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# 6. $\Phi$-$\Gamma$-modules and Galois cohomology

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## 6.0. Introduction

Let $G$ be a profinite group and $p$ a prime number.

**Definition.** A finitely generated $\mathbb{Z}_p$-module $V$ endowed with a continuous $G$-action is called a $\mathbb{Z}_p$-adic representation of $G$. Such representations form a category denoted by $\mathrm{Rep}_{\mathbb{Z}_p}(G)$; its subcategory $\mathrm{Rep}_{\mathbb{F}_p}(G)$ (respectively $\mathrm{Rep}_{p\text{-tor}}(G)$) of mod $p$ representations (respectively $p$-torsion representations) consists of the $V$ annihilated by $p$ (respectively a power of $p$).

**Problem.** To calculate in a simple explicit way the cohomology groups $H^i(G,V)$ of the representation $V$.

A method to solve it for $G = G_K$ ( $K$ is a local field) is to use Fontaine's theory of $\Phi$-$\Gamma$-modules and pass to a simpler Galois representation, paying the price of enlarging $\mathbb{Z}_p$ to the ring of integers of a two-dimensional local field. In doing this we have to replace linear with semi-linear actions.

In this paper we give an overview of the applications of such techniques in different situations. We begin with a simple example.

## 6.1. The case of a field of positive characteristic

Let $E$ be a field of characteristic $p$, $G = G_E$ and $\sigma\colon E^{\mathrm{sep}} \to E^{\mathrm{sep}}$, $\sigma(x) = x^p$ the absolute Frobenius map.

**Definition.** For $V \in \mathrm{Rep}_{\mathbb{F}_p}(G_E)$ put $D(V) := (E^{\mathrm{sep}} \otimes_{\mathbb{F}_p} V)^{G_E}$; $\sigma$ acts on $D(V)$ by acting on $E^{\mathrm{sep}}$.

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**Properties.**

(1) $\dim_E D(V) = \dim_{\mathbb{F}_p} V$;

(2) the "Frobenius" map $\varphi\colon D(V) \to D(V)$ induced by $\sigma \otimes \mathrm{id}_V$ satisfies:

a) $\varphi(\lambda x) = \sigma(\lambda)\varphi(x)$ for all $\lambda \in E,\ x \in D(V)$ (so $\varphi$ is $\sigma$-semilinear);

b) $\varphi(D(V))$ generates $D(V)$ as an $E$-vector space.

**Definition.** A finite dimensional vector space $M$ over $E$ is called an *étale* $\Phi$*-module* over $E$ if there is a $\sigma$-semilinear map $\varphi\colon M \to M$ such that $\varphi(M)$ generates $M$ as an $E$-vector space.

Étale $\Phi$-modules form an abelian category $\Phi M_E^{\text{ét}}$ (the morphisms are the linear maps commuting with the Frobenius $\varphi$).

**Theorem 1** (Fontaine, [F])**.** *The functor $V \to D(V)$ is an equivalence of the categories* $\mathrm{Rep}_{\mathbb{F}_p}(G_E)$ *and* $\Phi M_E^{\text{ét}}$.

We see immediately that $H^0(G_E, V) = V^{G_E} \simeq D(V)^{\varphi}$.

So in order to obtain an explicit description of the Galois cohomology of mod $p$ representations of $G_E$, we should try to derive in a simple manner the functor associating to an étale $\Phi$-module the group of points fixed under $\varphi$. This is indeed a much simpler problem because there is only one operator acting.

For $(M, \varphi) \in \Phi M_E^{\text{ét}}$ define the following complex of abelian groups:

$$C_1(M): \qquad 0 \to M \xrightarrow{\varphi - 1} M \to 0$$

($M$ stands at degree 0 and 1).

This is a functorial construction, so by taking the cohomology of the complex, we obtain a cohomological functor $(\mathcal{H}^i := H^i \circ C_1)_{i \in \mathbb{N}}$ from $\Phi M_E^{\text{ét}}$ to the category of abelian groups.

