*ISSN 1464-8997 (on line) 1464-8989 (printed)* **19**

Geometry & Topology Monographs
Volume 3: Invitation to higher local fields
Part I, section 2, pages 19–29

# 2. $p$-primary part of the Milnor $K$-groups and Galois cohomologies of fields of characteristic $p$

*Oleg Izhboldin*

## 2.0. Introduction

Let $F$ be a field and $F^{\mathrm{sep}}$ be the separable closure of $F$. Let $F^{\mathrm{ab}}$ be the maximal abelian extension of $F$. Clearly the Galois group $G^{\mathrm{ab}} = \mathrm{Gal}(F^{\mathrm{ab}}/F)$ is canonically isomorphic to the quotient of the absolute Galois group $G = \mathrm{Gal}(F^{\mathrm{sep}}/F)$ modulo the closure of its commutant. By Pontryagin duality, a description of $G^{\mathrm{ab}}$ is equivalent to a description of

$$\mathrm{Hom}_{\mathrm{cont}}\big(G^{\mathrm{ab}}, \mathbb{Z}/m\big) = \mathrm{Hom}_{\mathrm{cont}}\big(G, \mathbb{Z}/m\big) = H^1(F, \mathbb{Z}/m).$$

where $m$ runs over all positive integers. Clearly, it suffices to consider the case where $m$ is a power of a prime, say $m = p^i$. The main cohomological tool to compute the group $H^1(F, \mathbb{Z}/m)$ is a pairing

$$(\ ,\ )_m \colon H^1(F, \mathbb{Z}/m) \otimes K_n(F)/m \to H_m^{n+1}(F)$$

where the right hand side is a certain cohomological group discussed below.

Here $K_n(F)$ for a field $F$ is the $n$th Milnor $K$-group $K_n(F) = K_n^M(F)$ defined as

$$(F^*)^{\otimes n}/J$$

where $J$ is the subgroup generated by the elements of the form $a_1 \otimes \ldots \otimes a_n$ such that $a_i + a_j = 1$ for some $i \neq j$. We denote by $\{a_1, \ldots, a_n\}$ the class of $a_1 \otimes \ldots \otimes a_n$. Namely, $K_n(F)$ is the abelian group defined by the following
generators: symbols $\{a_1, \ldots, a_n\}$ with $a_1, \ldots, a_n \in F^*$
and relations:

$$\{a_1, \ldots, a_i a_i', \ldots a_n\} = \{a_1, \ldots, a_i, \ldots a_n\} + \{a_1, \ldots, a_i', \ldots a_n\}$$
$$\{a_1, \ldots, a_n\} = 0 \quad \text{if } a_i + a_j = 1 \text{ for some } i \text{ and } j \text{ with } i \neq j.$$

We write the group law additively.

Consider the following example (definitions of the groups will be given later).

**Example.** Let $F$ be a field and let $p$ be a prime integer. Assume that there is an integer $n$ with the following properties:
(i) the group $H_p^{n+1}(F)$ is isomorphic to $\mathbb{Z}/p$,
(ii) the pairing

$$( \, , \, )_p \colon H^1(F,\mathbb{Z}/p)\otimes K_n(F)/p \to H_p^{n+1}(F)\simeq \mathbb{Z}/p$$

is non-degenerate in a certain sense.
Then the $\mathbb{Z}/p$-linear space $H^1(F,\mathbb{Z}/p)$ is obviously dual to the $\mathbb{Z}/p$-linear space $K_n(F)/p$. On the other hand, $H^1(F,\mathbb{Z}/p)$ is dual to the $\mathbb{Z}/p$-space $G^{\mathrm{ab}}/(G^{\mathrm{ab}})^p$. Therefore there is an isomorphism

$$\Psi_{F,p}\colon K_n(F)/p \simeq G^{\mathrm{ab}}/(G^{\mathrm{ab}})^p.$$

It turns out that this example can be applied to computations of the group $G^{\mathrm{ab}}/(G^{\mathrm{ab}})^p$ for multidimensional local fields. Moreover, it is possible to show that the homomorphism $\Psi_{F,p}$ can be naturally extended to a homomorphism $\Psi_F\colon K_n(F)\to G^{\mathrm{ab}}$ (the so called reciprocity map). Since $G^{\mathrm{ab}}$ is a profinite group, it follows that the homomorphism $\Psi_F\colon K_n(F)\to G^{\mathrm{ab}}$ factors through the homomorphism $K_n(F)/DK_n(F)\to G^{\mathrm{ab}}$ where the group $DK_n(F)$ consists of all divisible elements:

$$DK_n(F) := \cap_{m\geqslant 1} mK_n(F).$$

This observation makes natural the following notation:

