ISSN 1464-8997 **455**

Geometry & Topology Monographs
Volume 2: Proceedings of the Kirbyfest
Pages 455–472

Homology stratifications and intersection homology

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Abstract A homology stratification is a filtered space with local homology groups constant on strata. Despite being used by Goresky and MacPherson [3] in their proof of topological invariance of intersection homology, homology stratifications do not appear to have been studied in any detail and their properties remain obscure. Here we use them to present a simplified version of the Goresky–MacPherson proof valid for PL spaces, and we ask a number of questions. The proof uses a new technique, homology general position, which sheds light on the (open) problem of defining generalised intersection homology.

AMS Classification 55N33, 57Q25, 57Q65; 18G35, 18G60, 54E20, 55N10, 57N80, 57P05

Keywords Permutation homology, intersection homology, homology stratification, homology general position

Rob Kirby has been a great source of encouragement. His help in founding the new electronic journal Geometry & Topology has been invaluable. It is a great pleasure to dedicate this paper to him.

1 Introduction

Homology stratifications are filtered spaces with local homology groups constant on strata; they include stratified sets as special cases. Despite being used by Goresky and MacPherson [3] in their proof of topological invariance of intersection homology, they do not appear to have been studied in any detail and their properties remain obscure. It is the purpose of this paper is to publicise these neglected but powerful tools. The main result is that the intersection homology groups of a PL homology stratification are given by singular cycles meeting the strata with appropriate dimension restrictions. Since any PL space has a natural intrinsic (topologically invariant) homology stratification, this gives a new proof of topological invariance for intersection homology, cf [5]. This new proof is in the spirit of the original proof of Goresky and MacPherson [3] who used a similar, but more technical, definition of homology stratification. It applies only to PL spaces, but these include all the cases of serious application (eg algebraic varieties). In the proof we introduce a new tool: a homology general position theorem for homology stratifications. This theorem sheds light on the (open) problem of defining intersection bordism and, more generally, generalised intersection homology.

The rest of this paper is arranged as follows. In section 2 we define *permutation homology groups*. These are groups $H_i^\pi(K)$ defined for any principal n–complex K and permutation $\pi \in \Sigma_{n+1}$. Permutation homology is a convenient device (implicit in Goresky and MacPherson [2]) for studying intersection homology. We prove that, for a PL manifold, all permutation homology groups are the same as ordinary homology groups. In section 3 we prove that the groups are PL invariant for *allowable* permutations by giving an equivalent singular definition (for a stratified set). This makes clear the connection with intersection homology. In section 4 we extend the arguments of section 2 to homology manifolds and in section 5 we define homology stratifications, extend the arguments of sections 3 and 4 to homology stratifications and deduce topological invariance. In section 6 we discuss the problem of defining intersection bordism (and more generally, generalised intersection homology) in the light of the ideas of previous sections. Finally in section 7 we ask a number of questions about homology stratifications.

2 Permutation homology

We refer to [9] for details of PL topology. Throughout the paper a *complex* will mean a locally finite simplicial complex and a *PL space* will mean a topological space equipped with a PL equivalence class of triangulations by complexes. Let K be a *principal n–complex*, ie, a complex in which each simplex is the face of an n–simplex. Let $K^{(1)}$ denote the (barycentric) first derived complex of K. Recall that $K^{(1)}$ is the subdivision of K with simplexes spanned by barycentres of simplexes of K; more precisely, if we denote the barycentre of a typical simplex $A_i \in K$ by a_i then a typical simplex of $K^{(1)}$ is of the form $(a_{i_0}, a_{i_1}, \ldots, a_{i_k})$ where $A_{i_0} < A_{i_1} < \ldots < A_{i_k}$ and $A_i < A_j$ means A_i is a face of A_j.

Now let $\pi \in \Sigma_{n+1}$, the symmetic group, ie, $\pi\colon \{0,1,\ldots,n\} \to \{0,1,\ldots,n\}$ is a permutation. Define subcomplexes K_i^π of $K^{(1)}$, $0 \le i \le n$, to comprise simplexes $(a_{i_0}, a_{i_1}, \ldots, a_{i_k})$ where $\dim(A_{i_s}) \in \{\pi(0), \ldots, \pi(i)\}$ for $0 \le s \le k$. In other words K_i^π is the full subcomplex of $K^{(1)}$ generated by the barycentres of simplexes of dimensions $\pi(0)$, $\pi(1)$ $\ldots$ $\pi(i)$. The definition implies that K_i^π

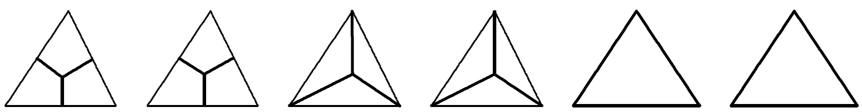
is a principal i–complex and that $K_i^\pi \subset K_{i+1}^\pi$ for each $0 \le i < n$. Here is an alternative description. K_0^π is the 0–complex which comprises the barycentres of the $\pi(0)$–dimensional simplexes of K and in general we can describe K_i^π inductively as follows. To obtain K_i^π from K_{i-1}^π, attach for each simplex A_s of dimension $\pi(i)$ the cone with vertex a_s and base $K_{i-1}^\pi \cap \mathrm{lk}(a_s, K^{(1)})$.

Examples (cf Goresky and MacPherson [2, pages 145–147])

(1) If $\pi = \mathrm{id}$ then $K_i^\pi = K^i$ the i–skeleton of K.

(2) If $\pi(k) = n - k$ for $0 \le k \le n$ then $K_i^\pi = (DK)^i$ the dual i–skeleton of K.

