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Four-manifolds, geometries and knots

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Preface

Every closed surface admits a geometry of constant curvature, and may be classified topologically either by its fundamental group or by its Euler characteristic and orientation character. It is generally expected that all closed 3-manifolds have decompositions into geometric pieces, and are determined up to homeomorphism by invariants associated with the fundamental group (whereas the Euler characteristic is always 0). In dimension 4 the Euler characteristic and fundamental group are largely independent, and the class of closed 4-manifolds which admit a geometric decomposition is rather restricted. For instance, there are only 11 such manifolds with finite fundamental group. On the other hand, many complex surfaces admit geometric structures, as do all the manifolds arising from surgery on twist spun simple knots.

The goal of this book is to characterize algebraically the closed 4-manifolds that fibre nontrivially or admit geometries, or which are obtained by surgery on 2-knots, and to provide a reference for the topology of such manifolds and knots. In many cases the Euler characteristic, fundamental group and Stiefel-Whitney classes together form a complete system of invariants for the homotopy type of such manifolds, and the possible values of the invariants can be described explicitly. If the fundamental group is elementary amenable we may use topological surgery to obtain classifications up to homeomorphism. Surgery techniques also work well "stably" in dimension 4 (i.e., modulo connected sums with copies of $S^2 \times S^2$). However, in our situation the fundamental group may have nonabelian free subgroups and the Euler characteristic is usually the minimal possible for the group, and it is not known whether s-cobordisms between such 4-manifolds are always topologically products. Our strongest results are characterizations of manifolds which fibre homotopically over S^1 or an aspherical surface (up to homotopy equivalence) and infrasolvmanifolds (up to homeomorphism). As a consequence 2-knots whose groups are poly-Z are determined up to Gluck reconstruction and change of orientations by their groups alone.

We shall now outline the chapters in somewhat greater detail. The first chapter is purely algebraic; here we summarize the relevant group theory and present the notions of amenable group, Hirsch length of an elementary amenable group, finiteness conditions, criteria for the vanishing of cohomology of a group with coefficients in a free module, Poincaré duality groups, and Hilbert modules over the von Neumann algebra of a group. The rest of the book may be divided into three parts: general results on homotopy and surgery (Chapters 2-6), geometries and geometric decompositions (Chapters 7-13), and 2-knots (Chapters 14-18).

Some of the later arguments are applied in microcosm to 2-complexes and PD_3-complexes in Chapter 2, which presents equivariant cohomology, L^2-Betti numbers and Poincaré duality. Chapter 3 gives general criteria for two closed 4-manifolds to be homotopy equivalent, and we show that a closed 4-manifold M is aspherical if and only if $\pi_1(M)$ is a PD_4-group of type FF and $\chi(M)=\chi(\pi)$. We show that if the universal cover of a closed 4-manifold is finitely dominated then it is contractible or homotopy equivalent to S^2 or S^3 or the fundamental group is finite. We also consider at length the relationship between fundamental group and Euler characteristic for closed 4-manifolds. In Chapter 4 we show that a closed 4-manifold M fibres homotopically over S^1 with fibre a PD_3-complex if and only if $\chi(M)=0$ and $\pi_1(M)$ is an extension of Z by a finitely presentable normal subgroup. (There remains the problem of recognizing which PD_3-complexes are homotopy equivalent to 3-manifolds). The dual problem of characterizing the total spaces of S^1-bundles over 3-dimensional bases seems more difficult. We give a criterion that applies under some restrictions on the fundamental group. In Chapter 5 we characterize the homotopy types of total spaces of surface bundles. (Our results are incomplete if the base is RP^2). In particular, a closed 4-manifold M is simple homotopy equivalent to the total space of an F-bundle over B (where B and F are closed surfaces and B is aspherical) if and only if $\chi(M)=\chi(B)\chi(F)$ and $\pi_1(M)$ is an extension of $\pi_1(B)$ by a normal subgroup isomorphic to $\pi_1(F)$. (The extension should split if $F=RP^2$). Any such extension is the fundamental group of such a bundle space; the bundle is determined by the extension of groups in the aspherical cases and by the group and Stiefel-Whitney classes if the fibre is S^2 or RP^2. This characterization is improved in Chapter 6, which considers Whitehead groups and obstructions to constructing s-cobordisms via surgery.

The next seven chapters consider geometries and geometric decompositions. Chapter 7 introduces the 4-dimensional geometries and demonstrates the limitations of geometric methods in this dimension. It also gives a brief outline of the connections between geometries, Seifert fibrations and complex surfaces. In Chapter 8 we show that a closed 4-manifold M is homeomorphic to an infrasolvmanifold if and only if $\chi(M)=0$ and $\pi_1(M)$ has a locally nilpotent normal subgroup of Hirsch length at least 3, and two such manifolds are homeomorphic if and only if their fundamental groups are isomorphic. Moreover $\pi_1(M)$ is then a torsion free virtually poly-Z group of Hirsch length 4 and every such group is the fundamental group of an infrasolvmanifold. We also consider in detail the question of when such a manifold is the mapping torus of a self homeomorphism of a 3-manifold, and give a direct and elementary derivation of the fundamental groups of flat 4-manifolds. At the end of this chapter we show that all orientable 4-dimensional infrasolvmanifolds are determined up to diffeomorphism by their fundamental groups. (The corresponding result in other dimensions was known).

Chapters 9-12 consider the remaining 4-dimensional geometries, grouped according to whether the model is homeomorphic to R^4, $S^2 \times R^2$, $S^3 \times R$ or is compact. Aspherical geometric 4-manifolds are determined up to s-cobordism by their homotopy type. However there are only partial characterizations of the groups arising as fundamental groups of $\mathbb{H}^2 \times \mathbb{E}^2$-, $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$-, $\mathbb{H}^3 \times \mathbb{E}^1$- or $\mathbb{H}^2 \times \mathbb{H}^2$-manifolds, while very little is known about $\mathbb{H}^4$- or $\mathbb{H}^2(\mathbb{C})$-manifolds. We show that the homotopy types of manifolds covered by $S^2 \times R^2$ are determined up to finite ambiguity by their fundamental groups. If the fundamental group is torsion free such a manifold is s-cobordant to the total space of an S^2-bundle over an aspherical surface. The homotopy types of manifolds covered by $S^3 \times R$ are determined by the fundamental group and first nonzero k-invariant; much is known about the possible fundamental groups, but less is known about which k-invariants are realized. Moreover, although the fundamental groups are all "good", so that in principle surgery may be used to give a classification up to homeomorphism, the problem of computing surgery obstructions seems very difficult. We conclude the geometric section of the book in Chapter 13 by considering geometric decompositions of 4-manifolds which are also mapping tori or total spaces of surface bundles, and we characterize the complex surfaces which fibre over S^1 or over a closed orientable 2-manifold.

The final five chapters are on 2-knots. Chapter 14 is an overview of knot theory; in particular it is shown how the classification of higher-dimensional knots may be largely reduced to the classification of knot manifolds. The knot exterior is determined by the knot manifold and the conjugacy class of a normal generator for the knot group, and at most two knots share a given exterior. An essential step is to characterize 2-knot groups. Kervaire gave homological conditions which characterize high dimensional knot groups and which 2-knot groups must satisfy, and showed that any high dimensional knot group with a presentation of deficiency 1 is a 2-knot group. Bridging the gap between the homological and combinatorial conditions appears to be a delicate task. In Chapter 15 we investigate 2-knot groups with infinite normal subgroups which have no noncyclic free subgroups. We show that under mild coherence hypotheses such 2-knot groups usually have nontrivial abelian normal subgroups, and we determine all 2-knot groups with finite commutator subgroup. In Chapter 16 we show that if there is an abelian normal subgroup of rank > 1 then the knot manifold is either s-cobordant to a $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$-manifold or is homeomorphic to an infrasolvmanifold.

In Chapter 17 we characterize the closed 4-manifolds obtained by surgery on certain 2-knots, and show that just eight of the 4-dimensional geometries are realised by knot manifolds. We also consider when the knot manifold admits a complex structure. The final chapter considers when a fibred 2-knot with geometric fibre is determined by its exterior. We settle this question when the monodromy has finite order or when the fibre is R^3/Z^3 or is a coset space of the Lie group Nil^3.

This book arose out of two earlier books of mine, on "*2-Knots and their Groups*" and "*The Algebraic Characterization of Geometric 4-Manifolds*", published by Cambridge University Press for the Australian Mathematical Society and for the London Mathematical Society, respectively.About a quarter of the present text has been taken from these books. [1] However the arguments have been improved in many cases, notably in using Bowditch's homological criterion for virtual surface groups to streamline the results on surface bundles, using L^2-methods instead of localization, completing the characterization of mapping tori, relaxing the hypotheses on torsion or on abelian normal subgroups in the fundamental group and in deriving the results on 2-knot groups from the work on 4-manifolds. The main tools used here beyond what can be found in *Algebraic Topology* [Sp] are cohomology of groups, equivariant Poincaré duality and (to a lesser extent) L^2-(co)homology. Our references for these are the books *Homological Dimension of Discrete Groups* [Bi], *Surgery on Compact Manifolds* [Wl] and *L^2-Invariants: Theory and Applications to Geometry and K-Theory* [Lü], respectively. We also use properties of 3-manifolds (for the construction of examples) and calculations of Whitehead groups and surgery obstructions.

This work has been supported in part by ARC small grants, enabling visits by Steve Plotnick, Mike Dyer, Charles Thomas and Fang Fuquan. I would like to thank them all for their advice, and in particular Steve Plotnick for the collaboration reported in Chapter 18. I would also like to thank Robert Bieri, Robin Cobb, Peter Linnell and Steve Wilson for their collaboration, and Warren Dicks, William Dunbar, Ross Geoghegan, F.T.Farrell, Ian Hambleton, Derek Holt, K.F.Lai, Eamonn O'Brien, Peter Scott and Shmuel Weinberger for their correspondance and advice on aspects of this work.

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[1] See the Acknowledment following this preface for a summary of the textual borrowings.

Acknowledgment

I wish to thank Cambridge University Press for their permission to use material from my earlier books [H1] and [H2]. The textual borrowings in each Chapter are outlined below.

1. §1, Lemmas 1.7 and 1.10 and Theorem 1.11, §6 (up to the discussion of $\chi(\pi)$), the first paragraph of §7 and Theorem 1.16 are from [H2:Chapter I]. (Lemma 1.1 is from [H1]). §3 is from [H2:Chapter VI].

2. §1, most of §4, part of §5 and §9 are from [H2:Chapter II and Appendix].

3. Lemma 3.1, Theorems 3.2, 3.7-3.9 and 3.12 and Corollaries 3.9.1-3.9.3 are from [H2:Chapter II]. (Theorems 3.9 and 3.12 have been improved).

4. The first half of §2, the statements of Corollaries 4.5.1-4.5.3, Theorem 4.6 and its Corollaries, and most of §8 are from [H2:Chapter III]. (Theorem 11 and the subsequent discussion have been improved).

5. Part of Lemma 5.15, Theorem 5.16 and §4-§5 are from [H2:Chapter IV]. (Theorem 5.19 and Lemmas 5.21 and 5.22 have been improved).

6. §1 (excepting Theorem 6.1), Theorem 6.12 and the proof of Theorem 6.14 are from [H2:Chapter V].

8. Part of Theorem 8.1, §6, most of §7 and §8 are from [H2:Chapter VI].

9. Theorems 9.1, 9.2 and 9.7 are from [H2:Chapter VI], with improvements.

10. Theorems 10.10-10.12 and §6 are largely from [H2:Chapter VII]. (Theorem 10.10 has been improved).

11. Theorem 11.1 is from [H2:Chapter II]. Lemma 11.3, §3 and the first three paragraphs of §5 are from [H2:Chapter VIII]. §6 is from [H2:Chapter IV].

12. The introduction, §1-§3, §5, most of §6 (from Lemma 12.5 onwards) and §7 are from [H2:Chapter IX], with improvements (particularly in §7).

14. §1-§5 are from [H1:Chapter I]. §6 and §7 are from [H1:Chapter II].

16. Most of §3 is from [H1:Chapter V].(Theorem 16.4 is new and Theorems 16.5 and 16.6 have been improved).

17. Lemma 2 and Theorem 7 are from [H1:Chapter VIII], while Corollary 17.6.1 is from [H1:Chapter VII]. The first two paragraphs of §8 and Lemma 17.12 are from [H2:Chapter X].

Part I

Manifolds and PD-complexes

Chapter 1

Group theoretic preliminaries

The key algebraic idea used in this book is to study the homology groups of covering spaces as modules over the group ring of the group of covering transformations. In this chapter we shall summarize the relevant notions from group theory, in particular, the Hirsch-Plotkin radical, amenable groups, Hirsch length, finiteness conditions, the connection between ends and the vanishing of cohomology with coefficients in a free module, Poincaré duality groups and Hilbert modules.

Our principal references for group theory are [Bi], [DD] and [Ro].

1.1 Group theoretic notation and terminology

We shall reserve the notation Z for the free (abelian) group of rank 1 (with a prefered generator) and $\mathbb{Z}$ for the ring of integers. Let $F(r)$ be the free group of rank r.

Let G be a group. Then G' and ζG denote the commutator subgroup and centre of G, respectively. The outer automorphism group of G is $Out(G) = Aut(G)/Inn(G)$, where $Inn(G) \cong G/\zeta G$ is the subgroup of $Aut(G)$ consisting of conjugations by elements of G. If H is a subgroup of G let $N_G(H)$ and $C_G(H)$ denote the normalizer and centralizer of H in G, respectively. The subgroup H is a *characteristic* subgroup of G if it is preserved under all automorphisms of G. In particular, $I(G) = \{g \in G \mid \exists n > 0,\ g^n \in G'\}$ is a characteristic subgroup of G, and the quotient $G/I(G)$ is a torsion free abelian group of rank $\beta_1(G)$. A group G is *indicable* if there is an epimorphism $p : G \to Z$, or if $G = 1$. The *normal closure* of a subset $S \subseteq G$ is $\langle\langle S\rangle\rangle_G$, the intersection of the normal subgroups of G which contain S.

If P and Q are classes of groups let PQ denote the class of ("P *by* Q") groups G which have a normal subgroup H in P such that the quotient G/H is in Q, and let ℓP denote the class of ("*locally-*P") groups such that each finitely generated subgroup is in the class P. In particular, if F is the class of finite groups ℓF is the class of *locally-finite* groups. In any group the union of all the locally-finite normal subgroups is the unique maximal locally-finite normal subgroup. Clearly there are no nontrivial homomorphisms from such a group to a torsion free group. Let *poly-P* be the class of groups with a finite composition series such that each subquotient is in P. Thus if Ab is the class of abelian groups poly-Ab is the class of solvable groups.

Let P be a class of groups which is closed under taking subgroups. A group is *virtually* P if it has a subgroup of finite index in P. Let vP be the class of groups which are virtually P. Thus a *virtually poly-Z* group is one which has a subgroup of finite index with a composition series whose factors are all infinite cyclic. The number of infinite cyclic factors is independent of the choice of finite index subgroup or composition series, and is called the *Hirsch length* of the group. We shall also say that a space virtually has some property if it has a finite regular covering space with that property.

If $p : G \to Q$ is an epimorphism with kernel N we shall say that G is an *extension of $Q = G/N$ by the normal subgroup N*. The action of G on N by conjugation determines a homomorphism from G to $Aut(N)$ with kernel $C_G(N)$ and hence a homomorphism from G/N to $Out(N) = Aut(N)/Inn(N)$. If $G/N \cong Z$ the extension splits: a choice of element t in G which projects to a generator of G/N determines a right inverse to p. Let θ be the automorphism of N determined by conjugation by t in G. Then G is isomorphic to the semidirect product $N \times_\theta Z$. Every automorphism of N arises in this way, and automorphisms whose images in $Out(N)$ are conjugate determine isomorphic semidirect products. In particular, $G \cong N \times Z$ if θ is an inner automorphism.

Lemma 1.1 *Let θ and ϕ automorphisms of a group G such that $H_1(\theta;\mathbb{Q})-1$ and $H_1(\phi;\mathbb{Q})-1$ are automorphisms of $H_1(G;\mathbb{Q}) = (G/G')\otimes\mathbb{Q}$. Then the semidirect products $\pi_\theta = G\times_\theta Z$ and $\pi_\phi = G \times_\phi Z$ are isomorphic if and only if θ is conjugate to ϕ or ϕ^{-1} in $Out(G)$.*

Proof Let t and u be fixed elements of π_θ and π_ϕ, respectively, which map to 1 in Z. Since $H_1(\pi_\theta;\mathbb{Q}) \cong H_1(\pi_\phi;\mathbb{Q}) \cong Q$ the image of G in each group is characteristic. Hence an isomorphism $h : \pi_\theta \to \pi_\phi$ induces an isomorphism $e : Z \to Z$ of the quotients, for some $e = \pm 1$, and so $h(t) = u^e g$ for some g in G. Therefore $h(\theta(h^{-1}(j)))) = h(th^{-1}(j)t^{-1}) = u^e gjg^{-1}u^{-e} = \phi^e(gjg^{-1})$ for all j in G. Thus θ is conjugate to ϕ^e in $Out(G)$.

Conversely, if θ and ϕ are conjugate in $Out(G)$ there is an f in $Aut(G)$ and a g in G such that $\theta(j) = f^{-1}\phi^e f(gjg^{-1})$ for all j in G. Hence $F(j) = f(j)$ for all j in G and $F(t) = u^e f(g)$ defines an isomorphism $F : \pi_\theta \to \pi_\phi$. □

1.2 Matrix groups

In this section we shall recall some useful facts about matrices over $\mathbb{Z}$.

Lemma 1.2 *Let p be an odd prime. Then the kernel of the reduction modulo (p) homomorphism from $SL(n,\mathbb{Z})$ to $SL(n,\mathbb{F}_p)$ is torsion free.*

Proof This follows easily from the observation that if A is an integral matrix and $k = p^v q$ with q not divisible by p then $(I+p^r A)^k \equiv I + kp^r A \ mod\ (p^{2r+v})$, and $kp^r \not\equiv 0 \ mod\ (p^{2r+v})$ if $r \geq 1$. $\square$

The corresponding result for $p = 2$ is that the kernel of reduction *mod* (4) is torsion free.

Since $SL(n,\mathbb{F}_p)$ has order $(\Pi_{j=0}^{j=n-1}(p^n - p^j))/(p-1)$, it follows that the order of any finite subgroup of $SL(n,\mathbb{Z})$ must divide the highest common factor of these numbers, as p varies over all odd primes. In particular, finite subgroups of $SL(2,\mathbb{Z})$ have order dividing 24, and so are solvable.

Let $A = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $B = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 1 \end{smallmatrix}\right)$ and $R = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Then $A^2 = B^3 = -I$ and $A^4 = B^6 = I$. The matrices A and R generate a dihedral group of order 8, while B and R generate a dihedral group of order 12.

Theorem 1.3 *Let G be a nontrivial finite subgroup of $GL(2,\mathbb{Z})$. Then G is conjugate to one of the cyclic groups generated by A, A^2, B, B^2, R or RA, or to a dihedral subgroup generated by one of the pairs $\{A,R\}$, $\{A^2,R\}$, $\{A^2,RA\}$, $\{B,R\}$, $\{B^2,R\}$ or $\{B^2,RB\}$.*

Proof If $M \in GL(2,\mathbb{Z})$ has finite order then its characteristic polynomial has cyclotomic factors. If the characteristic polynomial is $(X \pm 1)^2$ then $M = \mp I$. (This uses the finite order of M.) If the characteristic polynomial is $X^2 - 1$ then M is conjugate to R or RA. If the characteristic polynomial is X^2+1, $X^2 - X + 1$ or $X^2 + X + 1$ then M is irreducible, and the corresponding ring of algebraic numbers is a PID. Since any $\mathbb{Z}$-torsion free module over such a ring is free it follows easily that M is conjugate to A, B or B^2.

The normalizers in $SL(2,\mathbb{Z})$ of the subgroups generated by A, B or B^2 are easily seen to be finite cyclic. Since $G \cap SL(2,\mathbb{Z})$ is solvable it must be cyclic also. As it has index at most 2 in G the theorem follows easily. $\square$

Although the 12 groups listed in the theorem represent distinct conjugacy classes in $GL(2,\mathbb{Z})$, some of these conjugacy classes coalesce in $GL(2,\mathbb{R})$. (For instance, R and RA are conjugate in $GL(2,\mathbb{Z}[\frac{1}{2}])$.)

Corollary 1.3.1 *Let G be a locally finite subgroup of $GL(2,\mathbb{R})$. Then G is finite, and is conjugate to one of the above subgroups of $GL(2,\mathbb{Z})$.*

Proof Let L be a lattice in $\mathbb{R}^2$. If G is finite then $\cup_{g\in G} gL$ is a G-invariant lattice, and so G is conjugate to a subgroup of $GL(2,\mathbb{Z})$. In general, as the finite subgroups of G have bounded order G must be finite. □

The main results of this section follow also from the fact that $PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\langle\pm I\rangle$ is a free product $(Z/2Z)*(Z/3Z)$, generated by the images of A and B. (In fact $\langle A,B \mid A^2 = B^3, A^4 = 1\rangle$ is a presentation for $SL(2,\mathbb{Z})$.) Moreover $SL(2,\mathbb{Z})' \cong PSL(2,\mathbb{Z})'$ is freely generated by the images of $B^{-1}AB^{-2}A = \left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)$ and $B^{-2}AB^{-1}A = \left(\begin{smallmatrix}1&1\\1&2\end{smallmatrix}\right)$, while the abelianizations are generated by the images of $B^4A = \left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$. (See §6.2 of [Ro].)

Let $\Lambda = \mathbb{Z}[t,t^{-1}]$ be the ring of integral Laurent polynomials. The next theorem is a special case of a classical result of Latimer and MacDuffee.

Theorem 1.4 *There is a 1-1 correspondance between conjugacy classes of matrices in $GL(n,\mathbb{Z})$ with irreducible characteristic polynomial $\Delta(t)$ and isomorphism classes of ideals in $\Lambda/(\Delta(t))$. The set of such ideal classes is finite.*

Proof Let $A \in GL(n,\mathbb{Z})$ have characteristic polynomial $\Delta(t)$ and let $R = \Lambda/(\Delta(t))$. As $\Delta(A) = 0$, by the Cayley-Hamilton Theorem, we may define an R-module M_A with underlying abelian group Z^n by $t.z = A(z)$ for all $z \in Z^n$. As R is a domain and has rank n as an abelian group M_A is torsion free and of rank 1 as an R-module, and so is isomorphic to an ideal of R. Conversely every R-ideal arises in this way. The isomorphism of abelian groups underlying an R-isomorphism between two such modules M_A and M_B determines a matrix $C \in GL(n,\mathbb{Z})$ such that $CA = BC$. The final assertion follows from the Jordan-Zassenhaus Theorem. □

1.3 The Hirsch-Plotkin radical

The *Hirsch-Plotkin radical* $\sqrt{G}$ of a group G is its maximal locally-nilpotent normal subgroup; in a virtually poly-Z group every subgroup is finitely generated, and so $\sqrt{G}$ is then the maximal nilpotent normal subgroup. If H is normal in G then $\sqrt{H}$ is normal in G also, since it is a characteristic subgroup of H, and in particular it is a subgroup of $\sqrt{G}$.

For each natural number $q \geq 1$ let Γ_q be the group with presentation

$$\langle x, y, z \mid xz = zx,\ yz = zy,\ xy = z^q yx \rangle.$$

Every such group Γ_q is torsion free and nilpotent of Hirsch length 3.

Theorem 1.5 *Let G be a finitely generated torsion free nilpotent group of Hirsch length $h(G) \leq 4$. Then either*

(1) *G is free abelian; or*

(2) *$h(G) = 3$ and $G \cong \Gamma_q$ for some $q \geq 1$; or*

(3) *$h(G) = 4$, $\zeta G \cong Z^2$ and $G \cong \Gamma_q \times Z$ for some $q \geq 1$; or*

(4) *$h(G) = 4$, $\zeta G \cong Z$ and $G/\zeta G \cong \Gamma_q$ for some $q \geq 1$.*

In the latter case G has characteristic subgroups which are free abelian of rank 1, 2 and 3. In all cases G is an extension of Z by a free abelian normal subgroup.

Proof The centre ζG is nontrivial and the quotient $G/\zeta G$ is again torsion free, by Proposition 5.2.19 of [Ro]. We may assume that G is not abelian, and hence that $G/\zeta G$ is not cyclic. Hence $h(G/\zeta G) \geq 2$, so $h(G) \geq 3$ and $1 \leq h(\zeta G) \leq h(G) - 2$. In all cases ζG is free abelian.

If $h(G) = 3$ then $\zeta G \cong Z$ and $G/\zeta G \cong Z^2$. On choosing elements x and y representing a basis of $G/\zeta G$ and z generating ζG we quickly find that G is isomorphic to one of the groups Γ_q, and thus is an extension of Z by Z^2.

If $h(G) = 4$ and $\zeta G \cong Z^2$ then $G/\zeta G \cong Z^2$, so $G' \subseteq \zeta G$. Since G may be generated by elements x, y, t and u where x and y represent a basis of $G/\zeta G$ and t and u are central it follows easily that G' is infinite cyclic. Therefore ζG is not contained in G' and G has an infinite cyclic direct factor. Hence $G \cong Z \times \Gamma_q$, for some $q \geq 1$, and thus is an extension of Z by Z^3.

The remaining possibility is that $h(G) = 4$ and $\zeta G \cong Z$. In this case $G/\zeta G$ is torsion free nilpotent of Hirsch length 3. If $G/\zeta G$ were abelian G' would also be infinite cyclic, and the pairing from $G/\zeta G \times G/\zeta G$ into G' defined by the commutator would be nondegenerate and skewsymmetric. But there are no such pairings on free abelian groups of odd rank. Therefore $G/\zeta G \cong \Gamma_q$, for some $q \geq 1$.

Let $\zeta_2 G$ be the preimage in G of $\zeta(G/\zeta G)$. Then $\zeta_2 G \cong Z^2$ and is a characteristic subgroup of G, so $C_G(\zeta_2 G)$ is also characteristic in G. The quotient $G/\zeta_2 G$ acts by conjugation on $\zeta_2 G$. Since $Aut(Z^2) = GL(2,\mathbb{Z})$ is virtually free and $G/\zeta_2 G \cong \Gamma_q/\zeta\Gamma_q \cong Z^2$ and since $\zeta_2 G \neq \zeta G$ it follows that $h(C_G(\zeta_2 G)) = 3$. Since $C_G(\zeta_2 G)$ is nilpotent and has centre of rank ≥ 2 it is abelian, and so $C_G(\zeta_2 G) \cong Z^3$. The preimage in G of the torsion subgroup of $G/C_G(\zeta_2 G)$ is torsion free, nilpotent of Hirsch length 3 and virtually abelian and hence is abelian. Therefore $G/C_G(\zeta_2 G) \cong Z$. □

Theorem 1.6 *Let π be a torsion free virtually poly-Z group of Hirsch length 4. Then $h(\sqrt{\pi}) \geq 3$.*

Proof Let S be a solvable normal subgroup of finite index in π. Then the lowest nontrivial term of the derived series of S is an abelian subgroup which is characteristic in S and so normal in π. Hence $\sqrt{\pi} \neq 1$. If $h(\sqrt{\pi}) \leq 2$ then $\sqrt{\pi} \cong Z$ or Z^2. Suppose π has an infinite cyclic normal subgroup A. On replacing π by a normal subgroup σ of finite index we may assume that A is central and that σ/A is poly-Z. Let B be the preimage in σ of a nontrivial abelian normal subgroup of σ/A. Then B is nilpotent (since A is central and B/A is abelian) and $h(B) > 1$ (since $B/A \neq 1$ and σ/A is torsion free). Hence $h(\sqrt{\pi}) \geq h(\sqrt{\sigma}) > 1$.

If π has a normal subgroup $N \cong Z^2$ then $Aut(N) \cong GL(2,\mathbb{Z})$ is virtually free, and so the kernel of the natural map from π to $Aut(N)$ is nontrivial. Hence $h(C_\pi(N)) \geq 3$. Since $h(\pi/N) = 2$ the quotient π/N is virtually abelian, and so $C_\pi(N)$ is virtually nilpotent.

In all cases we must have $h(\sqrt{\pi}) \geq 3$. □

1.4 Amenable groups

The class of *amenable* groups arose first in connection with the Banach-Tarski paradox. A group is amenable if it admits an invariant mean for bounded $\mathbb{C}$-valued functions [Pi]. There is a more geometric characterization of finitely presentable amenable groups that is more convenient for our purposes. Let X be a finite cell-complex with universal cover $\widetilde{X}$. Then $\widetilde{X}$ is an increasing union of finite subcomplexes $X_j \subseteq X_{j+1} \subseteq \widetilde{X} = \cup_{n\geq 1} X_n$ such that X_j is the union of $N_j < \infty$ translates of some fundamental domain D for $G = \pi_1(X)$. Let N'_j be the number of translates of D which meet the frontier of X_j in $\widetilde{X}$. The sequence $\{X_j\}$ is a *Følner exhaustion* for $\widetilde{X}$ if $\lim(N'_j/N_j) = 0$, and $\pi_1(X)$ is amenable if and only if $\widetilde{X}$ has a Følner exhaustion. This class contains all finite groups and Z, and is closed under the operations of extension, increasing union, and under the formation of sub- and quotient groups. (However nonabelian free groups are not amenable.)

The subclass EA generated from finite groups and Z by the operations of extension and increasing union is the class of *elementary amenable* groups. We may construct this class as follows. Let $U_0 = 1$ and U_1 be the class of finitely generated virtually abelian groups. If U_α has been defined for some ordinal α let $U_{\alpha+1} = (\ell U_\alpha)U_1$ and if U_α has been defined for all ordinals less than some limit ordinal β let $U_\beta = \cup_{\alpha<\beta}U_\alpha$. Let κ be the first uncountable ordinal. Then $EA = \ell U_\kappa$.

This class is well adapted to arguments by transfinite induction on the ordinal $\alpha(G) = \min\{\alpha | G \in U_\alpha\}$. It is closed under extension (in fact $U_\alpha U_\beta \subseteq U_{\alpha+\beta}$) and increasing union, and under the formation of sub- and quotient groups. As U_κ contains every countable elementary amenable group, $U_\lambda = \ell U_\kappa = EA$ if $\lambda > \kappa$. Torsion groups in EA are locally finite and elementary amenable free groups are cyclic. Every locally-finite by virtually solvable group is elementary amenable; however this inclusion is proper.

For example, let Z^∞ be the free abelian group with basis $\{x_i \mid i \in Z\}$ and let G be the subgroup of $Aut(Z^\infty)$ generated by $\{e_i \mid i \in Z\}$, where $e_i(x_i) = x_i + x_{i+1}$ and $e_i(x_j) = x_j$ if $j \neq i$. Then G is the increasing union of subgroups isomorphic to groups of upper triangular matrices, and so is locally nilpotent. However it has no nontrivial abelian normal subgroups. If we let ϕ be the automorphism of G defined by $\phi(e_i) = e_{i+1}$ for all i then $G \times_\phi Z$ is a finitely generated torsion free elementary amenable group which is not virtually solvable.

It can be shown (using the Følner condition) that finitely generated groups of subexponential growth are amenable. The class SA generated from such groups by extensions and increasing unions contains EA (since finite groups and finitely generated abelian groups have polynomial growth), and is the largest class of groups over which topological surgery techniques are known to work in dimension 4 [FT95]. Is every amenable group in SA? There is a finitely presentable group in SA which is not elementary amenable [Gr98].

A group is *restrained* if it has no noncyclic free subgroup. Amenable groups are restrained, but there are finitely presentable restrained groups which are not amenable [OS01]. There are also infinite finitely generated torsion groups. (See §14.2 of [Ro].) These are restrained, but are not elementary amenable. No known example is also finitely presentable.

1.5 Hirsch length

In this section we shall use transfinite induction to extend the notion of Hirsch length (as a measure of the size of a solvable group) to elementary amenable groups, and to establish the basic properties of this invariant.

Lemma 1.7 *Let G be a finitely generated infinite elementary amenable group. Then G has normal subgroups $K < H$ such that G/H is finite, H/K is free abelian of positive rank and the action of G/H on H/K by conjugation is effective.*

Proof We may show that G has a normal subgroup K such that G/K is an infinite virtually abelian group, by transfinite induction on $\alpha(G)$. We may assume that G/K has no nontrivial finite normal subgroup. If H is a subgroup of G which contains K and is such that H/K is a maximal abelian normal subgroup of G/K then H and K satisfy the above conditions. □

In particular, finitely generated infinite elementary amenable groups are virtually indicable.

If G is in U_1 let $h(G)$ be the rank of an abelian subgroup of finite index in G. If $h(G)$ has been defined for all G in U_α and H is in ℓU_α let

$$h(H) = \text{l.u.b.}\{h(F) | F \leq H,\ F \in U_\alpha\}.$$

Finally, if G is in $U_{\alpha+1}$, so has a normal subgroup H in ℓU_α with G/H in U_1, let $h(G) = h(H) + h(G/H)$.

Theorem 1.8 *Let G be an elementary amenable group. Then*

(1) *$h(G)$ is well defined;*

(2) *If H is a subgroup of G then $h(H) \leq h(G)$;*

(3) *$h(G) = \text{l.u.b.}\{h(F) \mid F\ is\ a\ finitely\ generated\ subgroup\ of\ G\}$;*

(4) *if H is a normal subgroup of G then $h(G) = h(H) + h(G/H)$.*

Proof We shall prove all four assertions simultaneously by induction on $\alpha(G)$. They are clearly true when $\alpha(G) = 1$. Suppose that they hold for all groups in U_α and that $\alpha(G) = \alpha + 1$. If G is in LU_α so is any subgroup, and (1) and (2) are immediate, while (3) follows since it holds for groups in U_α and since each finitely generated subgroup of G is a U_α-subgroup. To prove (4) we may assume that $h(H)$ is finite, for otherwise both $h(G)$ and $h(H) + h(G/H)$ are

∞, by (2). Therefore by (3) there is a finitely generated subgroup $J \leq H$ with $h(J) = h(H)$. Given a finitely generated subgroup Q of G/H we may choose a finitely generated subgroup F of G containing J and whose image in G/H is Q. Since F is finitely generated it is in U_α and so $h(F) = h(H)+h(Q)$. Taking least upper bounds over all such Q we have $h(G) \geq h(H) + h(G/H)$. On the other hand if F is any U_α-subgroup of G then $h(F) = h(F \cap H) + h(FH/H)$, since (4) holds for F, and so $h(G) \leq h(H) + h(G/H)$, Thus (4) holds for G also.