**Theorem 2.** *The cohomological functor $(\mathcal{H}^i \circ D)_{i \in \mathbb{N}}$ can be identified with the Galois cohomology functor $(H^i(G_E, .))_{i \in \mathbb{N}}$ for the category* $\mathrm{Rep}_{\mathbb{F}_p}(G_E)$. *So, if $M = D(V)$ then $\mathcal{H}^i(M)$ provides a simple explicit description of $H^i(G_E, V)$.*

*Proof of Theorem 2.* We need to check that the cohomological functor $(\mathcal{H}^i)_{i \in \mathbb{N}}$ is universal; therefore it suffices to verify that for every $i \geqslant 1$ the functor $\mathcal{H}^i$ is effaceable: this means that for every $(M, \varphi_M) \in \Phi M_E^{\text{ét}}$ and every $x \in \mathcal{H}^i(M)$ there exists an embedding $u$ of $(M, \varphi_M)$ in $(N, \varphi_N) \in \Phi M_E^{\text{ét}}$ such that $\mathcal{H}^i(u)(x)$ is zero in $\mathcal{H}^i(N)$. But this is easy: it is trivial for $i \geqslant 2$; for $i = 1$ choose an element $m$ belonging to the class $x \in M/(\varphi - 1)(M)$, put $N := M \oplus Et$ and extend $\varphi_M$ to the $\sigma$-semi-linear map $\varphi_N$ determined by $\varphi_N(t) := t + m$. □

## 6.2. $\Phi$-$\Gamma$-modules and $\mathbb{Z}_p$-adic representations

**Definition.** Recall that a Cohen ring is an absolutely unramified complete discrete valuation ring of mixed characteristic $(0, p > 0)$, so its maximal ideal is generated by $p$.

We describe a general formalism, explained by Fontaine in [F], which lifts the equivalence of categories of Theorem 1 in characteristic 0 and relates the $\mathbb{Z}_p$-adic representations of $G$ to a category of modules over a Cohen ring, endowed with a "Frobenius" map and a group action.

Let $R$ be an algebraically closed complete valuation (of rank 1) field of characteristic $p$ and let $H$ be a normal closed subgroup of $G$. Suppose that $G$ acts continuously on $R$ by ring automorphisms. Then $F := R^H$ is a perfect closed subfield of $R$.

For every integer $n \geq 1$, the ring $W_n(R)$ of Witt vectors of length $n$ is endowed with the product of the topology on $R$ defined by the valuation and then $W(R)$ with the inverse limit topology. Then the componentwise action of the group $G$ is continuous and commutes with the natural Frobenius $\sigma$ on $W(R)$. We also have $W(R)^H = W(F)$.

Let $E$ be a closed subfield of $F$ such that $F$ is the completion of the $p$-radical closure of $E$ in $R$. Suppose there exists a Cohen subring $\mathcal{O}_{\mathcal{E}}$ of $W(R)$ with residue field $E$ and which is stable under the actions of $\sigma$ and of $G$. Denote by $\mathcal{O}_{\widehat{\mathcal{E}_{\mathrm{ur}}}}$ the completion of the integral closure of $\mathcal{O}_{\mathcal{E}}$ in $W(R)$: it is a Cohen ring which is stable by $\sigma$ and $G$, its residue field is the separable closure of $E$ in $R$ and $(\mathcal{O}_{\widehat{\mathcal{E}_{\mathrm{ur}}}})^H = \mathcal{O}_{\mathcal{E}}$.

The natural map from $H$ to $G_E$ is an isomorphism if and only if the action of $H$ on $R$ induces an isomorphism from $H$ to $G_F$. We suppose that this is the case.

**Definition.** Let $\Gamma$ be the quotient group $G/H$. An étale $\Phi$-$\Gamma$-module over $\mathcal{O}_{\mathcal{E}}$ is a finitely generated $\mathcal{O}_{\mathcal{E}}$-module endowed with a $\sigma$-semi-linear Frobenius map $\varphi: M \to M$ and a continuous $\Gamma$-semi-linear action of $\Gamma$ commuting with $\varphi$ such that the image of $\varphi$ generates the module $M$.