(3) For $n = 2$ the possibilities for a 2–simplex intersected with K_0^π and K_1^π are illustrated in figure 1.



(02) (012) (12) (021) Id (10)

Figure 1

(4) For $n = 3$ the intersection of a 3–simplex with K_0^π, K_1^π and K_2^π is shown in figure 2 for various π.

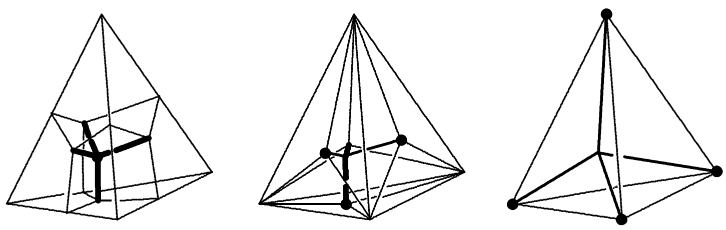


Figure 2

Definition The i^{th} *permutation homology group*, $H_i^\pi(K)$, of K is the i^{th} homology group of the chain complex:

$$\dots \longrightarrow H_{i+1}(K_{i+1}^\pi, K_i^\pi) \xrightarrow{\partial} H_i(K_i^\pi, K_{i-1}^\pi) \xrightarrow{\partial} H_{i-1}(K_{i-1}^\pi, K_{i-2}^\pi) \longrightarrow \dots$$

where the boundary homomorphisms come from boundaries in the homology exact sequencies of the appropriate triples. Cohomology groups $H_\pi^i(K)$ are defined similarly. The definition also extends to any generalised homology theory; but see the discussion in section 7.

Using a standard diagram chase (and the fact that homology groups vanish above the dimension of the complex) we have:

Proposition 2.1 $H_i^\pi(K) \cong Im(H_i(K_i^\pi) \to H_i(K_{i+1}^\pi))$ □

It follows that $H_i^\pi(K)$ can be described as i–cycles in $|K_i^\pi|$ modulo homologies in $|K_{i+1}^\pi|$ and we are at liberty to use singular or simplicial cycles and homologies. By releasing the restriction on cycles and boundaries we get a natural map ϕ: $H_i^\pi(K) \to H_i(K)$.

Proposition 2.2 *If $|K|$ is a PL manifold then the natural map ϕ: $H_i^\pi(K) \to H_i(K)$ is an isomorphism.*

Proof The vertices of $K^{(1)}$ not used in the construction of K_i^π consist of barycentres of simplexes A with $\dim(A) \notin \pi[0,i]$ and we denote by CK_i^π the full subcomplex (of dimension $n-i-1$) generated by these unused vertices. This can also be defined as follows: write $\bar\pi(k) = n - \pi(k)$ then $CK_i^\pi := K_{n-i-1}^{\bar\pi}$. Note that $|K_i^\pi| \cap |CK_i^\pi| = \emptyset$ and any simplex of $K^{(1)}$ may be uniquely expressed as a join of a simplex of K_i^π with a simplex of CK^π. Now an i–cycle in $|K|$ may be pushed off $|CK^\pi|$ by general position and then it can be pushed down join lines into $|K_i^\pi|$. Similarly homologies can be pushed off $|CK_{i+1}^\pi|$ into $|K_{i+1}^\pi|$. □

3 PL invariance

Now let $d_{i,j}^\pi$ be $|\pi[0,i] \cap [0,j]| - 1$, ie, one less than the number of integers $\leq i$ which have image under π which is $\leq j$.

The following facts are readily checked:

Lemma 3.1

(1) *The integers $d_{i,j}^\pi$ satisfy $d_{i,j}^\pi \leq \min(i,j)$, $d_{n,j}^\pi = j$, $d_{i,n}^\pi = i$, $d_{i,j}^\pi - d_{i-1,j}^\pi = 0$ or 1, $d_{i,j}^\pi - d_{i,j-1}^\pi = 0$ or 1.*

(2) *The integers $d_{i,j}^\pi$ determine the permutation π.*

(3) *$d_{i,j}^\pi$ is the dimension of $K_i^\pi \cap K^j$ where K^j is the j–skeleton of K.* □

We now use the integers $d^\pi_{i,j}$ to define *singular permutation homology* for a filtered space.

Define a *(geometric) n–cycle* (often called a *pseudo-manifold*) to be a compact oriented PL n–manifold with singularity of codimension ≥ 2. This is the natural picture for a (glued-up) singular cycle. A *cycle with boundary* is a compact oriented PL manifold with boundary and singularity of codimension ≥ 2, which meets the boundary in codimension ≥ 2. In other words if P is a cycle with boundary then ∂P is a cycle of one lower dimension. By a *(geometric) singular cycle* (P, f) in a space X we mean a geometric n–cycle P and a map $f\colon P \to X$. A *(geometric) singular homology* (Q, F) between singular cycles (P, f), (P', f') is a cycle Q with boundary isomorphic to $P \cup -P'$ such that $F|P = f$, $F|P' = f'$. It is well known that (singular) homology can be described as geometric singular homology classes of geometric singular cycles. There is a similar description for relative singular homology. A *relative* singular cycle (P, f) in a pair of spaces (X, A) is a geometric cycle P with boundary ∂P and a map of pairs $f\colon (P, \partial P) \to (X, A)$. A *relative homology* (Q, F) between relative cycles is a cycle Q with boundary isomorphic to $P \cup -P' \cup Z$, where Z is a cycle with boundary $\partial P \cup -\partial P'$, and F is a map of pairs $(Q, Z) \to (X, A)$ such that $F|P = f$, $F|P' = f'$. We shall refer to Z as the homology *restricted to the boundary*. From now singular cycles and homologies will all be geometric and we shall omit "geometric".