**Definition** (cf. section 6 of Part I)**.** For a field $F$ and integer $n\geqslant 0$ set

$$K_n^t(F) := K_n(F)/DK_n(F),$$

where $DK_n(F) := \bigcap_{m\geqslant 1} mK_n(F)$.

The group $K_n^t(F)$ for a higher local field $F$ endowed with a certain topology (cf. section 6 of this part of the volume) is called a topological Milnor $K$-group $K^{\mathrm{top}}(F)$ of $F$.

The example shows that computing the group $G^{\mathrm{ab}}$ is closely related to computing the groups $K_n(F)$, $K_n^t(F)$, and $H_m^{n+1}(F)$. The main purpose of this section is to explain some basic properties of these groups and discuss several classical conjectures. Among the problems, we point out the following:

- discuss $p$-torsion and cotorsion of the groups $K_n(F)$ and $K_n^t(F)$,
- study an analogue of Satz 90 for the groups $K_n(F)$ and $K_n^t(F)$,
- compute the group $H_m^{n+1}(F)$ in two “classical” cases where $F$ is either the rational function field in one variable $F=k(t)$ or the formal power series $F=k((t))$.

We shall consider in detail the case (so called “non-classical case”) of a field $F$ of characteristic $p$ and $m=p$.

## 2.1. Definition of $H_m^{n+1}(F)$ and pairing $(\ ,\ )_m$

To define the group $H_m^{n+1}(F)$ we consider three cases depending on the characteristic of the field $F$.

**Case 1** (Classical)**.** Either $\operatorname{char}(F) = 0$ or $\operatorname{char}(F) = p$ is prime to $m$.

In this case we set

$$H_m^{n+1}(F) := H^{n+1}(F, \mu_m^{\otimes n}).$$

The Kummer theory gives rise to the well known natural isomorphism $F^*/F^{*m} \to H^1(F, \mu_m)$. Denote the image of an element $a \in F^*$ under this isomorphism by $(a)$. The cup product gives the homomorphism

$$\underbrace{F^* \otimes \cdots \otimes F^*}_{n} \to H^n(F, \mu_m^{\otimes n}), \qquad a_1 \otimes \cdots \otimes a_n \to (a_1, \ldots, a_n)$$

where $(a_1, \ldots, a_n) := (a_1) \cup \cdots \cup (a_n)$. It is well known that the element $(a_1, \ldots, a_n)$ is zero if $a_i + a_j = 1$ for some $i \neq j$. From the definition of the Milnor $K$-group we get the homomorphism

$$\eta_m \colon K_n^M(F)/m \to H^n(F, \mu_m^{\otimes n}), \qquad \{a_1, \ldots, a_n\} \to (a_1, \ldots, a_n).$$

Now, we define the pairing $(\ ,\ )_m$ as the following composite

$$H^1(F, \mathbb{Z}/m) \otimes K_n(F)/m \xrightarrow{\text{id} \otimes \eta_m} H^1(F, \mathbb{Z}/m) \otimes H^n(F, \mu_m^{\otimes n}) \xrightarrow{\cup} H_m^{n+1}(F, \mu_m^{\otimes n}).$$

**Case 2.** $\operatorname{char}(F) = p \neq 0$ and $m$ is a power of $p$.

To simplify the exposition we start with the case $m = p$. Set

$$H_p^{n+1}(F) = \operatorname{coker}\bigl(\Omega_F^n \xrightarrow{\wp} \Omega_F^n / d\Omega_F^{n-1}\bigr)$$

where

$$d(a\,db_2 \wedge \cdots \wedge db_n) = da \wedge db_2 \wedge \cdots \wedge db_n,$$
$$\wp\Bigl(a\frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_n}{b_n}\Bigr) = (a^p - a)\,\frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_n}{b_n} + d\Omega_F^{n-1}$$

($\wp = \mathrm{C}^{-1} - 1$ where $\mathrm{C}^{-1}$ is the inverse Cartier operator defined in subsection 4.2). The pairing $(\ ,\ )_p$ is defined as follows:

$$(\ ,\ )_p \colon F/\wp(F) \times K_n(F)/p \to H_p^{n+1}(F),$$
$$(a, \{b_1, \ldots, b_n\}) \mapsto a\frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_n}{b_n}$$

where $F/\wp(F)$ is identified with $H^1(F, \mathbb{Z}/p)$ via Artin–Schreier theory.