Now suppose that G is not in LU_α, but has a normal subgroup K in LU_α such that G/K is in U_1. If K_1 is another such subgroup then (4) holds for K and K_1 by the hypothesis of induction and so $h(K) = h(K \cap K_1) + h(KK_1/K)$. Since we also have $h(G/K) = h(G/KK_1)+h(KK_1/K)$ and $h(G/K_1) = h(G/KK_1)+h(KK_1/K_1)$ it follows that $h(K_1)+h(G/K_1) = h(K)+h(G/K)$ and so $h(G)$ is well defined. Property (2) follows easily, as any subgroup of G is an extension of a subgroup of G/K by a subgroup of K. Property (3) holds for K by the hypothesis of induction. Therefore if $h(K)$ is finite K has a finitely generated subgroup J with $h(J) = h(K)$. Since G/K is finitely generated there is a finitely generated subgroup F of G containing J and such that $FK/K = G/K$. Clearly $h(F) = h(G)$. If $h(K)$ is infinite then for every $n \geq 0$ there is a finitely generated subgroup J_n of K with $h(J_n) \geq n$. In either case, (3) also holds for G. If H is a normal subgroup of G then H and G/H are also in $U_{\alpha+1}$, while $H \cap K$ and $KH/H = K/H \cap K$ are in LU_α and $HK/K = H/H \cap K$ and G/HK are in U_1. Therefore

$$\begin{aligned} h(H) + h(G/H) &= h(H \cap K) + h(HK/K) + h(HK/H) + h(G/HK) \\ &= h(H \cap K) + h(HK/H) + h(HK/K) + h(G/HK). \end{aligned}$$

Since K is in LU_α and G/K is in U_1 this sum gives $h(G) = h(K) + h(G/K)$ and so (4) holds for G. This completes the inductive step. □

Let $\Lambda(G)$ be the maximal locally-finite normal subgroup of G.

Theorem 1.9 *There are functions d and M from $\mathbb{Z}_{\geq 0}$ to $\mathbb{Z}_{\geq 0}$ such that if G is an elementary amenable group of Hirsch length at most h and $\Lambda(G)$ is its maximal locally finite normal subgroup then $G/\Lambda(G)$ has a maximal solvable normal subgroup of derived length at most $d(h)$ and index at most $M(h)$.*

Proof We argue by induction on h. Since an elementary amenable group has Hirsch length 0 if and only if it is locally finite we may set $d(0) = 0$ and $M(0) = 1$. assume that the result is true for all such groups with Hirsch length at most h and that G is an elementary amenable group with $h(G) = h + 1$.

Suppose first that G is finitely generated. Then by Lemma 1.7 there are normal subgroups $K < H$ in G such that G/H is finite, H/K is free abelian of rank $r \geq 1$ and the action of G/H on H/K by conjugation is effective. (Note that $r = h(G/K) \leq h(G) = h + 1$.) Since the kernel of the natural map from $GL(r, \mathbb{Z})$ to $GL(r, \mathbb{F}_3)$ is torsion free, by Lemma 1.2, we see that G/H embeds in $GL(r, \mathbb{F}_3)$ and so has order at most 3^{r^2}. Since $h(K) = h(G) - r \leq h$ the inductive hypothesis applies for K, so it has a normal subgroup L containing $\Lambda(K)$ and of index at most $M(h)$ such that $L/\Lambda(K)$ has derived length at most $d(h)$ and is the maximal solvable normal subgroup of $K/\Lambda(K)$. As $\Lambda(K)$ and L are characteristic in K they are normal in G. (In particular, $\Lambda(K) = K \cap \Lambda(G)$.) The centralizer of K/L in H/L is a normal solvable subgroup of G/L with index at most $[K : L]![G : H]$ and derived length at most 2. Set $M(h + 1) = M(h)!3^{(h+1)^2}$ and $d(h + 1) = M(h + 1) + 2 + d(h)$. Then $G.\Lambda(G)$ has a maximal solvable normal subgroup of index at most the centralizer of K/L in H/L).

In general, let $\{G_i \mid i \in I\}$ be the set of finitely generated subgroups of G. By the above argument G_i has a normal subgroup H_i containing $\Lambda(G_i)$ and such that $H_i/\Lambda(G_i)$ is a maximal normal solvable subgroup of $G_i/\Lambda(G_i)$ and has derived length at most $d(h + 1)$ and index at most $M(h + 1)$. Let $N = \max\{[G_i : H_i] \mid i \in I\}$ and choose $\alpha \in I$ such that $[G_\alpha : H_\alpha] = N$. If $G_i \geq G_\alpha$ then $H_i \cap G_\alpha \leq H_\alpha$. Since $[G_\alpha : H_\alpha] \leq [G_\alpha : H_i \cap G_\alpha] = [H_i G_\alpha : H_i] \leq [G_i : H_i]$ we have $[G_i : H_i] = N$ and $H_i \geq H_\alpha$. It follows easily that if $G_\alpha \leq G_i \leq G_j$ then $H_i \leq H_j$.

Set $J = \{i \in I \mid H_\alpha \leq H_i\}$ and $H = \cup_{i \in J} H_i$. If $x, y \in H$ and $g \in G$ then there are indices i, k and $k \in J$ such that $x \in H_i$, $y \in H_j$ and $g \in G_k$. Choose $l \in J$ such that G_l contains $G_i \cup G_j \cup G_k$. Then xy^{-1} and gxg^{-1} are in $H_l \leq H$, and so H is a normal subgroup of G. Moreover if $x_1, \dots, x_N$ is a set of coset representatives for H_α in G_α then it remains a set of coset representatives for H in G, and so $[G; H] = N$.

Let D_i be the $d(h + 1)^{th}$ derived subgroup of H_i. Then D_i is a locally-finite normal subgroup of G_i and so, bu an argument similar to that of the above paragraph $\cup_{i \in J} D_i$ is a locally-finite normal subgroup of G. Since it is easily seen that the $d(h + 1)^{th}$ derived subgroup of H is contained in $\cup_{i \in J} D_i$ (as each iterated commutator involves only finitely many elements of H) it follows that $H\Lambda(G)/\Lambda(G) \cong H/H \cap \Lambda(G)$ is solvable and of derived length at most $d(h + 1)$. □

The above result is from [HL92]. The argument can be simplified to some extent if G is countable and torsion-free. (In fact a virtually solvable group of finite Hirsch length and with no nontrivial locally-finite normal subgroup must be countable, by Lemma 7.9 of [Bi]. Moreover its Hirsch-Plotkin radical is nilpotent and the quotient is virtually abelian, by Proposition 5.5 of [BH72].)

Lemma 1.10 *Let G be an elementary amenable group. If $h(G) = \infty$ then for every $k > 0$ there is a subgroup H of G with $k < h(H) < \infty$.*

Proof We shall argue by induction on $\alpha(G)$. The result is vacuously true if $\alpha(G) = 1$. Suppose that it is true for all groups in U_α and G is in ℓU_α. Since $h(G) = \text{l.u.b.}\{h(F)|F \leq G,\ F \in U_\alpha\}$ either there is a subgroup F of G in U_α with $h(F) = \infty$, in which case the result is true by the inductive hypothesis, or $h(G)$ is the least upper bound of a set of natural numbers and the result is true. If G is in $U_{\alpha+1}$ then it has a normal subgroup N which is in ℓU_α with quotient G/N in U_1. But then $h(N) = h(G) = \infty$ and so N has such a subgroup. □

Theorem 1.11 *Let G be a countable elementary amenable group of finite cohomological dimension. Then $h(G) \leq c.d.G$ and G is virtually solvable.*

Proof Since $c.d.G < \infty$ the group G is torsion free. Let H be a subgroup of finite Hirsch length. Then H is virtually solvable and $c.d.H \leq c.d.G$ so $h(H) \leq c.d.G$. The theorem now follows from Theorem 1.9 and Lemma 1.10. □

1.6 Modules and finiteness conditions

Let G be a group and $w : G \to Z/2Z$ a homomorphism, and let R be a commutative ring. Then $\bar{g} = (-1)^{w(g)}g^{-1}$ defines an anti-involution on $R[G]$. If L is a left $R[G]$-module $\overline{L}$ shall denote the *conjugate* right $R[G]$-module with the same underlying R-module and $R[G]$-action given by $l.g = \bar{g}.l$, for all $l \in L$ and $g \in G$. (We shall also use the overline to denote the conjugate of a right $R[G]$-module.) The conjugate of a free left (right) module is a free right (left) module of the same rank.

We shall also let Z^w denote the G-module with underlying abelian group Z and G-action given by $g.n = (-1)^{w(g)}n$ for all g in G and n in Z.

Lemma 1.12 [Wl65] *Let G and H be groups such that G is finitely presentable and there are homomorphisms $j : H \to G$ and $\rho : G \to H$ with $\rho j = id_H$. Then H is also finitely presentable.*

Proof Since G is finitely presentable there is an epimorphism $p : F \to G$ from a free group $F(X)$ with a finite basis X onto G, with kernel the normal closure of a finite set of relators R. We may choose elements w_x in $F(X)$ such that $j\rho p(x) = p(w_x)$, for all x in X. Then ρ factors through the group K with presentation $\langle X \mid R, x^{-1}w_x, \forall x \in X\rangle$, say $\rho = vu$. Now uj is clearly onto, while $vuj = \rho j = id_H$, and so v and uj are mutually inverse isomomorphisms. Therefore $H \cong K$ is finitely presentable. □

A group G is FP_n if the augmentation $\mathbb{Z}[G]$-module Z has a projective resolution which is finitely generated in degrees $\leq n$, and it is FP if it has finite cohomological dimension and is FP_n for $n = c.d.G$. It is FF if moreover Z has a finite resolution consisting of finitely generated free $\mathbb{Z}[G]$-modules. "Finitely generated" is equivalent to FP_1, while "finitely presentable" implies FP_2. Groups which are FP_2 are also said to be *almost finitely presentable.* (There are FP groups which are not finitely presentable [BB97].) An elementary amenable group G is FP_∞ if and only if it is virtually FP, and is then virtually constructible and solvable of finite Hirsch length [Kr93].

If the augmentation $\mathbb{Q}[\pi]$-module Q has a finite resolution F_* by finitely generated projective modules then $\chi(\pi) = \Sigma(-1)^i dim_{\mathbb{Q}}(\mathbb{Q}\otimes_\pi F_i)$ is independent of the resolution. (If π is the fundamental group of an aspherical finite complex K then $\chi(\pi) = \chi(K)$.) We may extend this definition to groups σ which have a subgroup π of finite index with such a resolution by setting $\chi(\sigma) = \chi(\pi)/[\sigma : \pi]$. (It is not hard to see that this is well defined.)

Let P be a finitely generated projective $\mathbb{Z}[\pi]$-module. Then P is a direct summand of $\mathbb{Z}[\pi]^r$, for some $r \geq 0$, and so is the image of some idempotent $r \times r$-matrix M with entries in $\mathbb{Z}[\pi]$. The *Kaplansky rank* $\kappa(P)$ is the coefficient of $1 \in \pi$ in the trace of M. It depends only on P and is strictly positive if $P \neq 0$. The group π satisfies the *Weak Bass Conjecture* if $\kappa(P) = dim_{\mathbb{Q}}Q\otimes_\pi P$. This conjecture has been confirmed for linear groups, solvable groups and groups of cohomological dimension ≤ 2 over $\mathbb{Q}$. (See [Dy87, Ec86, Ec96] for further details.)

The following result from [BS78] shall be useful.

Theorem 1.13 (Bieri-Strebel) *Let G be an FP_2 group such that G/G' is infinite. Then G is an HNN extension with finitely generated base and associated subgroups.*

Proof (Sketch – We shall assume that G is finitely presentable.) Let $h : F(m) \to G$ be an epimorphism, and let $g_i = h(x_i)$ for $1 \leq i \leq m$. We may assume that g_m has infinite order modulo the normal closure of $\{g_i \mid 1 \leq i < m\}$. Since G is finitely presentable the kernel of h is the normal closure of finitely many relators, of weight 0 in the letter x_m. Each such relator is a product of powers of conjugates of the generators $\{x_i \mid 1 \leq i < m\}$ by powers of x_m. Thus we may assume the relators are contained in the subgroup generated by $\{x_m^j x_i x_m^{-j} \mid 1 \leq i \leq m, -p \leq j \leq p\}$, for some sufficiently large p. Let U be the subgroup of G generated by $\{g_m^j g_i g_m^{-j} \mid 1 \leq i \leq m, -p \leq j < p\}$, and let $V = g_m U g_m^{-1}$. Let B be the subgroup of G generated by $U \cup V$ and let $\tilde{G}$ be the HNN extension with base B and associated subgroups U and V presented by $\tilde{G} = \langle B, s \mid sus^{-1} = \tau(u)\ \forall u \in U\rangle$, where $\tau : U \to V$ is the isomorphism determined by conjugation by g_m in G. There are obvious epimorphisms $\xi : F(m+1) \to \tilde{G}$ and $\psi : \tilde{G} \to G$ with composite h. It is easy to see that $\mathrm{Ker}(h) \leq \mathrm{Ker}(\xi)$ and so $\tilde{G} \cong G$. □

In particular, if G is restrained then it is an ascending HNN extension.

A ring R is *weakly finite* if every onto endomorphism of R^n is an isomorphism, for all $n \geq 0$. (In [H2] the term "SIBN ring" was used instead.) Finitely generated stably free modules over weakly finite rings have well defined ranks, and the rank is strictly positive if the module is nonzero. Skew fields are weakly finite, as are subrings of weakly finite rings. If G is a group its complex group algebra $\mathbb{C}[G]$ is weakly finite, by a result of Kaplansky. (See [Ro84] for a proof.)

A ring R is *(regular) coherent* if every finitely presentable left R-module has a (finite) resolution by finitely generated projective R-modules, and is *(regular) noetherian* if moreover every finitely generated R-module is finitely presentable. A group G is regular coherent or regular noetherian if the group ring $R[G]$ is regular coherent or regular noetherian (respectively) for any regular noetherian ring R. It is coherent as a *group* if all its finitely generated subgroups are finitely presentable.

Lemma 1.14 *If G is a group such that $\mathbb{Z}[G]$ is coherent then every finitely generated subgroup of G is FP_∞.*

Proof Let H be a subgroup of G. Since $\mathbb{Z}[H] \leq \mathbb{Z}[G]$ is a faithfully flat ring extension a left $\mathbb{Z}[H]$-module is finitely generated over $\mathbb{Z}[H]$ if and only if the induced module $\mathbb{Z}[G] \otimes_H M$ is finitely generated over $\mathbb{Z}[G]$. It follows by induction on n that M is FP_n over $\mathbb{Z}[H]$ if and only if $\mathbb{Z}[G] \otimes_H M$ is FP_n over $\mathbb{Z}[G]$.

If H is finitely generated then the augmentation $\mathbb{Z}[H]$-module Z is finitely presentable over $\mathbb{Z}[H]$. Hence $\mathbb{Z}[G] \otimes_H Z$ is finitely presentable over $\mathbb{Z}[G]$, and so is FP_∞ over $\mathbb{Z}[G]$, since that ring is coherent. Hence Z is FP_∞ over $\mathbb{Z}[H]$, i.e., H is FP_∞. □

Thus if either G is coherent (as a group) or $\mathbb{Z}[G]$ is coherent (as a ring) every finitely generated subgroup of G is FP_2. As the latter condition shall usually suffice for our purposes below, we shall say that such a group is *almost coherent*. The connection between these notions has not been much studied.

The class of groups whose integral group ring is regular coherent contains the trivial group and is closed under generalised free products and HNN extensions with amalgamation over subgroups whose group rings are regular noetherian, by Theorem 19.1 of [Wd78]. If $[G:H]$ is finite and G is torsion free then $\mathbb{Z}[G]$ is regular coherent if and only if $\mathbb{Z}[H]$ is. In particular, free groups and surface groups are coherent and their integral group rings are regular coherent, while (torsion free) virtually poly-Z groups are coherent and their integral group rings are (regular) noetherian.

1.7 Ends and cohomology with free coefficients

A finitely generated group G has 0, 1, 2 or infinitely many ends. It has 0 ends if and only if it is finite, in which case $H^0(G;\mathbb{Z}[G])\cong Z$ and $H^q(G;\mathbb{Z}[G])=0$ for $q>0$. Otherwise $H^0(G;\mathbb{Z}[G])=0$ and $H^1(G;\mathbb{Z}[G])$ is a free abelian group of rank $e(G)-1$, where $e(G)$ is the number of ends of G [Sp49]. The group G has more than one end if and only if it is either a nontrivial generalised free product with amalgamation $G\cong A*_C B$ or an HNN extension $A*_C\phi$ where C is a finite group. In particular, it has two ends if and only if it is virtually Z if and only if it has a (maximal) finite normal subgroup F such that the quotient G/F is either infinite cyclic (Z) or infinite dihedral ($D=(Z/2Z)*(Z/2Z)$). (See [DD].)

Lemma 1.15 *Let N be a finitely generated restrained group. Then N is either finite or virtually Z or has one end.*

Proof Groups with infinitely many ends have noncyclic free subgroups. □

It follows that a countable restrained group is either elementary amenable of Hirsch length at most 1 or it is an increasing union of finitely generated, one-ended subgroups.

If G is a group with a normal subgroup N, and A is a left $\mathbb{Z}[G]$-module there is a *Lyndon-Hochschild-Serre spectral sequence* (LHSSS) for G as an extension of G/N by N and with coefficients A:

$$E_2 = H^p(G/N; H^q(N;A)) \Rightarrow H^{p+q}(G;A),$$

the r^{th} differential having bidegree $(r, 1-r)$. (See Section 10.1 of [Mc].)

Theorem 1.16 [Ro75] *If G has a normal subgroup N which is the union of an increasing sequence of subgroups N_n such that $H^s(N_n;\mathbb{Z}[G]) = 0$ for $s \le r$ then $H^s(G;\mathbb{Z}[G]) = 0$ for $s \le r$.*

Proof Let $s \le r$. Let f be an s-cocycle for N with coefficients $\mathbb{Z}[G]$, and let f_n denote the restriction of f to a cocycle on N_n. Then there is an $(s-1)$-cochain g_n on N_n such that $\delta g_n = f_n$. Since $\delta(g_{n+1}|_{N_n} - g_n) = 0$ and $H^{s-1}(N_n;\mathbb{Z}[G]) = 0$ there is an $(s-2)$-cochain h_n on N_n with $\delta h_n = g_{n+1}|_{N_n} - g_n$. Choose an extension h_n' of h_n to N_{n+1} and let $\hat{g}_{n+1} = g_{n+1} - \delta h_n'$. Then $\hat{g}_{n+1}|_{N_n} = g_n$ and $\delta \hat{g}_{n+1} = f_{n+1}$. In this way we may extend g_0 to an $(s-1)$-cochain g on N such that $f = \delta g$ and so $H^s(N;\mathbb{Z}[G]) = 0$. The LHSSS for G as an extension of G/N by N, with coefficients $\mathbb{Z}[G]$, now gives $H^s(G;\mathbb{Z}[G]) = 0$ for $s \le r$. ∎

Corollary 1.16.1 *The hypotheses are satisfied if N is the union of an increasing sequence of FP_r subgroups N_n such that $H^s(N_n;\mathbb{Z}[N_n]) = 0$ for $s \le r$. In particular, if N is the union of an increasing sequence of finitely generated, one-ended subgroups then G has one end.*

Proof We have $H^s(N_n;\mathbb{Z}[G]) = H^s(N_n;\mathbb{Z}[N_n]) \otimes \mathbb{Z}[G/N_n] = 0$, for all $s \le r$ and all n, since N_n is FP_r. ∎

In particular, G has one end if N is a countable elementary amenable group and $h(N) > 1$, by Lemma 1.15.

The following results are Theorems 8.8 of [Bi] and Theorem 0.1 of [BG85], respectively.

Theorem (Bieri) *Let G be a nonabelian group with $c.d.G = n$. Then $c.d.\zeta G \le n-1$, and if ζG has rank $n-1$ then G' is free.* ∎

Theorem (Brown-Geoghegan) *Let G be an HNN extension $B*_\phi$ in which the base H and associated subgroups I and $\phi(I)$ are FP_n. If the homomorphism from $H^q(B;Z[G])$ to $H^q(I;Z[G])$ induced by restriction is injective for some $q \le n$ then the corresponding homomorphism in the Mayer-Vietoris sequence is injective, so $H^q(G;Z[G])$ is a quotient of $H^{q-1}(I;Z[G])$.* ∎

The second cohomology of a group with free coefficients ($H^2(G;R[G])$, $R=\mathbb{Z}$ or a field) shall play an important role in our investigations.

Theorem (Farrell) *Let G be a finitely presentable group. If G has an element of infinite order and $R=\mathbb{Z}$ or is a field then $H^2(G;R[G])$ is either 0 or R or is not finitely generated.* $\square$

Farrell also showed in [Fa74] that if $H^2(G;\mathbb{F}_2[G])\cong Z/2Z$ then every finitely generated subgroup of G with one end has finite index in G. Hence if G is also torsion free then subgroups of infinite index in G are locally free. Bowditch has since shown that such groups are virtually the fundamental groups of aspherical closed surfaces ([Bo99] - see §8 below).

We would also like to know when $H^2(G;\mathbb{Z}[G])$ is 0 (for G finitely presentable). In particular, we expect this to the case if G is an ascending HNN extension over a finitely generated, one-ended base, or if G has an elementary amenable, normal subgroup E such that either $h(E)=1$ and G/E has one end or $h(E)=2$ and $[G:E]=\infty$ or $h(E)\geq 3$. However our criteria here at present require finiteness hypotheses, either in order to apply an LHSSS argument or in the form of coherence.

Theorem 1.17 *Let G be a finitely presentable group with an almost coherent, locally virtually indicable, restrained normal subgroup E. Suppose that either E is abelian of rank 1 and G/E has one end or that E has a finitely generated, one-ended subgroup and G is not elementary amenable of Hirsch length 2. Then $H^s(G;\mathbb{Z}[G])=0$ for $s\leq 2$.*

Proof If E is abelian of positive rank and G/E has one end then G is 1-connected at ∞ and so $H^s(G;\mathbb{Z}[G])=0$ for $s\leq 2$, by Theorem 1 of [Mi87], and so $H^s(G;\mathbb{Z}[G])=0$ for $s\leq 2$, by [GM86].

We may assume henceforth that E is an increasing union of finitely generated one-ended subgroups $E_n\subseteq E_{n+1}\dots\subseteq E=\cup E_n$. Since E is locally virtually indicable there are subgroups $F_n\leq E_n$ such that $[E_n:F_n]<\infty$ and which map onto Z. Since E is almost coherent these subgroups are FP_2. Hence they are HNN extensions over FP_2 bases H_n, by Theorem 1.13, and the extensions are ascending, since E is restrained. Since E_n has one end H_n has one or two ends.

If H_n has two ends then E_n is elementary amenable and $h(E_n)=2$. Therefore if H_n has two ends for all n then $[E_{n+1}:E_n]<\infty$, E is elementary amenable and $h(E) = 2$. If $[G : E] < \infty$ then G is elementary amenable and $h(G) = 2$, and so we may assume that $[G : E] = \infty$. If E is finitely generated then it is FP_2 and so $H^s(G; \mathbb{Z}[G]) = 0$ for $s \leq 2$, by an LHSSS argument. This is also the case if E is not finitely generated, for then $H^s(E; \mathbb{Z}[G]) = 0$ for $s \leq 2$, by the argument of Theorem 3.3 of [GS81], and we may again apply an LHSSS argument. (The hypothesis of [GS81] that "each G_n is FP and $c.d.G_n = h$" can be relaxed to "each G_n is FP_h".)

Otherwise we may assume that H_n has one end, for all $n \geq 1$. In this case $H^s(F_n; \mathbb{Z}[F_n]) = 0$ for $s \leq 2$, by the Theorem of Brown and Geoghegan. Therefore $H^s(G; \mathbb{Z}[G]) = 0$ for $s \leq 2$, by Theorem 1.16. □

The theorem applies if E is almost coherent and elementary amenable, and either $h(E) = 2$ and $[G : E] = \infty$ or $h(E) \geq 3$, since elementary amenable groups are restrained and locally virtually indicable. It also applies if $E = \sqrt{G}$ is large enough, since finitely generated nilpotent groups are virtually poly-Z. A similar argument shows that if $h(\sqrt{G}) \geq r$ then $H^s(G; \mathbb{Z}[G]) = 0$ for $s < r$. If moreover $[G : \sqrt{G}] = \infty$ then $H^r(G; \mathbb{Z}[G]) = 0$ also.

Are the hypotheses that E be almost coherent and locally virtually indicable necessary? Is it sufficient that E be restrained and be an increasing union of finitely generated, one-ended subgroups?

Theorem 1.18 *Let $G = B*_\phi$ be an HNN extension with FP_2 base B and associated subgroups I and $\phi(I) = J$, and which has a restrained normal subgroup $N \leq \langle\langle B \rangle\rangle$. Then $H^s(G; \mathbb{Z}[G]) = 0$ for $s \leq 2$ if either*

(1) *the HNN extension is ascending and $B = I \cong J$ has one end;*

(2) *N is locally virtually Z and G/N has one end; or*

(3) *N has a finitely generated subgroup with one end.*

Proof The first assertion follows immediately from the Brown-Geogeghan Theorem.

Let t be the stable letter, so that $tit^{-1} = \phi(i)$, for all $i \in I$. Suppose that $N \cap J \neq N \cap B$, and let $b \in N \cap B - J$. Then $b^t = t^{-1}bt$ is in N, since N is normal in G. Let a be any element of $N \cap B$. Since N has no noncyclic free subgroup there is a word $w \in F(2)$ such that $w(a, b^t) = 1$ in G. It follows from Britton's Lemma that a must be in I and so $N \cap B = N \cap I$. In particular, N is the increasing union of copies of $N \cap B$.

Hence G/N is an HNN extension with base $B/N\cap B$ and associated subgroups $I/N\cap I$ and $J/N\cap J$. Therefore if G/N has one end the latter groups are infinite, and so B, I and J each have one end. If N is virtually Z then $H^s(G;\mathbb{Z}[G])=0$ for $s\leq 2$, by an LHSSS argument. If N is locally virtually Z but is not finitely generated then it is the increasing union of a sequence of two-ended subgroups and $H^s(N;\mathbb{Z}[G])=0$ for $s\leq 1$, by Theorem 3.3 of [GS81]. Since $H^2(B;\mathbb{Z}[G])\cong H^0(B;H^2(N\cap B;\mathbb{Z}[G]))$ and $H^2(I;\mathbb{Z}[G])\cong H^0(I;H^2(N\cap I;\mathbb{Z}[G]))$, the restriction map from $H^2(B;\mathbb{Z}[G])$ to $H^2(I;\mathbb{Z}[G])$ is injective. If N has a finitely generated, one-ended subgroup N_1, we may assume that $N_1\leq N\cap B$, and so B, I and J also have one end. Moreover $H^s(N\cap B;\mathbb{Z}[G])=0$ for $s\leq 1$, by Theorem 1.16. We again see that the restriction map from $H^2(B;\mathbb{Z}[G])$ to $H^2(I;\mathbb{Z}[G])$ is injective. The result now follows in these cases from the Theorem of Brown and Geoghegan. □

1.8 Poincaré duality groups

A group G is a *PD_n-group* if it is FP, $H^p(G;\mathbb{Z}[G])=0$ for $p\neq n$ and $H^n(G;\mathbb{Z}[G])\cong Z$. The "dualizing module" $H^n(G;\mathbb{Z}[G])=Ext^n_{\mathbb{Z}[G]}(Z,\mathbb{Z}[G])$ is a right $\mathbb{Z}[G]$-module; the group is *orientable* (or is a *PD_n^+-group*) if it acts trivially on the dualizing module, i.e., if $H^n(G;\mathbb{Z}[G])$ is isomorphic to the augmentation module Z. (See [Bi].)

The only PD_1-group is Z. Eckmann, Linnell and Müller showed that every PD_2-group is the fundamental group of a closed aspherical surface. (See Chapter VI of [DD].) Bowditch has since found a much stronger result, which must be close to the optimal characterization of such groups [Bo99].

Theorem (Bowditch) *Let G be an almost finitely presentable group and F a field. Then G is virtually a PD_2-group if and only if $H^2(G;F[G])$ has a 1-dimensional G-invariant subspace.* □

In particular, this theorem applies if $H^2(G;\mathbb{Z}[G])\cong Z$. for then the image of $H^2(G;\mathbb{Z}[G])$ in $H^2(G;\mathbb{F}_2[G])$ under reduction *mod* (2) is such a subspace.

The following result from [St77] corresponds to the fact that an infinite covering space of a PL n-manifold is homotopy equivalent to a complex of dimension $<n$.

Theorem (Strebel) *Let H be a subgroup of infinite index in a PD_n-group G. Then c.d.$H<n$.* □

If R is a subring of S, A is a left R-module and C is a left S-module then the abelian groups $Hom_R(C|_R, A)$ and $Hom_S(C, Hom_R(S|_R, A))$ are naturally isomorphic, where $C|_R$ and $S|_R$ are the left R-modules underlying C and S respectively. (The maps I and J defined by $I(f)(c)(s) = f(sc)$ and $J(\theta)(c) = \theta(c)(1)$ for $f : C \to A$ and $\theta : C \to Hom_R(S, A)$ are mutually inverse isomorphisms.) When K is a subgroup of π and $R = \mathbb{Z}[K]$ and $S = \mathbb{Z}[\pi]$ these isomorphisms give rise to Shapiro's lemma. In our applications π/K shall usually be infinite cyclic and S is then a twisted Laurent extension of R.

Theorem 1.19 *Let π be a PD_n-group with an FP_r normal subgroup K such that $G = \pi/K$ is a PD_{n-r} group and $2r \geq n-1$. Then K is a PD_r-group.*

Proof It shall suffice to show that $H^s(K; F) = 0$ for any free $\mathbb{Z}[K]$-module F and all $s > r$, for then $c.d.K = r$ and the result follows from Theorem 9.11 of [Bi]. Let $W = Hom_{\mathbb{Z}[K]}(\mathbb{Z}[\pi], F)$ be the $\mathbb{Z}[\pi]$-module coinduced from F. Then $H^s(K; F) \cong H^s(\pi; W) \cong H_{n-s}(\pi; \overline{W})$, by Shapiro's lemma and Poincaré duality. As a $\mathbb{Z}[K]$-module $\overline{W} \cong F^G$ (the direct product of $|G|$ copies of F), and so $H_q(K; \overline{W}) = 0$ for $0 < q \leq r$ (since K is FP_r), while $H_0(K; \overline{W}) \cong A^G$, where $A = H_0(K; F)$. Moreover $A^G \cong Hom_{\mathbb{Z}}(\mathbb{Z}[G], A)$ as a $\mathbb{Z}[G]$-module, and so is coinduced from a module over the trivial group. Therefere if $n - s \leq r$ the LHSSS gives $H^s(K; F) \cong H_{n-s}(G; A^G)$. Poincaré duality for G and another application of Shapiro's lemma now give $H^s(K; F) \cong H^{s-r}(G; A^G) \cong H^{s-r}(1; A) = 0$, if $s > r$. □

If the quotient is poly-Z we can do somewhat better.

Theorem 1.20 *Let π be a PD_n-group which is an extension of Z by a normal subgroup K which is $FP_{[n/2]}$. Then K is a PD_{n-1}-group.*

Proof It is sufficient to show that $\varinjlim H^q(K; M_i) = 0$ for any direct system $\{M_i\}_{i \in I}$ with limit 0 and for all $q \leq n-1$, for then K is FP_{n-1} [Br75], and the result again follows from Theorem 9.11 of [Bi]. Since K is $FP_{[n/2]}$ we may assume $q > n/2$. We have $H^q(K; M_i) \cong H^q(\pi; W_i) \cong H_{n-q}(\pi; \overline{W_i})$, where $W_i = Hom_{\mathbb{Z}[K]}(\mathbb{Z}[\pi], M_i)$, by Shapiro's lemma and Poincaré duality. The LHSSS for π as an extension of Z by K reduces to short exact sequences

$$0 \to H_0(\pi/K; H_s(K; \overline{W_i})) \to H_s(\pi; \overline{W_i}) \to H_1(\pi/K; H_{s-1}(K; \overline{W_i})) \to 0.$$

As a $\mathbb{Z}[K]$-module $W_i \cong (M_i)^{\pi/K}$ (the direct product of countably many copies of M_i). Since K is $FP_{[n/2]}$ homology commutes with direct products in this range, and so $H_s(K;\overline{W_i}) = H_s(K;\overline{M_i})^{\pi/K}$ if $s \leq n/2$. As π/K acts on this module by shifting the entries we see that $H_s(\pi;\overline{W_i}) \cong H_{s-1}(K;\overline{M_i})$ if $s \leq n/2$, and the result now follows easily. □

A similar argument shows that if π is a PD_n-group and $\phi : \pi \to Z$ is any epimorphism then $c.d.\mathrm{Ker}(\phi) < n$. (This weak version of Strebel's Theorem suffices for some of the applications below.)

Corollary 1.20.1 *If a PD_n-group π is an extension of a virtually poly-Z group Q by an $FP_{[n/2]}$ normal subgroup K then K is a $PD_{n-h(Q)}$-group.* □

1.9 Hilbert modules

Let π be a countable group and let $\ell^2(\pi)$ be the Hilbert space completion of $\mathbb{C}[\pi]$ with respect to the inner product given by $(\Sigma a_g g, \Sigma b_h h) = \Sigma a_g \overline{b_g}$. Left and right multiplication by elements of π determine left and right actions of $\mathbb{C}[\pi]$ as bounded operators on $\ell^2(\pi)$. The (left) von Neumann algebra $\mathcal{N}(\pi)$ is the algebra of bounded operators on $\ell^2(\pi)$ which are $\mathbb{C}[\pi]$-linear with respect to the left action. By the Tomita-Takesaki theorem this is also the bicommutant in $B(\ell^2(\pi))$ of the right action of $\mathbb{C}[\pi]$, i.e., the set of operators which commute with every operator which is right $\mathbb{C}[\pi]$-linear. (See pages 45-52 of [Su].) We may clearly use the canonical involution of $\mathbb{C}[\pi]$ to interchange the roles of left and right in these definitions.