Let $\bar{X} = \{X_0 \subset X_1 \subset \ldots \subset X_n\}$ be a filtered space where X_j has (nominal) dimension j. We refer to $X_j - X_{j-1}$ as the j^{th} *stratum* of $\bar{X}$ even though we are not assuming that $\bar{X}$ is a stratified set and we often abbreviate X_n to X. Define the *singular permutation homology group* $SH^\pi_i(\bar{X})$ to be the group generated by singular i–cycles (P, f) in X such that $f^{-1}(X_j)$ is a PL subset of dimension $\leq d^\pi_{i,j}$ modulo homologies (W, F) such that $F^{-1}(X_j)$ is a PL subset of dimension $\leq d^\pi_{i+1,j}$. There is a similar definition of *relative* singular permutation homology groups.

Remark 3.2 If X is a PL space filtered by PL subsets then there is no loss in assuming that the maps f and F in the definition are PL. This is because any map can be approximated by a PL map and it can be checked that this can be done preserving the (PL) preimages of the closures of the strata.[1]

[1] In the standard proof of the simplicial approximation theorem [6, pages 37–39], suppose that $f\colon K \to L$ is a map such that $f^{-1}(L_0) = K_0$ (subcomplexes). By subdividing if necessary assume that L_0 is a full subcomplex of L. Suppose that K is sufficiently subdivided for the simplicial approximation to be defined. When constructing the simplicial approximation g, choose images of vertices not in K_0 to be not in L_0 then $g^{-1}(L_0) = K_0$.

A permutation π is *allowable* if the integers $d^\pi_{i,j}$ satisfy the further condition:

$$d^\pi_{i+1,j} = d^\pi_{i,j} + 1 \quad \text{if} \quad 0 \le d^\pi_{i,j} < j \qquad (*)$$

We shall see that intersection homology groups are precisely the groups SH^π_i for allowable π.

More generally if $\bar{X}$ is a filtered space, define π to be $\bar{X}$*–allowable* if $(*)$ holds for all j such that $X_j - X_{j-1} \ne \emptyset$.

It can readily be verified that singular permutation homology has an excision property for allowable permutations (proved by cutting cycles and homologies along codimension 1 subsets—allowability is needed so that the "constant" homology is a homology in SH^π).

Now recall that any PL space X (of dimension n) has a natural PL stratification $\bar{X} = \{X^0 \subset X^1 \subset \ldots \subset X^n\}$ where $X_i - X_{i-1}$ is a PL i–manifold. For any PL stratification $\bar{X}$ of X, proposition 2.1 and lemma 3.1 provide a natural map $\psi\colon H^\pi_i(X) \to SH^\pi_i(\bar{X})$.

The following theorem generalises theorem 2.2 and implies PL invariance for allowable permutations.

Theorem 3.3 $\psi\colon H^\pi_i(X) \to SH^\pi_i(\bar{X})$ *is an isomorphism where $\bar{X}$ is any PL stratification of X and π is $\bar{X}$–allowable.*

Proof To see that ψ is onto we generalise the proof of 2.2. Triangulate X by K say and let (P, f) be a singular i–cycle representing an element of $SH^\pi_i(X)$. By remark 3.2 we may assume that f is PL; then working inductively over the strata of X we push $\mathrm{im}(f)$ off $|CK^\pi_i|$ (and hence into $|K^\pi_i|$) using general position and extending to higher strata using the local product structure of the stratification. Notice that the condition that π is $\bar{X}$–allowable is needed to ensure that the homologies given by these moves have the correct dimension restrictions. A similar argument (applied to homologies) shows that ψ is 1–1. □

Connection with intersection homology

The definition of singular permutation homology is very reminiscent of the definition of intersection homology. Indeed we can describe the connection precisely as follows. Recall from Goresky and MacPherson [2] or King [5] that a *perversity* is a sequence $\bar{p} = \{0 = p_0 \le p_1 \le p_2 \le \ldots \le p_n\}$[2] where

[2] Goresky and MacPherson have the additional condition $p_0 = p_1 = p_2 = 0$ and King has no condition on p_0. However if $p_i > i$ then the intersection condition is vacuous, so we may as well assume $p_0 = 0$.

$p_{i+1} - p_i \leq 1$. Intersection homology (cf [2, page 138]) is defined exactly like singular permutation homology with $d^\pi_{i,j}$ replaced by $i+j-n+p_{n-j}$. However by using simplicial homology it can be seen that the intersection of an i–cycle with a j–dimensional PL subset can always be assumed to have dimension $\leq j$ and a similar remark applies to homologies. Thus we get exactly the same groups if $d^\pi_{i,j}$ is replaced by $\min(j\,,\, i+j-n+p_{n-j})$. We now explain how to find a (unique) permutation π for which $d^\pi_{i,j}$ has this value.

Define a permutation $\pi \in \Sigma_{n+1}$ to be V*–shaped* if $\pi|[0,u]$ is monotone decreasing and $\pi|[u,n]$ is monotone increasing, where $0 \leq u \leq n$ is the unique number such that $\pi(u) = 0$. It is easy to see that a V–shaped permutation is uniquely determined by the subset $S_\pi = \pi[0, u-1] \subset \{1, 2, \ldots, n\}$. We shall see that perversities correspond to V–shaped permutations. Given a perversity $\bar{p}$, define $S = \{j : 0 < j \leq n, p_{n-j} = p_{n-j+1}\}$ and consider the V–shaped permutation π with $S_\pi = S$. Then inspecting the graph of π it can readily be seen that $d^\pi_{i,j} = \min(j, i - q_j)$ where $q_j = |S_\pi \cap [j+1, n]|$. But from the definition of S_π, $q_j = |k : j < k \leq n, p_{n-k} = p_{n-k+1}|$, and substituting c for $n-k$ we have $q_j = |c : 0 \leq c < n-j, p_c = p_{c+1}| = n - j - p_{n-j}$ and hence $d^\pi_{i,j} = \min(j, i + j - n + p_{n-j})$ as required.