To define the group $H_{p^i}^{n+1}(F)$ for an arbitrary $i \geqslant 1$ we note that the group $H_p^{n+1}(F)$ is the quotient group of $\Omega_F^n$. In particular, generators of the group $H_p^{n+1}(F)$ can be written in the form $a db_1 \wedge \cdots \wedge db_n$. Clearly, the natural homomorphism

$$F \otimes \underbrace{F^* \otimes \cdots \otimes F^*}_{n} \to H_p^{n+1}(F), \qquad a \otimes b_1 \otimes \cdots \otimes b_n \mapsto a\frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_n}{b_n}$$

is surjective. Therefore the group $H_p^{n+1}(F)$ is naturally identified with the quotient group $F \otimes F^* \otimes \cdots \otimes F^*/J$. It is not difficult to show that the subgroup $J$ is generated by the following elements:

$(a^p - a) \otimes b_1 \otimes \cdots \otimes b_n$,
$a \otimes a \otimes b_2 \otimes \cdots \otimes b_n$,
$a \otimes b_1 \otimes \cdots \otimes b_n$, where $b_i = b_j$ for some $i \neq j$.

This description of the group $H_p^{n+1}(F)$ can be easily generalized to define $H_{p^i}^{n+1}(F)$ for an arbitrary $i \geqslant 1$. Namely, we define the group $H_{p^i}^{n+1}(F)$ as the quotient group

$$W_i(F) \otimes \underbrace{F^* \otimes \cdots \otimes F^*}_{n} / J$$

where $W_i(F)$ is the group of Witt vectors of length $i$ and $J$ is the subgroup of $W_i(F) \otimes F^* \otimes \cdots \otimes F^*$ generated by the following elements:

$(\mathbf{F}(w) - w) \otimes b_1 \otimes \cdots \otimes b_n$,
$(a, 0, \ldots, 0) \otimes a \otimes b_2 \otimes \cdots \otimes b_n$,
$w \otimes b_1 \otimes \cdots \otimes b_n$, where $b_i = b_j$ for some $i \neq j$.

The pairing $(\ ,\ )_{p^i}$ is defined as follows:

$$(\ ,\ )_p \colon W_i(F)/\wp(W_i(F)) \times K_n(F)/p^i \to H_{p^i}^{n+1}(F),$$
$$(w, \{b_1, \ldots, b_n\}) \mapsto w \otimes b_1 \otimes \cdots \otimes b_n$$

where $\wp = \mathbf{F} - \mathrm{id} \colon W_i(F) \to W_i(F)$ and the group $W_i(F)/\wp(W_i(F))$ is identified with $H^1(F, \mathbb{Z}/p^i)$ via Witt theory. This completes definitions in Case 2.

**Case 3.** $\mathrm{char}\,(F) = p \neq 0$ and $m = m'p^i$ where $m' > 1$ is an integer prime to $p$ and $i \geqslant 1$.

The groups $H_{m'}^{n+1}(F)$ and $H_{p^i}^{n+1}(F)$ are already defined (see Cases 1 and 2). We define the group $H_m^{n+1}(F)$ by the following formula:

$$H_m^{n+1}(F) := H_{m'}^{n+1}(F) \oplus H_{p^i}^{n+1}(F)$$

Since $H^1(F, \mathbb{Z}/m) \simeq H^1(F, \mathbb{Z}/m') \oplus H^1(F, \mathbb{Z}/p^i)$ and $K_n(F)/m \simeq K_n(F)/m' \oplus K_n(F)/p^i$, we can define the pairing $(\ ,\ )_m$ as the direct sum of the pairings $(\ ,\ )_{m'}$ and $(\ ,\ )_{p^i}$. This completes the definition of the group $H_m^{n+1}(F)$ and of the pairing $(\ ,\ )_m$.