If $e \in \pi$ is the unit element we may define the von Neumann trace on $\mathcal{N}(\pi)$ by the inner product $tr(f) = (f(e), e)$. This extends to square matrices over $\mathcal{N}(\pi)$ by taking the sum of the traces of the diagonal entries. A *Hilbert* $\mathcal{N}(\pi)$-*module* is a Hilbert space M with a unitary left π-action which embeds isometrically and π-equivariantly into the completed tensor product $H\widehat{\otimes}\ell^2(\pi)$ for some Hilbert space H. It is finitely generated if we may take $H \cong \mathbb{C}^n$ for some integer n. (In this case we do not need to complete the ordinary tensor product over $\mathbb{C}$.) A *morphism* of Hilbert $\mathcal{N}(\pi)$-modules is a π-equivariant bounded linear operator $f : M \to N$. It is a *weak isomorphism* if it is injective and has dense image. A bounded π-linear operator on $\ell^2(\pi)^n = \mathbb{C}^n \otimes \ell^2(\pi)$ is represented by a matrix whose entries are in $\mathcal{N}(\pi)$. The von Neumann dimension of a finitely generated Hilbert $\mathcal{N}(\pi)$-module M is the real number $dim_{\mathcal{N}(\pi)}(M) = tr(P) \in [0, \infty)$, where P is any projection operator on $H \otimes \ell^2(\pi)$ with image π-isometric to M. In particular, $dim_{\mathcal{N}(\pi)}(M) = 0$ if and only if $M = 0$. The notions of finitely generated Hilbert $\mathcal{N}(\pi)$-module and finitely generated projective $\mathcal{N}(\pi)$-module are essentially equivalent, and arbitrary $\mathcal{N}(\pi)$-modules have well-defined dimensions in $[0,\infty]$ [Lü].

A sequence of bounded maps between Hilbert $\mathcal{N}(\pi)$-modules

$$M \xrightarrow{j} N \xrightarrow{p} P$$

is *weakly exact at* N if $\mathrm{Ker}(p)$ is the closure of $\mathrm{Im}(j)$. If $0 \to M \to N \to P \to 0$ is weakly exact then j is injective, $\mathrm{Ker}(p)$ is the closure of $\mathrm{Im}(j)$ and $\mathrm{Im}(p)$ is dense in P, and $dim_{\mathcal{N}(\pi)}(N) = dim_{\mathcal{N}(\pi)}(M) + dim_{\mathcal{N}(\pi)}(P)$. A finitely generated *Hilbert* $\mathcal{N}(\pi)$*-complex* C_* is a chain complex of finitely generated Hilbert $\mathcal{N}(\pi)$-modules with bounded $\mathbb{C}[\pi]$-linear operators as differentials. The *reduced* L^2-homology is defined to be $\bar{H}_p^{(2)}(C_*) = \mathrm{Ker}(d_p)/\overline{\mathrm{Im}(d_{p+1})}$. The p^{th} L^2-Betti number of C_* is then $dim_{\mathcal{N}(\pi)}\bar{H}_p^{(2)}(C_*)$. (As the images of the differentials need not be closed the *un*reduced L^2-homology modules $H_p^{(2)}(C_*) = \mathrm{Ker}(d_p)/\mathrm{Im}(d_{p+1})$ are not in general Hilbert modules.)

See [Lü] for more on modules over von Neumann algebras and L^2 invariants of complexes and manifolds.

[In this book L^2-Betti number arguments shall replace the localization arguments used in [H2]. However we shall recall the definition of *safe extension* used there. An extension of rings $\mathbb{Z}[G] < \Phi$ is a safe extension if it is faithfully flat, Φ is weakly finite and $\Phi \otimes_{\mathbb{Z}[G]} \mathbb{Z} = 0$. It was shown there that if a group has a nontrivial elementary amenable normal subgroup whose finite subgroups have bounded order and which has no nontrivial finite normal subgroup then $\mathbb{Z}[G]$ has a safe extension.]

Chapter 2

2-Complexes and PD_3-complexes

This chapter begins with a review of the notation we use for (co)homology with local coefficients and of the universal coefficient spectral sequence. We then define the L^2-Betti numbers and present some useful vanishing theorems of Lück and Gromov. These invariants are used in §3, where they are used to estimate the Euler characteristics of finite $[\pi, m]$-complexes and to give a converse to the Cheeger-Gromov-Gottlieb Theorem on aspherical finite complexes. Some of the arguments and results here may be regarded as representing in microcosm the bulk of this book; the analogies and connections between 2-complexes and 4-manifolds are well known. We then review Poincaré duality and PD_n-complexes. In §5-§9 we shall summarize briefly what is known about the homotopy types of PD_3-complexes.

2.1 Notation

Let X be a connected cell complex and let $\widetilde{X}$ be its universal covering space. If H is a normal subgroup of $G = \pi_1(X)$ we may lift the cellular decomposition of X to an equivariant cellular decomposition of the corresponding covering space X_H. The cellular chain complex C_* of X_H with coefficients in a commutative ring R is then a complex of left $R[G/H]$-modules, with respect to the action of the covering group G/H. Moreover C_* is a complex of free modules, with bases obtained by choosing a lift of each cell of X. If X is a finite complex G is finitely presentable and these modules are finitely generated. If X is finitely dominated, i.e., is a retract of a finite complex Y, then G is a retract of $\pi_1(Y)$ and so is finitely presentable, by Lemma 1.12. Moreover the chain complex C_* of the universal cover is chain homotopy equivalent over $R[G]$ to a complex of finitely generated projective modules [Wl65].

The i^{th} *equivariant homology* module of X with coefficients $R[G/H]$ is the left module $H_i(X; R[G/H]) = H_i(C_*)$, which is clearly isomorphic to $H_i(X_H; R)$ as an R-module, with the action of the covering group determining its $R[G/H]$-module structure. The i^{th} *equivariant cohomology* module of X with coefficients $R[G/H]$ is the right module $H^i(X; R[G/H]) = H^i(C^*)$, where $C^* =$

$Hom_{R[G/H]}(C_*, R[G/H])$ is the associated cochain complex of right $R[G/H]$-modules. More generally, if A and B are right and left $\mathbb{Z}[G/H]$-modules (respectively) we may define $H_j(X;A) = H_j(A \otimes_{\mathbb{Z}[G/H]} C_*)$ and $H^{n-j}(X;B) = H^{n-j}(Hom_{\mathbb{Z}[G/H]}(C_*, B))$. There is a *Universal Coefficient Spectral Sequence* (UCSS) relating equivariant homology and cohomology:

$$E_2^{pq} = Ext^q_{R[G/H]}(H_p(X;R[G/H]), R[G/H]) \Rightarrow H^{p+q}(X;R[G/H]),$$

with r^{th} differential d_r of bidegree $(1-r, r)$.

If J is a normal subgroup of G which contains H there is also a *Cartan-Leray* spectral sequence relating the homology of X_H and X_J:

$$E^2_{pq} = Tor_p^{R[G/H]}(H_q(X;R[G/H]), R[G/J]) \Rightarrow H_{p+q}(X;R[G/J]),$$

with r^{th} differential d^r of bidegree $(-r, r-1)$. (See [Mc] for more details on these spectral sequences.)

If M is a cell complex let $c_M : M \to K(\pi_1(M), 1)$ denote the classifying map for the fundamental group and let $f_M : M \to P_2(M)$ denote the second stage of the Postnikov tower for M. (Thus $c_M = c_{P_2(M)} f_M$.) A map $f : X \to K(\pi_1(M), 1)$ lifts to a map from X to $P_2(M)$ if and only if $f^* k_1(M) = 0$, where $k_1(M)$ is the first k-invariant of M in $H^3(\pi_1(M); \pi_2(M))$. In particular, if $k_1(M) = 0$ then $c_{P_2(M)}$ has a cross-section. The *algebraic 2-type* of M is the triple $[\pi, \pi_2(M), k_1(M)]$. Two such triples $[\pi, \Pi, \kappa]$ and $[\pi', \Pi', \kappa']$ (corresponding to M and M', respectively) are equivalent if there are isomorphisms $\alpha : \pi \to \pi'$ and $\beta : \Pi \to \Pi'$ such that $\beta(gm) = \alpha(g)\beta(m)$ for all $g \in \pi$ and $m \in \Pi$ and $\beta_* \kappa = \alpha^* \kappa'$ in $H^3(\pi, \alpha^* \Pi')$. Such an equivalence may be realized by a homotopy equivalence of $P_2(M)$ and $P_2(M')$. (The reference [Ba] gives a detailed treatment of Postnikov factorizations of nonsimple maps and spaces.)

Throughout this book *closed manifold* shall mean compact, connected TOP manifold without boundary. Every closed manifold has the homotopy type of a finite Poincaré duality complex [KS].

2.2 L^2-Betti numbers

Let X be a finite complex with fundamental group π. The *L^2-Betti numbers* of X are defined by $\beta_i^{(2)}(X) = dim_{\mathcal{N}(\pi)}(\bar{H}_2^{(2)}(\widetilde{X}))$ where the L^2-homology $\bar{H}_i^{(2)}(\widetilde{X}) = \bar{H}_i(C_*^{(2)})$ is the reduced homology of the Hilbert $\mathcal{N}(\pi)$-complex $C_*^{(2)} = \ell^2 \otimes C_*(\widetilde{X})$ of square summable chains on $\widetilde{X}$ [At76]. They are multiplicative in finite covers, and for $i = 0$ or 1 depend only on π. (In particular,

$\beta_0^{(2)}(\pi) = 0$ if π is infinite.) The alternating sum of the L^2-Betti numbers is the Euler characteristic $\chi(X)$ [At76]. The usual Betti numbers of a space or group with coefficients in a field F shall be denoted by $\beta_i(X;F) = dim_F H_i(X;F)$ (or just $\beta_i(X)$, if $F = \mathbb{Q}$).

It may be shown that $\beta_i^{(2)}(X) = dim_{\mathcal{N}(\pi)} H_i(\mathcal{N}(\pi) \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{X}))$, and this formulation of the definition applies to arbitrary complexes (see [CG86], [Lü]). (However we may have $\beta_i^{(2)}(X) = \infty$.) These numbers are finite if X is finitely dominated, and the Euler characteristic formula holds if also π satisfies the Strong Bass Conjecture [Ec96]. In particular, $\beta_i^{(2)}(\pi) = dim_{\mathcal{N}(\pi)} H_i(\pi; \mathcal{N}(\pi))$ is defined for any group, and $\beta_2^{(2)}(\pi_1(X)) \leq \beta_2^{(2)}(X)$. (See Theorems 1.35 and 6.54 of [Lü].)

Lemma 2.1 *Let $\pi = H*_\phi$ be a finitely presentable group which is an ascending HNN extension with finitely generated base H. Then $\beta_1^{(2)}(\pi) = 0$.*

Proof Let t be the stable letter and let H_n be the subgroup generated by H and t^n, and suppose that H is generated by g elements. Then $[\pi : H_n] = n$, so $\beta_1^{(2)}(H_n) = n\beta_1^{(2)}(\pi)$. But each H_n is also finitely presentable and generated by $g+1$ elements. Hence $\beta_1^{(2)}(H_n) \leq g+1$, and so $\beta_1^{(2)}(\pi) = 0$. □

In particular, this lemma holds if π is an extension of Z by a finitely generated normal subgroup. We shall only sketch the next theorem (from Chapter 7 of [Lü]) as we do not use it in an essential way. (See however Theorems 5.8 and 9.9.)

Theorem 2.2 (Lück) *Let π be a group with a finitely generated infinite normal subgroup Δ such that π/Δ has an element of infinite order. Then $\beta_1^{(2)}(\pi) = 0$.*

Proof (Sketch) Let $\rho \leq \pi$ be a subgroup containing Δ such that $\rho/\Delta \cong Z$. The terms in the line $p + q = 1$ of the homology LHSSS for ρ as an extension of Z by Δ with coefficients $\mathcal{N}(\rho)$ have dimension 0, by Lemma 2.1. Since $dim_{\mathcal{N}(\rho)} M = dim_{\mathcal{N}(\pi)}(\mathcal{N}(\pi) \otimes_{\mathcal{N}(\rho)} M)$ for any $\mathcal{N}(\rho)$-module M the corresponding terms for the LHSSS for π as an extension of π/Δ by Δ with coefficients $\mathcal{N}(\pi)$ also have dimension 0 and the theorem follows. □

Gaboriau has shown that the hypothesis "π/Δ has an element of infinite order" can be relaxed to "π/Δ is infinite" [Ga00]. A similar argument gives the following result.

Theorem 2.3 *Let π be a group with an infinite subnormal subgroup N such that $\beta_i^{(2)}(N)=0$ for all $i\leq s$. Then $\beta_i^{(2)}(\pi)=0$ for all $i\leq s$.*

Proof Suppose first that N is normal in π. If $[\pi:N]<\infty$ the result follows by multiplicativity of the L^2-Betti numbers, while if $[\pi:N]=\infty$ it follows from the LHSSS with coefficients $\mathcal{N}(\pi)$. We may then induct up a subnormal chain to obtain the theorem. $\square$

In particular, we obtain the following result from page 226 of [Gr]. (Note also that if A is an amenable ascendant subgroup of π then its normal closure in π is amenable.)

Corollary 2.3.1 (Gromov) *Let π be a group with an infinite amenable normal subgroup A. Then $\beta_i^{(2)}(\pi)=0$ for all i.*

Proof If A is an infinite amenable group $\beta_i^{(2)}(A)=0$ for all i [CG86]. $\square$

2.3 2-Complexes and finitely presentable groups

If a group π has a finite presentation P with g generators and r relators then the *deficiency* of P is $\mathrm{def}(P)=g-r$, and $\mathrm{def}(\pi)$ is the maximal deficiency of all finite presentations of π. Such a presentation determines a finite 2-complex $C(P)$ with one 0-cell, g 1-cells and r 2-cells and with $\pi_1(C(P))\cong\pi$. Clearly $\mathrm{def}(P)=1-\chi(P)=\beta_1(C(P))-\beta_2(C(P))$ and so $\mathrm{def}(\pi)\leq\beta_1(\pi)-\beta_2(\pi)$. Conversely every finite 2-complex with one 0-cell arises in this way. In general, any connected finite 2-complex X is homotopy equivalent to one with a single 0-cell, obtained by collapsing a maximal tree T in the 1-skeleton $X^{[1]}$.

We shall say that π has *geometric dimension at most* 2, written $g.d.\pi\leq 2$, if it is the fundamental group of a finite aspherical 2-complex.

Theorem 2.4 *Let X be a connected finite 2-complex with fundamental group π. Then $\chi(X)\geq\beta_2^{(2)}(\pi)-\beta_1^{(2)}(\pi)$. If $\chi(X)=-\beta_1^{(2)}(\pi)$ then X is aspherical and $\pi\neq 1$.*

Proof The lower bound follows from the Euler characteristic formula $\chi(X)=\beta_0^{(2)}(X)-\beta_1^{(2)}(X)+\beta_2^{(2)}(X)$, since $\beta_i^{(2)}(\pi)=\beta_i^{(2)}(X)$ for $i=0$ and 1 and $\beta_2^{(2)}(\pi)\leq\beta_2^{(2)}(X)$. Since X is 2-dimensional $\pi_2(X)=H_2(\widetilde{X};\mathbb{Z})$ is a subgroup of $\bar{H}_2^{(2)}(\widetilde{X})$. If $\chi(X)=-\beta_1^{(2)}(\pi)$ then $\beta_0^{(2)}(X)=0$, so π is infinite, and $\beta_2^{(2)}(X)=0$, so $\bar{H}_2^{(2)}(\widetilde{X})=0$. Therefore $\pi_2(X)=0$ and so X is aspherical. $\square$

Corollary 2.4.1 *Let π be a finitely presentable group. Then* $\mathrm{def}(\pi) \leq 1 + \beta_1^{(2)}(\pi) - \beta_2^{(2)}(\pi)$. *If* $\mathrm{def}(\pi) = 1 + \beta_1^{(2)}(\pi)$ *then* $g.d.\pi \leq 2$.

Let $G = F(2)\times F(2)$. Then $g.d.G = 2$ and $\mathrm{def}(G) \leq \beta_1(G)-\beta_2(G) = 0$. Hence $\langle u, v, x, y \mid ux = xu,\ uy = yu,\ vx = xv,\ vy = yv\rangle$ is an optimal presentation, and $\mathrm{def}(G) = 0$. The subgroup N generated by u, vx^{-1} and y is normal in G and $G/N \cong Z$, so $\beta_1^{(2)}(G) = 0$, by Lemma 2.1. Thus asphericity need not imply equality in Theorem 2.4, in general.

Theorem 2.5 *Let π be a finitely presentable group such that $\beta_1^{(2)}(\pi) = 0$. Then $\mathrm{def}(\pi) \leq 1$, with equality if and only if $g.d.\pi \leq 2$ and $\beta_2(\pi) = \beta_1(\pi) - 1$.*

Proof The upper bound and the necessity of the conditions follow from Theorem 2.4. Conversely, if they hold and X is a finite aspherical 2-complex with $\pi_1(X) \cong \pi$ then $\chi(X) = 1 - \beta_1(\pi) + \beta_2(\pi) = 0$. After collapsing a maximal tree in X we may assume it has a single 0-cell, and then the presentation read off the 1- and 2-cells has deficiency 1. □

This theorem applies if π is a finitely presentable group which is an ascending HNN extension with finitely generated base H, or has an infinite amenable normal subgroup. In the latter case, the condition $\beta_2(\pi) = \beta_1(\pi)-1$ is redundant. For suppose that X is a finite aspherical 2-complex with $\pi_1(X) \cong \pi$. If π has an infinite amenable normal subgroup then $\beta_i^{(2)}(\pi) = 0$ for all i, by Theorem 2.3, and so $\chi(X) = 0$.

[Similarly, if $\mathbb{Z}[\pi]$ has a safe extension Ψ and C_* is the equivariant cellular chain complex of the universal cover $\widetilde{X}$ then $\Psi \otimes_{\mathbb{Z}[\pi]} C_*$ is a complex of free left Ψ-modules with bases corresponding to the cells of X. Since Ψ is a safe extension $H_i(X;\Psi) = \Psi\otimes_{\mathbb{Z}[\pi]}H_i(X;\mathbb{Z}[\pi]) = 0$ for all i, and so again $\chi(X) = 0$.]

Corollary 2.5.1 *Let π be a finitely presentable group which is an extension of Z by an FP_2 normal subgroup N and such that $\mathrm{def}(\pi) = 1$. Then N is free.*

Proof This follows from Corollary 8.6 of [Bi]. □

The subgroup N of $F(2) \times F(2)$ defined after the Corollary to Theorem 2.4 is finitely generated, but is not free, as u and y generate a rank two abelian subgroup. (Thus N is not FP_2 and $F(2) \times F(2)$ is not almost coherent.)

The next result is a version of the "Tits alternative" for coherent groups of cohomological dimension 2. For each $m \in Z$ let $Z*_m$ be the group with presentation $\langle a, t \mid tat^{-1} = a^m\rangle$. (Thus $Z*_0 \cong Z$ and $Z*_{-1} \cong Z \times_{-1} Z$.)

Theorem 2.6 *Let π be a finitely generated group such that $c.d.\pi = 2$. Then $\pi \cong Z*_m$ for some $m \neq 0$ if and only if it is almost coherent and restrained and π/π' is infinite.*

Proof The conditions are easily seen to be necessary. Conversely, if π is almost coherent and π/π' is infinite π is an HNN extension with almost finitely presentable base H, by Theorem 1.13. The HNN extension must be ascending as π has no noncyclic free subgroup. Hence $H^2(\pi;\mathbb{Z}[\pi])$ is a quotient of $H^1(H;\mathbb{Z}[\pi]) \cong H^1(H;\mathbb{Z}[H]) \otimes \mathbb{Z}[\pi/H]$, by the Brown-Geoghegan Theorem. Now $H^2(\pi;\mathbb{Z}[\pi]) \neq 0$, since $c.d.\pi = 2$, and so $H^1(H;\mathbb{Z}[H]) \neq 0$. Since H is restrained it must have two ends, so $H \cong Z$ and $\pi \cong Z*_m$ for some $m \neq 0$. ∎

Does this remain true without any such coherence hypothesis?

Corollary 2.6.1 *Let π be an FP_2 group. Then the following are equivalent:*

(1) *$\pi \cong Z*_m$ for some $m \in Z$;*

(2) *π is torsion free, elementary amenable and $h(\pi) \leq 2$;*

(3) *π is elementary amenable and $c.d.\pi \leq 2$;*

(4) *π is elementary amenable and $\mathrm{def}(\pi) = 1$; and*

(5) *π is almost coherent and restrained and $\mathrm{def}(\pi) = 1$.*

Proof Condition (1) clearly implies the others. Suppose (2) holds. We may assume that $h(\pi) = 2$ and $h(\sqrt{\pi}) = 1$ (for otherwise $\pi \cong Z$, $Z^2 = Z*_1$ or $Z*_{-1}$). Hence $h(\pi/\sqrt{\pi}) = 1$, and so $\pi/\sqrt{\pi}$ is an extension of Z or D by a finite normal subgroup. If $\pi/\sqrt{\pi}$ maps onto D then $\pi \cong A *_C B$, where $[A : C] = [B : C] = 2$ and $h(A) = h(B) = h(C) = 1$, and so $\pi \cong Z \times_{-1} Z$. But then $h(\sqrt{\pi}) = 2$. Hence we may assume that π maps onto Z, and so π is an ascending HNN extension with finitely generated base H, by Theorem 1.13. Since H is torsion free, elementary amenable and $h(H) = 1$ it must be infinite cyclic and so (2) implies (1). If $\mathrm{def}(\pi) = 1$ then π is an ascending HNN extension with finitely generated base, so $\beta_1^{(2)}(\pi) = 0$, by Lemma 2.1. Hence (4) and (5) each imply (3) by Theorem 2.5, together with Theorem 2.6. Finally (3) implies (2), by Theorem 1.11. ∎

In fact all finitely generated solvable groups of cohomological dimension 2 are as in this corollary [Gi79]. Are these conditions also equivalent to "π is almost coherent and restrained and $c.d.\pi \leq 2$"? Note also that if $\mathrm{def}(\pi) > 1$ then π has noncyclic free subgroups [Ro77].

Let $\mathcal{X}$ be the class of groups of finite graphs of groups, all of whose edge and vertex groups are infinite cyclic. Kropholler has shown that a finitely generated, noncyclic group G is in $\mathcal{X}$ if and only if $c.d.G = 2$ and G has an infinite cyclic subgroup H which meets all its conjugates nontrivially. Moreover G is then coherent, one ended and $g.d.G = 2$ [Kr90'].

Theorem 2.7 *Let π be a finitely generated group such that $c.d.\pi = 2$. If π has a nontrivial normal subgroup E which either is almost coherent, locally virtually indicable and restrained or is elementary amenable then π is in $\mathcal{X}$ and either $E \cong Z$ or π/π' is infinite and π' is abelian.*

Proof Let F be a finitely generated subgroup of E. Then F is metabelian, by Theorem 2.6 and its Corollary, and so all words in E of the form $[[g,h],[g',h']]$ are trivial. Hence E is metabelian also. Therefore $A = \sqrt{E}$ is nontrivial, and as A is characteristic in E it is normal in π. Since A is the union of its finitely generated subgroups, which are torsion free nilpotent groups of Hirsch length ≤ 2, it is abelian. If $A \cong Z$ then $[\pi : C_\pi(A)] \leq 2$. Moreover $C_\pi(A)'$ is free, by Bieri's Theorem. If $C_\pi(A)'$ is cyclic then $\pi \cong Z^2$ or $Z \times_{-1} Z$; if $C_\pi(A)'$ is nonabelian then $E = A \cong Z$. Otherwise $c.d.A = c.d.C_\pi(A) = 2$ and so $C_\pi(A) = A$, by Bieri's Theorem. If A has rank 1 then $Aut(A)$ is abelian, so $\pi' \leq C_\pi(A)$ and π is metabelian. If $A \cong Z^2$ then π/A is isomorphic to a subgroup of $GL(2,\mathbb{Z})$, and so is virtually free. As A together with an element $t \in \pi$ of infinite order modulo A would generate a subgroup of cohomological dimension 3, which is impossible, the quotient π/A must be finite. Hence $\pi \cong Z^2$ or $Z \times_{-1} Z$. In all cases π is in $\mathcal{X}$, by Theorem C of [Kr90']. $\square$

If $c.d.\pi = 2$, $\zeta\pi \neq 1$ and π is nonabelian then $\zeta\pi \cong Z$ and π' is free, by Bieri's Theorem. On the evidence of his work on 1-relator groups Murasugi conjectured that if G is a finitely presentable group other than Z^2 and $\mathrm{def}(G) \geq 1$ then $\zeta G \cong Z$ or 1, and is trivial if $\mathrm{def}(G) > 1$, and he verified this for classical link groups [Mu65]. Theorems 2.3, 2.5 and 2.7 together imply that if ζG is infinite then $\mathrm{def}(G) = 1$ and $\zeta G \cong Z$.

It remains an open question whether every finitely presentable group of cohomological dimension 2 has geometric dimension 2. The following partial answer to this question was first obtained by W.Beckmann under the additional assumption that the group was FF (cf. [Dy87']).

Theorem 2.8 *Let π be a finitely presentable group. Then $g.d.\pi \leq 2$ if and only if $c.d.\pi \leq 2$ and $\mathrm{def}(\pi) = \beta_1(\pi) - \beta_2(\pi)$.*

Proof The necessity of the conditions is clear. Suppose that they hold and that $C(P)$ is the 2-complex corresponding to a presentation for π of maximal deficiency. The cellular chain complex of $\widetilde{C(P)}$ gives an exact sequence

$$0 \to K = \pi_2(C(P)) \to \mathbb{Z}[\pi]^r \to \mathbb{Z}[\pi]^g \to \cdots \to \mathbb{Z}[\pi] \to 0.$$

As $c.d.\pi \leq 2$ the image of $\mathbb{Z}[\pi]^r$ in $\mathbb{Z}[\pi]^g$ is projective, by Schanuel's Lemma. Therefore the inclusion of K into $\mathbb{Z}[\pi]^r$ splits, and K is projective. Moreover $dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}[\pi]} K) = 0$, and so $K = 0$, since the Weak Bass Conjecture holds for π [Ec86]. Hence $\widetilde{C(P)}$ is contractible, and so $C(P)$ is aspherical. □

The arguments of this section may easily be extended to other highly connected finite complexes. A $[\pi, m]_f$-complex is a finite m-dimensional complex X with $\pi_1(X) \cong \pi$ and with $(m-1)$-connected universal cover $\widetilde{X}$. Such a $[\pi, m]_f$-complex X is aspherical if and only if $\pi_m(X) = 0$. In that case we shall say that π has geometric dimension at most m, written $g.d.\pi \leq m$.

Theorem 2.4′ *Let X be a $[\pi, m]_f$-complex and suppose that $\beta_i^{(2)}(\pi) = 0$ for $i < m$. Then $(-1)^m \chi(X) \geq 0$. If $\chi(X) = 0$ then X is aspherical.* □

In general the implication in the statement of this theorem cannot be reversed. For $S^1 \vee S^1$ is an aspherical $[F(2), 1]_f$-complex and $\beta_0^{(2)}(F(2)) = 0$, but $\chi(S^1 \vee S^1) = -1 \neq 0$.

One of the applications of L^2-cohomology in [CG86] was to show that if X is a finite aspherical complex such that $\pi_1(X)$ has an infinite amenable normal subgroup A then $\chi(X) = 0$. (This generalised a theorem of Gottlieb, who assumed that A was a central subgroup [Go65].) We may similarly extend Theorem 2.5 to give a converse to the Cheeger-Gromov extension of Gottlieb's Theorem.

Theorem 2.5′ *Let X be a $[\pi, m]_f$-complex and suppose that π has an infinite amenable normal subgroup. Then X is aspherical if and only if $\chi(X) = 0$.* □

2.4 Poincaré duality

The main reason for studying PD-complexes is that they represent the homotopy theory of manifolds. However they also arise in situations where the geometry does not immediately provide a corresponding manifold. For instance, under suitable finiteness assumptions an infinite cyclic covering space of a closed

4-manifold with Euler characteristic 0 will be a PD_3-complex, but need not be homotopy equivalent to a closed 3-manifold (see Chapter 11).

A PD_n-*complex* is a finitely dominated cell complex which satisfies Poincaré duality of formal dimension n with local coefficients. It is *finite* if it is homotopy equivalent to a finite cell complex. (It is most convenient for our purposes below to require that PD_n-complexes be finitely dominated. If a CW-complex X satisfies local duality then $\pi_1(X)$ is FP_2, and X is finitely dominated if and only if $\pi_1(X)$ is finitely presentable [Br72, Br75]. Ranicki uses the broader definition in his book [Rn].) All the PD_n-complexes that we consider shall be assumed to be connected.

Let P be a PD_n-complex and C_* be the cellular chain complex of $\widetilde{P}$. Then the Poincaré duality isomorphism may also be described in terms of a chain homotopy equivalence from $\overline{C^*}$ to C_{n-*}, which induces isomorphisms from $H^j(\overline{C^*})$ to $H_{n-j}(C_*)$, given by cap product with a generator $[P]$ of $H_n(P;Z^{w_1(P)}) = H_n(\bar{Z}\otimes_{\mathbb{Z}[\pi_1(P)]} C_*)$. (Here the first Stiefel-Whitney class $w_1(P)$ is considered as a homomorphism from $\pi_1(P)$ to $Z/2Z$.) From this point of view it is easy to see that Poincaré duality gives rise to ($\mathbb{Z}$-linear) isomorphisms from $H^j(P;B)$ to $H_{n-j}(P;\bar{B})$, where B is any left $\mathbb{Z}[\pi_1(P)]$-module of coefficients. (See [Wl67] or Chapter II of [Wl] for further details.) If P is a Poincaré duality complex then the L^2-Betti numbers also satisfy Poincaré duality. (This does not require that P be finite or orientable!)

A finitely presentable group is a PD_n-group (as defined in Chapter 2) if and only if $K(G,1)$ is a PD_n-complex. For every $n \geq 4$ there are PD_n-groups which are not finitely presentable [Da98].

Dwyer, Stolz and Taylor have extended Strebel's Theorem to show that if H is a subgroup of infinite index in $\pi_1(P)$ then the corresponding covering space P_H has homological dimension $< n$; hence if moreover $n \neq 3$ then P_H is homotopy equivalent to a complex of dimension $< n$ [DST96].

2.5 PD_3-complexes

In this section we shall summarize briefly what is known about PD_n-complexes of dimension at most 3. It is easy to see that a connected PD_1-complex must be homotopy equivalent to S^1. The 2-dimensional case is already quite difficult, but has been settled by Eckmann, Linnell and Müller, who showed that every PD_2-complex is homotopy equivalent to a closed surface. (See Chapter VI of [DD]. This result has been further improved by Bowditch's Theorem.)

There are PD_3-complexes with finite fundamental group which are not homotopy equivalent to any closed 3-manifold [Th77]. On the other hand, Turaev's Theorem below implies that every PD_3-complex with torsion free fundamental group is homotopy equivalent to a closed 3-manifold if every PD_3-group is a 3-manifold group. The latter is so if the Hirsch-Plotkin radical of the group is nontrivial (see §7 below), but remains open in general.

The fundamental triple of a PD_3-complex P is $(\pi_1(P), w_1(P), c_{P*}[P])$. This is a complete homotopy invariant for such complexes.

Theorem (Hendriks) *Two PD_3-complexes are homotopy equivalent if and only if their fundamental triples are isomorphic.* □

Turaev has characterized the possible triples corresponding to a given finitely presentable group and orientation character, and has used this result to deduce a basic splitting theorem [Tu90].

Theorem (Turaev) *A PD_3-complex is irreducible with respect to connected sum if and only if its fundamental group is indecomposable with respect to free product.* □

Wall has asked whether every PD_3-complex whose fundamental group has infinitely many ends is a proper connected sum [Wl67]. Since the fundamental group of a PD_3-complex is finitely presentable it is the fundamental group of a finite graph of (finitely generated) groups in which each vertex group has at most one end and each edge group is finite, by Theorem VI.6.3 of [DD]. Starting from this observation, Crisp has given a substantial partial answer to Wall's question [Cr00].

Theorem (Crisp) *Let X be an indecomposable PD_3^+-complex. If $\pi_1(X)$ is not virtually free then it has one end, and so X is aspherical.* □

With Turaev's theorem this implies that the fundamental group of any PD_3-complex is virtually torsion free, and that if X is irreducible and π has more than one end then it is virtually free. There remains the possibility that, for instance, the free product of two copies of the symmetric group on 3 letters with amalgamation over a subgroup of order 2 may be the fundamental group of an orientable PD_3-complex. (It appears difficult in practice to apply Turaev's work to the question of whether a given group can be the fundamental group of a PD_3-complex.)

2.6 The spherical cases

The possible PD_3-complexes with finite fundamental group are well understood (although it is not yet completely known which are homotopy equivalent to 3-manifolds).

Theorem 2.9 [Wl67] *Let X be a PD_3-complex with finite fundamental group F. Then*

(1) $\widetilde{X} \simeq S^3$, *F has cohomological period dividing 4 and X is orientable;*

(2) *the first nontrivial k-invariant $k(X)$ generates $H^4(F;\mathbb{Z}) \cong Z/|F|Z$.*

(3) *the homotopy type of X is determined by F and the orbit of $k(M)$ under $Out(F) \times \{\pm 1\}$.*

Proof Since the universal cover $\widetilde{X}$ is also a finite PD_3-complex it is homotopy equivalent to S^3. A standard Gysin sequence argument shows that F has cohomological period dividing 4. Suppose that X is nonorientable, and let C be a cyclic subgroup of F generated by an orientation reversing element. Let $\tilde{Z}$ be the nontrivial infinite cyclic $\mathbb{Z}[C]$-module. Then $H^2(X_C;\tilde{Z}) \cong H_1(X_C;\mathbb{Z}) \cong C$, by Poincaré duality. But $H^2(X_C;\tilde{Z}) \cong H^2(C;\tilde{Z}) = 0$, since the classifying map from $X_C = \widetilde{X}/C$ to $K(C,1)$ is 3-connected. Therefore X must be orientable and F must act trivially on $\pi_3(X) \cong H_3(\widetilde{X};\mathbb{Z})$.