It is not hard to see, from graphical considerations, that V–shaped permutations are precisely the same as allowable permutations. Thus the singular permutation homology groups for allowable permutations are precisely the intersection homology groups. Further it can be seen that, given an $\bar{X}$–allowable permutation, there is an allowable permutation with the same values of $d^\pi_{i,j}$ for all j such that $X_j - X_{j-1} \neq \emptyset$. Thus the $\bar{X}$–allowable singular permutation groups of $\bar{X}$ are the intersection homology groups of $\bar{X}$. Thus although permutation homology gives a richer set of definitions than intersection homology, in the cases where the groups are PL invariant (which we shall see are the same as the cases where the groups are topologically invariant) the groups defined are the intersection homology groups.

In section 5 we will need to consider the permutation π' of $\{0, 1, \ldots, n-1\}$ associated to a permutation π of $\{0, 1, \ldots, n\}$, defined as follows: remove 0 from the codomain of π and $\pi^{-1}(0)$ from the domain. This gives a bijection between two ordered sets of size n. Identify each with $\{0, 1, \ldots, n-1\}$ by the unique order-preserving bijection. The resulting permutation is π'. We call π' the *reduction* of π. If π is allowable then so is π' and in terms of perversities, the operation corresponds to ignoring the final term of the perversity sequence. It can be checked that, in terms of the d's, π' is defined by $d^{\pi'}_{i-1,j-1} = d^\pi_{i,j} - 1$.

4 Homology general position

Recall that a PL space M is a *homology n–manifold* if $H_i(M, M - x) = 0$ for $i < n$ and $H_n(M, M - x) = \mathbb{Z}$ for all $x \in M$ or equivalently if the link of each point in M is a homology $(n-1)$–sphere.

The purpose of this section is to generalise proposition 2.2 to homology manifolds.

Proposition 4.1 *If M is a homology manifold then the natural map ϕ: $H_i^\pi(M) \to H_i(M)$ is an isomorphism.*

The proof is very similar to the proof of 2.2. However the key point in the proof (the application of PL general position) does not work in a homology manifold. In general it is not possible to homotope a map of an i–dimensional set in a homology manifold M off a codimension $i+1$ subset. However we only need to move off by a *homology* and this can be done.

Theorem 4.2 (Homology general position) *Suppose that M is a homology n–manifold and $Y \subset M$ a PL subset of dimension y. Suppose that (P, f) is a singular cycle in M of dimension q where $q + y < n$. Then there is a singular homology (Q, F) between (P, f) and (P', f') such that $f'(P') \cap Y = \emptyset$.*

Furthermore the "move" can be assumed to be arbitrarily small in the sense that $F(Q)$ is contained within an arbitrarily small neighbourhood of $f(P)$.

There is a version of the theorem which applies to cycles with boundary:

Addendum *Suppose that P has boundary ∂P then there is a relative singular homology (Q, F) between (P, f) and (P', f') such that $f'(P') \cap Y = \emptyset$. Further the moves on both P and ∂P can be assumed to be small, ie, $F(Q)$ is contained within an arbitrarily small neighbourhood of $f(P)$ and $F(Z)$ is contained within an arbitrarily small neighbourhood of $f(\partial P)$ where Z is the restriction of the homology to the boundary.*

There is also a relative version of the theorem, which we leave the reader to prove: *If $f(\partial P) \cap Y = \emptyset$ then we can assume that the homology fixes the boundary in the sense that $Z \cong \partial P \times I$ and $F|Z$ is F composed with projection on ∂P.*

Proof We observe that if, in the small version of the addendum, $\partial P = \emptyset$ then $Z = \emptyset$ and the addendum reduces to the main theorem. Thus we just have to prove the addendum. (By contrast the non-small version of the addendum is vacuous, since there is always a relative homology to the empty cycle!)

The proof of the addendum is by induction on n (this is the *main induction*; there will be a subsidiary induction). Using the fact that M is a PL space and Y a PL subset, we may cover M by cones (denoted C_i, with bases denoted B_i) with the property that each C_i is contained in a larger cone C_i^+ of the form $C_i \cup B_i \times I$ and such that $Y \cap C_i^+$ is a subcone. Furthermore we can assume that each C_i^+ has small diameter and that the C_i^+ form $n+1$ disjoint subfamilies, ie, two cones in the same family do not meet. This implies that any subset of more than $n+1$ of the C_i^+ has empty intersection. (This is seen as follows. Choose a triangulation K of M such that Y is a subcomplex and let $K^{(2)}$ be the second derived. Define the C_i to be small neighbourhoods of the vertex stars $\mathrm{st}(v_i, K^{(2)})$ for vertices $v_i \in K^{(1)}$. Define the C_i^+ to be slightly larger neighbourhoods. Smallness is achieved by taking K to have small mesh and the subfamilies correspond to the dimension of the simplex of K of which v_i is the barycentre.) Since M is a homology manifold, the cones C_i are in fact homology n–balls and their bases C_i are homology $(n-1)$–spheres.

