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2. Adelic constructions for direct images of differentials and symbols

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2.0. Introduction

Let X be a smooth algebraic surface over a perfect field k.

Consider pairs $x \in C$, x is a closed point of X, C is either an irreducible curve on X which is smooth at x, or an irreducible analytic branch near x of an irreducible curve on X. As in the previous section 1 for every such pair $x \in C$ we get a two-dimensional local field $K_{x,C}$.

If X is a projective surface, then from the adelic description of Serre duality on X there is a local decomposition for the trace map $H^2(X, \Omega^2_X) \to k$ by using a two-dimensional residue map $\operatorname{res}_{K_{x,C}/k(x)}: \Omega^2_{K_{x,C}/k(x)} \to k(x)$ (see [P1]).

From the adelic interpretation of the divisors intersection index on X there is a similar local decomposition for the global degree map from the group $CH^2(X)$ of algebraic cycles of codimension 2 on X modulo the rational equivalence to $\mathbb{Z}$ by means of explicit maps from $K_2(K_{x,C})$ to $\mathbb{Z}$ (see [P3]).

Now we pass to the relative situation. Further assume that X is any smooth surface, but there are a smooth curve S over k and a smooth projective morphism $f: X \to S$ with connected fibres. Using two-dimensional local fields and explicit maps we describe in this section a local decomposition for the maps

$$f_*: H^n(X, \Omega^2_X) \to H^{n-1}(S, \Omega^1_S), \quad f_*: H^n(X, \mathcal{K}_2(X)) \to H^{n-1}(S, \mathcal{K}_1(S))$$

where $\mathcal{K}$ is the Zariski sheaf associated to the presheaf $U \to K(U)$. The last two groups have the following geometric interpretation:

$$H^n(X, \mathcal{K}_2(X)) = CH^2(X, 2-n), \quad H^{n-1}(S, \mathcal{K}_1(S)) = CH^1(S, 2-n)$$

where $CH^2(X, 2-n)$ and $CH^1(S, 1-n)$ are higher Chow groups on X and S (see [B]). Note also that $CH^2(X,0) = CH^2(X)$, $CH^1(S,0) = CH^1(S) = \operatorname{Pic}(S)$, $CH^1(S,1) = H^0(S, \mathcal{O}^*_S)$.

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Let $s = f(x) \in S$. There is a canonical embedding $f^*\colon K_s \to K_{x,C}$ where K_s is the quotient of the completion of the local ring of S at s.

Consider two cases:

(1) $C \neq f^{-1}(s)$. Then $K_{x,C}$ is non-canonically isomorphic to $k(C)_x((t_C))$ where $k(C)_x$ is the completion of $k(C)$ at x and t_C is a local equation of C near x.

(2) $C = f^{-1}(s)$. Then $K_{x,C}$ is non-canonically isomorphic to $k(x)((u))((t_s))$ where $\{u = 0\}$ is a transversal curve at x to $f^{-1}(s)$ and $t_s \in K_s$ is a local parameter at s, i.e. $k(s)((t_s)) = K_s$.

2.1. Local constructions for differentials

Definition. For $K = k((u))((t))$ let $U = u^i k[[u,t]]dk[[u,t]] + t^j k((u))[[t]]dk((u))[[t]]$ be a basis of neighbourhoods of zero in $\Omega^1_{k((u))[[t]]/k}$ (compare with 1.4.1 of Part I). Let $\widetilde{\Omega}^1_K = \Omega^1_{K/k}/(K \cdot \cap U)$ and $\widetilde{\Omega}^n_K = \wedge^n \widetilde{\Omega}^1_K$. Similarly define $\widetilde{\Omega}^n_{K_s}$.

Note that $\widetilde{\Omega}^2_{K_{x,C}}$ is a one-dimensional space over $K_{x,C}$; and $\widetilde{\Omega}^n_{K_{x,C}}$ does not depend on the choice of a system of local parameters of $\widehat{\mathcal{O}}_x$, where $\widehat{\mathcal{O}}_x$ is the completion of the local ring of X at x.