The image μ of the orientation class of X generates $H_3(F;\mathbb{Z}) \cong Z/|F|Z$, and corresponds to the first nonzero k-invariant under the isomorphism $H_3(F;\mathbb{Z}) \cong H^4(F;\mathbb{Z})$ [Wl67]. Inner automorphisms of F act trivially on $H^4(F;\mathbb{Z})$, while changing the orientation of X corresponds to multiplication by -1. Thus the orbit of $k(M)$ under $Out(F) \times \{\pm 1\}$ is the significant invariant.

We may construct the third stage of the Postnikov tower for X by adjoining cells of dimension greater than 4 to X. The natural inclusion $j : X \to P_3(X)$ is then 4-connected. If X_1 is another such PD_3-complex and $\theta : \pi_1(X_1) \to F$ is an isomorphism which identifies the k-invariants then there is a 4-connected map $j_1 : X_1 \to P_3(X)$ inducing θ, which is homotopic to a map with image in the 4-skeleton of $P_3(X)$, and so there is a map $h : X_1 \to X$ such that j_1 is homotopic to jh. The map h induces isomorphisms on π_i for $i \leq 3$, since j and j_1 are 4-connected, and so the lift $\tilde{h} : \widetilde{X}_1 \simeq S^3 \to \widetilde{X} \simeq S^3$ is a homotopy equivalence, by the theorems of Hurewicz and Whitehead. Thus h is itself a homotopy equivalence. □

The list of finite groups with cohomological period dividing 4 is well known. Each such group F and generator $k \in H^4(F;\mathbb{Z})$ is realized by some PD_3^+-complex [Sw60, Wl67]. (See also Chapter 11 below.) In particular, there is an unique homotopy type of PD_3-complexes with fundamental group the symmetric group S_3, but there is no 3-manifold with this fundamental group.

The fundamental group of a PD_3-complex P has two ends if and only if $\widetilde{P} \simeq S^2$, and then P is homotopy equivalent to one of the four $\mathbb{S}^2 \times \mathbb{E}^1$-manifolds $S^2 \times S^1$, $S^2 \tilde{\times} S^1$, $RP^2 \times S^1$ or $RP^3 \sharp RP^3$. The following simple lemma leads to an alternative characterization.

Lemma 2.10 *Let P be a finite dimensional complex with fundamental group π and such that $H_q(\widetilde{P};\mathbb{Z}) = 0$ for all $q > 2$. If C is a cyclic subgroup of π then $H_{s+3}(C;\mathbb{Z}) \cong H_s(C;\pi_2(P))$ for all $s \geq \dim(P)$.*

Proof Since $H_2(\widetilde{P};\mathbb{Z}) \cong \pi_2(P)$ and $\dim(\widetilde{P}/C) \leq \dim(P)$ this follows either from the Cartan-Leray spectral sequence for the universal cover of $\widetilde{P}/C$ or by devissage applied to the homology of $C_*(\widetilde{P})$, considered as a chain complex over $\mathbb{Z}[C]$. ☐

Theorem 2.11 *Let P be a PD_3-complex whose fundamental group π has a nontrivial finite normal subgroup N. Then either P is homotopy equivalent to $RP^2 \times S^1$ or π is finite.*

Proof We may clearly assume that π is infinite. Then $H_q(\widetilde{P};\mathbb{Z}) = 0$ for $q > 2$, by Poincaré duality. Let $\Pi = \pi_2(P)$. The augmentation sequence

$$0 \to A(\pi) \to \mathbb{Z}[\pi] \to Z \to 0$$

gives rise to a short exact sequence

$$0 \to Hom_{\mathbb{Z}[\pi]}(\mathbb{Z}[\pi],\mathbb{Z}[\pi]) \to Hom_{\mathbb{Z}[\pi]}(A(\pi),\mathbb{Z}[\pi]) \to H^1(\pi;\mathbb{Z}[\pi]) \to 0.$$

Let $f : A(\pi) \to \mathbb{Z}[\pi]$ be a homomorphism and ζ be a central element of π. Then $f.\zeta(i) = f(i)\zeta = \zeta f(i) = f(\zeta i) = f(i\zeta)$ and so $(f.\zeta - f)(i) = f(i(\zeta-1)) = if(\zeta-1)$ for all $i \in A(\pi)$. Hence $f.\zeta - f$ is the restriction of a homomorphism from $\mathbb{Z}[\pi]$ to $\mathbb{Z}[\pi]$. Thus central elements of π act trivially on $H^1(\pi;\mathbb{Z}[\pi])$.

If $n \in N$ the centraliser $\gamma = C_\pi(\langle n\rangle)$ has finite index in π, and so the covering space P_γ is again a PD_3-complex with universal covering space $\widetilde{P}$. Therefore $\Pi \cong \overline{H^1(\gamma;\mathbb{Z}[\gamma])}$ as a (left) $\mathbb{Z}[\gamma]$-module. In particular, Π is a free abelian group. Since n is central in γ it acts trivially on $H^1(\gamma;\mathbb{Z}[\gamma])$ and hence via

$w(n)$ on Π. Suppose first that $w(n) = 1$. Then Lemma 2.10 gives an exact sequence

$$0 \to Z/|n|Z \to \Pi \to \Pi \to 0,$$

where the right hand homomorphism is multiplication by $|n|$, since n has finite order and acts trivially on Π. As Π is torsion free we must have $n = 1$.

Therefore if $n \in N$ is nontrivial it has order 2 and $w(n) = -1$. In this case Lemma 2.10 gives an exact sequence

$$0 \to \Pi \to \Pi \to Z/2Z \to 0,$$

where the left hand homomorphism is multiplication by 2. Since Π is a free abelian group it must be infinite cyclic, and so $\widetilde{P} \simeq S^2$. The theorem now follows from Theorem 4.4 of [Wl67]. □

If $\pi_1(P)$ has a finitely generated infinite normal subgroup of infinite index then it has one end, and so P is aspherical. We shall discuss this case next.

2.7 PD_3-groups

If Wall's question has an affirmative answer, the study of PD_3-complexes reduces largely to the study of PD_3-groups. It is not yet known whether all such groups are 3-manifold groups. The fundamental groups of 3-manifolds which are finitely covered by surface bundles or which admit one of the geometries of aspherical Seifert type may be characterized among all PD_3-groups in simple group-theoretic terms.

Theorem 2.12 *Let G be a PD_3-group with a nontrivial almost finitely presentable normal subgroup N of infinite index. Then either*

(1) *$N \cong Z$ and G/N is virtually a PD_2-group; or*

(2) *N is a PD_2-group and G/N has two ends.*

Proof Let e be the number of ends of N. If N is free then $H^3(G;\mathbb{Z}[G]) \cong H^2(G/N;H^1(N;\mathbb{Z}[G]))$. Since N is finitely generated and G/N is FP_2 this is in turn isomorphic to $H^2(G/N;\mathbb{Z}[G/N])^{(e-1)}$. Since G is a PD_3-group we must have $e-1 = 1$ and so $N \cong Z$. We then have $H^2(G/N;\mathbb{Z}[G/N]) \cong H^3(G;\mathbb{Z}[G]) \cong Z$, so G/N is virtually a PD_2-group, by Bowditch's Theorem.

Otherwise $c.d.N = 2$ and so $e = 1$ or ∞. The LHSSS gives an isomorphism $H^2(G;\mathbb{Z}[G]) \cong H^1(G/N;\mathbb{Z}[G/N]) \otimes H^1(N;\mathbb{Z}[N]) \cong H^1(G/N;\mathbb{Z}[G/N])^{e-1}$.

Hence either $e = 1$ or $H^1(G/N;\mathbb{Z}[G/N]) = 0$. But in the latter case we have $H^3(G;\mathbb{Z}[G]) \cong H^2(G/N;\mathbb{Z}[G/N]) \otimes H^1(N;\mathbb{Z}[N])$ and so $H^3(G;\mathbb{Z}[G])$ is either 0 or infinite dimensional. Therefore $e = 1$, and so $H^3(G;\mathbb{Z}[G]) \cong H^1(G/N;\mathbb{Z}[G/N]) \otimes H^2(N;\mathbb{Z}[N])$. Hence G/N has two ends and $H^2(N;\mathbb{Z}[N]) \cong Z$, so N is a PD_2-group. □

We shall strengthen this result in Theorem 2.16 below.

Corollary 2.12.1 *A PD_3-complex P is homotopy equivalent to the mapping torus of a self homeomorphism of a closed surface if and only if there is an epimorphism $\phi : \pi_1(P) \to Z$ with finitely generated kernel.*

Proof This follows from Theorems 1.20, 2.11 and 2.12. □

If $\pi_1(P)$ is infinite and is a nontrivial direct product then P is homotopy equivalent to the product of S^1 with a closed surface.

Theorem 2.13 *Let G be a PD_3-group. Then every almost coherent, locally virtually indicable subgroup of G is either virtually solvable or contains a noncyclic free subgroup.*

Proof Let S be a restrained, locally virtually indicable subgroup of G. Suppose first that S has finite index in G, and so is again a PD_3-group. Since S is virtually indicable we may assume without loss of generality that $\beta_1(S) > 0$. Then S is an ascending HNN extension $H*_\phi$ with finitely generated base. Since G is almost coherent H is finitely presentable, and since $H^3(S;\mathbb{Z}[S]) \cong Z$ it follows from Lemma 3.4 of [BG85] that H is normal in S and $S/H \cong Z$. Hence H is a PD_2-group, by Theorem 1.20. Since H has no noncyclic free subgroup it is virtually Z^2 and so S and G are virtually poly-Z.

If $[G : S] = \infty$ then $c.d.S \leq 2$, by Strebel's Theorem. As the finitely generated subgroups of S are virtually indicable they are metabelian, by Theorem 2.6 and its Corollary. Hence S is metabelian also. □

As the fundamental groups of virtually Haken 3-manifolds are coherent and locally virtually indicable, this implies the "Tits alternative" for such groups [EJ73]. In fact solvable subgroups of infinite index in 3-manifold groups are virtually abelian. This remains true if $K(G, 1)$ is a finite PD_3-complex, by Corollary 1.4 of [KK99]. Does this hold for all PD_3-groups?

A slight modification of the argument gives the following corollary.

Corollary 2.13.1 *A PD_3-group G is virtually poly-Z if and only if it is coherent, restrained and has a subgroup of finite index with infinite abelianization.* □

If $\beta_1(G) \geq 2$ the hypothesis of coherence is redundant, for there is then an epimorphism $p: G \to Z$ with finitely generated kernel, by [BNS87], and Theorem 1.20 requires only that H be finitely generated.

The argument of Theorem 2.13 and its corollary extend to show by induction on m that a PD_m-group is virtually poly-Z if and only if it is restrained and every finitely generated subgroup is FP_{m-1} and virtually indicable.

Theorem 2.14 *Let G be a PD_3-group. Then G is the fundamental group of an aspherical Seifert fibred 3-manifold or a $\mathbb{S}ol^3$-manifold if and only if $\sqrt{G} \neq 1$. Moreover*

(1) $h(\sqrt{G}) = 1$ *if and only if G is the group of an $\mathbb{H}^2 \times \mathbb{E}^1$- or $\widetilde{\mathbb{SL}}$-manifold;*

(2) $h(\sqrt{G}) = 2$ *if and only if G is the group of a $\mathbb{S}ol^3$-manifold;*

(3) $h(\sqrt{G}) = 3$ *if and only if G is the group of an $\mathbb{E}^3$- or $\mathbb{N}il^3$-manifold.*

Proof The necessity of the conditions is clear. (See [Sc83'], or §2 and §3 of Chapter 7 below.) Certainly $h(\sqrt{G}) \leq c.d.\sqrt{G} \leq 3$. Moreover $c.d.\sqrt{G} = 3$ if and only if $[G : \sqrt{G}]$ is finite, by Strebel's Theorem. Hence G is virtually nilpotent if and only if $h(\sqrt{G}) = 3$. If $h(\sqrt{G}) = 2$ then $\sqrt{G}$ is locally abelian, and hence abelian. Moreover $\sqrt{G}$ must be finitely generated, for otherwise $c.d\sqrt{G} = 3$. Thus $\sqrt{G} \cong Z^2$ and case (2) follows from Theorem 2.12.

Suppose now that $h(\sqrt{G}) = 1$ and let $C = C_G(\sqrt{G})$. Then $\sqrt{G}$ is torsion free abelian of rank 1, so $Aut(\sqrt{G})$ is isomorphic to a subgroup of $\mathbb{Q}^\times$. Therefore G/C is abelian. If G/C is infinite then $c.d.C \leq 2$ by Strebel's Theorem and $\sqrt{G}$ is not finitely generated, so C is abelian, by Bieri's Theorem, and hence G is solvable. But then $h(\sqrt{G}) > 1$, which is contrary to our hypothesis. Therefore G/C is isomorphic to a finite subgroup of $\mathbb{Q}^\times \cong Z^\infty \oplus (Z/2Z)$ and so has order at most 2. In particular, if A is an infinite cyclic subgroup of $\sqrt{G}$ then A is normal in G, and so G/A is virtually a PD_2-group, by Theorem 2.12. If G/A is a PD_2-group then G is the fundamental group of an S^1-bundle over a closed surface. In general, a finite torsion free extension of the fundamental group of a closed Seifert fibred 3-manifold is again the fundamental group of a closed Seifert fibred 3-manifold, by [Sc83] and Section 63 of [Zi]. □

The heart of this result is the deep theorem of Bowditch. The weaker characterization of fundamental groups of $\mathbb{S}ol^3$-manifolds and aspherical Seifert fibred 3-manifolds as PD_3-groups G such that $\sqrt{G} \neq 1$ and G has a subgroup of finite index with infinite abelianization is much easier to prove [H2]. There is as yet no comparable characterization of the groups of $\mathbb{H}^3$-manifolds, although it may be conjectured that these are exactly the PD_3-groups with no noncyclic abelian subgroups. (Note also that it remains an open question whether every closed $\mathbb{H}^3$-manifold is finitely covered by a mapping torus.)

$\mathbb{N}il^3$- and $\widetilde{\mathbb{SL}}$-manifolds are orientable, and so their groups are PD_3^+-groups. This can also be seen algebraically, as every such group has a characteristic subgroup H which is a nonsplit central extension of a PD_2^+-group β by Z. An automorphism of such a group H must be orientation preserving.

Theorem 2.14 implies that if a PD_3-group G is not virtually poly-Z then its maximal elementary amenable normal subgroup is Z or 1. For this subgroup is virtually solvable, by Theorem 1.11, and if it is nontrivial then so is $\sqrt{G}$.

Lemma 2.15 *Let G be a PD_3-group with subgroups H and J such that H is almost finitely presentable, has one end and is normal in J. Then either $[J : H]$ or $[G : J]$ is finite.*

Proof Suppose that $[J : H]$ and $[G : H]$ are both infinite. Since H has one end it is not free and so $c.d.H = c.d.J = 2$, by Strebel's Theorem. Hence there is a free $\mathbb{Z}[J]$-module W such that $H^2(J; W) \neq 0$, by Proposition 5.1 of [Bi]. Since H is FP_2 and has one end $H^q(H; W) = 0$ for $q = 0$ or 1 and $H^2(H; W)$ is an induced $\mathbb{Z}[J/H]$-module. Since $[J : H]$ is infinite $H^0(J/H; H^2(H; W)) = 0$, by Lemma 8.1 of [Bi]. The LHSSS for J as an extension of J/H by H now gives $H^r(J; W) = 0$ for $r \leq 2$, which is a contradiction. □

Theorem 2.16 *Let G be a PD_3-group with a nontrivial almost finitely presentable subgroup H which is subnormal and of infinite index in G. Then either H is infinite cyclic and is normal in G or G is virtually poly-Z or H is a PD_2-group, $[G : N_G(H)] < \infty$ and $N_G(H)/H$ has two ends.*

Proof Since H is subnormal in G there is a finite increasing sequence $\{J_i \mid 0 \leq i \leq n\}$ of subgroups of G with $J_0 = H$, J_i normal in J_{i+1} for each $i < n$ and $J_n = G$. Since $[G : H] = \infty$ either $c.d.H = 2$ or H is free, by Strebel's Theorem. Suppose first that $c.d.H = 2$. Let $k = \min\{i \mid [J_i : H] = \infty\}$. Then H has finite index in J_{k-1}, which therefore is also FP_2. Suppose that $c.d.J_k = 2$. If K is a finitely generated subgroup of J_k which contains J_{k-1}

then $[K : J_{k-1}]$ is finite, by Corollary 8.6 of [Bi], and so J_k is the union of a strictly increasing sequence of finite extensions of J_{k-1}. But it follows from the Kurosh subgroup theorem that the number of indecomposable factors in such intermediate groups must be strictly decreasing unless one is indecomposable (in which case all are). (See Lemma 1.4 of [Sc76].) Thus J_{k-1} is indecomposable, and so has one end (since it is torsion free but not infinite cyclic). Therefore $[G : J_k] < \infty$, by Lemma 2.15, and so J_k is a PD_3-group. Since J_{k-1} is finitely generated, normal in J_k and $[J_{k-1} : H] < \infty$ it follows easily that $[J_k : N_{J_k}(H)] < \infty$. Therefore $[G : N_G(H)] < \infty$ and so H is a PD_2-group and $N_G(H)/H$ has two ends, by Theorem 2.12.

Next suppose that $H \cong Z$. Since $\sqrt{J_i}$ is characteristic in J_i it is normal in J_{i+1}, for each $i < n$. A finite induction now shows that $H \leq \sqrt{G}$. Therefore either $\sqrt{G} \cong Z$, so $H \cong Z$ and is normal in G, or G is virtually poly-Z, by Theorem 2.14.

Suppose finally that G has a finitely generated noncyclic free subnormal subgroup. We may assume that $\{J_i \mid 0 \leq i \leq n\}$ is a chain of minimal length n among subnormal chains with $H = J_0$ a finitely generated noncyclic free group. In particular, $[J_1 : H] = \infty$, for otherwise J_1 would also be a finitely generated noncyclic free group. We may also assume that H is maximal in the partially ordered set of finitely generated free normal subgroups of J_1. (Note that ascending chains of such subgroups are always finite, for if $F(r)$ is a nontrivial normal subgroup of a free group G then G is also finitely generated, of rank s say, and and $[G : F](1 - s) = 1 - r$.)

Since J_1 has a finitely generated noncyclic free normal subgroup of infinite index it is not free, and nor is it a PD_3-group. Therefore $c.d.J_1 = 2$. The kernel of the homomorphism from J_1/H to $Out(H)$ determined by the conjugation action of J_1 on H is $HC_{J_1}(H)/H$, which is isomorphic to $C_{J_1}(H)$ since $\zeta H = 1$. As $Out(H)$ is virtually of finite cohomological dimension and $c.d.C_{J_1}(H)$ is finite $v.c.d.J_1/H < \infty$. Therefore $c.d.J_1 = c.d.H + v.c.d.J_1/H$, by Theorem 5.6 of [Bi], so $v.c.d.J_1/H = 1$ and J_1/H is virtually free.

If g normalizes J_1 then $HH^g/H = H^g/H \cap H^g$ is a finitely generated normal subgroup of J_1/H and so either has finite index or is finite. (Here $H^g = gHg^{-1}$.) In the former case J_1/H would be finitely presentable (since it is then an extension of a finitely generated virtually free group by a finitely generated free normal subgroup) and as it is subnormal in G it must be a PD_2-group, by our earlier work. But PD_2-groups do not have finitely generated noncyclic free normal subgroups. Therefore HH^g/H is finite and so $HH^g = H$, by the maximality of H. Since this holds for any $g \in J_2$ the subgroup H is normal in J_2 and so is the initial term of a subnormal chain of length $n-1$ terminating with G, contradicting the minimality of n. Therefore G has no finitely generated noncyclic free subnormal subgroups. □

The theorem as stated can be proven without appeal to Bowditch's Theorem (used here for the cases when $H \cong Z$) [BH91].

If H is a PD_2-group $N_G(H)$ is the fundamental group of a 3-manifold which is double covered by the mapping torus of a surface homeomorphism. There are however $\mathbb{N}il^3$-manifolds with no normal PD_2-subgroup (although they always have subnormal copies of Z^2).

Theorem 2.17 *Let G be a PD_3-group with an almost finitely presentable subgroup H which has one end and is of infinite index in G. Let $H_0 = H$ and $H_{i+1} = N_G(H_i)$ for $i \geq 0$. Then $\widehat{H} = \cup H_i$ is almost finitely presentable and has one end, and either $c.d.\widehat{H} = 2$ and $N_G(\widehat{H}) = \widehat{H}$ or $[G : \widehat{H}] < \infty$ and G is virtually the group of a surface bundle.*

Proof If $c.d.H_i = 2$ for all $i \geq 0$ then $[H_{i+1} : H_i] < \infty$ for all $i \geq 0$, by Lemma 2.15. Hence $h.d.\widehat{H} = 2$, by Theorem 4.7 of [Bi]. Therefore $[G : \widehat{H}] = \infty$, so $c.d.\widehat{H} = 2$ also. Hence $\widehat{H}$ is finitely generated, and so $\widehat{H} = H_i$ for i large, by Theorem 3.3 of [GS81]. In particular, $N_G(\widehat{H}) = \widehat{H}$.

Otherwise let $k = \max\{i \mid c.d.H_i = 2\}$. Then H_k is FP_2 and has one end and $[G : H_{k+1}] < \infty$, so G is virtually the group of a surface bundle, by Theorem 2.12 and the observation preceding this theorem. □

Corollary 2.17.1 *If G has a subgroup H which is a PD_2-group with $\chi(H) = 0$ (respectively, < 0) then either it has such a subgroup which is its own normalizer in G or it is virtually the group of a surface bundle.*

Proof If $c.d.\widehat{H} = 2$ then $[\widehat{H} : H] < \infty$, so $\widehat{H}$ is a PD_2-group, and $\chi(H) = [\widehat{H} : H]\chi(\widehat{H})$. □

2.8 Subgroups of PD_3-groups and 3-manifold groups

The central role played by incompressible surfaces in the geometric study of Haken 3-manifolds suggests strongly the importance of studying subgroups of infinite index in PD_3-groups. Such subgroups have cohomological dimension ≤ 2, by Strebel's Theorem.

There are substantial constraints on 3-manifold groups and their subgroups. Every finitely generated subgroup of a 3-manifold group is the fundamental group of a compact 3-manifold (possibly with boundary) [Sc73], and thus is finitely presentable and is either a 3-manifold group or has finite geometric dimension 2 or is a free group. All 3-manifold groups have *Max-c* (every strictly increasing sequence of centralizers is finite), and solvable subgroups of infinite index are virtually abelian [Kr90a]. If the Thurston Geometrization Conjecture is true every aspherical closed 3-manifold is Haken, hyperbolic or Seifert fibred. The groups of such 3-manifolds are residually finite [He87], and the centralizer of any element in the group is finitely generated [JS79]. Thus solvable subgroups are virtually poly-Z.

In contrast, any group of finite geometric dimension 2 is the fundamental group of a compact aspherical 4-manifold with boundary, obtained by attaching 1- and 2-handles to D^4. On applying the orbifold hyperbolization technique of Gromov, Davis and Januszkiewicz [DJ91] to the boundary we see that each such group embeds in a PD_4-group. Thus the question of which such groups are subgroups of PD_3-groups is critical. (In particular, which $\mathcal{X}$-groups are subgroups of PD_3-groups?)

The Baumslag-Solitar groups $\langle x, t \mid tx^pt^{-1} = x^q\rangle$ are not hopfian, and hence not residually finite, and do not have *Max-c*. As they embed in PD_4-groups there are such groups which are not residually finite and do not have *Max-c*. The product of two nonabelian PD_2^+-groups contains a copy of $F(2) \times F(2)$, and so is a PD_4^+-group which is not almost coherent.

Kropholler and Roller have shown that $F(2) \times F(2)$ is not a subgroup of any PD_3-group [KR89]. They have also proved some strong splitting theorems for PD_n-groups. Let G be a PD_3-group with a subgroup $H \cong Z^2$. If G is residually finite then it is virtually split over a subgroup commensurate with H [KR88]. If $\sqrt{G} = 1$ then G splits over an $\mathcal{X}$-group [Kr93]; if moreover G has *Max-c* then it splits over a subgroup commensurate with H [Kr90].

The geometric conclusions of Theorem 2.14 and the coherence of 3-manifold groups suggest that Theorems 2.12 and 2.16 should hold under the weaker hypothesis that N be finitely generated. (Compare Theorem 1.20.)

Is there a characterization of virtual PD_3-groups parallel to Bowditch's Theorem? (It may be relevant that homology n-manifolds are manifolds for $n \leq 2$. High dimensional analogues are known to be false. For every $k \geq 6$ there are FP_k groups G with $H^k(G; \mathbb{Z}[G]) \cong Z$ but which are not virtually torsion free [FS93].)

2.9 $\pi_2(P)$ as a $\mathbb{Z}[\pi]$-module

The cohomology group $H^2(P;\pi_2(P))$ arises in studying homotopy classes of self homotopy equivalences of P. Hendriks and Laudenbach showed that if N is a P^2-irreducible 3-manifold and $\pi_1(N)$ is virtually free then $H^2(N;\pi_2(N)) \cong Z$, and otherwise $H^2(N;\pi_2(N)) = 0$ [HL74]. Swarup showed that if N is a 3-manifold which is the connected sum of a 3-manifold whose fundamental group is free of rank r with $s \geq 1$ aspherical 3-manifolds then $\pi_2(N)$ is a finitely generated free $\mathbb{Z}[\nu]$-module of rank $2r+s-1$ [Sw73]. We shall give direct homological arguments using Schanuel's Lemma to extend these results to PD_3-complexes with torsion free fundamental group.

Theorem 2.18 *Let N be a PD_3-complex with torsion free fundamental group ν. Then*

(1) *$c.d.\nu \leq 3$;*

(2) *the $\mathbb{Z}[\nu]$-module $\pi_2(N)$ is finitely presentable and has projective dimension at most 1;*

(3) *if ν is a nontrivial free group then $H^2(N;\pi_2(N)) \cong Z$;*

(4) *if ν is not a free group then $\pi_2(N)$ is projective and $H^2(N;\pi_2(N)) = 0$;*

(5) *if ν is not a free group then any two of the conditions "ν is FF", "N is homotopy equivalent to a finite complex" and "$\pi_2(N)$ is stably free" imply the third.*

Proof We may clearly assume that $\nu \neq 1$. The PD_3-complex N is homotopy equivalent to a connected sum of aspherical PD_3-complexes and a 3-manifold with free fundamental group, by Turaev's Theorem. Therefore ν is a corresponding free product, and so it has cohomological dimension at most 3 and is FP. Since N is finitely dominated the equivariant chain complex of the universal covering space $\widetilde{N}$ is chain homotopy equivalent to a complex

$$0 \to C_3 \to C_2 \to C_1 \to C_0 \to 0$$

of finitely generated projective left $\mathbb{Z}[\nu]$-modules. Then the sequences

$$0 \to Z_2 \to C_2 \to C_1 \to C_0 \to Z \to 0$$

and

$$0 \to C_3 \to Z_2 \to \pi_2(N) \to 0$$

are exact, where Z_2 is the module of 2-cycles in C_2. Since ν is FP and $c.d.\nu \leq 3$ Schanuel's Lemma implies that Z_2 is projective and finitely generated. Hence $\pi_2(N)$ has projective dimension at most 1, and is finitely presentable.

It follows easily from the UCSS and Poincaré duality that $\pi_2(N)$ is isomorphic to $\overline{H^1(\nu;\mathbb{Z}[\nu])}$ and that there is an exact sequence

$$H^3(\nu;\mathbb{Z}[\nu]) \to H^3(N;\mathbb{Z}[\nu]) \to Ext^1_{\mathbb{Z}[\nu]}(\pi_2(N),\mathbb{Z}[\nu]) \to 0 \tag{2.1}$$

The $w_1(N)$-twisted augmentation homomorphism from $\mathbb{Z}[\nu]$ to $\bar{Z}$ which sends $g \in \nu$ to $w_1(N)(g)$ induces an isomorphism from $H^3(N;\mathbb{Z}[\nu])$ to $H^3(N;\bar{Z}) \cong Z$. If ν is free the first term in this sequence is 0, and so $Ext^1_{\mathbb{Z}[\nu]}(\pi_2(N),\mathbb{Z}[\nu]) \cong Z$. (In particular, $\pi_2(N)$ has projective dimension 1.) There is also a short exact sequence of left modules

$$0 \to \mathbb{Z}[\nu]^r \to \mathbb{Z}[\nu] \to Z \to 0,$$

where r is the rank of ν. On dualizing we obtain the sequence of right modules

$$0 \to \mathbb{Z}[\nu] \to \mathbb{Z}[\nu]^r \to H^1(\nu;\mathbb{Z}[\nu]) \to 0.$$

The long exact sequence of homology with these coefficients includes an exact sequence

$$0 \to H_1(N;H^1(\nu;\mathbb{Z}[\nu])) \to H_0(N;\mathbb{Z}[\nu]) \to H_0(N;\mathbb{Z}[\nu]^r)$$

in which the right hand map is 0, and so $H_1(N;H^1(\nu;\mathbb{Z}[\nu])) \cong H_0(N;\mathbb{Z}[\nu]) = Z$. Hence $H^2(N;\pi_2(N) \cong H_1(N;\bar{\pi}_2(N)) = H_1(N;H^1(\nu;\mathbb{Z}[\nu])) \cong Z$, by Poincaré duality.

If ν is *not* free then the map $H^3(\nu;\mathbb{Z}[\nu]) \to H^3(N;\mathbb{Z}[\nu])$ in sequence 2.1 above is onto, as can be seen by comparison with the corresponding sequence with coefficients $\bar{Z}$. Therefore $Ext^1_{\mathbb{Z}[\nu]}(\pi_2(N),\mathbb{Z}[\nu]) = 0$. Since $\pi_2(N)$ has a short resolution by finitely generated projective modules, it follows that it is in fact projective. As $H^2(N;\mathbb{Z}[\nu]) = H_1(N;\mathbb{Z}[\nu]) = 0$ it follows that $H^2(N;P) = 0$ for any projective $\mathbb{Z}[\nu]$-module P. Hence $H^2(N;\pi_2(N)) = 0$.

The final assertion follows easily from the fact that if $\pi_2(N)$ is projective then $Z_2 \cong \pi_2(N) \oplus C_3$. □

If ν is not torsion free then the projective dimension of $\pi_2(N)$ is infinite. Does the result of [HL74] extend to all PD_3-complexes?

Chapter 3

Homotopy invariants of PD_4-complexes

The homotopy type of a 4-manifold M is largely determined (through Poincaré duality) by its algebraic 2-type and orientation character. In many cases the formally weaker invariants $\pi_1(M)$, $w_1(M)$ and $\chi(M)$ already suffice. In §1 we give criteria in such terms for a degree-1 map between PD_4-complexes to be a homotopy equivalence, and for a PD_4-complex to be aspherical. We then show in §2 that if the universal covering space of a PD_4-complex is homotopy equivalent to a finite complex then it is either compact, contractible, or homotopy equivalent to S^2 or S^3. In §3 we obtain estimates for the minimal Euler characteristic of PD_4-complexes with fundamental group of cohomological dimension at most 2 and determine the second homotopy groups of PD_4-complexes realizing the minimal value. The class of such groups includes all surface groups and classical link groups, and the groups of many other (bounded) 3-manifolds. The minima are realized by s-parallelizable PL 4-manifolds. In the final section we shall show that if $\chi(M) = 0$ then $\pi_1(M)$ satisfies some stringent constraints.

3.1 Homotopy equivalence and asphericity

Many of the results of this section depend on the following lemma, in conjunction with use of the Euler characteristic to compute the rank of the surgery kernel. (This lemma and the following theorem derive from Lemmas 2.2 and 2.3 of [Wa].)

Lemma 3.1 *Let R be a ring and C_* be a finite chain complex of projective R-modules. If $H_i(C_*) = 0$ for $i < q$ and $H^{q+1}(Hom_R(C_*, B)) = 0$ for any left R-module B then $H_q(C_*)$ is projective. If moreover $H_i(C_*) = 0$ for $i > q$ then $H_q(C_*) \oplus \bigoplus_{i \equiv q+1\ (2)} C_i \cong \bigoplus_{i \equiv q\ (2)} C_i$.*

Proof We may assume without loss of generality that $q = 0$ and $C_i = 0$ for $i < 0$. We may factor $\partial_1 : C_1 \to C_0$ through $B = \mathrm{Im}\partial_1$ as $\partial_1 = j\beta$, where β is an epimorphism and j is the natural inclusion of the submodule

B. Since $j\beta\partial_2 = \partial_1\partial_2 = 0$ and j is injective $\beta\partial_2 = 0$. Hence β is a 1-cocycle of the complex $Hom_R(C_*, B)$. Since $H^1(Hom_R(C_*, B)) = 0$ there is a homomorphism $\sigma : C_0 \to B$ such that $\beta = \sigma\partial_1 = \sigma j\beta$. Since β is an epimorphism $\sigma j = id_B$ and so B is a direct summand of C_0. This proves the first assertion.

The second assertion follows by an induction on the length of the complex. □

Theorem 3.2 *Let N and M be finite PD_4-complexes. A map $f : M \to N$ is a homotopy equivalence if and only if $\pi_1(f)$ is an isomorphism, $f^*w_1(N) = w_1(M)$, $f_*[M] = \pm[N]$ and $\chi(M) = \chi(N)$.*

Proof The conditions are clearly necessary. Suppose that they hold. Up to homotopy type we may assume that f is a cellular inclusion of finite cell complexes, and so M is a subcomplex of N. We may also identify $\pi_1(M)$ with $\pi = \pi_1(N)$. Let $C_*(M)$, $C_*(N)$ and D_* be the cellular chain complexes of $\widetilde{M}$, $\widetilde{N}$ and $(\widetilde{N}, \widetilde{M})$, respectively. Then the sequence

$$0 \to C_*(M) \to C_*(N) \to D_* \to 0$$

is a short exact sequence of finitely generated free $\mathbb{Z}[\pi]$-chain complexes.