We shall "move" (P, f) by a series of moves each supported by one of the cones C_i^+ and with the property that if $\partial P \cap C_i^+$ is empty before the move, then it still is after the move. We number the subfamilies $1, \ldots, n+1$ and we order the moves so that all the moves corresponding to cones in the subfamily 1 occur first and then subfamily 2 and so on. Thus if each C_i^+ has diameter smaller than $\frac{\epsilon}{n+1}$ then the set of moves corresponding to subfamily i is supported by the $\frac{\epsilon}{n+1}$–neighbourhood of $f(P)$ and the whole move is supported by the ϵ–neighbourhood of $f(P)$ with similar properties for the restriction to the boundary. The individual moves are defined by a subsidiary inductive process which we now describe.

By remark 3.2 we may assume that f is PL. By compactness of $f(P)$ choose a finite subset $\mathcal{C} = \{C_1, C_2, \ldots, C_t\}$ of cones so that $\bigcup \mathcal{C}$ is a neighbourhood of $Y \cap f(P)$ and with the order compatible with the order on the subfamilies. Define $Y_j = Y \cap (C_1 \cup \ldots \cup C_j)$.

Suppose that we have already moved (P, f) so that $f(P) \cap Y_j = \emptyset$ and so that $\bigcup \mathcal{C}$ is still a neighbourhood of $Y \cap f(P)$. We shall explain how to move (P, f) off Y in $C = C_{j+1}$ by a move supported in C^+ so that $\bigcup \mathcal{C}$ remains a neighbourhood of $Y \cap f(P)$ and the property that $f(P) \cap Y_j = \emptyset$ is preserved. The result is that $f(P) \cap Y_{j+1} = \emptyset$. This inductive process starts trivially and ends with $P \cap Y_t = P \cap Y = \emptyset$ proving the theorem.

For the induction step we have to move (P, f) off Y in C. We start by applying (genuine) transversality to B. By transversality we may assume that $f^{-1}(B)$ is a bicollared subcomplex R of P of dimension $q-1$ which is therefore a cycle (possibly with boundary) cutting P into two cycles with boundary P_0 and P_1 where $P_1 = f^{-1}C$. Note that ∂P also splits at $f^{-1}(B)$ into two cycles with boundary S_0 and S_1 with $\partial R = \partial S_0 = \partial S_1$ where $S_1 \subset P_1$.[3]

We now need to consider two cases.

Case 1 : $S_1 \neq \emptyset$ In this case there is a very easy move which achieves the required result. Let P_1^+ be a small neighbourhood of P_1 in P and P_0^- the corresponding shrunk copy of P_0. We "move" (P, f) to $(P_0^-, f|)$ by excising P_1^+. More precisely, we regard $(P \times I, f \circ \mathrm{proj})$ as a relative homology between (P, f) and $(P_0^-, f|)$ by setting Z (the homology restricted to the boundary) equal to $\partial P \times I \cup P_1^+ \times \{1\}$. If we now let $(P_0^-, f|)$ be the new (P, f) the required properties are clear.

Case 2 : $S_1 = \emptyset$ In this case the easy move described in case 1 would be fallacious, because we have $\partial P \cap C^+$ non-empty after the move whilst it could well be empty before the move and the restriction to the boundary of the entire process would not be small. We now use the fact that M is a homology manifold. Since $\partial R = \partial S_1 = \emptyset$, R is a cycle and further $(R, f|)$ bounds $(P_1, f|)$ in C. Since C is a homology ball with boundary B a homology sphere of dimension bigger than $q-1$, there is a cycle (P_2, f_2) with boundary (R, f) in B and a cycle with boundary (Q, F) in C with boundary $(P_1 \cup P_2, f| \cup f_2)$. Extend Q by a collar on P to give a homology between (P, f) and $(P_0 \cup_R P_2, f| \cup f_2)$. This is the first move. At this point we use the main induction hypothesis. By induction we may make a second move of (P_2, f_2) off Y in B to (P_2', f_2') say. Using collars this extends to a move of $(P_0 \cup_R P_2, f| \cup f_2)$ to (P', f') say where ${f'}^{-1}(B) = P_2'$. It is clear that $f'(P') \cap Y \cap C = \emptyset$ and it remains to check that $f'(P') \cap Y_j = \emptyset$ and that $\bigcup \mathcal{C}$ is still a neighbourhood of $Y \cap f'(P')$. But before the start of the induction step $f(P) \cap Y_j = \emptyset$ and since these two are compact they start a definite distance apart; now the two moves which may have affected this were (1) the application of genuine transversality to B and (2) the (inductive) move of P_2 off Y in B, both of which may be assumed to be arbitrarily small and hence not affect $f(P) \cap Y_j = \emptyset$. $\bigcup \mathcal{C}$ remains a neighbourhood of $Y \cap f'(P')$ for similar reasons. $f(P) \cap Y$ starts a definite

[3]The transversality theorem being used here is elementary. Projecting onto the collar coordinate we have to make a PL map g say, from P to an interval, transverse to an interior point. But we may assume that g is simplicial and, by inspection, a simplicial map to an interval is tranverse to all points other than vertices. So we just compose g with a small movement in the collar direction so that B does not project to a vertex.

distance from the frontier of $\bigcup \mathcal{C}$ and the same smallness considerations imply that this property is preserved. □

Proof of proposition 4.1 The analogue of the proof of proposition 2.2 now proceeds with obvious changes. Define CK_i^π as before. Then by homology general position we can move an i–cycle in M off $|CK_i^\pi|$ by a homology and hence by pushing down join lines we can move it into $|K_i^\pi|$. Similarly a homology can be moved into $|K_{i+1}^\pi|$. □

5 Homology stratifications

Let $x \in X$ a PL space and let h be any (possibly generalised or permutation) homology theory. Then for each y close to x there is a natural map $q\colon h_*(X, X-x) \to h_*(X, X-y)$. This is because $X - x \to X - \mathrm{st}(x)$ is a homeomorphism where $\mathrm{st}(x)$ denotes a small star of x in X. So define $q\colon h_*(X, X-x) \cong h_*(X, X-\mathrm{st}(x)) \xrightarrow{j} h_*(X, X-y)$ where $y \in \mathrm{st}(x)$ and j is induced by inclusion.