Definition. For $K = k((u))((t))$ and $\omega = \sum_i \omega_i(u) \wedge t^i dt = \sum_i u^i du \wedge \omega_i'(t) \in \widetilde{\Omega}^2_K$ put

$$\operatorname{res}_t(\omega) = \omega_{-1}(u) \in \widetilde{\Omega}^1_{k((u))},$$
$$\operatorname{res}_u(\omega) = \omega'_{-1}(t) \in \widetilde{\Omega}^1_{k((t))}.$$

Define a relative residue map

$$f_*^{x,C}\colon \widetilde{\Omega}^2_{K_{x,C}} \to \widetilde{\Omega}^1_{K_s}$$

as

$$f_*^{x,C}(\omega) = \begin{cases} \operatorname{Tr}_{k(C)_x/K_s} \operatorname{res}_{t_C}(\omega) & \text{if } C \neq f^{-1}(s) \\ \operatorname{Tr}_{k(x)((t_s))/K_s} \operatorname{res}_u(\omega) & \text{if } C = f^{-1}(s). \end{cases}$$

The relative residue map doesn't depend on the choice of local parameters.

Theorem (reciprocity laws for relative residues). *Fix $x \in X$. Let $\omega \in \widetilde{\Omega}^2_{K_x}$ where K_x is the minimal subring of $K_{x,C}$ which contains $k(X)$ and $\widehat{\mathcal{O}}_x$. Then*

$$\sum_{C \ni x} f_*^{x,C}(\omega) = 0.$$

Fix $s \in S$*. Let* $\omega \in \widetilde{\Omega}^2_{K_F}$ *where* K_F *is the completion of* $k(X)$ *with respect to the discrete valuation associated with the curve* $F = f^{-1}(s)$*. Then*

$$\sum_{x \in F} f_*^{x,F}(\omega) = 0.$$

See [O].

2.2. The Gysin map for differentials

Definition. In the notations of subsection 1.2.1 in the previous section put

$$\Omega^1_{\mathbb{A}_S} = \{(f_s dt_s) \in \prod_{s \in S} \widetilde{\Omega}^1_{K_s}, \quad v_s(f_s) \geqslant 0 \text{ for almost all } s \in S\}$$

where t_s is a local parameter at s, v_s is the discrete valuation associated to t_s and K_s is the quotient of the completion of the local ring of S at s. For a divisor I on S define

$$\Omega_{\mathbb{A}_S}(I) = \{(f_s) \in \Omega^1_{\mathbb{A}_S} : v_s(f_s) \geqslant -v_s(I) \quad \text{for all } s \in S\}.$$

Recall that the n-th cohomology group of the following complex

$$\begin{array}{ccc} \Omega^1_{k(S)/k} \oplus \Omega^1_{\mathbb{A}_S}(0) & \longrightarrow & \Omega^1_{\mathbb{A}_S} \\ (f_0, f_1) & \longmapsto & f_0 + f_1. \end{array}$$

is canonically isomorphic to $H^n(S, \Omega^1_S)$ (see [S, Ch.II]).

The sheaf Ω^2_X is invertible on X. Therefore, Parshin's theorem (see [P1]) shows that similarly to the previous definition and definition in 1.2.2 of the previous section for the complex $\Omega^2(\mathcal{A}_X)$

$$\begin{array}{ccccc} \Omega^2_{A_0} \oplus \Omega^2_{A_1} \oplus \Omega^2_{A_2} & \longrightarrow & \Omega^2_{A_{01}} \oplus \Omega^2_{A_{02}} \oplus \Omega^2_{A_{12}} & \longrightarrow & \Omega^2_{A_{012}} \\ (f_0, f_1, f_2) & \longmapsto & (f_0 + f_1, f_2 - f_0, -f_1 - f_2) & & \\ & & (g_1, g_2, g_3) & \longmapsto & g_1 + g_2 + g_3 \end{array}$$

where

$$\Omega^2_{A_i} \subset \Omega^2_{A_{ij}} \subset \Omega^2_{A_{012}} = \Omega^2_{\mathbb{A}_X} = {\prod_{x \in C}}' \widetilde{\Omega}^2_{K_{x,C}} \subset \prod_{x \in C} \widetilde{\Omega}^2_{K_{x,C}}$$

there is a canonical isomorphism

$$H^n(\Omega^2(\mathcal{A}_X)) \simeq H^n(X, \Omega^2_X).$$

Using the reciprocity laws above one can deduce:

Theorem. *The map* $f_* = \sum_{C\ni x, f(x)=s} f_*^{x,C}$ *from* $\Omega^2_{\mathbb{A}_X}$ *to* $\Omega^1_{\mathbb{A}_S}$ *is well defined. It maps the complex* $\Omega^2(\mathcal{A}_X)$ *to the complex*

$$0 \longrightarrow \Omega^1_{k(S)/k} \oplus \Omega^1_{\mathbb{A}_S}(0) \longrightarrow \Omega^1_{\mathbb{A}_S}.$$

It induces the map $f_*\colon H^n(X,\Omega^2_X) \to H^{n-1}(S,\Omega^1_S)$ *of 2.0.*

See [O].

2.3. Local constructions for symbols

Assume that k is of characteristic 0.

Theorem. *There is an explicitly defined symbolic map*

$$f_*(\ ,\ )_{x,C}\colon K^*_{x,C} \times K^*_{x,C} \to K^*_s$$

(*see remark below*) *which is uniquely determined by the following properties*

$$N_{k(x)/k(s)}\, t_{K_{x,C}}(\alpha,\beta,f^*\gamma) = t_{K_s}(f_*(\alpha,\beta)_{x,C},\gamma) \quad \textit{for all } \alpha,\beta\in K^*_{x,C},\ \gamma\in K^*_s$$

where $t_{K_{x,C}}$ *is the tame symbol of the two-dimensional local field* $K_{x,C}$ *and* t_{K_s} *is the tame symbol of the one-dimensional local field* K_s (*see 6.4.2 of Part I*);

$$\mathrm{Tr}_{\,k(x)/k(s)}(\alpha,\beta,f^*(\gamma)]_{K_{x,C}} = (f_*(\alpha,\beta)_{x,C},\gamma]_{K_s} \quad \textit{for all } \alpha,\beta\in K^*_{x,C},\ \gamma\in K_s$$

where $(\alpha,\beta,\gamma]_{K_{x,C}} = \mathrm{res}_{K_{x,C}/k(x)}(\gamma d\alpha/\alpha\wedge d\beta/\beta)$ *and*
$(\alpha,\beta]_{K_s} = \mathrm{res}_{K_s/k(s)}(\alpha d\beta/\beta)$.

The map $f_*(\ ,\ )_{x,C}$ induces the map

$$f_*(\ ,\ )_{x,C}\colon K_2(K_{x,C}) \to K_1(K_s).$$

Corollary (reciprocity laws)**.** *Fix a point* $s\in S$. *Let* $F = f^{-1}(s)$.
Let $\alpha,\beta\in K^*_F$. *Then*

$$\prod_{x\in F} f_*(\alpha,\beta)_{x,F} = 1.$$

Fix a point $x\in F$. *Let* $\alpha,\beta\in K^*_x$. *Then*

$$\prod_{C\ni x} f_*(\alpha,\beta)_{x,C} = 1.$$

Remark. If $C \neq f^{-1}(s)$ then $f_*(\ ,\ )_{x,C} = N_{k(C)_x/K_s}\, t_{K_{x,C}}$ where $t_{K_{x,C}}$ is the tame symbol with respect to the discrete valuation of rank 1 on $K_{x,C}$.

If $C = f^{-1}(s)$ then $f_*(\ ,\ )_{x,C} = N_{k(x)((t_s))/K_s}(\ ,\ )_f$ where $(\ ,\ )_f^{-1}$ coincides with Kato's residue homomorphism [K, §1]. An explicit formula for $(\ ,\ )_f$ is constructed in [O, Th.2].

2.4. The Gysin map for Chow groups

Assume that k is of arbitrary characteristic.

Definition. Let $K_2'(\mathbb{A}_X)$ be the subset of all $(f_{x,C}) \in K_2(K_{x,C})$, $x \in C$ such that

(a) $f_{x,C} \in K_2(\mathcal{O}_{x,C})$ for almost all irreducible curves C where $\mathcal{O}_{x,C}$ is the ring of integers of $K_{x,C}$ with respect to the discrete valuation of rank 1 on it;

(b) for all irreducible curves $C \subset X$, all integers $r \geqslant 1$ and almost all points $x \in C$

$$f_{x,C} \in K_2(\mathcal{O}_{x,C}, \mathcal{M}_C^r) + K_2(\widehat{\mathcal{O}}_x[t_C^{-1}]) \subset K_2(K_{x,C})$$

where $\mathcal{M}_C$ is the maximal ideal of $\mathcal{O}_{x,C}$ and
$K_2(A, J) = \ker(K_2(A) \to K_2(A/J))$.