By the projection formula $f_*(f^*a \cap [M]) = a \cap f_*[M] = \pm a \cap [N]$ for any cohomology class $a \in H^*(N; \mathbb{Z}[\pi])$. Since M and N satisfy Poincaré duality it follows that f induces split surjections on homology and split injections on cohomology. Hence $H_q(D_*)$ is the "surgery kernel" in degree $q-1$, and the duality isomorphisms induce isomorphisms from $H^r(Hom_{\mathbb{Z}[\pi]}(D_*, B))$ to $H_{6-r}(\overline{D_*} \otimes B)$, where B is any left $\mathbb{Z}[\pi]$-module. Since f induces isomorphisms on homology and cohomology in degrees ≤ 1, with any coefficients, the hypotheses of Lemma 3.1 are satisfied for the $\mathbb{Z}[\pi]$-chain complex D_*, with $q = 3$, and so $H_3(D_*) = \mathrm{K}er(\pi_2(f))$ is projective. Moreover $H_3(D_*) \oplus \bigoplus_{i\ odd} D_i \cong \bigoplus_{i\ even} D_i$. Thus $H_3(D_*)$ is a stably free $\mathbb{Z}[\pi]$-module of rank $\chi(E, M) = \chi(M) - \chi(E) = 0$ and so it is trivial, as $\mathbb{Z}[\pi]$ is weakly finite, by a theorem of Kaplansky (see [Ro84]). Therefore f is a homotopy equivalence. □

If M and N are merely finitely dominated, rather than finite, then $H_3(D_*)$ is a finitely generated projective $\mathbb{Z}[\pi]$-module such that $H_3(D_*) \otimes_{\mathbb{Z}[\pi]} Z = 0$. If the Wall finiteness obstructions satisfy $f_*\sigma(M) = \sigma(N)$ in $\tilde{K}_0(\mathbb{Z}[\pi])$ then $H_3(D_*)$ is stably free, and the theorem remains true. This additional condition is redundant if π satisfies the Weak Bass Conjecture. (Similar comments apply elsewhere in this section.)

Corollary 3.2.1 *Let N be orientable. Then a map $f : N \to N$ which induces automorphisms of $\pi_1(N)$ and $H_4(N;\mathbb{Z})$ is a homotopy equivalence.* □

In the aspherical cases we shall see that we can relax the hypothesis that the classifying map have degree ± 1.

Lemma 3.3 *Let M be a PD_4-complex with fundamental group π. Then there is an exact sequence*

$$0 \to H^2(\pi;\mathbb{Z}[\pi]) \to \overline{\pi_2(M)} \to Hom_{\mathbb{Z}[\pi]}(\pi_2(M),\mathbb{Z}[\pi]) \to H^3(\pi;\mathbb{Z}[\pi]) \to 0.$$

Proof Since $H_2(M;\mathbb{Z}[\pi]) \cong \pi_2(M)$ and $H^3(M;\mathbb{Z}[\pi]) \cong H_1(\widetilde{M};\mathbb{Z}) = 0$, this follows from the UCSS and Poincaré duality. □

Exactness of much of this sequence can be derived without the UCSS. The middle arrow is the composite of a Poincaré duality isomorphism and the evaluation homomorphism. Note also that $Hom_{\mathbb{Z}[\pi]}(\pi_2(M),\mathbb{Z}[\pi])$ may be identified with $H^0(\pi;H^2(\widetilde{M};\mathbb{Z})\otimes\mathbb{Z}[\pi])$, the π-invariant subgroup of the cohomology of the universal covering space. When π is finite the sequence reduces to an isomorphism $\pi_2(M) \cong \overline{Hom_{\mathbb{Z}[\pi]}(\pi_2(M),\mathbb{Z}[\pi])}$.

Let $ev^{(2)} : H^2_{(2)}(\widetilde{M}) \to Hom_{\mathbb{Z}[\pi]}(\pi_2(M),\ell^2(\pi))$ be the evaluation homomorphism defined on the *un*reduced L^2-cohomology by $ev^{(2)}(f)(z) = \Sigma f(g^{-1}z)g$ for all 2-cycles z and square summable 2-cocycles f. Much of the next theorem is implicit in [Ec94].

Theorem 3.4 *Let M be a finite PD_4-complex with fundamental group π. Then*

1. *if $\beta_1^{(2)}(\pi) = 0$ then $\chi(M) \geq 0$;*
2. $\mathrm{Ker}(ev^{(2)})$ *is closed;*
3. *if $\chi(M) = \beta_1^{(2)}(\pi) = 0$ then $c_M^* : H^2(\pi;\mathbb{Z}[\pi]) \to H^2(M;\mathbb{Z}[\pi]) \cong \overline{\pi_2(M)}$ is an isomorphism.*

Proof Since M is a PD_4-complex $\chi(M) = 2\beta_0^{(2)}(\pi) - 2\beta_1^{(2)}(\pi) + \beta_2^{(2)}(M)$. Hence $\chi(M) \geq \beta_2^{(2)}(M) \geq 0$ if $\beta_1^{(2)}(\pi) = 0$.

Let $z \in C_2(\widetilde{M})$ be a 2-cycle and $f \in C_2^{(2)}(\widetilde{M})$ a square-summable 2-cocycle. As $||ev^{(2)}(f)(z)||_2 \leq ||f||_2||z||_2$, the map $f \mapsto ev^{(2)}(f)(z)$ is continuous, for fixed z. Hence if $f = lim f_n$ and $ev^{(2)}(f_n) = 0$ for all n then $ev^{(2)}(f) = 0$.

The inclusion $\mathbb{Z}[\pi] < \ell^2(\pi)$ induces a homomorphism from the exact sequence of Lemma 3.3 to the corresponding sequence with coefficients $\ell^2(\pi)$. The module $H^2(M;\ell^2(\pi))$ may be identified with the *un*reduced L^2-cohomology, and $ev^{(2)}$ may be viewed as mapping $H_2^{(2)}(\widetilde{M})$ to $H^2(\widetilde{M};\mathbb{Z}) \otimes \ell^2(\pi)$ [Ec94]. As $\widetilde{M}$ is 1-connected the induced homomorphism from $H^2(\widetilde{M};\mathbb{Z}) \otimes \mathbb{Z}[\pi]$ to $H^2(\widetilde{M};\mathbb{Z}) \otimes \ell^2(\pi)$ is injective. As $ev^{(2)}(\delta g)(z) = ev^{(2)}(g)(\partial z) = 0$ for any square summable 1-chain g and $\mathrm{Ker}(ev^{(2)})$ is closed $ev^{(2)}$ factors through the reduced L^2-cohomology $\bar{H}^2_{(2)}(\widetilde{M})$. In particular, it is 0 if $\beta_1^{(2)}(\pi) = \chi(M) = 0$. Hence the middle arrow of the sequence in Lemma 3.3 is also 0 and c_M^* is an isomorphism. □

A related argument gives a complete and natural criterion for asphericity for closed 4-manifolds.

Theorem 3.5 *Let M be a finite PD_4-complex with fundamental group π. Then M is aspherical if and only if $H^s(\pi;\mathbb{Z}[\pi]) = 0$ for $s \leq 2$ and $\beta_2^{(2)}(M) = \beta_2^{(2)}(\pi)$.*

Proof The conditions are clearly necessary. Suppose that they hold. Then as $\beta_i^{(2)}(M) = \beta_i^{(2)}(\pi)$ for $i \leq 2$ the classifying map $c_M : M \to K(\pi,1)$ induces weak isomorphisms on reduced L^2-cohomology $\bar{H}^i_{(2)}(\pi) \to \bar{H}^i_{(2)}(\widetilde{M})$ for $i \leq 2$.

The natural homomorphism $h : H^2(M;\ell^2(\pi)) \to H^2(\widetilde{M};\mathbb{Z}) \otimes \ell^2(\pi)$ factors through $\bar{H}^2_{(2)}(\widetilde{M})$. The induced homomorphism is a homomorphism of Hilbert modules and so has closed kernel. But the image of $\bar{H}^2_{(2)}(\pi)$ is dense in $\bar{H}_2^{(2)}(\widetilde{M})$ and is in this kernel. Hence $h = 0$. Since $H^2(\pi;\mathbb{Z}[\pi]) = 0$ the homomorphism from $H^2(M;\mathbb{Z}[\pi])$ to $H^2(\widetilde{M};\mathbb{Z}) \otimes \mathbb{Z}[\pi]$ obtained by forgetting $\mathbb{Z}[\pi]$-linearity is injective. Hence the composite homomorphism from $H^2(M;\mathbb{Z}[\pi])$ to $H^2(\widetilde{M};\mathbb{Z}) \otimes \ell^2(\pi)$ is also injective. But this composite may also be factored as the natural map from $H^2(M;\mathbb{Z}[\pi])$ to $H^2(M;\ell^2(\pi))$ followed by h. Hence $H^2(M;\mathbb{Z}[\pi]) = 0$ and so M is aspherical, by Poincaré duality. □

Corollary 3.5.1 *M is aspherical if and only if π is an FF PD_4-group and $\chi(M) = \chi(\pi)$.* □

This also follows immediately from Theorem 3.2, if also $\beta_2(\pi) \neq 0$. For we may assume that M and π are orientable, after passing to the subgroup $Ker(w_1(M)) \cap Ker(w_1(\pi))$, if necessary. As $H_2(c_M;\mathbb{Z})$ is an epimorphism it is an isomorphism, and so c_M must have degree ± 1, by Poincaré duality.

Corollary 3.5.2 *If $\chi(M)=\beta_1^{(2)}(\pi)=0$ and $H^s(\pi;\mathbb{Z}[\pi])=0$ for $s\leq 2$ then M is aspherical and π is a PD_4-group.* ∎

Corollary 3.5.3 *If $\pi\cong Z^r$ then $\chi(M)\geq 0$, with equality only if $r=1$, 2 or 4.*

Proof If $r>2$ then $H^s(\pi;\mathbb{Z}[\pi])=0$ for $s\leq 2$. ∎

Is it possible to replace the hypothesis "$\beta_2^{(2)}(M)=\beta_2^{(2)}(\pi)$" in Theorem 3.5 by "$\beta_2(M^+)=\beta_2(\mathrm{Ker}w_1(M))$", where $p_+ : M^+\to M$ is the orientation cover? It is easy to find examples to show that the homological conditions on π cannot be relaxed further.

Theorem 3.5 implies that if π is a PD_4-group and $\chi(M)=\chi(\pi)$ then $c_{M*}[M]$ is nonzero. If we drop the condition $\chi(M)=\chi(\pi)$ this need not be true. Given any finitely presentable group G there is a closed orientable 4-manifold M with $\pi_1(M)\cong G$ and such that $c_{M*}[M]=0$ in $H_4(G;\mathbb{Z})$. We may take M to be the boundary of a regular neighbourhood N of some embedding in $\mathbb{R}^5$ of a finite 2-complex K with $\pi_1(K)\cong G$. As the inclusion of M into N is 2-connected and K is a deformation retract of N the classifying map c_M factors through c_K and so induces the trivial homomorphism on homology in degrees >2. However if M and π are orientable and $\beta_2(M)<2\beta_2(\pi)$ then c_M must have nonzero degree, for the image of $H^2(\pi;\mathbb{Q})$ in $H^2(M;\mathbb{Q})$ then cannot be self-orthogonal under cup-product.

Theorem 3.6 *Let π be a PD_4-group with a finite $K(\pi,1)$-complex and such that $\chi(\pi)=0$. Then $\mathrm{def}(\pi)\leq 0$.*

Proof Suppose that π has a presentation of deficiency >0, and let X be the corresponding 2-complex. Then $\beta_2^{(2)}(\pi)-\beta_1^{(2)}(\pi)\leq\beta_2^{(2)}(X)-\beta_1^{(2)}(\pi)=\chi(X)\leq 0$. We also have $\beta_2^{(2)}(\pi)-2\beta_1^{(2)}(\pi)=\chi(\pi)=0$. Hence $\beta_1^{(2)}(\pi)=\beta_2^{(2)}(\pi)=\chi(X)=0$. Therefore X is aspherical, by Theorem 2.4, and so $c.d.\pi\leq 2$. But this contradicts the hypothesis that π is a PD_4-group. ∎

Is $\mathrm{def}(\pi)\leq 0$ for any PD_4-group π? This bound is best possible for groups with $\chi=0$, since there is a poly-Z group $Z^3\times_A Z$, where $A\in SL(3,\mathbb{Z})$, with presentation $\langle s,x,\mid sxs^{-1}x=xsxs^{-1},\ s^3x=xs^3\rangle$.

The hypothesis on orientation characters in Theorem 3.2 is often redundant.

Theorem 3.7 *Let $f : M \to N$ be a 2-connected map between finite PD_4-complexes with $\chi(M) = \chi(N)$. If $H^2(N;\mathbb{F}_2) \neq 0$ then $f^*w_1(N) = w_1(M)$, and if moreover N is orientable and $H^2(N;\mathbb{Q}) \neq 0$ then f is a homotopy equivalence.*

Proof Since f is 2-connected $H^2(f;\mathbb{F}_2)$ is injective, and since $\chi(M) = \chi(N)$ it is an isomorphism. Since $H^2(N;\mathbb{F}_2) \neq 0$, the nondegeneracy of Poincaré duality implies that $H^4(f;\mathbb{F}_2) \neq 0$, and so f is a $\mathbb{F}_2$-(co)homology equivalence. Since $w_1(M)$ is characterized by the Wu formula $x \cup w_1(M) = Sq^1 x$ for all x in $H^3(M;\mathbb{F}_2)$, it follows that $f^*w_1(N) = w_1(M)$.

If $H^2(N;\mathbb{Q}) \neq 0$ then $H^2(N;\mathbb{Z})$ has positive rank and $H^2(N;\mathbb{F}_2) \neq 0$, so N orientable implies M orientable. We may then repeat the above argument with integral coefficients, to conclude that f has degree ± 1. The result then follows from Theorem 3.2. □

The argument breaks down if, for instance, $M = S^1\tilde{\times}S^3$ is the nonorientable S^3-bundle over S^1, $N = S^1 \times S^3$ and f is the composite of the projection of M onto S^1 followed by the inclusion of a factor.

We would like to replace the hypotheses above that there be a map $f : M \to N$ realizing certain isomorphisms by weaker, more algebraic conditions. If M and N are closed 4-manifolds with isomorphic algebraic 2-types then there is a 3-connected map $f : M \to P_2(N)$. The restriction of such a map to $M_o = M\backslash D^4$ is homotopic to a map $f_o : M_o \to N$ which induces isomorphisms on π_i for $i \leq 2$. In particular, $\chi(M) = \chi(N)$. Thus if f_o extends to a map from M to N we may be able to apply Theorem 3.2. However we usually need more information on how the top cell is attached. The characteristic classes and the equivariant intersection pairing on $\pi_2(M)$ are the obvious candidates.

The following criterion arises in studying the homotopy types of circle bundles over 3-manifolds. (See Chapter 4.)

Theorem 3.8 *Let E be a finite PD_4-complex with fundamental group π and suppose that $H^4(f_E;Z^{w_1(E)})$ is a monomorphism. A finite PD_4-complex M is homotopy equivalent to E if and only if there is an isomorphism θ from $\pi_1(M)$ to π such that $w_1(M) = w_1(E)\theta$, there is a lift $\hat{c} : M \to P_2(E)$ of θc_M such that $\hat{c}_*[M] = \pm f_{E*}[E]$ and $\chi(M) = \chi(E)$.*

Proof The conditions are clearly necessary. Conversely, suppose that they hold. We shall adapt to our situation the arguments of Hendriks in analyzing the obstructions to the existence of a degree 1 map between PD_3-complexes realizing a given homomorphism of fundamental groups. For simplicity of notation we shall write $\tilde{Z}$ for $Z^{w_1(E)}$ and also for $Z^{w_1(M)}(=\theta^*\tilde{Z})$, and use θ to identify $\pi_1(M)$ with π and $K(\pi_1(M),1)$ with $K(\pi,1)$. We may suppose the sign of the fundamental class $[M]$ is so chosen that $\hat{c}_*[M]=f_{E*}[E]$.

Let $E_o=E\backslash D^4$. Then $P_2(E_o)=P_2(E)$ and may be constructed as the union of E_o with cells of dimension ≥ 4. Let

$$h:\tilde{Z}\otimes_{\mathbb{Z}[\pi]}\pi_4(P_2(E_o),E_o)\to H_4(P_2(E_o),E_o;\tilde{Z})$$

be the $w_1(E)$-twisted relative Hurewicz homomorphism, and let ∂ be the connecting homomorphism from $\pi_4(P_2(E_o),E_o)$ to $\pi_3(E_o)$ in the exact sequence of homotopy for the pair $(P_2(E_o),E_o)$. Then h and ∂ are isomorphisms since f_{E_o} is 3-connected, and so the homomorphism $\tau_E:H_4(P_2(E);\tilde{Z})\to\tilde{Z}\otimes_{\mathbb{Z}[\pi]}\pi_3(E_o)$ given by the composite of the inclusion

$$H_4(P_2(E);\tilde{Z})=H_4(P_2(E_o);\tilde{Z})\to H_4(P_2(E_o),E_o;\tilde{Z})$$

with h^{-1} and $1\otimes_{\mathbb{Z}[\pi]}\partial$ is a monomorphism. Similarly $M_o=M\backslash D^4$ may be viewed as a subspace of $P_2(M_o)$ and there is a monomorphism τ_M from $H_4(P_2(M);\tilde{Z})$ to $\tilde{Z}\otimes_{\mathbb{Z}[\pi]}\pi_3(M_o)$. These monomorphisms are natural with respect to maps defined on the 3-skeleta (i.e., E_o and M_o).

The classes $\tau_E(f_{E*}[E])$ and $\tau_M(f_{M*}[M])$ are the images of the primary obstructions to retracting E onto E_o and M onto M_o, under the Poincaré duality isomorphisms from $H^4(E,E_o;\pi_3(E_o))$ to $H_0(E\backslash E_o;\tilde{Z}\otimes_{\mathbb{Z}[\pi]}\pi_3(E_o))=\tilde{Z}\otimes_{\mathbb{Z}[\pi]}\pi_3(E_o)$ and $H^4(M,M_o;\pi_3(M_o))$ to $\tilde{Z}\otimes_{\mathbb{Z}[\pi]}\pi_3(M_o)$, respectively. Since M_o is homotopy equivalent to a cell complex of dimension ≤ 3 the restriction of $\hat{c}$ to M_o is homotopic to a map from M_o to E_o. Let $\hat{c}_\sharp$ be the homomorphism from $\pi_3(M_o)$ to $\pi_3(E_o)$ induced by $\hat{c}|M_o$. Then $(1\otimes_{\mathbb{Z}[\pi]}\hat{c}_\sharp)\tau_M(f_{M*}[M])=\tau_E(f_{E*}[E])$. It follows as in [Hn77] that the obstruction to extending $\hat{c}|M_o:M_o\to E_o$ to a map d from M to E is trivial.

Since $f_{E*}d_*[M]=\hat{c}_*[M]=f_{E*}[E]$ and f_{E*} is a monomorphism in degree 4 the map d has degree 1, and so is a homotopy equivalence, by Theorem 3.2. ∎

If there is such a lift $\hat{c}$ then $c_M^*\theta^*k_1(E)=0$ and $\theta_*c_{M*}[M]=c_{E*}[E]$.

3.2 Finitely dominated covering spaces

In this section we shall show that if a PD_4-complex has an infinite regular covering space which is finitely dominated then either the complex is aspherical or its universal covering space is homotopy equivalent to S^2 or S^3. In Chapters 4 and 5 we shall see that such manifolds are close to being total spaces of fibre bundles.

Theorem 3.9 *Let M be a PD_4-complex with fundamental group π. Suppose that $p:\widehat{M}\to M$ is a regular covering map, with covering group $G=Aut(p)$, and such that $\widehat{M}$ is finitely dominated. Then*

(1) *G has finitely many ends;*

(2) *if $\widehat{M}$ is acyclic then it is contractible and M is aspherical;*

(3) *if G has one end and $\pi_1(\widehat{M})$ is infinite and FP_3 then M is aspherical and $\widehat{M}$ is homotopy equivalent to an aspherical closed surface or to S^1;*

(4) *if G has one end and $\pi_1(\widehat{M})$ is finite but $\widehat{M}$ is not acyclic then $\widehat{M}\simeq S^2$ or RP^2;*

(5) *G has two ends if and only if $\widehat{M}$ is a PD_3-complex.*

Proof We may clearly assume that G is infinite and that M is orientable. As $\mathbb{Z}[G]$ has no nonzero left ideal (i.e., submodule) which is finitely generated as an abelian group $Hom_{\mathbb{Z}[G]}(H_p(\widehat{M};\mathbb{Z}),\mathbb{Z}[G])=0$ for all $p\geq 0$, and so the bottom row of the UCSS for the covering p is 0. From Poincaré duality and the UCSS we find that $H^1(G;\mathbb{Z}[G])\cong\overline{H_3(\widehat{M};\mathbb{Z})}$. As this group is finitely generated, and as G is infinite, G has one or two ends.

If $\widehat{M}$ is acyclic then G is a PD_4-group and so $\widehat{M}$ is a PD_0-complex, hence contractible, by [Go79]. Hence M is aspherical.

Suppose that G has one end. Then $H_3(\widehat{M};\mathbb{Z})=H_4(\widehat{M};\mathbb{Z})=0$. Since $\widehat{M}$ is finitely dominated the chain complex $C_*(\widetilde{M})$ is chain homotopy equivalent over $\mathbb{Z}[\pi_1(\widehat{M})]$ to a complex D_* of finitely generated projective $\mathbb{Z}[\pi_1(\widehat{M})]$-modules. If $\pi_1(\widehat{M})$ is FP_3 then the aumentation $\mathbb{Z}[\pi_1(\widehat{M})]$-module Z has a free resolution P_* which is finitely generated in degrees ≤ 3. On applying Schanuel's Lemma to the exact sequences

$$0\to Z_2\to D_2\to D_1\to D_0\to Z\to 0$$

and

$$0\to\partial P_3\to P_2\to P_1\to P_0\to Z\to 0$$

derived from these two chain complexes we find that Z_2 is finitely generated as a $\mathbb{Z}[\pi_1(\widehat{M})]$-module. Hence $\Pi=\pi_2(M)=\pi_2(\widehat{M})$ is also finitely generated as a $\mathbb{Z}[\pi_1(\widehat{M})]$-module and so $Hom_\pi(\Pi,\mathbb{Z}[\pi])=0$. If moreover $\pi_1(\widehat{M})$ is infinite then $H^s(\pi;\mathbb{Z}[\pi])=0$ for $s\leq 2$, so $\Pi=0$, by Lemma 3.3, and M

is aspherical. A spectral sequence corner argument then shows that either $H^2(G;\mathbb{Z}[G])\cong Z$ and $\widehat{M}$ is homotopy equivalent to an aspherical closed surface or $H^2(G;\mathbb{Z}[G])=0$, $H^3(G;\mathbb{Z}[G])\cong Z$ and $\widehat{M}\simeq S^1$. (See the following theorem.)

If $\pi_1(\widehat{M})$ is finite but $\widehat{M}$ is not acyclic then the universal covering space $\widetilde{M}$ is also finitely dominated but not contractible, and $\Pi=H_2(\widetilde{M};\mathbb{Z})$ is a nontrivial finitely generated abelian group, while $H_3(\widetilde{M};\mathbb{Z})=H_4(\widetilde{M};\mathbb{Z})=0$. If C is a finite cyclic subgroup of π there are isomorphisms $H_{n+3}(C;\mathbb{Z})\cong H_n(C;\Pi)$, for all $n\geq 4$, by Lemma 2.10. Suppose that C acts trivially on Π. Then if n is odd this isomorphism reduces to $0=\Pi/|C|\Pi$. Since Π is finitely generated, this implies that multiplication by $|C|$ is an isomorphism. On the other hand, if n is even we have $Z/|C|Z\cong\{a\in\Pi\mid |C|a=0\}$. Hence we must have $C=1$. Now since Π is finitely generated any torsion subgroup of $Aut(\Pi)$ is finite. (Let T be the torsion subgroup of Π and suppose that $\Pi/T\cong Z^r$. Then the natural homomorphism from $Aut(\Pi)$ to $Aut(\Pi/T)$ has finite kernel, and its image is isomorphic to a subgroup of $GL(r,\mathbb{Z})$, which is virtually torsion free.) Hence as π is infinite it must have elements of infinite order. Since $H^2(\pi;\mathbb{Z}[\pi])\cong\bar{\Pi}$, by Lemma 3.3, it is a finitely generated abelian group. Therefore it must be infinite cyclic, by Corollary 5.2 of [Fa74]. Hence $\widetilde{M}\simeq S^2$ and $\pi_1(\widehat{M})$ has order at most 2, so $\widehat{M}\simeq S^2$ or RP^2.

Suppose now that $\widehat{M}$ is a PD_3-complex. After passing to a finite covering of M, if necessary, we may assume that $\widehat{M}$ is orientable. Then $H^1(G;\mathbb{Z}[G])\cong H_3(\widehat{M};\mathbb{Z})$, and so G has two ends. Conversely, if G has two ends we may assume that $G\cong Z$, after passing to a finite covering of M, if necessary. Hence $\widehat{M}$ is a PD_3-complex, by [Go79] again. (See Theorem 4.5 for an alternative argument, with weaker, algebraic hypotheses.) □

Is the hypothesis in (3) that $\pi_1(\widehat{M})$ be FP_3 redundant?

Corollary 3.9.1 *The covering space $\widehat{M}$ is homotopy equivalent to a closed surface if and only if it is finitely dominated, $H^2(G;\mathbb{Z}[G])\cong Z$ and $\pi_1(\widehat{M})$ is FP_3.* □

In this case M has a finite covering space which is homotopy equivalent to the total space of a surface bundle over an aspherical closed surface. (See Chapter 5.)

Corollary 3.9.2 *The covering space $\widehat{M}$ is homotopy equivalent to S^1 if and only if it is finitely dominated, G has one end, $H^2(G;\mathbb{Z}[G])=0$ and $\pi_1(\widehat{M})$ is a nontrivial finitely generated free group.*

Proof If $\widehat{M} \simeq S^1$ then it is finitely dominated and M is aspherical, and the conditions on G follow from the LHSSS. The converse follows from part (3) of the theorem, since a nontrivial finitely generated free group is infinite and FP. □

In fact any finitely generated free normal subgroup F of a PD_n-group π must be infinite cyclic. For $\pi/C_\pi(F)$ embeds in $Out(F)$, so $v.c.d.\pi/C_\pi(F) \leq v.c.d.Out(F(r)) < \infty$. If F is nonabelian then $C_\pi(F) \cap F = 1$ and so $c.d.\pi/F < \infty$. Since F is finitely generated π/F is FP_∞. Hence we may apply Theorem 9.11 of [Bi], and an LHSSS corner argument gives a contradiction.

In the simply connected case "finitely dominated", "homotopy equivalent to a finite complex" and "having finitely generated homology" are all equivalent.

Corollary 3.9.3 *If $H_*(\widetilde{M};\mathbb{Z})$ is finitely generated then either M is aspherical or $\widetilde{M}$ is homotopy equivalent to S^2 or S^3 or $\pi_1(M)$ is finite.* □

We shall examine the spherical cases more closely in Chapters 10 and 11. (The arguments in these chapters may apply also to PD_n-complexes with universal covering space homotopy equivalent to S^{n-1} or S^{n-2}. The analogues in higher codimensions appear to be less accessible.)

The "finitely dominated" condition is used only to ensure that the chain complex of the covering is chain homotopy equivalent over $\mathbb{Z}[\pi_1(\widehat{M})]$ to a finite projective complex. Thus when M is aspherical this condition can be relaxed slightly. The following variation on the aspherical case shall be used in Theorem 4.8, but belongs most naturally here.

Theorem 3.10 *Let N be a nontrivial FP_3 normal subgroup of infinite index in a PD_4-group π, and let $G = \pi/N$. Then either*

(1) *N is a PD_3-group and G has two ends;*

(2) *N is a PD_2-group and G is virtually a PD_2-group; or*

(3) *$N \cong Z$, $H^s(G;\mathbb{Z}[G]) = 0$ for $s \leq 2$ and $H^3(G;\mathbb{Z}[G]) \cong Z$.*

Proof Since $c.d.N < 4$, by Strebel's Theorem, N and hence G are FP. The E_2 terms of the LHS spectral sequence with coefficients $\mathbb{Q}[\pi]$ can then be expressed as $E_2^{pq} = H^p(G;\mathbb{Q}[G]) \otimes H^q(N;\mathbb{Q}[N])$. If $H^j(\pi/N;\mathbb{Q}[\pi/N])$ and $H^k(N;\mathbb{Q}[N])$ are the first nonzero such cohomology groups then E_2^{jk} persists to E_∞ and hence $j + k = 4$. Therefore $H^j(G;\mathbb{Q}[G]) \otimes H^{4-j}(N;\mathbb{Q}[N]) \cong Q$.

Hence $H^j(G;\mathbb{Q}[G]) \cong H^{4-j}(N;\mathbb{Q}[N]) \cong Q$. In particular, G has one or two ends and N is a PD_{4-j}-group over $\mathbb{Q}$ [Fa75]. If G has two ends then it is virtually Z, and then N is a PD_3-group (over $\mathbb{Z}$) by Theorem 9.11 of [Bi]. If $H^2(N;\mathbb{Q}[N]) \cong H^2(G;\mathbb{Q}[G]) \cong Q$ then N and G are virtually PD_2-groups, by Bowditch's Theorem. Since N is torsion free it is then in fact a PD_2-group. The only remaining possibility is (3). □

In case (1) π has a subgroup of index ≤ 2 which is a semidirect product $H \times_\theta Z$ with $N \leq H$ and $[H:N] < \infty$. Is it sufficient that N be FP_2? Must the quotient π/N be virtually a PD_3-group in case (3)?

Corollary 3.10.1 *If K is FP_2 and is subnormal in N where N is an FP_3 normal subgroup of infinite index in the PD_4-group π then K is a PD_k-group for some $k < 4$.*

Proof This follows from Theorem 3.10 together with Theorem 2.16. □

What happens if we drop the hypothesis that the covering be regular? It can be shown that a closed 3-manifold has a finitely dominated infinite covering space if and only if its fundamental group has one or two ends. We might conjecture that if a closed 4-manifold M has a finitely dominated infinite covering space $\widehat{M}$ then either M is aspherical or the universal covering space $\widetilde{M}$ is homotopy equivalent to S^2 or S^3 or M has a finite covering space which is homotopy equivalent to the mapping torus of a self homotopy equivalence of a PD_3-complex. (In particular, $\pi_1(M)$ has one or two ends.) In [Hi94'] we extend the arguments of Theorem 3.9 to show that if $\pi_1(\widehat{M})$ is FP_3 and subnormal in π the only other possibility is that $\pi_1(\widehat{M})$ has two ends, $h(\sqrt{\pi}) = 1$ and $H^2(\pi;\mathbb{Z}[\pi])$ is not finitely generated. This paper also considers in more detail FP subnormal subgroups of PD_4-groups, corresponding to the aspherical case.

3.3 Minimizing the Euler characteristic

It is well known that every finitely presentable group is the fundamental group of some closed orientable 4-manifold. Such manifolds are far from unique, for the Euler characteristic may be made arbitrarily large by taking connected sums with simply connected manifolds. Following Hausmann and Weinberger [HW85] we may define an invariant $q(\pi)$ for any finitely presentable group π by

$$q(\pi) = \min\{\chi(M) | M \textit{ is a } PD_4 \textit{ complex with } \pi_1(M) \cong \pi\}.$$

We may also define related invariants q^X where the minimum is taken over the class of PD_4-complexes whose normal fibration has an X-reduction. There are the following basic estimates for q^{SG}, which is defined in terms of PD_4^+-complexes.

Lemma 3.11 *Let π be a finitely presentable group with a subgroup H of finite index and let F be a field. Then*

(1) $1-\beta_1(H;F)+\beta_2(H;F)\leq[\pi:H](1-\mathrm{def}\pi)$;
(2) $2-2\beta_1(H;F)+\beta_2(H;F)\leq[\pi:H]q^{SG}(\pi)$;
(3) $q^{SG}(\pi)\leq 2(1-\mathrm{def}(\pi))$;
(4) *if* $H^4(\pi;F)=0$ *then* $q^{SG}(\pi)\geq 2(1-\beta_1(\pi;F)+\beta_2(\pi;F))$.

Proof Let C be the 2-complex corresponding to a presentation for π of maximal deficiency and let C_H be the covering space associated to the subgroup H. Then $\chi(C)=1-\mathrm{def}\pi$ and $\chi(C_H)=[\pi:H]\chi(\pi)$. Condition (1) follows since $\beta_1(H;F)=\beta_1(C_H;F)$ and $\beta_2(H;F)\leq\beta_2(C_H;F)$.

Condition (2) follows similarly on considering the Euler characteristics of a PD_4^+-complex M with $\pi_1(M)\cong\pi$ and of the associated covering space M_H.

The boundary of a regular neighbourhood of a PL embedding of C in R^5 is a closed orientable 4-manifold realizing the upper bound in (3).

The image of $H^2(\pi;F)$ in $H^2(M;F)$ has dimension $\beta_2(\pi;F)$, and is self-annihilating under cup-product if $H^4(\pi;F)=0$. In that case $\beta_2(M;F)\geq 2\beta_2(\pi;F)$, which implies (4). □

Condition (2) was used in [HW85] to give examples of finitely presentable superperfect groups which are not fundamental groups of homology 4-spheres. (See Chapter 14 below.)

If π is a finitely presentable, orientable PD_4-group we see immediately that $q^{SG}(\pi)\geq\chi(\pi)$. Multiplicativity then implies that $q(\pi)=\chi(\pi)$ if $K(\pi,1)$ is a finite PD_4-complex.

For groups of cohomological dimension at most two we can say more.