Let $h_*^{\mathrm{loc}}(X)$ denote the collection $\{h_*(X, X-x) : x \in X\}$ of local homology groups of X. Let $Y \subset X$ define $h_*^{\mathrm{loc}}(X)$ to be *locally constant* on Y at $x \in Y$ if q is an isomorphism for $y \in Y$ and y close to x.

Comment This definition is independent of the PL structure on X. If X' denotes X with a different PL structure then we can find a star $\mathrm{st}(x, X') \subset \mathrm{st}(x, X)$ and then q factors as $h_*(X, X-x) \cong h_*(X, X-\mathrm{st}(x, X')) \cong h_*(X, X-\mathrm{st}(x, X)) \xrightarrow{j} h_*(X, X-y)$ and it can be seen that q and q' (the analogous map for X') coincide.

Further the definition makes sense for a wider class of spaces than PL spaces—essentially any space with locally contractible neighbourhoods—for example locally cone-like topologically stratified sets (Siebenmann's CS sets [10]).

Definition A filtered PL space $\bar{X} = \{X_0 \subset X_1 \subset \ldots \subset X_n\}$ is an h–*stratification* if $h_*^{\mathrm{loc}}(X_n)$ is locally constant on $X_j - X_{j-1}$ for each $j \leq n$. If h is singular permutation homology SH^π then we call it a π–*stratification.*

A locally trivial filtration with strata homology manifolds (eg a triangulated CS set) is an h–stratification for all h. However note that h–stratifications are weaker than any definition of topological stratification (eg Hughes [7], Quinn [8]). For example a homology manifold (with just one stratum) is an

h–stratification for all h but, if not a topological manifold, is not a topological stratification. There are several sensible alternative definitions of homology stratifications, see the discussion in section 7.

Now any principal complex X of dimension n has an *instrinsic h–stratification* defined inductively as follows. Set $X_n = X$ and define X_{n-1} by $x \notin X_{n-1}$ if $h_*^{\rm loc}(X)$ is locally constant at x. If $h_*^{\rm loc}$ is locally constant at a point in the interior of a simplex σ then it is locally constant on the open star of σ. It follows that X_{n-1} is a subcomplex of X of dimension $\leq n-1$. In general suppose X_j is defined. Define the subcomplex $X_{j-1} \subset X_j$ by $x \notin X_{j-1}$ if x is in some j–simplex in X_j and $h_*^{\rm loc}(X)$ is locally constant at x on X_j. It can be seen that X_j is a subcomplex of X of dimension $\leq j$.

By definition $\bar{X} = \{X_0 \subset X_1 \subset \ldots \subset X_n\}$ is an h–stratification. Further the stratification is topologically invariant since the conditions which define strata are independent of the PL structure by the comment made above.

Topological invariance

Topological invariance of intersection (ie allowable permutation) homology is proved by combining the arguments of sections 3 and 4. The key result follows.

Main theorem 5.1 *Let $\bar{X}$ be a π–stratification where π is $\bar{X}$–allowable. Then the natural map $\psi\colon H_i^\pi(X) \to SH_i^\pi(\bar{X})$ is an isomorphism.*

Topological invariance follows at once by applying the theorem to the (topologically invariant) instrinsic π–stratification. The proof is analogous to the proof of 3.3 and 4.1 using the following stratified homology general position theorem.

Theorem 5.2 (Stratified homology general position) *Suppose that $\bar{X}$ is a π–stratification where π is X–allowable. Suppose that (P,f) is a singular p–cycle in $SH_*^\pi(X)$ and suppose that $Y \subset X_n$ is a PL subset such that $\dim(Y \cap X_j) + d_{p,j}^\pi < j$ for each $0 \leq j \leq n$. Then there is a singular homology (Q,F) in $SH_*^\pi(X)$ between (P,f) and (P',f') such that $f'(P') \cap Y = \emptyset$.*

Furthermore the “move” can be assumed to be arbitrarily small in the sense that $F(Q)$ is contained within an arbitrarily small neighbourhood of $f(P)$.

The theorem has a version for cycles with boundary analogous to the addendum to theorem 4.2:

Addendum *Suppose that $\bar{X}$ and Y are as in the main theorem and (P, f) is a singular p–cycle with boundary in X which satisfies the dimension restrictions for a cycle in $SH_*^\pi(X)$. Then there is a relative singular homology (Q, F) which satisfies the dimension restrictions for a homology in $SH_*^\pi(X)$ between (P, f) and (P', f') such that $f'(P') \cap Y = \emptyset$. Further the moves on both P and ∂P can be assumed to be small, ie, $F(Q)$ is contained within an arbitrarily small neighbourhood of $f(P)$ and $F(Z)$ is contained within an arbitrarily small neighbourhood of $f(\partial P)$ where Z is the restriction of the homology to the boundary.*

There is also an analogous relative version of the theorem which we leave the reader to state and prove.

Proof The theorem is very similar to the proof of theorem 4.2 with M replaced by X and we shall sketch the proof paying careful attention only to the places where there is a substantive difference. We merely have to prove the addendum and we use induction on n. As before we may cover X by small cones $C_i \subset C_i^+$ (with the base of C_i denoted B_i) which form $n+1$ disjoint subfamilies and such that Y meets each in a subcone and such that the local filtration follows the cone structure. (In this proof the cones are not homology balls and the bases are not homology spheres.)

