note we introduce an inverse transformation called *blowing up*. Amenability can be defined using blowing up; moreover the connection with differentiation gives a useful characterisation and implies that the class of amenable t–shapes is strictly larger than the class considered by Clifford and Goldstein.

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1 Introduction

Let G be a group. An expression of the form

$$r = g_1 t^{\varepsilon_1} g_2 t^{\varepsilon_2} g_3 \cdots t^{\varepsilon_k} = 1, \qquad (1)$$

where $k \geq 1$, $g_i \in G$ and $\varepsilon = \pm 1$, is called an *equation* over G in the *variable* t with *coefficients* $g_1, g_2, \ldots, g_k$. The equation is said to have a *solution* if G embeds in a group H containing an element t for which (1) holds. This is equivalent to saying that the natural map

$$G \longrightarrow \frac{G * \langle t \rangle}{\langle\langle r = 1 \rangle\rangle}$$

is injective.

The equation is said to be *reduced* if it contains no subword tt^{-1} or $t^{-1}t$ (ie each coefficient which separates a pair t, t^{-1} is non-trivial). The equation is said to be *cyclically reduced* if all cyclic permutations are reduced and, unless explicitly stated otherwise, all equations are assumed to be cyclically reduced.

The *t–shape* of the word r is the sequence $t^{\varepsilon_1} t^{\varepsilon_2} \cdots t^{\varepsilon_k}$.

We use the abbreviated notation t^m for the sequence $tt \cdots t$ (m times) and t^{-m} for the sequence $t^{-1}t^{-1} \cdots t^{-1}$ (m times). We call the t–shape t^m ($m \in \mathbb{Z}$, $m \neq 0$) a *power* shape. If a t–shape is not a power then after cyclic permutation it can be written in the form

$$t^{r_1} t^{-r_2} t^{r_3} \cdots t^{-r_u}; \ u > 1$$

where each r_i is positive.

The sum $\varepsilon = r_1 - r_2 + \ldots - r_u$ is called the *degree* of the t–shape. The sum $w = r_1 + r_2 + \ldots + r_u$ is called the *width* of the t–shape. Note that the width is the length of the corresponding equation.

We call a cyclic t–shape *good* if any corresponding equation with torsion-free coefficients has a solution.

Conjecture [5] *All t–shapes are good.*

The conjecture is a special case of the adjunction problem [6] and for a brief history, see the introduction to [2]. The torsion-free condition is necessary because the t–shape tt^{-1} is good [3] but for example the equation $ata^2t^{-1} = 1$ has no solution over a group in which a has order 4.

The conjecture is known to be true in many cases. Levin [5] has proved that power shapes are good (without the torsion-free hypothesis). Klyachko [4] has proved that t–shapes of degree ± 1 are good. Furthermore both Clifford and Goldstein [1] and ourselves [2] have extended Klyachko's results to larger classes of t–shapes. The class of good t–shapes in [1] are characterised in terms of the *Magnus derivative* and for definitiveness we will call them *CG–good*. The class of good t–shapes in [2] are called *amenable*. No usable characterisation of amenability was given in [2] and it is the purpose of this note to supply such a characterisation and to compare the two classes.

The rest of the paper is organised as follows. In the next section (section 2) we review the Magnus derivative (an operation on t–shapes which we refer to simply as *differentiation*) and define the class of CG–good shapes. In section 3 we define another operation on t–shapes called *blowing up* and prove that it is the inverse of differentiation. Finally in section 4 we give two simple characterisations of amenable shapes. The first in terms of blowing up and the second, similar to the characterisation of CG–good shapes, in terms of differentiation. We conclude that the class of amenable shapes is strictly larger than the class of CG–good shapes.

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2 The Magnus derivative

Let $T = t^{\epsilon_1} t^{\epsilon_2} \cdots t^{\epsilon_w}$, where $\epsilon_i = \pm 1$, be a t–shape. We regard T as a cyclic t–shape and we define the cyclic t–shape $D(T)$, the *Magnus derivative* or simply *derivative* of T, as follows.

Arrange the signs of the exponent powers around a circle. The t–shape is well defined by this up to cyclic symmetry. Between each occurence of $+, +$ insert a new $+$, between each occurence of $-, -$ insert a new $-$ and in all other cases do nothing. Now delete the original signs. The remaining cyclic sequence of signs defines a new t–shape, $D(T)$.