Theorem 3.12 *Let M be a finite PD_4-complex with fundamental group π. Suppose that $c.d._{\mathbb{Q}}\pi\leq 2$ and $\chi(M)=2\chi(\pi)=2(1-\beta_1(\pi;\mathbb{Q})+\beta_2(\pi;\mathbb{Q}))$. Then $\pi_2(M)\cong\overline{H^2(\pi;\mathbb{Z}[\pi])}$. If moreover $c.d.\pi\leq 2$ the chain complex of the universal covering space $\widetilde{M}$ is determined up to chain homotopy equivalence over $\mathbb{Z}[\pi]$ by π.*

Proof Let $A_Q(\pi)$ be the augmentation ideal of $\mathbb{Q}[\pi]$. Then there are exact sequences

$$0 \to A_Q(\pi) \to \mathbb{Q}[\pi] \to Q \to 0 \tag{3.1}$$

and

$$0 \to P \to \mathbb{Q}[\pi]^g \to A_Q(\pi) \to 0. \tag{3.2}$$

where P is a finitely generated projective module. We may assume that that $\pi \neq 1$, i.e., that π is infinite, and that M is a finite 4-dimensional cell complex. Let C_* be the cellular chain complex of $\widetilde{M}$, with coefficients $\mathbb{Q}$, and let $H_i = H_i(C_*) = H_i(\widetilde{M};\mathbb{Q})$ and $H^t = H^t(Hom_{\mathbb{Q}[\pi]}(C_*, \mathbb{Q}[\pi]))$. Since $\widetilde{M}$ is simply connected and π is infinite, $H_0 \cong Q$ and $H_1 = H_4 = 0$. Poincaré duality gives further isomorphisms $H^1 \cong \overline{H_3}$, $H^2 \cong \overline{H_2}$, $H^3 = 0$ and $H^4 \cong \overline{Q}$.

The chain complex C_* breaks up into exact sequences:

$$0 \to C_4 \to Z_3 \to H_3 \to 0, \tag{3.3}$$

$$0 \to Z_3 \to C_3 \to Z_2 \to H_2 \to 0, \tag{3.4}$$

$$0 \to Z_2 \to C_2 \to C_1 \to C_0 \to Q \to 0. \tag{3.5}$$

We shall let $e^iN = Ext^i_{\mathbb{Q}[\pi]}(N, \mathbb{Q}[\pi])$, to simplify the notation in what follows. The UCSS gives isomorphisms $H^1 \cong e^1Q$ and $e^1H_2 = e^2H_3 = 0$ and another exact sequence:

$$0 \to e^2Q \to H^2 \to e^0H_2 \to 0. \tag{3.6}$$

Applying Schanuel's Lemma to the sequences 3.1, 3.2 and 3.5 we obtain $Z_2 \oplus C_1 \oplus \mathbb{Q}[\pi] \oplus P \cong C_2 \oplus C_0 \oplus \mathbb{Q}[\pi]^g$, so Z_2 is a finitely generated projective module. Similarly, Z_3 is projective, since $\mathbb{Q}[\pi]$ has global dimension at most 2. Since π is finitely presentable it is accessible, and hence e^1Q is finitely generated as a $\mathbb{Q}[\pi]$-module, by Theorems IV.7.5 and VI.6.3 of [DD]. Therefore Z_3 is also finitely generated, since it is an extension of $H_3 \cong \overline{e^1Q}$ by C_4. Dualizing the sequence 3.4 and using the fact that $e^1H_2 = 0$ we obtain an exact sequence of right modules

$$0 \to e^0H_2 \to e^0Z_2 \to e^0C_3 \to e^0Z_3 \to e^2H_2 \to 0. \tag{3.7}$$

Since duals of finitely generated projective modules are projective it follows that e^0H_2 is projective. Hence the sequence 3.6 gives $H^2 \cong e^0H_2 \oplus e^2Q$.

Dualizing the sequences 3.1 and 3.2, we obtain exact sequences of right modules

$$0 \to \mathbb{Q}[\pi] \to e^0A_Q(\pi) \to e^1Q \to 0 \tag{3.8}$$

and

$$0 \to e^0A_Q(\pi) \to \mathbb{Q}[\pi]^g \to e^0P \to e^2Q \to 0. \tag{3.9}$$

Applying Schanuel's Lemma twice more, to the pairs of sequences 3.3 and the conjugate of 3.8 (using $H_3 \cong \overline{e^1 Q}$) and to 3.4 and the conjugate of 3.9 (using $H_2 \cong \overline{e^0 H_2} \oplus \overline{e^2 Q}$) and putting all together, we obtain isomorphisms

$$Z_3 \oplus (\mathbb{Q}[\pi]^{2g} \oplus C_0 \oplus C_2 \oplus C_4) \cong Z_3 \oplus (\mathbb{Q}[\pi]^2 \oplus P \oplus \overline{e^0 P} \oplus C_1 \oplus C_3 \oplus \overline{e^0 H_2}).$$

On tensoring with the augmentation module we find that

$$dim_{\mathbb{Q}}(\mathbb{Q} \otimes_\pi \overline{e^0 H_2}) + dim_{\mathbb{Q}}(\mathbb{Q} \otimes_\pi P) + dim_{\mathbb{Q}}(\mathbb{Q} \otimes_\pi \overline{e^0 P}) = \chi(M) + 2g - 2.$$

Now

$$dim_{\mathbb{Q}}(\mathbb{Q} \otimes_\pi P) = dim_{\mathbb{Q}}(\mathbb{Q} \otimes_\pi \overline{e^0 P}) = g + \beta_2(\pi;\mathbb{Q}) - \beta_1(\pi;\mathbb{Q}),$$

so $dim_{\mathbb{Q}}(\mathbb{Q} \otimes_\pi \overline{e^0 H_2}) = \chi(M) - 2\chi(\pi) = 0$. Hence $e^0 H_2 = 0$, since π satisfies the Weak Bass Conjecture [Ec86]. As $Hom_{\mathbb{Z}[\pi]}(H_2(\widetilde{M};\mathbb{Z}), \mathbb{Z}[\pi]) \leq e^0 H_2$ it follows from Lemma 3.3 that $\pi_2(M) \cong H_2(\widetilde{M};\mathbb{Z}) \cong \overline{H^2(\pi;\mathbb{Z}[\pi])}$.

If $c.d.\pi \leq 2$ then $e^1 Z$ has a short finite projective resolution, and hence so does Z_3 (via sequence 3.2). The argument can then be modified to work over $\mathbb{Z}[\pi]$. As Z_1 is then projective, the integral chain complex of $\widetilde{M}$ is the direct sum of a projective resolution of Z with a projective resolution of $\pi_2(M)$ with degree shifted by 2. □

There are many natural examples of such manifolds for which $c.d._{\mathbb{Q}}\pi \leq 2$ and $\chi(M) = 2\chi(\pi)$ but π is not torsion free. (See Chapters 10 and 11.) However all the known examples satisfy $v.c.d.\pi \leq 2$.

Similar arguments may be used to prove the following variations.

Addendum *Suppose that $c.d._S\pi \leq 2$ for some subring $S \leq \mathbb{Q}$. Then $q(\pi) \geq 2(1 - \beta_1(\pi;S) + \beta_2(\pi;S))$. If moreover the augmentation $S[\pi]$-module S has a finitely generated free resolution then $S \otimes \pi_2(M)$ is stably isomorphic to $\overline{H^2(\pi;S[\pi])}$.* □

Corollary 3.12.1 *If $H_2(\pi;\mathbb{Q}) \neq 0$ the Hurewicz homomorphism from $\pi_2(M)$ to $H_2(M;\mathbb{Q})$ is nonzero.*

Proof By the addendum to the theorem, $H_2(M;\mathbb{Q})$ has dimension at least $2\beta_2(\pi)$, and so cannot be isomorphic to $H_2(\pi;\mathbb{Q})$ unless both are 0. □

Corollary 3.12.2 *If $\pi = \pi_1(P)$ where P is an aspherical finite 2-complex then $q(\pi) = 2\chi(P)$. The minimum is realized by an s-parallelizable PL 4-manifold.*

Proof If we choose a PL embedding $j : P \to \mathbb{R}^5$, the boundary of a regular neighbourhood N of $j(P)$ is an s-parallelizable PL 4-manifold with fundamental group π and with Euler characteristic $2\chi(P)$. $\square$

By Theorem 2.8 a finitely presentable group is the fundamental group of an aspherical finite 2-complex if and only if it has cohomological dimension ≤ 2 and is efficient, i.e. has a presentation of deficiency $\beta_1(\pi;\mathbb{Q}) - \beta_2(\pi;\mathbb{Q})$. It is not known whether every finitely presentable group of cohomological dimension 2 is efficient.

In Chapter 5 we shall see that if P is an aspherical closed surface and M is a closed 4-manifold with $\pi_1(M) \cong \pi$ then $\chi(M) = q(\pi)$ if and only if M is homotopy equivalent to the total space of an S^2-bundle over P. The homotopy types of such minimal 4-manifolds for π may be distinguished by their Stiefel-Whitney classes. Note that if π is orientable then $S^2 \times P$ is a minimal 4-manifold for π which is both s-parallelizable and also a projective algebraic complex surface. Note also that the conjugation of the module structure in the theorem involves the orientation character of M which may differ from that of the PD_2-group π.

Corollary 3.12.3 *If π is the group of an unsplittable μ-component 1-link then $q(\pi) = 0$.* $\square$

If π is the group of a μ-component n-link with $n \geq 2$ then $H_2(\pi;\mathbb{Q}) = 0$ and so $q(\pi) \geq 2(1-\mu)$, with equality if and only if π is the group of a 2-link. (See Chapter 14.)

Corollary 3.12.4 *If π is an extension of Z by a finitely generated free normal subgroup then $q(\pi) = 0$.* $\square$

In Chapter 4 we shall see that if M is a closed 4-manifold with $\pi_1(M)$ such an extension then $\chi(M) = q(\pi)$ if and only if M is homotopy equivalent to a manifold which fibres over S^1 with fibre a closed 3-manifold with free fundamental group, and then π and $w_1(M)$ determine the homotopy type.

Finite generation of the normal subgroup is essential; $F(2)$ is an extension of Z by $F(\infty)$, and $q(F(2)) = 2\chi(F(2)) = -2$.

Let π be the fundamental group of a closed orientable 3-manifold. Then $\pi \cong F*\nu$ where F is free of rank r and ν has no infinite cyclic free factors. Moreover $\nu = \pi_1(N)$ for some closed orientable 3-manifold N. If M_0 is the closed 4-manifold obtained by surgery on $\{n\} \times S^1$ in $N \times S^1$ then $M = M_0\sharp(\sharp^r(S^1 \times S^3)$

is a smooth s-parallelisable 4-manifold with $\pi_1(M) \cong \pi$ and $\chi(M) = 2(1-r)$. Hence $q^{SG}(\pi) = 2(1-r)$, by Lemma 3.11.

The arguments of Theorem 3.12 give stronger results in this case also.

Theorem 3.13 *Let M be a finite PD_4-complex whose fundamental group π is a PD_3-group such that $w_1(\pi) = w_1(M)$. Then $\chi(M) > 0$ and $\pi_2(M)$ is stably isomorphic to the augmentation ideal $A(\pi)$ of $\mathbb{Z}[\pi]$.*

Proof The cellular chain complex for the universal covering space of M gives exact sequences

$$0 \to C_4 \to C_3 \to Z_2 \to H_2 \to 0 \tag{3.10}$$

and

$$0 \to Z_2 \to C_2 \to C_1 \to C_0 \to Z \to 0. \tag{3.11}$$

Since π is a PD_3-group the augmentation module Z has a finite projective resolution of length 3. On comparing sequence 3.11 with such a resolution and applying Schanuel's lemma we find that Z_2 is a finitely generated projective $\mathbb{Z}[\pi]$-module. Since π has one end, the UCSS reduces to an exact sequence

$$0 \to H^2 \to e^0H_2 \to e^3Z \to H^3 \to e^1H_2 \to 0 \tag{3.12}$$

and isomorphisms $H^4 \cong e^2H_2$ and $e^3H_2 = e^4H_2 = 0$. Poincaré duality implies that $H^3 = 0$ and $H^4 \cong \overline{Z}$. Hence sequence 3.12 reduces to

$$0 \to H^2 \to e^0H_2 \to e^3Z \to 0 \tag{3.13}$$

and $e^1H_2 = 0$. Hence on dualizing the sequence 3.10 we get an exact sequence of right modules

$$0 \to e^0H_2 \to e^0Z_2 \to e^0C_3 \to e^0C_4 \to e^2H_2 \to 0. \tag{3.14}$$

Schanuel's lemma again implies that e^0H_2 is a finitely generated projective module. Therefore we may splice together 3.10 and the conjugate of 3.13 to get

$$0 \to C_4 \to C_3 \to Z_2 \to \overline{e^0H_2} \to Z \to 0. \tag{3.15}$$

(Note that we have used the hypothesis on $w_1(M)$ here.) Applying Schanuel's lemma once more to the pair of sequences 3.11 and 3.15 we obtain

$$C_0 \oplus C_2 \oplus C_4 \oplus Z_2 \cong \overline{e^0H_2} \oplus C_1 \oplus C_3 \oplus Z_2.$$

Hence $\overline{e^0H_2}$ is stably free, of rank $\chi(M)$. Since sequence 3.15 is exact $\overline{e^0H_2}$ maps onto Z, and so $\chi(M) > 0$. Since π is a PD_3-group, $e^3Z \cong \overline{Z}$ and so the final assertion follows from sequence 3.13 and Schanuel's Lemma. □

Corollary 3.13.1 $1 \leq q(\pi) \leq 2$.

Proof If M is a finite PD_4-complex with $\pi_1(M) \cong \pi$ then the covering space associated to the kernel of $w_1(M) - w_1(\pi)$ satisfies the condition on w_1. Since the condition $\chi(M) > 0$ is invariant under passage to finite covers, $q(\pi) \geq 1$.

Let N be a PD_3-complex with fundamental group π. We may suppose that $N = N_o \cup D^3$, where $N_o \cap D^3 = S^2$. Let $M = N_o \times S^1 \cup S^2 \times D^2$. Then M is a finite PD_4-complex, $\chi(M) = 2$ and $\pi_1(M) \cong \pi$. Hence $q(\pi) \leq 2$. □

Can Theorem 3.13 be extended to all torsion free 3-manifold groups, or more generally to all free products of PD_3-groups?

A simple application of Schanuel's Lemma to $C_*(\tilde{M})$ shows that if M is a finite PD_4-complex with fundamental group π such that $c.d.\pi \leq 4$ and $e(\pi) = 1$ then $\pi_2(M)$ has projective dimension at most 2. If moreover π is an FF PD_4-group and c_M has degree 1 then $\pi_2(M)$ is stably free of rank $\chi(M) - \chi(\pi)$, by the argument of Lemma 3.1 and Theorem 3.2.

There has been some related work estimating the difference $\chi(M) - |\sigma(M)|$ where M is a closed orientable 4-manifold M with $\pi_1(M) \cong \pi$ and where $\sigma(M)$ is the signature of M. In particular, this difference is always ≥ 0 if $\beta_1^{(2)}(\pi) = 0$. (See [JK93] and §3 of Chapter 7 of [Lü].) The minimum value of this difference ($p(\pi) = \min\{\chi(M) - |\sigma(M)|\}$) is another numerical invariant of π, which is studied in [Ko94].

3.4 Euler Characteristic 0

In this section we shall consider the interaction of the fundamental group and Euler characteristic from another point of view. We shall assume that $\chi(M) = 0$ and show that if π is an ascending HNN extension then it satisfies some very stringent conditions. The groups $Z*_m$ shall play an important role. We shall approach our main result via several lemmas.

We begin with a simple observation relating Euler characteristic and fundamental group which shall be invoked in several of the later chapters. Recall that if G is a group then $I(G)$ is the minimal normal subgroup such that $G/I(G)$ is free abelian.

Lemma 3.14 *Let M be a PD_4-complex with $\chi(M) \leq 0$. If M is orientable then $H^1(M;\mathbb{Z}) \neq 0$ and so $\pi = \pi_1(M)$ maps onto Z. If $H^1(M;\mathbb{Z}) = 0$ then π maps onto D.*

Proof The covering space M_W corresponding to $W = \mathrm{Ker}(w_1(M))$ is orientable and $\chi(M_W) = 2 - 2\beta_1(M_W) + \beta_2(M_W) = [\pi : W]\chi(M) \leq 0$. Therefore $\beta_1(W) = \beta_1(M_W) > 0$ and so $W/I(W) \cong Z^r$ for some $r > 0$. Since $I(W)$ is characteristic in W it is normal in π. As $[\pi : W] \leq 2$ it follows easily that $\pi/I(W)$ maps onto Z or D. □

Note that if $M = RP^4 \sharp RP^4$, then $\chi(M) = 0$ and $\pi_1(M) \cong D$, but $\pi_1(M)$ does not map onto Z.

Lemma 3.15 *Let M be a PD_4^+-complex such that $\chi(M) = 0$ and $\pi = \pi_1(M)$ is an extension of $Z*_m$ by a finite normal subgroup F, for some $m \neq 0$. Then the abelian subgroups of F are cyclic. If $F \neq 1$ then π has a subgroup of finite index which is a central extension of $Z*_n$ by a nontrivial finite cyclic group, where n is a power of m.*

Proof Let $\widehat{M}$ be the infinite cyclic covering space corresponding to the subgroup $I(\pi)$. Since M is compact and $\Lambda = \mathbb{Z}[Z]$ is noetherian the groups $H_i(\widehat{M};\mathbb{Z}) = H_i(M;\Lambda)$ are finitely generated as Λ-modules. Since M is orientable, $\chi(M) = 0$ and $H_1(M;\mathbb{Z})$ has rank 1 they are Λ-torsion modules, by the Wang sequence for the projection of $\widehat{M}$ onto M. Now $H_2(\widehat{M};\mathbb{Z}) \cong Ext^1_\Lambda(I(\pi)/I(\pi)',\Lambda)$, by Poincaré duality. There is an exact sequence

$$0 \to T \to I(\pi)/I(\pi)' \to I(Z*_m) \cong \Lambda/(t-m) \to 0,$$

where T is a finite Λ-module. Therefore $Ext^1_\Lambda(I(\pi)/I(\pi)',\Lambda) \cong \Lambda/(t-m)$ and so $H_2(I(\pi);\mathbb{Z})$ is a quotient of $\Lambda/(mt-1)$, which is isomorphic to $Z[\frac{1}{m}]$ as an abelian group. Now $I(\pi)/\mathrm{Ker}(f) \cong Z[\frac{1}{m}]$ also, and $H_2(Z[\frac{1}{m}];\mathbb{Z}) \cong Z[\frac{1}{m}] \wedge Z[\frac{1}{m}] = 0$ (see page 334 of [Ro]). Hence $H_2(I(\pi);\mathbb{Z})$ is finite, by an LHSSS argument, and so is cyclic, of order relatively prime to m.

Let t in π generate $\pi/I(\pi) \cong Z$. Let A be a maximal abelian subgroup of F and let $C = C_\pi(A)$. Then $q = [\pi : C]$ is finite, since F is finite and normal in π. In particular, t^q is in C and C maps onto Z, with kernel J, say. Since J is an extension of $Z[\frac{1}{m}]$ by a finite normal subgroup its centre ζJ has finite index in J. Therefore the subgroup G generated by ζJ and t^q has finite index in π, and there is an epimorphism f from G onto $Z*_{m^q}$, with kernel A. Moreover $I(G) = f^{-1}(I(Z*_{m^q}))$ is abelian, and is an extension of $Z[\frac{1}{m}]$ by the finite abelian group A. Hence it is isomorphic to $A \oplus Z[\frac{1}{m}]$ (see page 106 of [Ro]). Now $H_2(I(G);\mathbb{Z})$ is cyclic of order prime to m. On the other hand $H_2(I(G);\mathbb{Z}) \cong (A \wedge A) \oplus (A \otimes Z[\frac{1}{m}])$ and so A must be cyclic.

If $F \neq 1$ then A is cyclic, nontrivial, central in G and $G/A \cong Z*_{m^q}$. □

Lemma 3.16 *Let M be a finite PD_4-complex with fundamental group π. Suppose that π has a nontrivial finite cyclic central subgroup F with quotient $G=\pi/F$ such that $g.d.G=2$, $e(G)=1$ and $\mathrm{def}(G)=1$. Then $\chi(M)\geq 0$. If $\chi(M)=0$ and $\mathbb{F}_p[G]$ is a weakly finite ring for some prime p dividing $|F|$ then π is virtually Z^2.*

Proof Let $\widehat{M}$ be the covering space of M with group F, and let $\Xi=\mathbb{F}_p[G]$. Let $C_*=C_*(M;\Xi)=\mathbb{F}_p\otimes C_*(M)$ be the equivariant cellular chain complex of $\widehat{M}$ with coefficients $\mathbb{F}_p$, and let c_q be the number of q-cells of M, for $q\geq 0$. Let $H_p=H_p(M;\Xi)=H_p(\widehat{M};\mathbb{F}_p)$. For any left Ξ-module H let $e^qH=Ext^q_\Xi(H,\Xi)$.

Suppose first that M is orientable. Since $\widehat{M}$ is a connected open 4-manifold $H_0=\mathbb{F}_p$ and $H_4=0$, while $H_1\cong\mathbb{F}_p$ also. Since G has one end Poincaré duality and the UCSS give $H_3=0$ and $e^2H_2\cong\mathbb{F}_p$, and an exact sequence

$$0\to e^2\mathbb{F}_p\to\overline{H_2}\to e^0H_2\to e^2H_1\to\overline{H_1}\to e^1H_2\to 0.$$

In particular, $e^1H_2\cong\mathbb{F}_p$ or is 0. Since $g.d.G=2$ and $\mathrm{def}(G)=1$ the augmentation module has a resolution

$$0\to\Xi^r\to\Xi^{r+1}\to\Xi\to\mathbb{F}_p\to 0.$$

The chain complex C_* gives four exact sequences

$$\begin{aligned} &0\to Z_1\to C_1\to C_0\to\mathbb{F}_p\to 0,\\ &0\to Z_2\to C_2\to Z_1\to\mathbb{F}_p\to 0,\\ &0\to B_2\to Z_2\to H_2\to 0\\ \text{and}\quad &0\to C_4\to C_3\to B_2\to 0.\end{aligned}$$

Using Schanuel's Lemma several times we find that the cycle submodules Z_1 and Z_2 are stably free, of stable ranks c_1-c_0 and $c_2-c_1+c_0$, respectively. Dualizing the last two sequences gives two new sequences

$$\begin{aligned} &0\to e^0B_2\to e^0C_3\to e^0C_4\to e^1B_2\to 0\\ \text{and}\quad &0\to e^0H_2\to e^0Z_2\to e^0B_2\to e^1H_2\to 0,\end{aligned}$$

and an isomorphism $e^1B_2\cong e^2H_2\cong\mathbb{F}_p$. Further applications of Schanuel's Lemma show that e^0B_2 is stably free of rank c_3-c_4, and hence that e^0H_2 is stably free of rank $c_2-c_1+c_0-(c_3-c_4)=\chi(M)$. (Note that we do not need to know whether $e^1H_2\cong\mathbb{F}_p$ or is 0, at this point.) Since Ξ maps onto the field $\mathbb{F}_p$ the rank must be non-negative, and so $\chi(M)\geq 0$.

If $\chi(M)=0$ and $\Xi=\mathbb{F}_p[G]$ is a weakly finite ring then $e^0H_2=0$ and so $e^2\mathbb{F}_p=e^2H_1$ is a submodule of $\mathbb{F}_p\cong\overline{H_1}$. Moreover it cannot be 0, for otherwise the UCSS would give $H_2=0$ and then $H_1=0$, which is impossible. Therefore $e^2\mathbb{F}_p\cong\mathbb{F}_p$.

If M is nonorientable and $p>2$ the above argument applies to the orientation cover, since p divides $|\mathrm{Ker}(w_1(M)|_F)|$, and Euler characteristic is multiplicative in finite covers. If $p=2$ a similar argument applies directly without assuming that M is orientable.

Since G is torsion free and indicable it must be a PD_2-group, by Theorem V.12.2 of [DD]. Since $\mathrm{def}(G)=1$ it follows that G is virtually Z^2, and hence that π is also virtually Z^2. □

We may now give the main result of this section.

Theorem 3.17 *Let M be a finite PD_4-complex whose fundamental group π is an ascending HNN extension with finitely generated base B. Then $\chi(M)\geq 0$, and hence $q(\pi)\geq 0$. If $\chi(M)=0$ and B is FP_2 and finitely ended then either π has two ends or has a subgroup of finite index which is isomorphic to Z^2 or $\pi\cong Z*_m$ or $Z*_m\tilde{\times}(Z/2Z)$ for some $m\neq 0$ or ±1 or M is aspherical.*

Proof The L^2 Euler characteristic formula gives $\chi(M)=\beta_2^{(2)}(M)\geq 0$, since $\beta_i^{(2)}(M)=\beta_i^{(2)}(\pi)=0$ for $i=0$ or 1, by Lemma 2.1.

Let $\phi:B\to B$ be the monomorphism determining $\pi\cong B*_\phi$. If B is finite then ϕ is an automorphism and so π has two ends. If B is FP_2 and has one end then $H^s(\pi;\mathbb{Z}[\pi])=0$ for $s\leq 2$, by the Brown-Geoghegan Theorem. If moreover $\chi(M)=0$ then M is aspherical, by Corollary 3.5.1.

If B has two ends then it is an extension of Z or D by a finite normal subgroup F. As ϕ must map F isomorphically to itself, F is normal in π, and is the maximal finite normal subgroup of π. Moreover $\pi/F\cong Z*_m$, for some $m\neq 0$, if $B/F\cong Z$, and is a semidirect product $Z*_m\tilde{\times}(Z/2Z)$, with a presentation $\langle a,t,u\mid tat^{-1}=a^m,\ tut^{-1}=ua^r,\ u^2=1,\ uau=a^{-1}\rangle$, for some $m\neq 0$ and some $r\in Z$, if $B/F\cong D$. (On replacing t by $a^{[r/2]}t$, if necessary, we may assume that $r=0$ or 1.)

Suppose first that M is orientable, and that $F\neq 1$. Then π has a subgroup σ of finite index which is a central extension of $Z*_{m^q}$ by a finite cyclic group, for some $q\geq 1$, by Lemma 3.15. Let p be a prime dividing q. Since $Z*_{m^q}$ is a torsion free solvable group the ring $\Xi=\mathbb{F}_p[Z*_{m^q}]$ has a skew field of fractions

L, which as a right Ξ-module is the direct limit of the system $\{\Xi_\theta \mid 0 \neq \theta \in \Xi\}$, where each $\Xi_\theta = \Xi$, the index set is ordered by right divisibility ($\theta \leq \phi\theta$) and the map from Ξ_θ to $\Xi_{\phi\theta}$ sends ξ to $\phi\xi$ [KLM88]. In particular, Ξ is a weakly finite ring and so σ is torsion free, by Lemma 3.16. Therefore $F = 1$.

If M is nonorientable then $w_1(M)|_F$ must be injective, and so another application of Lemma 3.16 (with $p = 2$) shows again that $F = 1$. □

Is M still aspherical if B is assumed only finitely generated and one ended?

Corollary 3.17.1 *Let M be a finite PD_4-complex such that $\chi(M) = 0$ and $\pi = \pi_1(M)$ is almost coherent and restrained. Then either π has two ends or is virtually Z^2 or $\pi \cong Z*_m$ or $Z*_m \tilde{\times}(Z/2Z)$ for some $m \neq 0$ or ± 1 or M is aspherical.*

Proof Let $\pi^+ = \mathrm{Ker}(w_1(M))$. Then π^+ maps onto Z, by Lemma 3.14, and so is an ascending HNN extension $\pi^+ \cong B*_\phi$ with finitely generated base B. Since π is almost coherent B is FP_2, and since π has no nonabelian free subgroup B has at most two ends. Hence Lemma 3.16 and Theorem 3.17 apply, so either π has two ends or M is aspherical or $\pi^+ \cong Z*_m$ or $Z *_m \tilde{\times}(Z/2Z)$ for some $m \neq 0$ or ± 1. In the latter case $\sqrt{\pi}$ is isomorphic to a subgroup of the additive rationals Q, and $\sqrt{\pi} = C_\pi(\sqrt{\pi})$. Hence the image of π in $Aut(\sqrt{\pi}) \leq Q^\times$ is infinite. Therefore π maps onto Z and so is an ascending HNN extension $B*_\phi$, and we may again use Theorem 3.17. □

Does this corollary remain true without the hypothesis that π be almost coherent?

There are nine groups which are virtually Z^2 and are fundamental groups of PD_4-complexes with Euler characteristic 0. (See Chapter 11.) Are any of the semidirect products $Z *_m \tilde{\times}(Z/2Z)$ realized by PD_4-complexes with $\chi = 0$? If π is restrained and M is aspherical must π be virtually poly-Z? (Aspherical 4-manifolds with virtually poly-Z fundamental groups are characterized in Chapter 8.)

Let G is a group with a presentation of deficiency d and $w : G \to \{\pm 1\}$ be a homomorphism, and let $\langle x_i,\ 1 \leq i \leq m \mid r_j,\ 1 \leq j \leq n\rangle$ be a presentation for G with $m - n = d$. We may assume that $w(x_i) = +1$ for $i \leq m - 1$. Let $X = \natural^m(S^1 \times D^3)$ if $w = 1$ and $X = (\natural^{m-1}(S^1 \times D^3))\natural(S^1 \tilde{\times} D^3)$ otherwise. The relators r_j may be represented by disjoint orientation preserving embeddings of S^1 in ∂X, and so we may attach 2-handles along product neighbourhoods, to get a bounded 4-manifold Y with $\pi_1(Y) = G$, $w_1(Y) = w$ and $\chi(Y) = 1 - d$. Doubling Y gives a closed 4-manifold M with $\chi(M) = 2(1-d)$ and $(\pi_1(M), w_1(M))$ isomorphic to (G, w).

Since the groups $Z*_m$ have deficiency 1 it follows that any homomorphism $w : Z*_m \to \{\pm 1\}$ may be realized as the orientation character of a closed 4-manifold with fundamental group $Z*_m$ and Euler characteristic 0. What other invariants are needed to determine the homotopy type of such a manifold?

Chapter 4

Mapping tori and circle bundles

Stallings showed that if M is a 3-manifold and $f : M \to S^1$ a map which induces an epimorphism $f_* : \pi_1(M) \to Z$ with infinite kernel K then f is homotopic to a bundle projection if and only if M is irreducible and K is finitely generated. Farrell gave an analogous characterization in dimensions ≥ 6, with the hypotheses that the homotopy fibre of f is finitely dominated and a torsion invariant $\tau(f) \in Wh(\pi_1(M))$ is 0. The corresponding results in dimensions 4 and 5 are constrained by the present limitations of geometric topology in these dimensions. (In fact there are counter-examples to the most natural 4-dimensional analogue of Farrell's theorem [We87].)

Quinn showed that the total space of a fibration with finitely dominated base and fibre is a Poincaré duality complex if and only if both the base and fibre are Poincaré duality complexes. (See [Go79] for a very elegant proof of this result.) The main result of this chapter is a 4-dimensional homotopy fibration theorem with hypotheses similar to those of Stallings and a conclusion similar to that of Quinn and Gottlieb.

The *mapping torus* of a self homotopy equivalence $f : X \to X$ is the space $M(f) = X \times [0,1]/\sim$, where $(x,0) \sim (f(x),1)$ for all $x \in X$. If X is finitely dominated then $\pi_1(M(f))$ is an extension of Z by a finitely presentable normal subgroup and $\chi(M(f)) = \chi(X)\chi(S^1) = 0$. We shall show that a finite PD_4-complex M is homotopy equivalent to such a mapping torus, with X a PD_3-complex, if and only if $\pi_1(M)$ is such an extension and $\chi(M) = 0$.

In the final section we consider instead bundles with fibre S^1. We give conditions for a 4-manifold to be homotopy equivalent to the total space of an S^1-bundle over a PD_3-complex, and show that these conditions are sufficient if the fundamental group of the PD_3-complex is torsion free but not free.

4.1 Some necessary conditions

Let E be a connected cell complex and let $f : E \to S^1$ be a map which induces an epimorphism $f_* : \pi_1(E) \to Z$, with kernel ν. The associated covering space with group ν is $E_\nu = E \times_{S^1} R = \{(x,y) \in E \times R \mid f(x) = e^{2\pi iy}\}$, and

$E \simeq M(\phi)$, where $\phi : E_\nu \to E_\nu$ is the generator of the covering group given by $\phi(x,y) = (x, y+1)$ for all (x,y) in E_ν. If E is a PD_4-complex and E_ν is finitely dominated then E_ν is a PD_3-complex, by Quinn's result. In particular, ν is FP_2 and $\chi(E) = 0$. The latter conditions characterize aspherical mapping tori, by the following theorem.

Theorem 4.1 *Let M be a finite PD_4-complex whose fundamental group π is an extension of Z by a finitely generated normal subgroup ν, and let M_ν be the infinite cyclic covering space corresponding to the subgroup ν. Then*

(1) *$\chi(M) \geq 0$, with equality if and only if $H_2(M_\nu;\mathbb{Q})$ is finitely generated;*

(2) *if $\chi(M) = 0$ then M is aspherical if and only if ν is infinite and $H^2(\pi;\mathbb{Z}[\pi]) = 0$;*

(3) *M_ν is an aspherical PD_3-complex if and only if $\chi(M) = 0$ and ν is almost finitely presentable and has one end.*

Proof Since M is a finite complex and $\mathbb{Q}\Lambda = \mathbb{Q}[t,t^{-1}]$ is noetherian the homology groups $H_q(M_\nu;\mathbb{Q})$ are finitely generated as $\mathbb{Q}\Lambda$-modules. Since ν is finitely generated they are finite dimensional as $\mathbb{Q}$-vector spaces if $q < 2$, and hence also if $q > 2$, by Poincaré duality. Now $H_2(M_\nu;\mathbb{Q}) \cong \mathbb{Q}^r \oplus (\mathbb{Q}\Lambda)^s$ for some $r, s \geq 0$, by the Structure Theorem for modules over a PID. It follows easily from the Wang sequence for the covering projection from M_ν to M, that $\chi(M) = s \geq 0$.

Since ν is finitely generated $\beta_1^{(2)}(\pi) = 0$, by Lemma 2.1. If M is aspherical then clearly ν is infinite and $H^2(\pi;\mathbb{Z}[\pi]) = 0$. Conversely, if these conditions hold then $H^s(\pi;\mathbb{Z}[\pi]) = 0$ for $s \leq 2$. Hence if moreover $\chi(M) = 0$ then M is aspherical, by Corollary 3.5.2.