It can be checked that the induced filtration on B_i is a π'–stratification; essentially this is because the local homology of C_i at B_i is the suspension of the local homology of B_i. In the following "cycle" means singular cycle in π or π'–homology as appropriate.

We define a finite subset $\mathcal{C} = \{C_1, C_2, \ldots, C_t\}$ such that $\bigcup \mathcal{C}$ is a neighbourhood of $Y \cap f(P)$ as before and we set up a subsidiary induction with exactly the same properties. The induction proceeds with no change at all for case 1. For case 2, which was the first place that properties of M were used, there are now two subcases to consider. Let c be the conepoint of C and let T (a subcone) be the intersection of the stratum of $\bar{X}$ containing c with C.

Case 2.1 $f(P) \supset T$ In this case, by the dimension hypotheses Y misses T and hence, since Y is a subcone of C^+ we have $Y \cap C^+ = \emptyset$, and there is nothing to do.

Case 2.2 *There is a point $x \in T, x \notin P$.* In this case, denote $C - B$ by C'. Now $SH_*^\pi(X, X - x) \cong SH_*^\pi(X, X - C')$ by the definition of π–stratification and hence using excision $SH_*^\pi(C, C - x) \cong SH_*^\pi(C, B)$. But $(P_1, f|)$ represents the zero class in the former group and hence in the latter. Thus there is a homology (Q, F) say in SH_*^π of (P_1, f) rel boundary to a class (P_2, f_2) say with $f_2(P_2) \subset B$. The proof now terminates exactly as in the previous proof. We use (Q, F) to move (P, f) to $(P_0 \cup_R P_2, f| \cup f_2)$ (the first move) and then we apply induction to move (P_2, f_2) off Y in B extending by collars as before to produce (P', f') (the second move). The required properties are checked as before. □

Proof of the main theorem The analogue of previous similar proofs now proceeds with obvious changes. Triangulate X by K and define CK_i^π as before. Then by stratified homology general position we can move an i–cycle in $SH^\pi(\bar{X})$ off $|CK_i^\pi|$ by a homology in $SH^\pi(\bar{X})$ and hence by pushing down join lines we can move it into $|K_i^\pi|$. Similarly a homology can be moved into $|K_{i+1}^\pi|$. □

6 Intersection bordism

We have given three equivalent definitions of permutation homology and we shall see shortly that there is a hidden fourth definition. All four generalise to give definitions of intersection bordism (and more generally of generalised intersection homology). Only two are the same for intersection bordism. We shall see that these two are topologically invariant.

The three equivalent definitions of the i^{th} permutation homology group were:

(1) The homology of the chain complex:

$$\ldots \longrightarrow H_{i+1}(K_{i+1}^\pi, K_i^\pi) \xrightarrow{\partial} H_i(K_i^\pi, K_{i-1}^\pi) \xrightarrow{\partial} H_{i-1}(K_{i-1}^\pi, K_{i-2}^\pi) \longrightarrow \ldots$$

(2) Cycles in K_i^π modulo homologies in K_{i+1}^π.

(3) Singular permutation homology of a stratified set, ie, singular i–cycles meeting strata of dimension j in dimension $\le d_{i,j}^\pi$ modulo homologies meeting strata of dimension j in dimension $\le d_{i+1,j}^\pi$.

The fourth equivalent definition follows from definition (2) using the property that K_i^π meets K^j in dimension $\le d_{i,j}^\pi$, see lemma 3.1:

(4) Singular i–cycles in K_i^π which meet K^j in dimension $\le d_{i,j}^\pi$ modulo homologies in K_{i+1}^π which meet K^j in dimension $\le d_{i+1,j}^\pi$.

Now let h denote smooth bordism then we can define permutation bordism theory (denoted h^π) in direct analogy to permutation homology in any of the four ways listed above.

There are natural maps between the four definitions of h^π as follows (3) $\leftarrow$ (4) $\to$ (2) $\to$ (1). We shall see shortly that (4) $\to$ (3) is an isomorphism. There is no reason to expect that either of (4) $\to$ (2) $\to$ (1) are isomorphisms. To prove (2) $\to$ (1) is an isomorphism for homology the fact that homology groups vanish above the dimension of the complex is used; this is false for bordism. To prove that (4) $\to$ (2) is an isomorphism another fact special to homology is used, namely that a cycle can be assumed to be simplicial and hence a subcomplex. Again this is in general false for bordism. In favour of the two equivalent definitions (3) and (4) we have the following result.

Theorem 6.1 *Definitions* (3) *and* (4) *are equivalent for bordism and define a topological invariant of* X.

Sketch of proof Stratified homology general position (theorem 5.2) can be extended in two ways (1) replace π–stratifications by h^π–stratifications and SH^π by Sh^π (ie definition (3) above) and (2) delete the condition $\dim(Y \cap X_j) + d^\pi_{p,j} < j$ and alter the conclusion to get $\dim(f'(P') \cap Y \cap X_j) \leq \dim(Y \cap X_j) + d^\pi_{p,j} - j$. The proof is the same with obvious changes. This implies that a cycle in Sh^π can be assumed to meet K^j in the appropriate dimension by applying the theorem with $Y = K^j$ and then the usual argument (make disjoint from CK^π and push into K^π) yields a cycle in definition (4). A similar argument applies to a homology and this proves that definitions (3) and (4) coincide.

Topological invariance follows by applying this to the instrinsic h^π–stratification. □

Remarks 1) Definition (4) is briefly considered by Goresky and MacPherson in [4, problem 1]. They do not state topological invariance but they point out that the definition is unlikely to yield any form of Poincaré duality. In defence of the definition we would observe that ordinary bordism has no Poincaré duality for manifolds (there is a duality between bordism and cobordism but none between bordism groups of complementary dimension). Thus there is no reason to expect a definition which generalises bordism of a manifold (intersection homology generalises ordinary homology of a manifold) to satisfy Poincaré duality.