For example $\quad tttt^{-1}tt^{-1}t^{-1}t \xrightarrow{D} ttt^{-1}t \xrightarrow{D} tt$.

The following is easy to prove.

Lemma *Let the cyclic t–shape T have degree $\epsilon(T)$ and width $w(T)$ then:*

1) $\epsilon(DT) = \epsilon(T)$.
2) $w(DT) \le w(T)$ *with equality if and only if T is empty or a power shape.*
3) $D(T) = T$ *if and only if T is empty or a power shape.*
4) $D^{n}(T)$ *is empty or a power shape if $n > w(T)/2$.*
5) *If $T = t^{r_1} t^{-r_2} t^{r_3} \cdots t^{-r_k}$, where $r_i \ge 1$, is not a power shape then* $DT = t^{r_1 - 1} t^{-r_2 + 1} \cdots t^{-r_k + 1}$. □

We can illustrate the effect of differentiation by looking at the *graph* of the t–shape $T = t^{\epsilon_1} t^{\epsilon_2} \cdots t^{\epsilon_w}$.

This is a function $f = f_T : [0, w] \to \mathbb{R}$ defined as follows. Define $f(0) = 0$ and for integers i in the range $0 < i \le w$ $f(i) = \epsilon_1 + \epsilon_2 + \ldots + \epsilon_i$. Extend f over the whole interval by piecewise-linear interpolation. Notice that the graph of the t–shape starts at $(0, 0)$ and finishes at (w, ϵ).

Figure 1 shows the graph of the example above and the effect of differentiation which ‘smooths off’ the peaks and troughs until a straight line graph is left.

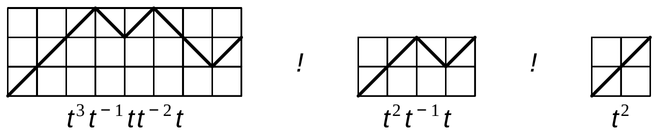

Figure 1: Di erentiation

A *clump* in a cyclic t{shape is de ned to be a maximal connected subsequence of the form t^m where $jmj > 1$. A *one-clump shape* is a shape with just one clump, which is not the whole sequence, ie, after possible cyclic permutation and inversion, a shape of the form $t^m t^{-1}(tt^{-1})^r$ where $m > 1$ and r 0. We can now de ne CG{good. A t{shape is *CG{good* if, after a (possibly empty) sequence of di erentiations it becomes a one-clump shape.

Theorem (Cli ord{Goldstein [1]) *All CG{good shapes are good.* □

3 Blowing up

We shall now introduce the notion of *blowing up* of a t{shape which was implicit in [2].

We consider non-cyclic t{shapes whose graphs start and end at level 0 and which lie between levels $-m$ and 0. Such a t{shape will be called an *m{block*. An m{block whose graph reaches level $-m$ at some point will be called a *full m{block*.

De nition *m{blow up* Start with a given cyclic t{shape. Between each pair $t^{-1}t$ (ie at local minima of the graph) insert a full m{block. Between other pairs insert a general m{block (see gure 2).

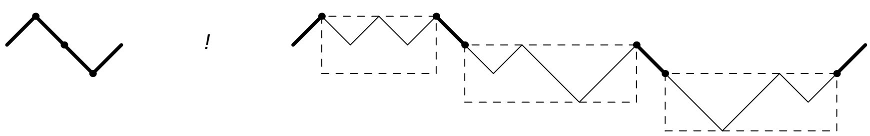

Figure 2: An example of a 2{blow-up

The de nition of blow up is not explicit in [2]. However we shall see later that it coincides with the concept of normal form given on page 69 of [2].

Notice that a 0{blow up of a shape T is the original shape T but that, in general, the result of blowing up depends on the choices of the blocks. We use the notation $B^m(T)$ for the set of m{blow ups of T and we abbreviate B^1 to B.

We now prove that blowing up is anti-di erentiation.

Lemma 3.1 $U\ 2\ B(T)$ *if and only if* $D(U) = T$.

Proof We give a graphical description of D. Start with the graph of a t{shape T. Introduce a new vertex halfway along each edge of the graph. At each local maximum (respectively minimum) join the new vertices just below (respectively above) and truncate. Now contract the horizontal edges and discard the old vertices. The result is the graph of $D(T)$.