If ν is FP_2 and has one end then $H^2(\pi;\mathbb{Z}[\pi]) \cong H^1(\nu;\mathbb{Z}[\nu]) = 0$, by the LHSSS. As M is aspherical ν is a PD_3-group, by Theorem 1.20, and therefore is finitely presentable, by Theorem 1.1 of [KK99]. Hence $M_\nu \simeq K(\nu,1)$ is finitely dominated and so is a PD_3-complex [Br72]. □

In particular, if $\chi(M) = 0$ then $q(\pi) = 0$. This observation and the bound $\chi(M) \geq 0$ were given in Theorem 3.17. (They also follow on counting bases for the cellular chain complex of M_ν and extending coefficients to $\mathbb{Q}(t)$.)

Let F be the orientable surface of genus 2. Then $M = F \times F$ is an aspherical closed 4-manifold, and $\pi \cong G \times G$ where $G = \pi_1(F)$ has a presentation $\langle a_1, a_2, b_1, b_2 \mid [a_1, b_1] = [a_2, b_2] \rangle$. The subgroup $\nu \leq \pi$ generated by the images of (a_1, a_1) and the six elements $(x, 1)$ and $(1, x)$, for $x = a_2$, b_1 or b_2, is normal in π and $\pi/\nu \cong Z$. However ν cannot be FP_2 since $\chi(\pi) = 4 \neq 0$. Is there an aspherical 4-manifold M such that $\pi_1(M)$ is an extension of Z by a finitely generated subgroup ν which is not FP_2 and with $\chi(M) = 0$? (Note that $H_2(\nu; \mathbb{Q})$ must be finitely generated, so showing that ν is not finitely related may require some finesse.)

If $H^2(\pi; \mathbb{Z}[\pi]) = 0$ then $H^1(\nu; \mathbb{Z}[\nu]) = 0$, by an LHSSS argument, and so ν must have one end, if it is infinite. Can the hypotheses of (2) above be replaced by "$\chi(M) = 0$ and ν has one end"? It can be shown that the finitely generated subgroup N of $F(2) \times F(2)$ defined after Theorem 2.4 has one end. However $H^2(F(2) \times F(2); \mathbb{Z}[F(2) \times F(2)]) \neq 0$. (Note that $q(F(2) \times F(2)) = 2$, by Corollary 3.12.2.)

4.2 Change of rings and cup products

In the next two sections we shall adapt and extend work of Barge in setting up duality maps in the equivariant (co)homology of covering spaces.

Let π be an extension of Z by a normal subgroup ν and fix an element t of π whose image generates π/ν. Let $\alpha : \nu \to \nu$ be the automorphism determined by $\alpha(h) = tht^{-1}$ for all h in ν. This automorphism extends to a ring automorphism (also denoted by α) of the group ring $\mathbb{Z}[\nu]$, and the ring $\mathbb{Z}[\pi]$ may then be viewed as a twisted Laurent extension, $\mathbb{Z}[\pi] = \mathbb{Z}[\nu]_\alpha[t, t^{-1}]$. The quotient of $\mathbb{Z}[\pi]$ by the two-sided ideal generated by $\{h - 1 | h \in \nu\}$ is isomorphic to Λ, while as a left module over itself $\mathbb{Z}[\nu]$ is isomorphic to $\mathbb{Z}[\pi]/\mathbb{Z}[\pi](t-1)$ and so may be viewed as a left $\mathbb{Z}[\pi]$-module. (Note that α is not a module automorphism unless t is central.)

If M is a left $\mathbb{Z}[\pi]$-module let $M|_\nu$ denote the underlying $\mathbb{Z}[\nu]$-module, and let $\hat{M} = Hom_{\mathbb{Z}[\nu]}(M|_\nu, \mathbb{Z}[\nu])$. Then $\hat{M}$ is a right $\mathbb{Z}[\nu]$-module via

$$(f\xi)(m) = f(m)\xi \text{ for all } \xi \in \mathbb{Z}[\nu],\ f \in \hat{M} \text{ and } m \in M.$$

If $M = \mathbb{Z}[\pi]$ then $\widehat{\mathbb{Z}[\pi]}$ is also a left $\mathbb{Z}[\pi]$-module via

$$(\phi t^r f)(\xi t^s) = \xi \alpha^{-s}(\phi) f(t^{s-r}) \text{ for all } f \in \widehat{\mathbb{Z}[\pi]},\ \phi, \xi \in \nu \text{ and } r, s \in Z.$$

As the left and right actions commute $\widehat{\mathbb{Z}[\pi]}$ is a $(\mathbb{Z}[\pi], \mathbb{Z}[\nu])$-bimodule. We may describe this bimodule more explicitly. Let $\mathbb{Z}[\nu][[t, t^{-1}]]$ be the set of doubly infinite power series $\Sigma_{n \in Z} t^n \phi_n$ with ϕ_n in $\mathbb{Z}[\nu]$ for all n in Z, with the obvious right $\mathbb{Z}[\nu]$-module structure, and with the left $\mathbb{Z}[\pi]$-module structure given by

$$\phi t^r (\Sigma t^n \phi_n) = \Sigma t^{n+r} \alpha^{-n-r}(\phi) \phi_n \text{ for all } \phi, \phi_n \in \mathbb{Z}[\nu] \text{ and } r \in Z.$$

(Note that even if $\nu = 1$ this module is not a ring in any natural way.) Then the homomorphism $j : \widehat{\mathbb{Z}[\pi]} \to \mathbb{Z}[\nu][[t, t^{-1}]]$ given by $j(f) = \Sigma t^n f(t^n)$ for all f in $\widehat{\mathbb{Z}[\pi]}$ is a $(\mathbb{Z}[\pi], \mathbb{Z}[\nu])$-bimodule isomorphism. (Indeed, it is clearly an isomorphism of right $\mathbb{Z}[\nu]$-modules, and we have defined the left $\mathbb{Z}[\pi]$-module structure on $\widehat{\mathbb{Z}[\pi]}$ by pulling back the one on $\mathbb{Z}[\nu][[t, t^{-1}]]$.)

For each f in $\hat{M}$ we may define a function $T_M f : M \to \widehat{\mathbb{Z}[\pi]}$ by the rule

$$(T_M f)(m)(t^n) = f(t^{-n}m) \text{ for all } m \in M \text{ and } n \in Z.$$

It is easily seen that $T_M f$ is $\mathbb{Z}[\pi]$-linear, and that $T_M : \hat{M} \to Hom_{\mathbb{Z}[\pi]}(M, \widehat{\mathbb{Z}[\pi]})$ is an isomorphism of abelian groups. (It is clearly a monomorphism, and if $g : M \to \widehat{\mathbb{Z}[\pi]}$ is $\mathbb{Z}[\pi]$-linear then $g = T_M f$ where $f(m) = g(m)(1)$ for all m in M. In fact if we give $Hom_{\mathbb{Z}[\pi]}(M, \widehat{\mathbb{Z}[\pi]})$ the natural right $\mathbb{Z}[\nu]$-module structure by $(\mu\phi)(m) = \mu(m)\phi$ for all $\phi \in \mathbb{Z}[\nu]$, $\mathbb{Z}[\pi]$-homomorphisms $\mu : M \to \widehat{\mathbb{Z}[\pi]}$ and $m \in M$ then T_M is an isomorphism of right $\mathbb{Z}[\nu]$-modules.) Thus we have a natural equivalence $T : Hom_{\mathbb{Z}[\nu]}(-|_\nu, \mathbb{Z}[\nu]) \Rightarrow Hom_{\mathbb{Z}[\pi]}(-, \widehat{\mathbb{Z}[\pi]})$ of functors from $\mathbf{Mod}_{\mathbb{Z}[\pi]}$ to $\mathbf{Mod}_{\mathbb{Z}[\nu]}$. If C_* is a chain complex of left $\mathbb{Z}[\pi]$-modules T induces natural isomorphisms from $H^*(C_*|_\nu; \mathbb{Z}[\nu]) = H^*(Hom_{\mathbb{Z}[\nu]}(C_*|_\nu, \mathbb{Z}[\nu])$ to $H^*(C_*; v) = H^*(Hom_{\mathbb{Z}[\pi]}(C_*, \widehat{\mathbb{Z}[\pi]}))$. In particular, since the forgetful functor $-|_\nu$ is exact and takes projectives to projectives there are isomorphisms from $Ext^*_{\mathbb{Z}[\nu]}(M|_\nu, \mathbb{Z}[\nu])$ to $Ext^*_{\mathbb{Z}[\pi]}(M, \widehat{\mathbb{Z}[\pi]})$ which are functorial in M.

If M and N are left $\mathbb{Z}[\pi]$-modules let $M \otimes N$ denote the tensor product over $\mathbb{Z}$ with the diagonal left π-action, defined by $g(m \otimes n) = gm \otimes gn$ for all $m \in M$, $n \in N$ and $g \in \pi$. The function $p_M : \Lambda \otimes M \to M$ defined by $p_M(\lambda \otimes m) = \lambda(1)m$ is then a $\mathbb{Z}[\pi]$-linear epimorphism.

We shall define products in cohomology by means of the $\mathbb{Z}[\pi]$-linear homomorphism $e : \Lambda \otimes \widehat{\mathbb{Z}[\pi]} \to \mathbb{Z}[\pi]$ given by

$$e(t^n \otimes f) = t^n f(t^n) \text{ for all } f \in \widehat{\mathbb{Z}[\pi]} \text{ and } n \in Z.$$

Let A_* be a Λ-chain complex and B_* a $\mathbb{Z}[\pi]$-chain complex and give the tensor product the total grading $A_* \otimes B_*$ and differential and the diagonal π-action. Let $e_\sharp$ be the change of coefficients homomorphism induced by e, and let $u \in H^p(A_*; \Lambda)$ and $v \in H^q(B_*; \widehat{\mathbb{Z}[\pi]})$. Then $u \otimes v \mapsto e_\sharp(u \times v)$ defines a pairing from $H^p(A_*; \Lambda) \otimes H^q(B_*; \widehat{\mathbb{Z}[\pi]})$ to $H^{p+q}(A_* \otimes B_*; \mathbb{Z}[\pi])$.

Now let A_* be the Λ-chain complex concentrated in degrees 0 and 1 with A_0 and A_1 free of rank 1, with bases $\{a_0\}$ and $\{a_1\}$, respectively, and with $\partial_1 : A_1 \to A_0$ given by $\partial_1(a_1) = (t-1)a_0$. Let $\eta_A : A_1 \to \Lambda$ be the isomorphism determined by $\eta_A(a_1) = 1$, and let $\alpha_A : A_0 \to \mathbb{Z}$ be the augmentation determined by $\alpha_A(a_0) = 1$. Then $[\eta_A]$ generates $H^1(A_*; \Lambda)$. Let B_* be a projective $\mathbb{Z}[\pi]$-chain complex and let $p_{B*} : A_* \otimes B_* \to B_*$ be the chain homotopy equivalence defined by $p_{Bj}((\lambda a_0) \otimes b_j) = \lambda(1)b_j$ and $p_{Bj}((\lambda a_1) \otimes b_{j-1}) = 0$, for all $\lambda \in \Lambda$, $b_{j-1} \in B_{j-1}$ and $b_j \in B_j$. Let $j_{B*} : B_* \to A_* \otimes B_*$ be a chain homotopy inverse to p_{B*}. Define a family of homomorphisms $h_{\mathbb{Z}[\pi]}$ from $H^q(B_*; \widehat{\mathbb{Z}[\pi]})$ to $H^{q+1}(B_*; \mathbb{Z}[\pi])$ by

$$h_{\mathbb{Z}[\pi]}([\phi]) = j_B^* e_\sharp([\eta_A] \times [\phi])$$

for $\phi : B_q \to \widehat{\mathbb{Z}[\pi]}$ such that $\phi\partial_{q+1} = 0$. Let $f : B_* \to B'_*$ be a chain homomorphism of projective $\mathbb{Z}[\pi]$-chain complexes. Then $h_{\mathbb{Z}[\pi]}([\phi f_q]) = f^* h_{\mathbb{Z}[\pi]}([\phi])$, and so these homomorphisms are functorial in B_*. In particular, if B_* is a projective resolution of the $\mathbb{Z}[\pi]$-module M we obtain homomorphisms $h_{\mathbb{Z}[\pi]} : Ext^q_{\mathbb{Z}[\pi]}(M, \widehat{\mathbb{Z}[\pi]}) \to Ext^{q+1}_{\mathbb{Z}[\pi]}(M, \mathbb{Z}[\pi])$ which are functorial in M.

Lemma 4.2 *Let M be a $\mathbb{Z}[\pi]$-module such that $M|_\nu$ is finitely generated as a $\mathbb{Z}[\nu]$-module. Then $h_{\mathbb{Z}[\pi]} : Hom_{\mathbb{Z}[\pi]}(M, \widehat{\mathbb{Z}[\pi]}) \to Ext^1_{\mathbb{Z}[\pi]}(M, \mathbb{Z}[\pi])$ is injective.*

Proof Let B_* be a projective resolution of the $\mathbb{Z}[\pi]$-module M and let $q : B_0 \to M$ be the defining epimorphism (so that $q\partial_1 = 0$). We may use composition with q to identify $Hom_{\mathbb{Z}[\pi]}(M, \widehat{\mathbb{Z}[\pi]})$ with the submodule of 0-cocycles in $Hom(B_*, \widehat{\mathbb{Z}[\pi]})$, and we set $h_{\mathbb{Z}[\pi]}(\phi) = h_{\mathbb{Z}[\pi]}([\phi q])$ for all $\phi : M \to \widehat{\mathbb{Z}[\pi]}$.

Suppose that $h_{\mathbb{Z}[\pi]}(\phi) = 0$ and let $g = \phi q : B_0 \to \widehat{\mathbb{Z}[\pi]}$. Then there is a $\mathbb{Z}[\pi]$-linear homomorphism $f : A_0 \otimes B_0 \to \mathbb{Z}[\pi]$ such that $e_\sharp([\eta_A] \times [g]) = \delta f$. We may write $g(b) = \Sigma t^n g_n(b) = \Sigma t^n g_0(t^{-n}b)$, where $g_0 : B_0 \to \mathbb{Z}[\nu]$ is $\mathbb{Z}[\nu]$-linear (and $g_0\partial_1 = 0$). We then have $g_0(b) = f((t-1)a_0 \otimes b)$ for all $b \in B_0$, while $f(1 \otimes \partial_1) = 0$. Let $k(b) = f(a_0 \otimes b)$ for $b \in B_0$. Then $k : B_0 \to \mathbb{Z}[\pi]$ is $\mathbb{Z}[\nu]$-linear, and $k\partial_1 = 0$, so k factors through M. In particular, $k(B_0)$ is finitely generated as a $\mathbb{Z}[\nu]$-submodule of $\mathbb{Z}[\pi]$. But as $\mathbb{Z}[\pi] = \bigoplus t^n \mathbb{Z}[\nu]$ and $g_0(b) = tk(t^{-1}b) - k(b)$ for all $b \in B_0$, this is only possible if $k = g_0 = 0$. Therefore $\phi = 0$ and so $h_{\mathbb{Z}[\pi]}$ is injective. $\square$

Let B_* be a projective $\mathbb{Z}[\pi]$-chain complex such that $B_j = 0$ for $j < 0$ and $H_0(B_*) \cong \mathbb{Z}$. Then there is a $\mathbb{Z}[\pi]$-chain homomorphism $\epsilon_{B*} : B_* \to A_*$ which induces an isomorphism $H_0(B_*) \cong H_0(A_*)$, and $\alpha_B = \alpha_A \epsilon_{B_0} : B_0 \to \mathbb{Z}$ is a generator of $H^0(B_*; \mathbb{Z})$. Let $\eta_B = \eta_A \epsilon_{B_1} : B_1 \to \Lambda$. If moreover $H_1(B_*) = 0$ then $H^1(B_*; \Lambda) \cong Z$ and is generated by $[\eta_B] = \epsilon_B^*([\eta_A])$

4.3 The case $\nu = 1$

When $\nu = 1$ (so $\mathbb{Z}[\pi] = \Lambda$) we shall show that h_Λ is an equivalence, and relate it to other more explicit homomorphisms. Let S be the multiplicative system in Λ consisting of monic polynomials with constant term ± 1. Let $Lexp(f, a)$ be the Laurent expansion of the rational function f about a. Then $\ell(f) = Lexp(f, \infty) - Lexp(f, 0)$ defines a homomorphism from the localization Λ_S to $\widehat{\Lambda} = \mathbb{Z}[[t, t^{-1}]]$, with kernel Λ. (Barge used a similar homomorphism to embed $\mathbb{Q}(t)/\Lambda$ in $\mathbb{Q}[[t, t^{-1}]]$ [Ba 80].) Let $\chi : \widehat{\Lambda} \to \mathbb{Z}$ be the additive homomorphism defined by $\chi(\Sigma t^n f_n) = f_0$. (This is a version of the "trace" function used by Trotter to relate Seifert forms and Blanchfield pairings on a knot module M [Tr78].)

Let M be a Λ-module which is finitely generated as an abelian group, and let N be its maximal finite submodule. Then M/N is $\mathbb{Z}$-torsion free and $Ann_\Lambda(M/N) = (\lambda_M)$, where λ_M is the minimal polynomial of t, considered as an automorphism of $(M/N)|_\mathbb{Z}$. (See Chapter 3 of [H3].) Since $M|_\mathbb{Z}$ is finitely generated $\lambda_M \in S$. The inclusion of Λ_S/Λ in $\mathbb{Q}(t)/\Lambda$ induces an isomorphism $D(M) = Hom_\Lambda(M, \Lambda_S/\Lambda) \cong Hom_\Lambda(M, \mathbb{Q}(t)/\Lambda)$. We shall show that $D(M)$ is naturally isomorphic to each of $\hat{D}(M) = Hom_\Lambda(M, \hat{\Lambda})$, $E(M) = Ext^1_\Lambda(M, \Lambda)$ and $F(M) = Hom_\mathbb{Z}(M|_\mathbb{Z}, \mathbb{Z})$.

Let $\ell_M : D(M) \to \hat{D}(M)$ and $\chi_M : \hat{D}(M) \to F(M)$ be the homomorphisms defined by composition with ℓ and χ, respectively. It is easily verified that χ_M and T_M are mutually inverse.

Let B_* be a projective resolution of M. If $\phi \in D(M)$ let $\phi_0 : B_0 \to \mathbb{Q}(t)$ be a lift of ϕ. Then $\phi_0 \partial_1$ has image in Λ, and so defines a homomorphism $\phi_1 : B_1 \to \Lambda$ such that $\phi_1 \partial_2 = 0$. Consideration of the short exact sequence of complexes

$$0 \to Hom_\Lambda(B_*, \Lambda) \to Hom_\Lambda(B_*, \mathbb{Q}(t)) \to Hom_\Lambda(B_*, \mathbb{Q}(t)/\Lambda) \to 0$$

shows that $\delta_M(\phi) = [\phi_1]$, where $\delta_M : D(M) \to E(M)$ is the Bockstein homomorphism associated to the coefficient sequence. (The extension corresponding to $\delta_M \phi$ is the pullback over ϕ of the sequence $0 \to \Lambda \to \mathbb{Q}(t) \to \mathbb{Q}(t)/\Lambda \to 0$.)

Lemma 4.3 *The natural transformation h_Λ is an equivalence, and $h_\Lambda \ell_M = \delta_M$.*

Proof The homomorphism j_M sending the image of g in $\Lambda/(\lambda_M)$ to the class of $g(\lambda_M)^{-1}$ in Λ_S/Λ induces an isomorphism $Hom_\Lambda(M, \Lambda/(\lambda_M)) \cong D(M)$.

Hence we may assume that $M = \Lambda/(\lambda)$ and it shall suffice to check that $h_\Lambda \ell_M(j_M) = \delta(j_M)$. Moreover we may extend coefficients to $\mathbb{C}$, and so we may reduce to the case $\lambda = (t-\alpha)^n$.

We may assume that B_1 and B_0 are freely generated by b_1 and b_0, respectively, and that $\partial(b_1) = \lambda b_0$. The chain homotopy equivalence j_{B*} may be defined by $j_0(b_0) = a_0 \otimes b_0$ and $j_1(b_1) = a_0 \otimes b_1 + \Sigma\beta_{pq}(t^p a_1) \otimes (t^q b_0)$, where $\Sigma\beta_{pq}x^p y^q = (\lambda(xy) - \lambda(y))/(x-1) = y\Sigma_{0\leq r<n}(xy-\alpha)^r(y-\alpha)^{n-r-1}$. (This formula arises naturally if we identify $\Lambda \otimes_{\mathbb{Z}} \Lambda$ with $\mathbb{Z}[x, y, x^{-1}, y^{-1}]$, with $t \in \Lambda$ acting via xy.) Note that $\delta(j_M)(b_1) = \beta_{0n} = 1$ and $\beta_{pq} = 0$ unless $0 \leq m < q \leq n$.

Now $h_\Lambda \ell_M(j_M)(b_1) = e_\sharp(\eta_A \times \ell_M(j_M))(j_*(b_1)) = \Sigma\beta_{pq}t^p\psi_{p-q}$, where ψ_{-r} is the coefficient of t^{-r} in $Lexp(\lambda^{-1}, \infty)$. Clearly $\psi_r = 0$ if $-n < r < 0$ and $\psi_{-n} = 1$, since $\lambda^{-1} = t^{-n}(1-\alpha t^{-1})^{-n}$. Hence $h_\Lambda \ell_M(j_M)(b_1) = \beta_{0n} = \delta(j_M)(b_1)$, and so $h_\Lambda \ell_M = \delta_M$, by linearity and functoriality.

Since δ is a natural equivalence and h_Λ is injective, by Lemma 4.2, h_Λ is also a natural equivalence. $\square$

It can be shown that the ring Λ_S defined above is a PID.

4.4 Duality in infinite cyclic covers

Let E, f and ν be as in §1, and suppose also that E is a PD_4-complex with $\chi(E) = 0$ and that ν is finitely generated and infinite. Let $C_* = C_*(\widetilde{E})$. Then $H_0(C_*) = Z$, $H_2(C_*) \cong \pi_2(E)$ and $H_q(C_*) = 0$ if $q \neq 0$ or 2, since $\widetilde{E}$ is simply connected and π has one end. Since $H_1(\bar{\Lambda} \otimes_{\mathbb{Z}[\pi]} C_*) = H_1(E_\nu; \mathbb{Z}) \cong \nu/\nu'$ is finitely generated as an abelian group, $Hom_{\mathbb{Z}[\pi]}(H_1(\bar{\Lambda} \otimes_{\mathbb{Z}[\pi]} C_*), \Lambda) = 0$. An elementary computation then shows that $H^1(C_*; \Lambda)$ is infinite cyclic, and generated by the class $\eta = \eta_C$ defined in §2. Let $[E]$ be a fixed generator of $H_4(\bar{Z} \otimes_{\mathbb{Z}[\pi]} C_*) \cong Z$, and let $[E_\nu] = \eta \cap [E]$ in $H_3(E_\nu; \mathbb{Z}) = H_3(\bar{\Lambda} \otimes_{\mathbb{Z}[\pi]} C_*) \cong Z$.

Since $\widetilde{E}$ is also the universal covering space of E_ν, the cellular chain complex for $\widetilde{E_\nu}$ is $C_*|_\nu$. In order to verify that E_ν is a PD_3-complex (with orientation class $[E_\nu]$) it shall suffice to show that (for each $p \geq 0$) the homomorphism η_p from $\overline{H^p(C_*; \mathbb{Z}[\nu])} = \overline{H^p(C_*; \widehat{\mathbb{Z}[\pi]})}$ to $\overline{H^{p+1}(C_*; \mathbb{Z}[\pi])}$ given by cup product with η is an isomorphism, by standard properties of cap and cup products. We may identify these cup products with the degree raising homomorphisms $h_{\mathbb{Z}[\pi]}$, by the following lemma.

Lemma 4.4 *Let X be a connected space with $\pi_1(X) \cong \pi$ and let $B_* = C_*(\tilde{X})$. Then $h_{\mathbb{Z}[\pi]}([\phi]) = [\eta_B] \cup [\phi]$.*

Proof The Alexander Whitney diagonal approximation d_* from B_* to $B_*\otimes B_*$ is π-equivariant, if the tensor product is given the diagonal left π-action, and we may take $j_{B*} = (\epsilon_B \otimes 1)d_*$ as a chain homotopy inverse to p_{B*}. Therefore $h_{\mathbb{Z}[\pi]}([\phi]) = d^* e_\sharp([\eta_B] \times [\phi]) = [\eta_B] \cup [\phi]$. $\square$

The cohomology modules $H^p(C_*;\mathbb{Z}[\nu])$ and $H^p(C_*;\mathbb{Z}[\pi])$ may be "computed" via the UCSS. Since cross product with a 1-cycle induces a degree 1 cochain homomorphism, the functorial homomorphisms $h_{\mathbb{Z}[\pi]}$ determine homomorphisms between these spectral sequences which are compatible with cup product with η on the limit terms. In each case the E_2^{p*} columns are nonzero only for $p = 0$ or 2. The E_2^{0*} terms of these spectral sequences involve only the cohomology of the groups and the homomorphisms between them may be identified with the maps arising in the LHSSS for π as an extension of Z by ν, under appropriate finiteness hypotheses on ν.

4.5 Homotopy mapping tori

In this section we shall apply the above ideas to the non-aspherical case. We use coinduced modules to transfer arguments about subgroups and covering spaces to contexts where Poincaré duality applies, and L^2-cohomology to identify $\pi_2(M)$, together with the above strategy of describing Poincaré duality for an infinite cyclic covering space in terms of cup product with a generator η of $H^1(M;\Lambda)$.

Note that most of the homology and cohomology groups defined below do not have natural module structures, and so the Poincaré duality isomorphisms are isomorphisms of abelian groups only.

Theorem 4.5 *A finite PD_4-complex M with fundamental group π is homotopy equivalent to the mapping torus of a self homotopy equivalence of a PD_3-complex if and only if $\chi(M) = 0$ and π is an extension of Z by a finitely presentable normal subgroup ν.*

Proof The conditions are clearly necessary, as observed in §1 above. Suppose conversely that they hold. Let M_ν be the infinite cyclic covering space of M with fundamental group ν, and let $\tau : M_\nu \to M_\nu$ be a covering transformation corresponding to a generator of $\pi/\nu \cong Z$. Then M is homotopy equivalent to the mapping torus $M(\tau)$. Moreover $H^1(M;\Lambda) \cong H^1(\pi;\Lambda)$ is infinite cyclic, since ν is finitely generated. Let $E^r_{p,q}(M_\nu)$ and $E^r_{p,q}(M)$ be the UCSS for the cohomology of M_ν with coefficients $\mathbb{Z}[\nu]$ and for that of M with coefficients

$\mathbb{Z}[\pi]$, respectively. A choice of generator η for $H^1(M;\Lambda)$ determines homomorphisms $h_{\mathbb{Z}[\pi]}: E^r_{p,q}(M_\nu) \to E^r_{p,q+1}(M)$, giving a homomorphism of bidegree $(0,1)$ between these spectral sequences corresponding to cup product with η on the abutments, by Lemma 4.4.

Suppose first that ν is finite. The UCSS and Poincaré duality then imply that $H_i(\widetilde{M};\mathbb{Z}) \cong Z$ for $i=0$ or 3 and is 0 otherwise. Hence $\widetilde{M} \simeq S^3$ and so $M_\nu = \widetilde{M}/\nu$ is a Swan complex for ν. (See Chapter 11 for more details.) Thus we may assume henceforth that ν is infinite. We must show that the cup product maps $\eta_p : H^p(M_\nu;\mathbb{Z}[\nu]) \to H^{p+1}(M;\mathbb{Z}[\pi])$ are isomorphisms, for $0 \le p \le 4$. If $p=0$ or 4 then all the groups are 0, and so η_0 and η_4 are isomorphisms.

Applying the isomorphisms defined in §8 of Chapter 1 to the cellular chain complex C_* of $\widetilde{M}$, we see that $H^q(M_\nu;A) \cong H^q(M;Hom_{\mathbb{Z}[\nu]}(\mathbb{Z}[\pi],A))$ is isomorphic to $\overline{H_{4-q}(M;Hom_{\mathbb{Z}[\nu]}(\mathbb{Z}[\pi],A))}$ for any local coefficient system (left $\mathbb{Z}[\nu]$-module) A on M_ν. Let $t \in \pi$ represent a generator of π/ν. Since multiplication by $t-1$ is surjective on $Hom_{\mathbb{Z}[\nu]}(\mathbb{Z}[\pi],A)$, the homology Wang sequence for the covering projection of M_ν onto M gives $H_0(M;Hom_{\mathbb{Z}[\nu]}(\mathbb{Z}[\pi],A))=0$. Hence $H^4(M_\nu;A)=0$ for any local coefficient system A, and so M_ν is homotopy equivalent to a 3-dimensional complex (see [Wl65]). (See also [DST96].)

Since π is an extension of Z by a finitely generated normal subgroup $\beta_1^{(2)}(\pi) = 0$, and so $\overline{\pi_2(M)} \cong H^2(M;\mathbb{Z}[\pi]) \cong H^2(\pi;\mathbb{Z}[\pi])$, by Theorem 3.4. Hence η_1 may be identified with the isomorphism $H^1(\nu;\mathbb{Z}[\nu]) \cong H^2(\pi;\mathbb{Z}[\pi])$ coming from the LHSSS for the extension. Moreover $\overline{\pi_2(M)}|_\nu \cong H^1(\nu;\mathbb{Z}[\nu])$ is finitely generated over $\mathbb{Z}[\nu]$, and so $Hom_{\mathbb{Z}[\pi]}(\pi_2(M),\mathbb{Z}[\pi])=0$. Therefore $H^3(\pi;\mathbb{Z}[\pi])=0$, by Lemma 3.3, and so the Wang sequence map $t-1 : H^2(\nu;\mathbb{Z}[\pi]) \to H^2(\nu;\mathbb{Z}[\pi])$ is onto. Since ν is FP_2 this cohomology group is isomorphic to $H^2(\nu;\mathbb{Z}[\nu]) \otimes_Z \mathbb{Z}[\pi/\nu]$, where $\mathbb{Z}[\pi/\nu] \cong \Lambda$ acts diagonally. It is easily seen that if $H^2(\nu;\mathbb{Z}[\nu])$ has a nonzero element h then $h\otimes 1$ is not divisible by $t-1$. Hence $H^2(\nu;\mathbb{Z}[\nu]) = 0$. The differential $d^3_{2,1}(M)$ is a monomorphism, since $H^3(M;\mathbb{Z}[\pi])=0$, and $h_{\mathbb{Z}[\pi]} : E^2_{2,0}(M_\nu) \to E^2_{2,1}(M)$ is a monomorphism by Lemma 4.2. Therefore $d^3_{2,0}(M_\nu)$ is also a monomorphism and so $H^2(M_\nu;\mathbb{Z}[\nu])=0$. Hence η_2 is an isomorphism.

It remains only to check that $H^3(M_\nu;\mathbb{Z}[\nu]) \cong Z$ and that η_3 is onto. Now $H^3(M_\nu;\mathbb{Z}[\nu]) \cong \overline{H_1(M;Hom_{\mathbb{Z}[\nu]}(\mathbb{Z}[\pi],\mathbb{Z}[\nu]))} = \overline{H_1(\pi;\mathbb{Z}[\nu]^{\pi/\nu})}$. (The exponent denotes direct product indexed by π/ν rather than fixed points!) The natural homomorphism from $H_1(\pi;\mathbb{Z}[\nu]^{\pi/\nu})$ to $H_1(\pi/\nu;H_0(\nu;\mathbb{Z}[\nu]^{\pi/\nu}))$ is onto, with kernel $H_0(\pi/\nu;H_1(\nu;\mathbb{Z}[\nu]^{\pi/\nu}))$, by the LHSSS for π. Since ν is finitely generated homology commutes with direct products in this range, and it follows that

$H_1(\pi;\mathbb{Z}[\nu]^{\pi/\nu}) \cong H_1(\pi/\nu;\mathbb{Z}^{\pi/\nu})$. Since $\pi/\nu \cong Z$ and acts by translation on the index set this homology group is Z. The homomorphisms from $H^3(M_\nu;\mathbb{Z}[\nu])$ to $H^3(M_\nu;\mathbb{Z})$ and from $H^4(M;\mathbb{Z}[\pi])$ to $H^4(M;\Lambda)$ induced by the augmentation homomomorphism and the epimorphism from $\mathbb{Z}[\pi]$ to $\mathbb{Z}[\pi/\nu] \cong \Lambda$ are epimorphisms, since M_ν and M are homotopy equivalent to 3- and 4-dimensional complexes, respectively. hence they are isomorphisms, since these cohomology modules are infiite cyclic as abelian groups. These isomorphisms form the vertical sides of a commutative square

$$\begin{tikzcd} H^3(M_\nu;\mathbb{Z}[\nu]) \arrow[r, "\eta_3"] \arrow[d, "\varepsilon"'] & H^4(M;\mathbb{Z}[\pi]) \arrow[d] \\ H^3(M_\nu;\mathbb{Z}) \arrow[r, "-\cup\eta"] & H^4(M;\Lambda). \end{tikzcd}$$

The lower horizontal edge is an isomorphism, by Lemma 4.3. Therefore η_3 is also an isomorphism.

Thus M_ν satisfies Poincaré duality of formal dimension 3 with local coefficients. Since $\pi_1(M_\nu) = \nu$ is finitely presentable M_ν is finitely dominated, and so is a PD_3-complex [Br72]. □

Note that M_ν need not be homotopy equivalent to a finite complex. If M is a *simple* PD_4-complex and a generator of $Aut(M_\nu/M) \cong \pi/\nu$ has finite order in the group of self homotopy equivalences of M_ν then M is finitely covered by a simple PD_4-complex homotopy equivalent to $M_\nu \times S^1$. In this case M_ν must be homotopy finite by [Rn86]. The hypothesis that M be finite is used in the proof of Theorem 3.4, but is probably not necessary here.

The hypothesis that ν be almost finitely presentable (FP_2) suffices to show that M_ν satisfies Poincaré duality with local coefficients. Finite presentability is used only to show that M_ν is finitely dominated. (Does the coarse Alexander duality argument of [KK99] used in part (3) of Theorem 4.1 extend to the non-aspherical case?) In view of the fact that 3-manifold groups are coherent, we might hope that the condition on ν could be weakened still further to require only that it be finitely generated.