This definition is similar to the definition of [P2].

Definition. Using the diagonal map of $K_2(K_C)$ to $\prod_{x\in C} K_2(K_{x\in C})$ and of $K_2(K_x)$ to $\prod_{C \ni x} K_2(K_{x\in C})$ put

$$K_2'(A_{01}) = K_2'(\mathbb{A}_X) \cap \text{image of} \prod_{C\subset X} K_2(K_C),$$
$$K_2'(A_{02}) = K_2'(\mathbb{A}_X) \cap \text{image of} \prod_{x\in X} K_2(K_x),$$
$$K_2'(A_{12}) = K_2'(\mathbb{A}_X) \cap \text{image of} \prod_{x\in C} K_2(\mathcal{O}_{x,C}),$$
$$K_2'(A_0) = K_2(k(X)),$$
$$K_2'(A_1) = K_2'(\mathbb{A}_X) \cap \text{image of} \prod_{C\subset X} K_2(\mathcal{O}_C),$$
$$K_2'(A_2) = K_2'(\mathbb{A}_X) \cap \text{image of} \prod_{x\in X} K_2(\widehat{\mathcal{O}}_x)$$

where $\mathcal{O}_C$ is the ring of integers of K_C.

Define the complex $K_2(\mathcal{A}_X)$:

$$K_2'(A_0) \oplus K_2'(A_1) \oplus K_2'(A_2) \to K_2'(A_{01}) \oplus K_2'(A_{02}) \oplus K_2'(A_{12}) \to K_2'(A_{012})$$
$$(f_0, f_1, f_2) \mapsto (f_0 + f_1, f_2 - f_0, -f_1 - f_2)$$
$$(g_1, g_2, g_3) \mapsto g_1 + g_2 + g_3$$

where $K_2'(A_{012}) = K_2'(\mathbb{A}_X)$.

Using the Gersten resolution from K-theory (see [Q, §7]) one can deduce:

Theorem. *There is a canonical isomorphism*

$$H^n(K_2(\mathcal{A}_X)) \simeq H^n(X, \mathcal{K}_2(X)).$$

Similarly one defines $K_1'(\mathbb{A}_S)$. From $H^1(S, \mathcal{K}_1(S)) = H^1(S, \mathcal{O}_S^*) = \mathrm{Pic}(S)$ (or from the approximation theorem) it is easy to see that the n-th cohomology group of the following complex

$$\begin{array}{ccc} K_1(k(S)) \oplus \sum_{s\in S} K_1(\widehat{\mathcal{O}}_s) & \longrightarrow & K_1'(\mathbb{A}_S) \\ (f_0, f_1) & \longmapsto & f_0 + f_1. \end{array}$$

is canonically isomorphic to $H^n(S, \mathcal{K}_1(S))$ (here $\widehat{\mathcal{O}}_s$ is the completion of the local ring of C at s).

Assume that k is of characteristic 0.

Using the reciprocity law above and the previous theorem one can deduce:

Theorem. *The map* $f_* = \sum_{C\ni x, f(x)=s} f_*(\ ,\ )_{x,C}$ *from* $K_2'(\mathbb{A}_X)$ *to* $K_1'(\mathbb{A}_S)$ *is well defined. It maps the complex* $K_2(\mathcal{A}_X)$ *to the complex*

$$0 \longrightarrow K_1(k(S)) \oplus \sum_{s\in S} K_1(\widehat{\mathcal{O}}_s) \longrightarrow K_1'(\mathbb{A}_S).$$

It induces the map $f_*\colon H^n(X, \mathcal{K}_2(X)) \to H^{n-1}(S, \mathcal{K}_1(S))$ *of 2.0.*

If $n = 2$*, then the last map is the direct image morphism* (*Gysin map*) *from* $CH^2(X)$ *to* $CH^1(S)$.

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