This process is illustrated in gure 3, where the new vertices are open dots and the old vertices are black dots.

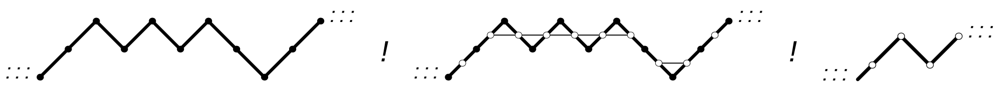

Figure 3: Graphical di erentiation

To see the connection with 1{blow ups consider the following alternative description. Introduce the new vertices as before but slide them up to the top of the edges. Discard all the locally minimal vertices of the graph of T and again reduce the resulting graph by contracting horizontal edges (see gure 4). In this description it is clear that the discarded pieces are precisely 1{blocks and the lemma follows. □

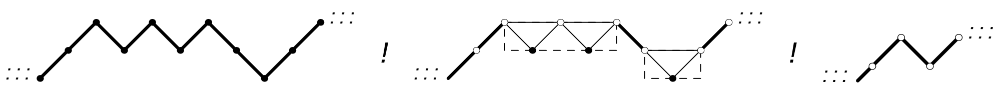

Figure 4: Di erentiation and 1{blow up

For the next lemma we need to extend di erentiation and blowing up to m{ blocks. If T is an m{block then we de ne an n{blow up by inserting full n{ blocks at local minima and general n{blocks at all other vertices, including the rst and last vertex (in other words we pre x and append a general n{block). It can then be seen that the n{blow up of an m{block is an $(m+n)${block and if the original block is full, then the blow up is also full.

We extend di erentiation by using the same rule as for cyclic t{shapes. In graphical terms it has the same meaning as in the last proof: Discard all the locally minimal vertices of the graph and reduce by contracting horizontal edges. The proof of the previous lemma then shows that B and D are inverse operations on m{blocks.

Lemma 3.2 (a) $B \quad B^m \quad B^{m+1}$ (b) $DB^{m+1} \quad B^m$.

Proof A 1–blow up of an m–blow up can be obtained by 1–blowing up the inserted m–blocks. Part (a) now follows from the remarks above. To see part (b) observe that D of a $(m+1)$–blow up is obtained by differentiating the inserted pieces and thus results in an m–blow up. □

Corollary 3.3 (a) $B \circ B^m = B^{m+1}$ (b) $B^n = B \circ \dots \circ B$ *(n factors)* (c) $B^n \circ B^m = B^{n+m}$.

Proof (a) By part (a) of lemma 3.2 we just have to show that if $U \in B^{m+1}(T)$ then $U \in B \circ B^m(T)$. But $D(U) \in B^m(T)$ by part (b), and $U \in B(D(U))$ by lemma 3.1 and hence $U \in B(D(U)) \subset B \circ B^m(T)$.

Parts (b) and (c) follow by induction. □

Corollary 3.4 *$U \in B^n(T)$ if and only if $D^n(U) = T$.*

Proof Repeat lemma 3.1 n times. □

We now turn to the connection of blowing up with the concept of normal form defined in [2].

On page 69 of [2] we define a word in *normal form* based on a particular cyclic t–shape T as a word obtained from T by inserting elements of certain subsets (X, J and Y defined on page 65) of the kernel of the exponential map $\varepsilon\colon G * \langle t\rangle \to \mathbb{Z}$ at *top* (between t and t^{-1}), *middle* (between t and t or t^{-1} and t^{-1}) and *bottom* (between t^{-1} and t) positions respectively. Inspecting the definitions of X, J and Y, it can be seen that this corresponds to inserting m–blocks and then allowing a controlled amount of cancellation. To be precise, define a *leading string* of an m–block to be an initial string $t^{-1}t^{-1}\dots t^{-1}$ and a *trailing string* to be a final string $tt\dots t$. Cancellation is allowed for specified leading and trailing strings of all blocks. The defining condition on X is that the graph of the corresponding block must meet level 0 after deletion of leading and trailing strings and the defining condition for Y is that the block must be full. There is no condition on J. We call the blocks corresponding to elements of X, J and Y, *top*, *middle* and *bottom* blocks, respectively and we denote the set of words in normal form based on the cyclic t–shape T by $NF(T)$.