Étale $\Phi$-$\Gamma$-modules over $\mathcal{O}_{\mathcal{E}}$ form an abelian category $\Phi\Gamma M^{\text{ét}}_{\mathcal{O}_{\mathcal{E}}}$ (the morphisms are the linear maps commuting with $\varphi$). There is a tensor product of $\Phi$-$\Gamma$-modules, the natural one. For two objects $M$ and $N$ of $\Phi\Gamma M^{\text{ét}}_{\mathcal{O}_{\mathcal{E}}}$ the $\mathcal{O}_{\mathcal{E}}$-module $\mathrm{Hom}_{\mathcal{O}_{\mathcal{E}}}(M, N)$ can be endowed with an étale $\Phi$-$\Gamma$-module structure (see [F]).

For every $\mathbb{Z}_p$-adic representation $V$ of $G$, let $D_H(V)$ be the $\mathcal{O}_{\mathcal{E}}$-module $(\mathcal{O}_{\widehat{\mathcal{E}_{\mathrm{ur}}}} \otimes_{\mathbb{Z}_p} V)^H$. It is naturally an étale $\Phi$-$\Gamma$-module, with $\varphi$ induced by the map $\sigma \otimes \mathrm{id}_V$ and $\Gamma$ acting on both sides of the tensor product. From Theorem 2 one deduces the following fundamental result:

**Theorem 3** (Fontaine, [F]). *The functor $V \to D_H(V)$ is an equivalence of the categories $\mathrm{Rep}_{\mathbb{Z}_p}(G)$ and $\Phi\Gamma M^{\text{ét}}_{\mathcal{O}_{\mathcal{E}}}$.*

**Remark.** If $E$ is a field of positive characteristic, $\mathcal{O}_{\mathcal{E}}$ is a Cohen ring with residue field $E$ endowed with a Frobenius $\sigma$, then we can easily extend the results of the whole subsection 6.1 to $\mathbb{Z}_p$-adic representations of $G$ by using Theorem 3 for $G = G_E$ and $H = \{1\}$.

## 6.3. A brief survey of the theory of the field of norms

For the details we refer to [W], [FV] or [F].

Let $K$ be a complete discrete valuation field of characteristic $0$ with perfect residue field $k$ of characteristic $p$. Put $G = G_K = \text{Gal}(K^{\text{sep}}/K)$.

Let $\mathbb{C}$ be the completion of $K^{\text{sep}}$, denote the extension of the discrete valuation $v_K$ of $K$ to $\mathbb{C}$ by $v_K$. Let $R^* = \varprojlim \mathbb{C}_n^*$ where $\mathbb{C}_n = \mathbb{C}$ and the morphism from $\mathbb{C}_{n+1}$ to $\mathbb{C}_n$ is raising to the $p$th power. Put $R := R^* \cup \{0\}$ and define $v_R((x_n)) = v_K(x_0)$. For $(x_n), (y_n) \in R$ define

$$(x_n) + (y_n) = (z_n) \qquad \text{where } z_n = \lim_m (x_{n+m} + y_{n+m})^{p^m} .$$

Then $R$ is an algebraically closed field of characteristic $p$ complete with respect to $v_R$ (cf. [W]). Its residue field is isomorphic to the algebraic closure of $k$ and there is a natural continuous action of $G$ on $R$. (Note that Fontaine denotes this field by Fr $R$ in [F]).

Let $L$ be a Galois extension of $K$ in $K^{\text{sep}}$. Recall that one can always define the ramification filtration on $\text{Gal}(L/K)$ in the upper numbering. Roughly speaking, $L/K$ is an arithmetically profinite extension if one can define the lower ramification subgroups of $G$ so that the classical relations between the two filtrations for finite extensions are preserved. This is in particular possible if $\text{Gal}(L/K)$ is a $p$-adic Lie group with finite residue field extension.