**Remark 1.** In the case $n = 1$ or $n = 2$ the group $H_m^n(F)$ can be determined as follows:

$$H_m^1(F) \simeq H^1(F, \mathbb{Z}/m) \qquad \text{and} \qquad H_m^2(F) \simeq {}_m\,\mathrm{Br}(F).$$

**Remark 2.** The group $H_m^{n+1}(F)$ is often denoted by $H^{n+1}(F, \mathbb{Z}/m\,(n))$.

## 2.2. The group $H^{n+1}(F)$

In the previous subsection we defined the group $H_m^{n+1}(F)$ and the pairing $(\ ,\ )_m$ for an arbitrary $m$. Now, let $m$ and $m'$ be positive integers such that $m'$ is divisible by $m$. In this case there exists a canonical homomorphism

$$i_{m,m'}: H_m^{n+1}(F) \to H_{m'}^{n+1}(F).$$

To define the homomorphism $i_{m,m'}$ it suffices to consider the following two cases:

**Case 1.** Either $\mathrm{char}\,(F) = 0$ or $\mathrm{char}\,(F) = p$ is prime to $m$ and $m'$.

This case corresponds to Case 1 in the definition of the group $H_m^{n+1}(F)$ (see subsection 2.1). We identify the homomorphism $i_{m,m'}$ with the homomorphism

$$H^{n+1}(F, \mu_m^{\otimes n}) \to H^{n+1}(F, \mu_{m'}^{\otimes n})$$

induced by the natural embedding $\mu_m \subset \mu_{m'}$.

**Case 2.** $m$ and $m'$ are powers of $p = \mathrm{char}\,(F)$.

We can assume that $m = p^i$ and $m' = p^{i'}$ with $i \leqslant i'$. This case corresponds to Case 2 in the definition of the group $H_m^{n+1}(F)$. We define $i_{m,m'}$ as the homomorphism induced by

$$W_i(F) \otimes F^* \otimes \dots F^* \to W_{i'}(F) \otimes F^* \otimes \dots F^*,$$
$$(a_1, \dots, a_i) \otimes b_1 \otimes \dots \otimes b_n \mapsto (0, \dots, 0, a_1, \dots, a_i) \otimes b_1 \otimes \dots \otimes b_n.$$

The maps $i_{m,m'}$ (where $m$ and $m'$ run over all integers such that $m'$ is divisible by $m$) determine the inductive system of the groups.

**Definition.** For a field $F$ and an integer $n$ set

$$H^{n+1}(F) = \varinjlim_m H_m^{n+1}(F).$$

**Conjecture 1.** *The natural homomorphism $H_m^{n+1}(F) \to H^{n+1}(F)$ is injective and the image of this homomorphism coincides with the $m$-torsion part of the group $H^{n+1}(F)$.*

This conjecture follows easily from the Milnor–Bloch–Kato conjecture (see subsection 4.1) in degree $n$. In particular, it is proved for $n \leqslant 2$. For fields of characteristic $p$ we have the following theorem.

**Theorem 1.** *Conjecture 1 is true if* $\mathrm{char}(F) = p$ *and* $m = p^i$.

## 2.3. Computing the group $H_m^{n+1}(F)$ for some fields

We start with the following well known result.

**Theorem 2** (classical)**.** *Let $F$ be a perfect field. Suppose that* $\mathrm{char}(F) = 0$ *or* $\mathrm{char}(F)$ *is prime to $m$. Then*

$$H_m^{n+1}\big(F((t))\big) \simeq H_m^{n+1}(F) \oplus H_m^n(F)$$
$$H_m^{n+1}\big(F(t)\big) \simeq H_m^{n+1}(F) \oplus \coprod_{\text{monic irred } f(t)} H_m^n\big(F[t]/f(t)\big).$$

It is known that we cannot omit the conditions on $F$ and $m$ in the statement of Theorem 2. To generalize the theorem to the arbitrary case we need the following notation. For a complete discrete valuation field $K$ and its maximal unramified extension $K_{\mathrm{ur}}$ define the groups $H^n_{m,\mathrm{ur}}(K)$ and $\widetilde{H}^n_m(K)$ as follows:

$$H^n_{m,\mathrm{ur}}(K) = \ker\big(H^n_m(K) \to H^n_m(K_{\mathrm{ur}})\big) \qquad \text{and} \qquad \widetilde{H}^n_m(K) = H^n_m(K)/H^n_{m,\mathrm{ur}}(K).$$

Note that for a field $K = F((t))$ we obviously have $K_{\mathrm{ur}} = F^{\mathrm{sep}}((t))$. We also note that under the hypotheses of Theorem 2 we have $H^n(K) = H^n_{m,\mathrm{ur}}(K)$ and $H^n(K) = 0$. The following theorem is due to Kato.