Lemma 3.5 $NF(T) = B^m(T)$.

Proof Blowing up corresponds to normal form with no cancellation allowed and hence $NF(T) \supset B^m(T)$. For the converse suppose that U is in normal form based on T and that for a particular top block D the leading t^{-1} is allowed to cancel. Define the $(m-1)$–block B by $D = t^{-1}BtC$ (see figure 5). Then figure 5 makes clear that U can also be obtained by appending B to the block inserted in the previous place and replacing D by C. After these substitutions there are fewer allowed cancellations.

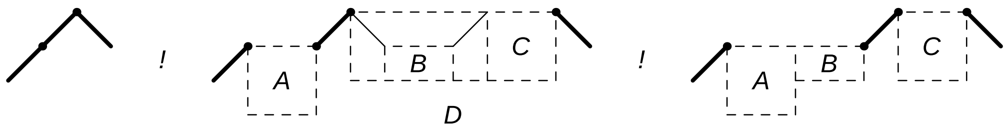

Figure 5: The simplification move

Similar arguments simplify the situation if cancellation takes place at the end of a top block or at either end of a middle block. (Notice that no cancellation can take place at bottom blocks.) Thus by repeating simplifications of this type a finite number of times, we see that U is an m–blow up of T. □

4 Amenability

We now recall the definition of *amenable* t–shapes from [2].

Recall that a *clump* in a cyclic t–shape is a maximal connected subsequence of the form t^m or t^{-m} where $m > 1$. These are said to have *order* m and $-m$ respectively. We call a clump of positive order an *up* clump and a clump of negative order a *down* clump. A t–shape is said to be *suitable* if it has exactly one up clump which is not the whole sequence and possibly some down clumps, or if it has exactly one down clump which is not the whole sequence and possibly some up clumps. It follows that, after a possible cyclic rotation or inversion, a suitable t–shape has the form

$$t^s t^{-r_0} t t^{-r_1} t \dots t t^{-r_k}$$

where $s > 1$, $k \geq 0$ and $r_i \geq 1$ for $i = 0, \dots, k$.

We now define amenable t–shapes. Using lemma 3.5 above we can rephrase the definition on page 69 of [2] as follows.

Definition *Amenable t–shapes* A t–shape which is the m–blow up of a suitable t–shape is called *amenable*.

Theorem (Fenn–Rourke [2]) *Amenable shapes are good.* □

We now turn to the characterisation of amenability. Using corollary 3.4, the definition of amenability says that a shape is amenable if and only if it eventually differentiates to a suitable shape. But now a suitable t–shape is either a one clump shape or differentiates to $t^s t^{-r}$ for some $r, s \geq 1$. This in turn either eventually differentiates to tt^{-1} or to $t^s t^{-1}$ or to tt^{-r} for some $r, s \geq 2$. Now the last two are one clump shapes and so we can see that a suitable shape either eventually di erentiates to a one clump shape or to tt^{-1}. To make the nal characterisation of amenability as simple as possible, we make the shape tt^{-1} an honorary amenable shape (it is good [3]) and then we have the following simple characterisation.

Theorem 4.1 (Characterisation of amenability) *A shape is amenable if and only if, after a (possibly empty) sequence of di erentiations, it becomes either a one-clump shape or the shape* tt^{-1}. □

Corollary 4.2 *Amenable shapes are a strictly larger class than CG{good shapes.* □

Final remarks (1) The class of amenable shapes which are not CG{good are precisely those which eventually di erentiate to tt^{-1}: an example would be $tt^{-1}t^2t^{-2}$. It seems that the methods of Cli ord and Goldstein can be extended with little extra work to the smaller class of shapes which eventually di erentiate to the shape t^2t^{-2}. However we cannot see how to extend their methods to cover all amenable shapes.

(2) The remark at the top of page 70 of [2], which was left unproven, can be quickly proved using theorem 4.1.

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School of Mathematical Sciences, Sussex University
Brighton, BN1 9QH, UK
and
Mathematics Institute, University of Warwick
Coventry, CV4 7AL, UK

Email: `R.A.Fenn@sussex.ac.uk, cpr@maths.warwick.ac.uk`

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