The field $R$ contains in a natural way the field of norms $N(L/K)$ of every arithmetically profinite extension $L$ of $K$ and the restriction of $v$ to $N(L/K)$ is a discrete valuation. The residue field of $N(L/K)$ is isomorphic to that of $L$ and $N(L/K)$ is stable under the action of $G$. The construction is functorial: if $L'$ is a finite extension of $L$ contained in $K^{\text{sep}}$, then $L'/K$ is still arithmetically profinite and $N(L'/K)$ is a separable extension of $N(L/K)$. The direct limit of the fields $N(L'/K)$ where $L'$ goes through all the finite extensions of $L$ contained in $K^{\text{sep}}$ is the separable closure $E^{\text{sep}}$ of $E = N(L/K)$. It is stable under the action of $G$ and the subgroup $G_L$ identifies with $G_E$. The field $E^{\text{sep}}$ is dense in $R$.

Fontaine described how to lift these constructions in characteristic $0$ when $L$ is the cyclotomic $\mathbb{Z}_p$-extension $K_\infty$ of $K$. Consider the ring of Witt vectors $W(R)$ endowed with the Frobenius map $\sigma$ and the natural componentwise action of $G$. Define the topology of $W(R)$ as the product of the topology defined by the valuation on $R$. Then one can construct a Cohen ring $\mathcal{O}_{\widehat{\mathcal{E}_{\text{ur}}}}$ with residue field $E^{\text{sep}}$ ($E = N(L/K)$) such that:

(i) $\mathcal{O}_{\widehat{\mathcal{E}_{\text{ur}}}}$ is stable by $\sigma$ and the action of $G$,

(ii) for every finite extension $L$ of $K_\infty$ the ring $(\mathcal{O}_{\widehat{\mathcal{E}_{\text{ur}}}})^{G_L}$ is a Cohen ring with residue field $E$.

Denote by $\mathcal{O}_{\mathcal{E}(K)}$ the ring $(\mathcal{O}_{\widehat{\mathcal{E}_{\text{ur}}}})^{G_{K_\infty}}$. It is stable by $\sigma$ and the quotient $\Gamma = G/G_{K_\infty}$ acts continuously on $\mathcal{O}_{\mathcal{E}(K)}$ with respect to the induced topology. Fix a topological generator $\gamma$ of $\Gamma$: it is a continuous ring automorphism commuting with $\sigma$. The fraction field of $\mathcal{O}_{\mathcal{E}(K)}$ is a two-dimensional standard local field (as defined in section 1 of Part I). If $\pi$ is a lifting of a prime element of $N(K_\infty/K)$ in $\mathcal{O}_{\mathcal{E}(K)}$ then the elements of $\mathcal{O}_{\mathcal{E}(K)}$ are the series $\sum_{i\in\mathbb{Z}} a_i\pi^i$, where the coefficients $a_i$ are in $W(k_{K_\infty})$ and converge $p$-adically to 0 when $i \to -\infty$.

## 6.4. Application of $\mathbb{Z}_p$-adic representations of $G$ to the Galois cohomology

If we put together Fontaine's construction and the general formalism of subsection 6.2 we obtain the following important result:

**Theorem 3'** (Fontaine, [F]). *The functor $V \to D(V) := (\mathcal{O}_{\widehat{\mathcal{E}_{\text{ur}}}} \otimes_{\mathbb{Z}_p} V)^{G_{K_\infty}}$ defines an equivalence of the categories* $\operatorname{Rep}_{\mathbb{Z}_p}(G)$ *and* $\Phi\Gamma M^{\text{ét}}_{\mathcal{O}_{\mathcal{E}(K)}}$.

Since for every $\mathbb{Z}_p$-adic representation of $G$ we have $H^0(G,V) = V^G \simeq D(V)^{\varphi}$, we want now, as in paragraph 6.1, compute explicitly the cohomology of the representation using the $\Phi$-$\Gamma$-module associated to $V$.