2) Let h be any connected generalised homology theory. Using the main result of [1] we can regard h as a generalised bordism theory (given by bordism classes of maps of suitable manifolds-with-singularity) and hence we can define permutation h–theory in analogy with permutation bordism as above. The analogue of the theorem is proved in exactly the same way. However it must be noted that this definition is dependent on the particular choice of representation for the theory as bordism with singularities (which in turn depends on a particular choice of CW structure for the spectrum). Thus this construction does not define h^π unambiguously.

7 Questions about homology stratifications

The following questions are asked in the spirit of a conference problem session. We have no clear idea how hard they are and indeed some may have simple answers which we failed to notice whilst writing them.

The simplest definition of homology stratification is given by using ordinary (integral) homology. Call such a stratification an *H–stratification*. Since, by the stable Whitehead theorem, a homology equivalence induces isomorphisms of all generalised homology groups, an H–stratification is an h–stratification for any generalised homology h. However this is not clear if h is intersection (ie allowable permutation) homology.

Question 1 *Is an H–stratification a π–stratification for allowable π? In other words, if the local homology groups are constant on strata, is the same true for local intersection homology groups?*

Question 1 is connected to the problem of characterising maps which induce isomorphisms of intersection homology groups in terms of ordinary homology. Here is a related question. We say that a map $f\colon X \to Y$ of filtered spaces (of dimensions n, m respectively) *respects* the filtration if $f^{-1}(Y_{m-k}) \subset X_{n-k}$ for each k. A map which respects the filtration induces a homomorphism $SH^\pi(X) \to SH^\sigma(Y)$, where π is a (repeated) reduction of σ or vice versa, (cf King [5; page 152]).

Question 2 *Suppose that $i\colon \bar{X} \subset \bar{Y}$ is an inclusion of filtered spaces which respects the filtration and induces isomorphisms of all ordinary homology groups for all strata and closures of strata. Does it follow that i induces isomorphism of intersection homology groups?*

Question 1 is also related to the problem of functoriality of intersection homology [4, problem 4]. Our main theorem gives an intrinsic definition of intersection homology namely singular permutation homology of the intrinsic π–stratification where π is the appropriate allowable permutation. By the remarks above question 2, a map which respects the intrinsic π–stratification induces a homomorphism $SH^\pi(X) \to SH^\sigma(Y)$. This is a somewhat circular characterisation of maps inducing homomorphisms of intersection homology, since they are characterised in terms of intersection homology; it is almost as circular as the characterisation given in [4, bottom of page 223]. If question 1 has a positive answer, then the characterisation becomes rather less circular: maps which respect the intrinsic H–stratification induce homomorphisms of intersection homology.

Question 3 *Is there a good geometric characterisation of maps which respect the intrinsic H–stratification? For example is it sensible to ask for a characterisation in terms of properties of point inverses?*

We have remarked that a locally trivial filtration with strata homology manifolds is an h–stratification for all h. The converse is easily seen to be false: glue three homology balls along a genuine ball in the boundary; the result is a homology stratification with the interior of the common boundary ball in one stratum, but is not necessarily locally trivial along that stratum. Indeed it is not clear that the strata of an H–stratification must be homology manifolds.

Question 4 *Are the strata of an H–stratification homology manifolds? Is the same true of a π–stratification for allowable π?*

We now turn to other (stronger) definitions of homology stratification. These all have the property that the strata are obviously homology manifolds. Goresky and MacPherson use a somewhat different definition of h–stratification. Their "canonical" $\bar{p}$–filtration [3, bottom of page 107] is defined exactly like our instrinsic h–stratification except that instead of our condition that $h_*^{\mathrm{loc}}(X)$ is locally constant on $X_j - X_{j-1}$ for each j they have two conditions: $h_*^{\mathrm{loc}}(X_j)$ *and* $h_*^{\mathrm{loc}}(X - X_j)$ are both locally constant on $X_j - X_{j-1}$ where h is intersection homology (the latter makes sense: they are using homology with infinite chains, the second local homology group is the same as $h_{*-1}(\mathrm{lk}(x, X) - \mathrm{lk}(x, X_j))$). The two conditions imply that $h_*^{\mathrm{loc}}(X)$ is locally constant. For ordinary homology if $h_*^{\mathrm{loc}}(X)$ and $h_*^{\mathrm{loc}}(X_j)$ are both locally constant then so is $h_*^{\mathrm{loc}}(X - X_j)$. For intersection homology this is not clear.

Definitions A *strong* h–stratification is one where $h_*^{\mathrm{loc}}(X)$ and $h_*^{\mathrm{loc}}(X_j)$ are both locally constant on $X_j - X_{j-1}$ for each j. A *GM–strong* h–stratification is one where $h_*^{\mathrm{loc}}(X - X_j)$ and $h_*^{\mathrm{loc}}(X_j)$ are both locally constant on $X_j - X_{j-1}$ for each j (this only makes sense for geometric theories for which the analogue of infinite chains is defined). A *very strong* h–stratification is one where $h_*^{\mathrm{loc}}(X_k)$ is locally constant on $X_j - X_{j-1}$ for each $k \geq j$.

Question 5 *What are the relationships between the definitions? Are the concepts of strong and GM–strong stratifications distinct? Are there examples of strong stratifications which are not very strong? Or indeed examples of stratifications which are not strong?*

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Received: 16 November 1998 Revised: 8 July 1999