**Theorem 3** (Kato, [K1, Th. 3 §0])**.** *Let $K$ be a complete discrete valuation field with residue field $k$. Then*

$$H^{n+1}_{m,\mathrm{ur}}(K) \simeq H^{n+1}_m(k) \oplus H^n_m(k).$$

*In particular,* $H^{n+1}_{m,\mathrm{ur}}(F((t))) \simeq H^{n+1}_m(F) \oplus H^n_m(F)$.

This theorem plays a key role in Kato's approach to class field theory of multidimensional local fields (see section 5 of this part).

To generalize the second isomorphism of Theorem 2 we need the following notation. Set

$$H^{n+1}_{m,\mathrm{sep}}(F(t)) = \ker\big(H^{n+1}_m(F(t)) \to H^{n+1}_m(F^{\mathrm{sep}}(t))\big) \text{ and}$$
$$\widetilde{H}^{n+1}_m(F(t)) = H^{n+1}_m(F(t))/H^{n+1}_{m,\mathrm{sep}}(F(t)).$$

If the field $F$ satisfies the hypotheses of Theorem 2, we have $H^{n+1}_{m,\mathrm{sep}}(F(t)) = H^{n+1}_m(F(t))$ and $\widetilde{H}^{n+1}_m(F(t)) = 0$.

In the general case we have the following statement.

**Theorem 4** (Izhboldin, [I2, Introduction]).

$$H^{n+1}_{m,\text{sep}}(F(t)) \simeq H^{n+1}_m(F) \oplus \coprod_{\text{monic irred } f(t)} H^n_m\left(F[t]/f(t)\right),$$
$$\widetilde{H}^{n+1}_m(F(t)) \simeq \coprod_v \widetilde{H}^{n+1}_m(F(t)_v)$$

*where $v$ runs over all normalized discrete valuations of the field $F(t)$ and $F(t)_v$ denotes the $v$-completion of $F(t)$.*

## 2.4. On the group $K_n(F)$

In this subsection we discuss the structure of the torsion and cotorsion in Milnor $K$-theory. For simplicity, we consider the case of prime $m = p$. We start with the following fundamental theorem concerning the quotient group $K_n(F)/p$ for fields of characteristic $p$.

**Theorem 5** (Bloch–Kato–Gabber, [BK, Th. 2.1]). *Let $F$ be a field of characteristic $p$. Then the differential symbol*

$$d_F\colon K_n(F)/p \to \Omega^n_F, \qquad \{a_1, \dots, a_n\} \mapsto \frac{da_1}{a_1} \wedge \dots \wedge \frac{da_n}{a_n}$$

*is injective and its image coincides with the kernel $\nu_n(F)$ of the homomorphism $\wp$ (for the definition see Case 2 of 2.1). In other words, the sequence*

$$0 \longrightarrow K_n(F)/p \xrightarrow{d_F} \Omega^n_F \xrightarrow{\wp} \Omega^n_F/d\Omega^{n-1}_F$$

*is exact.*

This theorem relates the Milnor $K$-group modulo $p$ of a field of characteristic $p$ with a submodule of the differential module whose structure is easier to understand. The theorem is important for Kato's approach to higher local class field theory. For a sketch of its proof see subsection A2 in the appendix to this section.

There exists a natural generalization of the above theorem for the quotient groups $K_n(F)/p^i$ by using De Rham–Witt complex ([BK, Cor. 2.8]).

Now, we recall well known Tate's conjecture concerning the torsion subgroup of the Milnor $K$-groups.

**Conjecture 2** (Tate). *Let $F$ be a field and $p$ be a prime integer.*
(i) *If* $\operatorname{char}(F) \neq p$ *and* $\zeta_p \in F$, *then* ${}_pK_n(F) = \{\zeta_p\} \cdot K_{n-1}(F)$.
(ii) *If* $\operatorname{char}(F) = p$ *then* ${}_pK_n(F) = 0$.