For every étale $\Phi$-$\Gamma$-module $(M,\varphi)$ define the following complex of abelian groups:

$$C_2(M): \qquad 0 \longrightarrow M \xrightarrow{\alpha} M \oplus M \xrightarrow{\beta} M \longrightarrow 0$$

where $M$ stands at degree 0 and 2,

$$\alpha(x) = ((\varphi-1)x, (\gamma-1)x), \quad \beta((y,z)) = (\gamma-1)y - (\varphi-1)z.$$

By functoriality, we obtain a cohomological functor $(\mathcal{H}^i := H^i \circ C_2)_{i\in\mathbb{N}}$ from $\Phi\Gamma M^{\text{ét}}_{\mathcal{O}_{\mathcal{E}(K)}}$ to the category of abelian groups.

**Theorem 4** (Herr, [H]). *The cohomological functor $(\mathcal{H}^i \circ D)_{i\in\mathbb{N}}$ can be identified with the Galois cohomology functor $(H^i(G,.))_{i\in\mathbb{N}}$ for the category* $\operatorname{Rep}_{p\text{-tor}}(G)$. *So, if $M = D(V)$ then $\mathcal{H}^i(M)$ provides a simple explicit description of $H^i(G,V)$ in the $p$-torsion case.*

*Idea of the proof of Theorem 4.* We have to check that for every $i \geqslant 1$ the functor $\mathcal{H}^i$ is effaceable. For every $p$-torsion object $(M,\varphi_M) \in \Phi\Gamma M^{\text{ét}}_{\mathcal{O}_{\mathcal{E}(K)}}$ and every $x \in \mathcal{H}^i(M)$ we construct an explicit embedding $u$ of $(M,\varphi_M)$ in a certain $(N,\varphi_N) \in \Phi\Gamma M^{\text{ét}}_{\mathcal{O}_{\mathcal{E}(K)}}$

such that $\mathcal{H}^i(u)(x)$ is zero in $\mathcal{H}^i(N)$. For details see [H]. The key point is of topological nature: we prove, following an idea of Fontaine in [F], that there exists an open neighbourhood of 0 in $M$ on which $(\varphi - 1)$ is bijective and use then the continuity of the action of $\Gamma$. □

As an application of theorem 4 we can prove the following result (due to Tate):

**Theorem 5.** *Assume that $k_K$ is finite and let $V$ be in $\mathrm{Rep}_{p\text{-tor}}(G)$. Without using class field theory the previous theorem implies that $H^i(G,V)$ are finite, $H^i(G,V) = 0$ for $i \geqslant 3$ and*

$$\sum_{i=0}^{2} l(H^i(G,V)) = -|K:\mathbb{Q}_p|\, l(V),$$

*where $l(\ )$ denotes the length over $\mathbb{Z}_p$.*

See [H].

**Remark.** Because the finiteness results imply that the Mittag–Leffler conditions are satisfied, it is possible to generalize the explicit construction of the cohomology and to prove analogous results for $\mathbb{Z}_p$ (or $\mathbb{Q}_p$)-adic representations by passing to the inverse limits.

## 6.5. A new approach to local class field theory

The results of the preceding paragraph allow us to prove without using class field theory the following:

**Theorem 6** (Tate's local duality)**.** *Let $V$ be in $\mathrm{Rep}_{p\text{-tor}}(G)$ and $n \in \mathbb{N}$ such that $p^nV = 0$. Put $V^*(1) := \mathrm{Hom}(V, \mu_{p^n})$. Then there is a canonical isomorphism from $H^2(G, \mu_{p^n})$ to $\mathbb{Z}/p^n$ and the cup product*

$$H^i(G,V) \times H^{2-i}(G,V^*(1)) \xrightarrow{\cup} H^2(G,\mu_{p^n}) \simeq \mathbb{Z}/p^n$$

*is a perfect pairing.*

It is well known that a proof of the local duality theorem of Tate without using class field theory gives a construction of the reciprocity map. For every $n \geqslant 1$ we have by duality a functorial isomorphism between the finite groups $\mathrm{Hom}(G, \mathbb{Z}/p^n) = H^1(G, \mathbb{Z}/p^n)$ and $H^1(G, \mu_{p^n})$ which is isomorphic to $K^*/(K^*)^{p^n}$ by Kummer theory. Taking the inverse limits gives us the $p$-part of the reciprocity map, the most difficult part.

*Sketch of the proof of Theorem 6.* ([H2]).

a) Introduction of differentials:

Let us denote by $\mathbf{\Omega}_c^1$ the $\mathcal{O}_{\mathcal{E}(K)}$-module of continuous differential forms of $\mathcal{O}_{\mathcal{E}}$ over $W(k_{K_\infty})$. If $\pi$ is a fixed lifting of a prime element of $E(K_\infty/K)$ in $\mathcal{O}_{\mathcal{E}(K)}$, then this module is free and generated by $d\pi$. Define the residue map from $\mathbf{\Omega}_c^1$ to $W(k_{K_\infty})$ by res $\left(\sum_{i\in\mathbb{Z}} a_i\pi^i d\pi\right) := a_{-1}$; it is independent of the choice of $\pi$.

b) Calculation of some $\Phi$-$\Gamma$-modules:

The $\mathcal{O}_{\mathcal{E}(K)}$-module $\mathbf{\Omega}_c^1$ is endowed with an étale $\Phi$-$\Gamma$-module structure by the following formulas: for every $\lambda \in \mathcal{O}_{\mathcal{E}(K)}$ we put:

$$p\varphi(\lambda d\pi) = \sigma(\lambda)d(\sigma(\pi)) \ , \ \gamma(\lambda d\pi) = \gamma(\lambda)d(\gamma(\pi)).$$

The fundamental fact is that there is a natural isomorphism of $\Phi$-$\Gamma$-modules over $\mathcal{O}_{\mathcal{E}(K)}$ between $D(\mu_{p^n})$ and the reduction $\mathbf{\Omega}_{c,n}^1$ of $\mathbf{\Omega}_c^1$ modulo $p^n$.

The étale $\Phi$-$\Gamma$-module associated to the representation $V^*(1)$ is $\widetilde{M} := \mathrm{Hom}(M, \mathbf{\Omega}_{c,n}^1)$, where $M = D(V)$. By composing the residue with the trace we obtain a surjective and continuous map $\mathrm{Tr}_n$ from $M$ to $\mathbb{Z}/p^n$. For every $f \in \widetilde{M}$, the map $\mathrm{Tr}_n \circ f$ is an element of the group $M^\vee$ of continuous group homomorphisms from $M$ to $\mathbb{Z}/p^n$. This gives in fact a group isomorphism from $\widetilde{M}$ to $M^\vee$ and we can therefore transfer the $\Phi$-$\Gamma$-module structure from $\widetilde{M}$ to $M^\vee$. But, since $k$ is finite, $M$ is locally compact and $M^\vee$ is in fact the Pontryagin dual of $M$.

c) Pontryagin duality implies local duality:

We simply dualize the complex $C_2(M)$ using Pontryagin duality (all arrows are strict morphisms in the category of topological groups) and obtain a complex:

$$C_2(M)^\vee : \qquad 0 \to M^\vee \xrightarrow{\beta^\vee} M^\vee \oplus M^\vee \xrightarrow{\alpha^\vee} M^\vee \to 0,$$

where the two $M^\vee$ are in degrees 0 and 2. Since we can construct an explicit quasi-isomorphism between $C_2(M^\vee)$ and $C_2(M)^\vee$, we easily obtain a duality between $\mathcal{H}^i(M)$ and $\mathcal{H}^{2-i}(M^\vee)$ for every $i \in \{0,1,2\}$.