This conjecture is trivial in the case where $n \leqslant 1$. In the other cases we have the following theorem.

**Theorem 6.** *Let $F$ be a field and $n$ be a positive integer.*
(1) *Tate's Conjecture holds if $n \leqslant 2$ (Suslin, [S]),*
(2) *Part (ii) of Tate's Conjecture holds for all $n$ (Izhboldin, [I1]).*

The proof of this theorem is closely related to the proof of Satz 90 for $K$-groups. Let us recall two basic conjectures on this subject.

**Conjecture 3** (Satz 90 for $K_n$)**.** *If $L/F$ is a cyclic extension of degree $p$ with the Galois group $G = \langle\sigma\rangle$ then the sequence*

$$K_n(L) \xrightarrow{1-\sigma} K_n(L) \xrightarrow{N_{L/F}} K_n(F)$$

*is exact.*

There is an analogue of the above conjecture for the quotient group $K_n(F)/p$. Fix the following notation till the end of this section:

**Definition.** For a field $F$ set

$$k_n(F) = K_n(F)/p.$$

**Conjecture 4** (Small Satz 90 for $k_n$)**.** *If $L/F$ is a cyclic extension of degree $p$ with the Galois group $G = \langle\sigma\rangle$, then the sequence*

$$k_n(F) \oplus k_n(L) \xrightarrow{i_{F/L} \oplus (1-\sigma)} k_n(L) \xrightarrow{N_{L/F}} k_n(F)$$

*is exact.*

The conjectures 2,3 and 4 are not independent:

**Lemma** (Suslin)**.** *Fix a prime integer $p$ and integer $n$. Then in the category of all fields* (*of a given characteristic*) *we have*

(*Small Satz 90 for $k_n$*) + (*Tate conjecture for ${}_pK_n$*) $\iff$ (*Satz 90 for $K_n$*).

*Moreover, for a given field $F$ we have*

(*Small Satz 90 for $k_n$*) + (*Tate conjecture for ${}_pK_n$*) $\Rightarrow$ (*Satz 90 for $K_n$*)

*and*

(*Satz 90 for $K_n$*) $\Rightarrow$ (*small Satz 90 for $k_n$*).

Satz 90 conjectures are proved for $n \leqslant 2$ (Merkurev-Suslin, [MS1]). If $p = 2$, $n = 3$, and $\mathrm{char}(F) \neq 2$, the conjectures were proved by Merkurev and Suslin [MS] and Rost. For $p = 2$ the conjectures follow from recent results of Voevodsky. For fields of characteristic $p$ the conjectures are proved for all $n$:

**Theorem 7** (Izhboldin, [I1]). *Let $F$ be a field of characteristic $p$ and $L/F$ be a cyclic extension of degree $p$. Then the following sequence is exact:*

$$0 \to K_n(F) \to K_n(L) \xrightarrow{1-\sigma} K_n(L) \xrightarrow{N_{L/F}} K_n(F) \to H_p^{n+1}(F) \to H_p^{n+1}(L)$$

## 2.5. On the group $K_n^t(F)$

In this subsection we discuss the same issues, as in the previous subsection, for the group $K_n^t(F)$.

**Definition.** Let $F$ be a field and $p$ be a prime integer. We set

$$DK_n(F) = \bigcap_{m\geqslant 1} mK_n(F) \quad \text{and} \quad D_pK_n(F) = \bigcap_{i\geqslant 0} p^iK_n(F).$$

We define the group $K_n^t(F)$ as the quotient group:

$$K_n^t(F) = K_n(F)/DK_n(F) = K_n(F)/\bigcap_{m\geqslant 1} mK_n(F).$$

The group $K_n^t(F)$ is of special interest for higher class field theory (see sections 6, 7 and 10). We have the following evident isomorphism (see also 2.0):

$$K_n^t(F) \simeq \operatorname{im}\left(K_n(F) \to \varprojlim_m K_n(F)/m\right).$$

The quotient group $K_n^t(F)/m$ is obviously isomorphic to the group $K_n(F)/m$. As for the torsion subgroup of $K_n^t(F)$, it is quite natural to state the same questions as for the group $K_n(F)$.