d) The canonical isomorphism from $\mathcal{H}^2(\mathbf{\Omega}_{c,n}^1)$ to $\mathbb{Z}/p^n$:

The map $\mathrm{Tr}_n$ from $\mathbf{\Omega}_{c,n}^1$ to $\mathbb{Z}/p^n$ factors through the group $\mathcal{H}^2(\mathbf{\Omega}_{c,n}^1)$ and this gives an isomorphism. But it is not canonical! In fact the construction of the complex $C_2(M)$ depends on the choice of $\gamma$. Fortunately, if we take another $\gamma$, we get a quasi-isomorphic complex and if we normalize the map $\mathrm{Tr}_n$ by multiplying it by the unit $-p^{v_p(\log \chi(\gamma))}/\log \chi(\gamma)$ of $\mathbb{Z}_p$, where log is the $p$-adic logarithm, $\chi$ the cyclotomic character and $v_p = v_{\mathbb{Q}_p}$, then everything is compatible with the change of $\gamma$.

e) The duality is given by the cup product:

We can construct explicit formulas for the cup product:

$$\mathcal{H}^i(M) \times \mathcal{H}^{2-i}(M^\vee) \xrightarrow{\cup} \mathcal{H}^2(\mathbf{\Omega}_{c,n}^1)$$

associated with the cohomological functor $(\mathcal{H}^i)_{i\in\mathbb{N}}$ and we compose them with the preceding normalized isomorphism from $\mathcal{H}^2(\Omega^1_{c,n})$ to $\mathbb{Z}/p^n$. Since everything is explicit, we can compare with the pairing obtained in c) and verify that it is the same up to a unit of $\mathbb{Z}_p$. □

**Remark.** Benois, using the previous theorem, deduced an explicit formula of Coleman's type for the Hilbert symbol and proved Perrin-Riou's formula for crystalline representations ([B]).

## 6.6. Explicit formulas for the generalized Hilbert symbol on formal groups

Let $K_0$ be the fraction field of the ring $W_0$ of Witt vectors with coefficients in a finite field of characteristic $p > 2$ and $\mathcal{F}$ a commutative formal group of finite height $h$ defined over $W_0$.

For every integer $n \geqslant 1$, denote by $\mathcal{F}[p^n]$ the $p^n$-torsion points in $\mathcal{F}(\mathcal{M}_C)$, where $\mathcal{M}_C$ is the maximal ideal of the completion $C$ of an algebraic closure of $K_0$. The group $\mathcal{F}[p^n]$ is isomorphic to $(\mathbb{Z}/p^n\mathbb{Z})^h$.

Let $K$ be a finite extension of $K_0$ contained in $K^{\text{sep}}$ and assume that the points of $\mathcal{F}[p^n]$ are defined over $K$. We then have a bilinear pairing:

$$( \ , \ ]_{\mathcal{F},n} \colon G_K^{\text{ab}} \times \mathcal{F}(\mathcal{M}_K) \to \mathcal{F}[p^n]$$

(see section 8 of Part I).

When the field $K$ contains a primitive $p^n$ th root of unity $\zeta_{p^n}$, Abrashkin gives an explicit description for this pairing generalizing the classical Brückner–Vostokov formula for the Hilbert symbol ([A]). In his paper he notices that the formula makes sense even if $K$ does not contain $\zeta_{p^n}$ and he asks whether it holds without this assumption. In a recent unpublished work, Benois proves that this is true.