**Question 1.** Are the $K^t$-analogue of Tate's conjecture and Satz 90 Conjecture true for the group $K_n^t(F)$?

If we know the (positive) answer to the corresponding question for the group $K_n(F)$, then the previous question is equivalent to the following:

**Question 2.** Is the group $DK_n(F)$ divisible?

At first sight this question looks trivial because the group $DK_n(F)$ consists of all divisible elements of $K_n(F)$. However, the following theorem shows that the group $DK_n(F)$ is not necessarily a divisible group!

**Theorem 8** (Izhboldin, [I3]). *For every $n \geqslant 2$ and prime $p$ there is a field $F$ such that* $\operatorname{char}(F) \neq p$, $\zeta_p \in F$ *and*

(1) *The group $DK_n(F)$ is not divisible, and the group $D_pK_2(F)$ is not $p$-divisible,*

(2) *The* $K^t$*-analogue of Tate's conjecture is false for* $K_n^t$*:*

$$_pK_n^t(F) \neq \{\zeta_p\} \cdot K_{n-1}^t(F).$$

(3) *The* $K^t$*-analogue of Hilbert 90 conjecture is false for group* $K_n^t(F)$*.*

**Remark 1.** The field $F$ satisfying the conditions of Theorem 8 can be constructed as the function field of some infinite dimensional variety over any field of characteristic zero whose group of roots of unity is finite.

Quite a different construction for irregular prime numbers $p$ and $F = \mathbb{Q}(\mu_p)$ follows from works of G. Banaszak [B].

**Remark 2.** If $F$ is a field of characteristic $p$ then the groups $D_pK_n(F)$ and $DK_n(F)$ are $p$-divisible. This easily implies that $_pK_n^t(F) = 0$. Moreover, Satz 90 theorem holds for $K_n^t$ in the case of cyclic $p$-extensions.

**Remark 3.** If $F$ is a multidimensional local fields then the group $K_n^t(F)$ is studied in section 6 of this volume. In particular, Fesenko (see subsections 6.3–6.8 of section 6) gives positive answers to Questions 1 and 2 for multidimensional local fields.

## References

[B] G. Banaszak, Generalization of the Moore exact sequence and the wild kernel for higher $K$-groups, Compos. Math., 86(1993), 281–305.

[BK] S. Bloch and K. Kato, $p$-adic étale cohomology, Inst. Hautes Études Sci. Publ. Math. 63, (1986), 107–152.

[F] I. Fesenko, Topological Milnor $K$-groups of higher local fields, section 6 of this volume.

[I1] O. Izhboldin, On $p$-torsion in $K_*^M$ for fields of characteristic $p$, Adv. Soviet Math., vol. 4, 129–144, Amer. Math. Soc., Providence RI, 1991

[I2] O. Izhboldin, On the cohomology groups of the field of rational functions, Mathematics in St.Petersburg, 21–44, Amer. Math. Soc. Transl. Ser. 2, vol. 174, Amer. Math. Soc., Providence, RI, 1996.

[I3] O. Izhboldin, On the quotient group of $K_2(F)$, preprint, www.maths.nott.ac.uk/personal/ibf/stqk.ps

[K1] K. Kato, Galois cohomology of complete discrete valuation fields, In Algebraic $K$-theory, Lect. Notes in Math. 967, Springer-Verlag, Berlin, 1982, 215–238.

[K2] K. Kato, Symmetric bilinear forms, quadratic forms and Milnor $K$-theory in characteristic two, Invent. Math. 66(1982), 493–510.

[MS1] A. S. Merkur'ev and A. A. Suslin, $K$-cohomology of Severi-Brauer varieties and the norm residue homomorphism, Izv. Akad. Nauk SSSR Ser. Mat. 46(1982); English translation in Math. USSR Izv. 21(1983), 307–340.

[MS2] A. S. Merkur'ev and A. A. Suslin, The norm residue homomorphism of degree three, Izv. Akad. Nauk SSSR Ser. Mat. 54(1990); English translation in Math. USSR Izv. 36(1991), 349–367.

[MS3] A. S. Merkur'ev and A. A. Suslin, The group $K_3$ for a field, Izv. Akad. Nauk SSSR Ser. Mat. 54(1990); English translation in Math. USSR Izv. 36(1991), 541–565.

[S] A. A. Suslin, Torsion in $K_2$ of fields, K-theory 1(1987), 5–29.