Suppose for simplicity that $K$ contains only $\zeta_p$. Abrashkin considers in his paper the extension $\widetilde{K} := K(\pi^{p^{-r}}, r \geqslant 1)$, where $\pi$ is a fixed prime element of $K$. It is not a Galois extension of $K$ but is arithmetically profinite, so by [W] one can consider the field of norms for it. In order not to loose information given by the roots of unity of order a power of $p$, Benois uses the composite Galois extension $L := K_\infty\widetilde{K}/K$ which is arithmetically profinite. There are several problems with the field of norms $N(L/K)$, especially it is not clear that one can lift it in characteristic 0 with its Galois action. So, Benois simply considers the completion $F$ of the $p$-radical closure of $E = N(L/K)$ and its separable closure $F^{\text{sep}}$ in $R$. If we apply what was explained in subsection 6.2 for $\Gamma = \text{Gal}(L/K)$, we get:

**Theorem 7.** *The functor* $V \to D(V) := (W(F^{\text{sep}}) \otimes_{\mathbb{Z}_p} V)^{G_L}$ *defines an equivalence of the categories* $\text{Rep}_{\mathbb{Z}_p}(G)$ *and* $\Phi\Gamma M^{\text{ét}}_{W(F)}$.

Choose a topological generator $\gamma'$ of $\mathrm{Gal}(L/K_\infty)$ and lift $\gamma$ to an element of $\mathrm{Gal}(L/\widetilde{K})$. Then $\Gamma$ is topologically generated by $\gamma$ and $\gamma'$, with the relation $\gamma\gamma' = (\gamma')^a\gamma$, where $a = \chi(\gamma)$ ( $\chi$ is the cyclotomic character). For $(M,\varphi) \in \Phi\Gamma M^{\text{ét}}_{W(F)}$ the continuous action of $\mathrm{Gal}(L/K_\infty)$ on $M$ makes it a module over the Iwasawa algebra $\mathbb{Z}_p[[\gamma' - 1]]$. So we can define the following complex of abelian groups:

$$C_3(M): \qquad 0 \to M_0 \xrightarrow{\alpha\mapsto A_0\alpha} M_1 \xrightarrow{\alpha\mapsto A_1\alpha} M_2 \xrightarrow{\alpha\mapsto A_2\alpha} M_3 \to 0$$

where $M_0$ is in degree 0, $M_0 = M_3 = M$, $M_1 = M_2 = M^3$,

$$A_0 = \begin{pmatrix} \varphi - 1 \\ \gamma - 1 \\ \gamma' - 1 \end{pmatrix}, A_1 = \begin{pmatrix} \gamma - 1 & 1-\varphi & 0 \\ \gamma' - 1 & 0 & 1 - \varphi \\ 0 & {\gamma'}^a - 1 & \delta - \gamma \end{pmatrix}, A_2 = ((\gamma')^a - 1 \;\; \delta - \gamma \;\; \varphi - 1)$$

and $\delta = ((\gamma')^a - 1)(\gamma' - 1)^{-1} \in \mathbb{Z}_p[[\gamma' - 1]]$.

As usual, by taking the cohomology of this complex, one defines a cohomological functor $(\mathcal{H}^i)_{i\in\mathbb{N}}$ from $\Phi\Gamma M^{\text{ét}}_{W(F)}$ in the category of abelian groups. Benois proves only that the cohomology of a $p$-torsion representation $V$ of $G$ injects in the groups $\mathcal{H}^i(D(V))$ which is enough to get the explicit formula. But in fact a stronger statement is true:

**Theorem 8.** *The cohomological functor $(\mathcal{H}^i \circ D)_{i\in\mathbb{N}}$ can be identified with the Galois cohomology functor $(H^i(G,.))_{i\in\mathbb{N}}$ for the category* $\mathrm{Rep}_{p\text{-tor}}(G)$.

*Idea of the proof.* Use the same method as in the proof of Theorem 4. It is only more technically complicated because of the structure of $\Gamma$. □

Finally, one can explicitly construct the cup products in terms of the groups $\mathcal{H}^i$ and, as in [B], Benois uses them to calculate the Hilbert symbol.

**Remark.** Analogous constructions (equivalence of category, explicit construction of the cohomology by a complex) seem to work for higher dimensional local fields. In particular, in the two-dimensional case, the formalism is similar to that of this paragraph; the group $\Gamma$ acting on the Φ-Γ-modules has the same structure as here and thus the complex is of the same form. This work is still in progress.

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