Some argument is needed above to show that η_2 is injective. If M_ν is homotopy equivalent to a 3-manifold with more than one aspherical summand then $H^1(\nu;\mathbb{Z}[\nu])$ is a nonzero free $\mathbb{Z}[\nu]$-module and so $Hom_{\mathbb{Z}[\nu]}(\Pi|_\nu, \mathbb{Z}[\nu]) \neq 0$.

A rather different proof of this theorem could be given using Ranicki's criterion for an infinite cyclic cover to be finitely dominated [Rn95] and the Quinn-Gottlieb theorem, if finitely generated stably free modules of rank 0 over the

Novikov rings $A_\pm = \mathbb{Z}[\nu]_\alpha((t^{\pm 1}))$ are trivial. (For $H_q(A_\pm \otimes_\pi C_*) = A_\pm \otimes_\pi H_*(C_*) = 0$ if $q \neq 2$, since $t-1$ is invertible in $A_\pm$. Hence $H_2(A_\pm \otimes_\pi C_*)$ is a stably free module of rank 0, by Lemma 3.1.)

An alternative strategy would be to show that $Lim_I H^q(M_\nu; A_i) = 0$ for any direct system with limit 0. We could then conclude that the cellular chain complex of $\widetilde{M} = \widetilde{M_\nu}$ is chain homotopy equivalent to a finite complex of finitely generated projective $Z[\nu]$-modules, and hence that M_ν is finitely dominated. Since ν is FP_2 this strategy applies easily when $q = 0$, 1, 3 or 4, but something else is needed when $q = 2$.

Corollary 4.5.1 *Let M be a PD_4-complex with $\chi(M) = 0$ and whose fundamental group π is an extension of Z by a normal subgroup $\nu \cong F(r)$. Then M is homotopy equivalent to a closed PL 4-manifold which fibres over the circle, with fibre $\sharp^r S^1 \times S^2$ if $w_1(M)|_\nu$ is trivial, and $\sharp^r S^1 \tilde{\times} S^2$ otherwise. The bundle is determined by the homotopy type of M.*

Proof By the theorem M_ν is a PD_3-complex with free fundamental group, and so is homotopy equivalent to $N = \sharp^r S^1 \times S^2$ if $w_1(M)|_\nu$ is trivial and to $\sharp^r S^1 \tilde{\times} S^2$ otherwise. Every self homotopy equivalence of a connected sum of S^2-bundles over S^1 is homotopic to a self-homeomorphism, and homotopy implies isotopy for such manifolds [La]. Thus M is homotopy equivalent to such a fibred 4-manifold, and the bundle is determined by the homotopy type of M. ∎

It is easy to see that the natural map from $Homeo(N)$ to $Out(F(r)$ is onto. If a self homeomorphism f of $N = \sharp^r S^1 \times S^2$ induces the trivial outer automorphism of $F(r)$ then f is homotopic to a product of twists about nonseparating 2-spheres [He]. How is this manifest in the topology of the mapping torus?

Since $c.d.\nu = 1$ and $c.d.\pi = 2$ the first k-invariants of M and N both lie in trivial groups, and so this Corollary also follows from Theorem 4.6 below.

Corollary 4.5.2 *Let M be a PD_4-complex with $\chi(M) = 0$ and whose fundamental group π is an extension of Z by a normal subgroup ν. If π has an infinite cyclic normal subgroup C which is not contained in ν then the covering space M_ν with fundamental group ν is a PD_3-complex.*

Proof We may assume without loss of generality that M is orientable and that C is central in π. Since $C \cap \nu = 1$ the subgroup $C\nu \cong C \times \nu$ has finite index in π. Thus by passing to a finite cover we may assume that $\pi = C \times \nu$. Hence ν is finitely presentable and so the Theorem applies. ∎

See [Hi89] for different proofs of Corollaries 4.5.1 and 4.5.2.

Since ν has one or two ends if it has an infinite cyclic normal subgroup, Corollary 4.5.2 remains true if $C \leq \nu$ and ν is finitely presentable. In this case ν is the fundamental group of a Seifert fibred 3-manifold, by Theorem 2.14.

Corollary 4.5.3 *Let M be a PD_4-complex with $\chi(M) = 0$ and whose fundamental group π is an extension of Z by an FP_2 normal subgroup ν. If ν is finite then it has cohomological period dividing 4. If ν has one end then M is aspherical and so π is a PD_4-group. If ν has two ends then $\nu \cong Z$, $Z \oplus (Z/2Z)$ or $D = (Z/2Z) * (Z/2Z)$. If moreover ν is finitely presentable the covering space M_ν with fundamental group ν is a PD_3-complex.*

Proof The final hypothesis is only needed if ν is one-ended, as finite groups and groups with two ends are finitely presentable. If ν is finite then $\widetilde{M} \simeq S^3$ and so the first assertion holds. (See Chapter 11 for more details.) If ν has one end then we may apply Theorem 4.1. If ν has two ends and its maximal finite normal subgroup is nontrivial then $\nu \cong Z \oplus (Z/2Z)$, by Theorem 2.11 (applied to the PD_3-complex M_ν). Otherwise $\nu \cong Z$ or D. □

In Chapter 6 we shall strengthen this Corollary to obtain a fibration theorem for 4-manifolds with torsion free elementary amenable fundamental group.

Our next result gives criteria (involving also the orientation character and first k-invariant) for an infinite cyclic cover of a closed 4-manifold M to be homotopy equivalent to a *particular* PD_3-complex N.

Theorem 4.6 *Let M be a PD_4-complex whose fundamental group π is an extension of Z by a torsion free normal subgroup ν which is isomorphic to the fundamental group of a PD_3-complex N. Then $\pi_2(M) \cong \pi_2(N)$ as $\mathbb{Z}[\nu]$-modules if and only if $Hom_{\mathbb{Z}[\pi]}(\pi_2(M), \mathbb{Z}[\pi]) = 0$. The infinite cyclic covering space M_ν with fundamental group ν is homotopy equivalent to N if and only if $w_1(M)|_\nu = w_1(N)$, $Hom_{\mathbb{Z}[\pi]}(\pi_2(M), \mathbb{Z}[\pi]) = 0$ and the images of $k_1(M)$ and $k_1(N)$ in $H^3(\nu; \pi_2(M)) \cong H^3(\nu; \pi_2(N))$ generate the same subgroup under the action of $Aut_{\mathbb{Z}[\nu]}(\pi_2(N))$.*

Proof If $\Pi = \pi_2(M)$ is isomorphic to $\pi_2(N)$ then it is finitely generated as a $\mathbb{Z}[\nu]$-module, by Theorem 2.18. As 0 is the only $\mathbb{Z}[\pi]$-submodule of $\mathbb{Z}[\pi]$ which is finitely generated as a $\mathbb{Z}[\nu]$-module it follows that $\Pi^* = Hom_{\mathbb{Z}[\pi]}(\pi_2(M), \mathbb{Z}[\pi])$ is trivial. It is then clear that the conditions must hold if M_ν is homotopy equivalent to N.

Suppose conversely that these conditions hold. If $\nu = 1$ then M_ν is simply connected and $\pi \cong Z$ has two ends. It follows immediately from Poincaré duality and the UCSS that $H_2(M_\nu;\mathbb{Z}) = \Pi \cong \overline{\Pi^*} = 0$ and that $H_3(M_\nu;\mathbb{Z}) \cong Z$. Therefore M_ν is homotopy equivalent to S^3. If $\nu \neq 1$ then π has one end, since it has a finitely generated infinite normal subgroup. The hypothesis that $\Pi^* = 0$ implies that $\Pi \cong \overline{H^2(\pi;\mathbb{Z}[\pi])}$, by Lemma 3.3. Hence $\Pi \cong \overline{H^1(\nu;\mathbb{Z}[\nu])}$ as a $\mathbb{Z}[\nu]$-module, by the LHSSS. (The overbar notation is unambiguous since $w_1(M)|_\nu = w_1(N)$.) But this is isomorphic to $\pi_2(N)$, by Poincaré duality for N. Since N is homotopy equivalent to a 3-dimensional complex the condition on the k-invariants implies that there is a map $f : N \to M_\nu$ which induces isomorphisms on fundamental group and second homotopy group. Since the homology of the universal covering spaces of these spaces vanishes above degree 2 the map f is a homotopy equivalence. □

We do not know whether the hypothesis on the k-invariants is implied by the other hypotheses.

Corollary 4.6.1 *Let M be a PD_4-complex whose fundamental group π is an extension of Z by a torsion free normal subgroup ν which is isomorphic to the fundamental group of a 3-manifold N whose irreducible factors are Haken, hyperbolic or Seifert fibred. Then M is homotopy equivalent to a closed PL 4-manifold which fibres over the circle with fibre N.*

Proof There is a homotopy equivalence $f : N \to M_\nu$, where N is a 3-manifold whose irreducible factors are as above, by Turaev's Theorem. (See §5 of Chapter 2.) Let $t : M_\nu \to M_\nu$ be the generator of the covering transformations. Then there is a self homotopy equivalence $u : N \to N$ such that $fu \sim tf$. As each irreducible factor of N has the property that self homotopy equivalences are homotopic to PL homeomorphisms (by [Hm], Mostow rigidity or [Sc83]), u is homotopic to a homeomorphism [HL74], and so M is homotopy equivalent to the mapping torus of this homeomorphism. □

All known PD_3-complexes with torsion free fundamental group are homotopy equivalent to connected sums of such 3-manifolds.

If the irreducible connected summands of the closed 3-manifold $N = \sharp_i N_i$ are P^2-irreducible and sufficiently large or have fundamental group Z then every self homotopy equivalence of N is realized by an unique isotopy class of homeomorphisms [HL74]. However if N is not aspherical then it admits nontrivial self-homeomorphisms ("rotations about 2-spheres") which induce the identity on ν, and so such bundles are not determined by the group alone.

Corollary 4.6.2 *Let M be a PD_4-complex whose fundamental group π is an extension of Z by a virtually torsion free normal subgroup ν. Then the infinite cyclic covering space M_ν with fundamental group ν is homotopy equivalent to a PD_3-complex if and only if ν is the fundamental group of a PD_3-complex N, $Hom_{\mathbb{Z}[\pi]}(\pi_2(M),\mathbb{Z}[\pi])=0$ and the images of $k_1(M)$ and $k_1(N)$ in $H^3(\nu_o;\pi_2(M))\cong H^3(\nu_o;\pi_2(N))$ generate the same subgroup under the action of $Aut_{\mathbb{Z}[\nu_o]}(\pi_2(N))$, where ν_o is a torsion free subgroup of finite index in ν.*

Proof The conditions are clearly necessary. Suppose that they hold. Let $\nu_1\subseteq\nu_o\cap\nu_+\cap\pi_+$ be a torsion free subgroup of finite index in ν, where $\pi_+ = \mathrm{K}erw_1(M)$ and $\nu_+ = \mathrm{K}erw_1(N)$, and let $t\in\pi$ generate π modulo ν. Then each of the conjugates $t^k\nu_1t^{-k}$ in π has the same index in ν. Since ν is finitely generated the intersection $\mu=\cap t^k\nu_1t^{-k}$ of all such conjugates has finite index in ν, and is clearly torsion free and normal in the subgroup ρ generated by μ and t. If $\{r_i\}$ is a transversal for ρ in π and $f:\pi_2(M)\to\mathbb{Z}[\rho]$ is a nontrivial $\mathbb{Z}[\rho]$-linear homomorphism then $g(m)=\Sigma r_if(r_i^{-1}m)$ defines a nontrivial element of $Hom_\pi(\pi_2(M),\mathbb{Z}[\pi])$. Hence $Hom_\rho(\pi_2(M),\mathbb{Z}[\rho])=0$ and so the covering spaces M_μ and N_μ are homotopy equivalent, by the theorem. It follows easily that M_ν is also a PD_3-complex. ∎

All PD_3-complexes have virtually torsion free fundamental group [Cr00].

4.6 Products

If $M = N\times S^1$, where N is a closed 3-manifold, then $\chi(M)=0$, Z is a direct factor of $\pi_1(M)$, $w_1(M)$ is trivial on this factor and the Pin^--condition $w_2=w_1^2$ holds. These conditions almost characterize such products up to homotopy equivalence. We need also a constraint on the other direct factor of the fundamental group.

Theorem 4.7 *Let M be a PD_4-complex whose fundamental group π has no 2-torsion. Then M is homotopy equivalent to a product $N\times S^1$, where N is a closed 3-manifold, if and only if $\chi(M)=0$, $w_2(M)=w_1(M)^2$ and there is an isomorphism $\theta:\pi\to\nu\times Z$ such that $w_1(M)\theta^{-1}|_Z=0$, where ν is a (2-torsion free) 3-manifold group.*

Proof The conditions are clearly necessary, since the Pin^--condition holds for 3-manifolds.

If these conditions hold then the covering space M_ν with fundamental group ν is a PD_3-complex, by Theorem 4.5 above. Since ν is a 3-manifold group and has no 2-torsion it is a free product of cyclic groups and groups of aspherical closed 3-manifolds. Hence there is a homotopy equivalence $h : M_\nu \to N$, where N is a connected sum of lens spaces and aspherical closed 3-manifolds, by Turaev's Theorem. (See §5 of Chapter 2.) Let ϕ generate the covering group $Aut(M/M_\nu) \cong Z$. Then there is a self homotopy equivalence $\psi : N \to N$ such that $\psi h \sim h\phi$, and M is homotopy equivalent to the mapping torus $M(\psi)$. We may assume that ψ fixes a basepoint and induces the identity on $\pi_1(N)$, since $\pi_1(M) \cong \nu \times Z$. Moreover ψ preserves the local orientation, since $w_1(M)\theta^{-1}|_Z = 0$. Since ν has no element of order 2 N has no two-sided projective planes and so ψ is homotopic to a rotation about a 2-sphere [Hn]. Since $w_2(M) = w_1(M)^2$ the rotation is homotopic to the identity and so M is homotopy equivalent to $N \times S^1$. □

Let ρ is an essential map from S^1 to $SO(3)$, and let $M = M(\tau)$, where $\tau : S^1 \times S^2 \to S^1 \times S^2$ is the twist map, given by $\tau(x, y) = (x, \rho(x)(y))$ for all (x, y) in $S^1 \times S^2$. Then $\pi_1(M) \cong Z \times Z$, $\chi(M) = 0$, and $w_1(M) = 0$, but $w_2(M) \neq w_1(M)^2 = 0$, so M is not homotopy equivalent to a product. (Clearly however $M(\tau^2) = S^1 \times S^2 \times S^1$.)

To what extent are the constraints on ν necessary? There are orientable 4-manifolds which are homotopy equivalent to products $N \times S^1$ where $\nu = \pi_1(N)$ is finite and is *not* a 3-manifold group. (See Chapter 11.) Theorem 4.1 implies that M is homotopy equivalent to a product of an aspherical PD_3-complex with S^1 if and only if $\chi(M) = 0$ and $\pi_1(M) \cong \nu \times Z$ where ν has one end.

There are 4-manifolds which are simple homotopy equivalent to $S^1 \times RP^3$ (and thus satisfy the hypotheses of our theorem) but which are not homeomorphic to mapping tori [We87].

4.7 Subnormal subgroups

In this brief section we shall give another characterization of aspherical 4-manifolds with finite covering spaces which are homotopy equivalent to mapping tori.

Theorem 4.8 *Let M be a PD_4-complex. Then M is aspherical and has a finite cover which is homotopy equivalent to a mapping torus if and only if $\chi(M) = 0$ and $\pi = \pi_1(M)$ has an FP_3 subnormal subgroup G of infinite index and such that $H^s(G; \mathbb{Z}[G]) = 0$ for $s \leq 2$. In that case G is a PD_3-group, $[\pi : N_\pi(G)] < \infty$ and $e(N_\pi(G)/G) = 2$.*

Proof The conditions are clearly necessary. Suppose that they hold. Let $G = G_0 < G_1 < \dots G_n = \pi$ be a subnormal chain of minimal length, and let $j = \min\{i \mid [G_{i+1} : G] = \infty\}$. Then $[G_j : G] < \infty$ and $\beta_1^{(2)}(G_{j+1}) = 0$ [Ga00]. A finite induction up the subnormal chain, using LHSSS arguments (with coefficients $\mathbb{Z}[\pi]$ and $\mathcal{N}(G_j)$, respectively) shows that $H^s(\pi;\mathbb{Z}[\pi]) = 0$ for $s \leq 2$ and that $\beta_1^{(2)}(\pi) = 0$. (See §2 of Chapter 2.) Hence M is aspherical, by Theorem 3.4.

On the other hand $H^s(G_{j+1}; W) = 0$ for $s \leq 3$ and any free $\mathbb{Z}[G_{j+1}]$-module W, so $c.d.G_{j+1} = 4$. Hence $[\pi : G_{j+1}] < \infty$, by Strebel's Theorem. Therefore G_{j+1} is a PD_4-group. Hence G_j is a PD_3-group and G_{j+1}/G_j has two ends, by Theorem 3.10. The theorem now follows easily, since $[G_j : G] < \infty$ and G_j has only finitely many subgroups of index $[G_j : G]$. □

The hypotheses on G could be replaced by "*G is a PD_3-group*", for then $[\pi : G] = \infty$, by Theorem 3.12.

We shall establish an analogous result for closed 4-manifolds M such that $\chi(M) = 0$ and $\pi_1(M)$ has a subnormal subgroup of infinite index which is a PD_2-group in Chapter 5.

4.8 Circle bundles

In this section we shall consider the "dual" situation, of 4-manifolds which are homotopy equivalent to the total space of a S^1-bundle over a 3-dimensional base N. Lemma 4.9 presents a number of conditions satisfied by such manifolds. (These conditions are not all independent.) Bundles $c_N^*\xi$ induced from S^1-bundles over $K(\pi_1(N), 1)$ are given equivalent characterizations in Lemma 4.10. In Theorem 4.11 we shall show that the conditions of Lemmas 4.9 and 4.10 characterize the homotopy types of such bundle spaces $E(c_N^*\xi)$, provided $\pi_1(N)$ is torsion free but not free.

Since $BS^1 \simeq K(Z, 2)$ any S^1-bundle over a connected base B is induced from some bundle over $P_2(B)$. For each epimorphism $\gamma : \mu \to \nu$ with cyclic kernel and such that the action of μ by conjugation on $\mathrm{Ker}(\gamma)$ factors through multiplication by ± 1 there is an S^1-bundle $p(\gamma) : X(\gamma) \to Y(\gamma)$ whose fundamental group sequence realizes γ and which is universal for such bundles; the total space $E(p(\gamma))$ is a $K(\mu, 1)$ space (cf. Proposition 11.4 of [Wl]).

Lemma 4.9 *Let $p : E \to B$ be the projection of an S^1-bundle ξ over a connected finite complex B. Then*

(1) *$\chi(E) = 0$;*

(2) *the natural map $p_* : \pi = \pi_1(E) \to \nu = \pi_1(B)$ is an epimorphism with cyclic kernel, and the action of ν on $\mathrm{Ker}(p_*)$ induced by conjugation in π is given by $w = w_1(\xi) : \pi_1(B) \to Z/2Z \cong \{\pm 1\} \leq Aut(\mathrm{Ker}(p_*))$;*

(3) *if B is a PD-complex $w_1(E) = p^*(w_1(B) + w)$;*

(4) *if B is a PD_3-complex there are maps $\hat{c} : E \to P_2(B)$ and $y : P_2(B) \to Y(p_*)$ such that $c_{P_2(B)} = c_{Y(p_*)}y$, $y\hat{c} = p(p_*)c_E$ and $(\hat{c}, c_E)_*[E] = \pm G(f_{B*}[B])$ where G is the Gysin homomorphism from $H_3(P_2(B); Z^{w_1(B)})$ to $H_4(P_2(E); Z^{w_1(E)})$;*

(5) *If B is a PD_3-complex $c_{E*}[E] = \pm G(c_{B*}[B])$, where G is the Gysin homomorphism from $H_3(\nu; Z^{w_B})$ to $H_4(\pi; Z^{w_E})$;*

(6) *$\mathrm{Ker}(p_*)$ acts trivially on $\pi_2(E)$.*

Proof Condition(1) follows from the multiplicativity of the Euler characteristic in a fibration. If α is any loop in B the total space of the induced bundle $\alpha^*\xi$ is the torus if $w(\alpha) = 0$ and the Klein bottle if $w(\alpha) = 1$ in $Z/2Z$; hence $gzg^{-1} = z^{\epsilon(g)}$ where $\epsilon(g) = (-1)^{w(p_*(g))}$ for g in $\pi_1(E)$ and z in $\mathrm{Ker}(p_*)$. Conditions (2) and (6) then follow from the exact homotopy sequence. If the base B is a PD-complex then so is E, and we may use naturality and the Whitney sum formula (applied to the Spivak normal bundles) to show that $w_1(E) = p^*(w_1(B) + w_1(\xi))$. (As $p^* : H^1(B; \mathbb{F}_2) \to H^1(E; \mathbb{F}_2)$ is a monomorphism this equation determines $w_1(\xi)$.)

Condition (4) implies (5), and follows from the observations in the paragraph preceding the lemma. (Note that the Gysin homomorphisms G in (4) and (5) are well defined, since $H_1(\mathrm{Ker}(\gamma); Z^{w_E})$ is isomorphic to Z^{w_B}, by (3).) □

Bundles with $\mathrm{Ker}(p_*) \cong Z$ have the following equivalent characterizations.

Lemma 4.10 *Let $p : E \to B$ be the projection of an S^1-bundle ξ over a connected finite complex B. Then the following conditions are equivalent:*

(1) *ξ is induced from an S^1-bundle over $K(\pi_1(B), 1)$ via c_B;*

(2) *for each map $\beta : S^2 \to B$ the induced bundle $\beta^*\xi$ is trivial;*

(3) *the induced epimorphism $p_* : \pi_1(E) \to \pi_1(B)$ has infinite cyclic kernel.*

If these conditions hold then $c(\xi) = c_B^*\Xi$, where $c(\xi)$ is the characteristic class of ξ in $H^2(B; Z^w)$ and Ξ is the class of the extension of fundamental groups in $H^2(\pi_1(B); Z^w) = H^2(K(\pi_1(B), 1); Z^w)$, where $w = w_1(\xi)$.

Proof Condition (1) implies condition (2) as for any such map β the composite $c_B\beta$ is nullhomotopic. Conversely, as we may construct $K(\pi_1(B),1)$ by adjoining cells of dimension ≥ 3 to B condition (2) implies that we may extend ξ over the 3-cells, and as S^1-bundles over S^n are trivial for all $n > 2$ we may then extend ξ over the whole of $K(\pi_1(B),1)$, so that (2) implies (1). The equivalence of (2) and (3) follows on observing that (3) holds if and only if $\partial\beta = 0$ for all such β, where ∂ is the connecting map from $\pi_2(B)$ to $\pi_1(S^1)$ in the exact sequence of homotopy for ξ, and on comparing this with the corresponding sequence for $\beta^*\xi$.

As the natural map from the set of S^1-bundles over $K(\pi,1)$ with $w_1 = w$ (which are classified by $H^2(K(\pi,1);Z^w)$) to the set of extensions of π by Z with π acting via w (which are classified by $H^2(\pi;Z^w)$) which sends a bundle to the extension of fundamental groups is an isomorphism we have $c(\xi) = c_B^*(\Xi)$. □

If N is a closed 3-manifold which has no summands of type $S^1\times S^2$ or $S^1\tilde{\times}S^2$ (i.e., if $\pi_1(N)$ has no infinite cyclic free factor) then every S^1-bundle over N with $w = 0$ restricts to a trivial bundle over any map from S^2 to N. For if ξ is such a bundle, with characteristic class $c(\chi)$ in $H^2(N;\mathbb{Z})$, and $\beta : S^2 \to N$ is any map then $\beta_*(c(\beta^*\xi)\cap[S^2]) = \beta_*(\beta^*c(\xi)\cap[S^2]) = c(\xi)\cap\beta_*[S^2] = 0$, as the Hurewicz homomorphism is trivial for such N. Since β_* is an isomorphism in degree 0 it follows that $c(\beta^*\xi) = 0$ and so $\beta^*\xi$ is trivial. (A similar argument applies for bundles with $w \neq 0$, provided the induced 2-fold covering space N^w has no summands of type $S^1\times S^2$ or $S^1\tilde{\times}S^2$.)

On the other hand, if η is the Hopf fibration the bundle with total space $S^1\times S^3$, base $S^1\times S^2$ and projection $id_{S^1}\times\eta$ has nontrivial pullback over any essential map from S^2 to $S^1\times S^2$, and is not induced from any bundle over $K(Z,1)$. Moreover, $S^1\times S^2$ is a 2-fold covering space of $RP^3\sharp RP^3$, and so the above hypothesis on summands of N is not stable under passage to 2-fold coverings (corresponding to a homomorphism w from $\pi_1(N)$ to $Z/2Z$).

Theorem 4.11 *Let M be a finite PD_4-complex and N a finite PD_3-complex whose fundamental group is torsion free but not free. Then M is homotopy equivalent to the total space of an S^1-bundle over N which satisfies the conditions of Lemma* 4.10 *if and only if*

(1) $\chi(M) = 0$;

(2) *there is an epimorphism* $\gamma : \pi = \pi_1(M) \to \nu = \pi_1(N)$ *with* $\mathrm{Ker}(\gamma) \cong Z$;

(3) $w_1(M) = (w_1(N) + w)\gamma$, *where* $w : \nu \to Z/2Z \cong Aut(\mathrm{Ker}(\gamma))$ *is determined by the action of* ν *on* $\mathrm{Ker}(\gamma)$ *induced by conjugation in* π;

(4) *$k_1(M) = \gamma^* k_1(N)$ (and so $P_2(M) \simeq P_2(N) \times_{K(\nu,1)} K(\pi,1)$);*

(5) *$f_{M*}[M] = \pm G(f_{N*}[N])$ in $H_4(P_2(M); Z^{w_1(M)})$, where G is the Gysin homomorphism in degree 3.*

If these conditions hold then M has minimal Euler characteristic for its fundamental group, i.e. $q(\pi) = 0$.

Remark The first three conditions and Poincaré duality imply that $\pi_2(M) \cong \gamma^*\pi_2(N)$, the $\mathbb{Z}[\pi]$-module with the same underlying group as $\pi_2(N)$ and with $\mathbb{Z}[\pi]$-action determined by the homomorphism γ.

Proof Since these conditions are homotopy invariant and hold if M is the total space of such a bundle, they are necessary. Suppose conversely that they hold. As ν is torsion free N is the connected sum of a 3-manifold with free fundamental group and some aspherical PD_3-complexes [Tu90]. As ν is not free there is at least one aspherical summand. Hence $c.d.\nu = 3$ and $H_3(c_N; Z^{w_1(N)})$ is a monomorphism.

Let $p(\gamma) : K(\pi,1) \to K(\nu,1)$ be the S^1-bundle corresponding to γ and let $E = N \times_{K(\nu,1)} K(\pi,1)$ be the total space of the S^1-bundle over N induced by the classifying map $c_N : N \to K(\nu,1)$. The bundle map covering c_N is the classifying map c_E. Then $\pi_1(E) \cong \pi = \pi_1(M)$, $w_1(E) = (w_1(N) + w)\gamma = w_1(M)$, as maps from π to $Z/2Z$, and $\chi(E) = 0 = \chi(M)$, by conditions (1) and (3). The maps c_N and c_E induce a homomorphism between the Gysin sequences of the S^1-bundles. Since N and ν have cohomological dimension 3 the Gysin homomorphisms in degree 3 are isomorphisms. Hence $H_4(c_E; Z^{w_1(E)})$ is a monomorphism, and so *a fortiori* $H_4(f_E; Z^{w_1(E)})$ is also a monomorphism.

Since $\chi(M) = 0$ and $\beta_1^{(2)}(\pi) = 0$, by Theorem 2.3, part (3) of Theorem 3.4 implies that $\pi_2(M) \cong \overline{H^2(\pi; \mathbb{Z}[\pi])}$. It follows from conditions (2) and (3) and the LHSSS that $\pi_2(M) \cong \pi_2(E) \cong \gamma^*\pi_2(N)$ as $\mathbb{Z}[\pi]$-modules. Conditions (4) and (5) then give us a map $(\hat{c}, c_M)$ from M to $P_2(E) = P_2(N) \times_{K(\nu,1)} K(\pi,1)$ such that $(\hat{c}, c_M)_*[M] = \pm f_{E*}[E]$. Hence M is homotopy equivalent to E, by Theorem 3.8.

The final assertion now follows from part (1) of Theorem 3.4. ∎

As $\pi_2(N)$ is a projective $\mathbb{Z}[\nu]$-module, by Theorem 2.18, it is homologically trivial and so $H_q(\pi; \gamma^*\pi_2(N) \otimes Z^{w_1(M)}) = 0$ if $q \geq 2$. Hence it follows from the spectral sequence for $c_{P_2(M)}$ that $H_4(P_2(M); Z^{w_1(M)})$ maps onto $H_4(\pi; Z^{w_1(M)})$, with kernel isomorphic to $H_0(\pi; \Gamma(\pi_2(M))) \otimes Z^{w_1(M)})$, where

$\Gamma(\pi_2(M)) = H_4(K(\pi_2(M),2);\mathbb{Z})$ is Whitehead's universal quadratic construction on $\pi_2(M)$ (see Chapter I of [Ba']). This suggests that there may be another formulation of the theorem in terms of conditions (1-3), together with some information on $k_1(M)$ and the intersection pairing on $\pi_2(M)$. If N is aspherical conditions (4) and (5) are vacuous or redundant.

Condition (4) is vacuous if ν is a free group, for then $c.d.\pi \leq 2$. In this case the Hurewicz homomorphism from $\pi_3(N)$ to $H_3(N;Z^{w_1(N)})$ is 0, and so $H_3(f_N;Z^{w_1(N)})$ is a monomorphism. The argument of the theorem would then extend if the Gysin map in degree 3 for the bundle $P_2(E) \to P_2(N)$ were a monomorphism. If $\nu = 1$ then M is orientable, $\pi \cong Z$ and $\chi(M) = 0$, so $M \simeq S^3 \times S^1$. In general, if the restriction on ν is removed it is not clear that there should be a degree 1 map from M to such a bundle space E.

It would be of interest to have a theorem with hypotheses involving only M, without reference to a model N. There is such a result in the aspherical case.

Theorem 4.12 *A finite PD_4-complex M is homotopy equivalent to the total space of an S^1-bundle over an aspherical PD_3-complex if and only if $\chi(M) = 0$ and $\pi = \pi_1(M)$ has an infinite cyclic normal subgroup A such that π/A has one end and finite cohomological dimension.*

Proof The conditions are clearly necessary. Conversely, suppose that they hold. Since π/A has one end $H^s(\pi/A;\mathbb{Z}[\pi/A]) = 0$ for $s \leq 1$ and so an LHSSS calculation gives $H^t(\pi;\mathbb{Z}[\pi]) = 0$ for $t \leq 2$. Moreover $\beta_1^{(2)}(\pi) = 0$, by Theorem 2.3. Hence M is aspherical and π is a PD_4-group, by Corollary 3.5.2. Since A is FP_∞ and $c.d.\pi/A < \infty$ the quotient π/A is a PD_3-group, by Theorem 9.11 of [Bi]. Therefore M is homotopy equivalent to the total space of an S^1-bundle over the PD_3-complex $K(\pi/A,1)$. □

Note that a finitely generated torsion free group has one end if and only if it is indecomposable as a free product and is neither infinite cyclic nor trivial.

In general, if M is homotopy equivalent to the total space of an S^1-bundle over some 3-manifold then $\chi(M) = 0$ and $\pi_1(M)$ has an infinite cyclic normal subgroup A such that $\pi_1(M)/A$ is virtually of finite cohomological dimension. Do these conditions characterize such homotopy types?

Chapter 5

Surface bundles

In this chapter we shall show that a closed 4-manifold M is homotopy equivalent to the total space of a fibre bundle with base and fibre closed surfaces if and only if the obviously necessary conditions on the Euler characteristic and fundamental group hold. When the base is S^2 we need also conditions on the characteristic classes of M, and when the base is RP^2 our results are incomplete. We shall defer consideration of bundles over RP^2 with fibre T or Kb and $\partial \neq 0$ to Chapter 11, and those with fibre S^2 or RP^2 to Chapter 12.

5.1 Some general results

If B, E and F are connected finite complexes and $p : E \to B$ is a Hurewicz fibration with fibre homotopy equivalent to F then $\chi(E) = \chi(B)\chi(F)$ and the long exact sequence of homotopy gives an exact sequence

$$\pi_2(B) \to \pi_1(F) \to \pi_1(E) \to \pi_1(B) \to 1$$

in which the image of $\pi_2(B)$ under the *connecting homomorphism* ∂ is in the centre of $\pi_1(F)$. (See page 51 of [Go68].) These conditions are clearly homotopy invariant.

Hurewicz fibrations with base B and fibre X are classified by homotopy classes of maps from B to the Milgram classifying space $BE(X)$, where $E(X)$ is the monoid of all self homotopy equivalences of X, with the compact-open topology [Mi67]. If X has been given a base point the evaluation map from $E(X)$ to X is a Hurewicz fibration with fibre the subspace (and submonoid) $E_0(X)$ of base point preserving self homotopy equivalences [Go68].

Let T and Kb denote the torus and Klein bottle, respectively.

Lemma 5.1 *Let F be an aspherical closed surface and B a closed smooth manifold. There are natural bijections from the set of isomorphism classes of smooth F-bundles over B to the set of fibre homotopy equivalence classes of Hurewicz fibrations with fibre F over B and to the set $\coprod_{[\xi]} H^2(B;\zeta\pi_1(F)^\xi)$, where the union is over conjugacy classes of homomorphisms $\xi : \pi_1(B) \to Out(\pi_1(F))$ and $\zeta\pi_1(F)^\xi$ is the $\mathbb{Z}[\pi_1(F)]$-module determined by ξ.*

Proof If $\zeta\pi_1(F) = 1$ the identity components of $Diff(F)$ and $E(F)$ are contractible [EE69]. Now every automorphism of $\pi_1(F)$ is realizable by a diffeomorphism and homotopy implies isotopy for self diffeomorphisms of surfaces. (See Chapter V of [ZVC].) Therefore $\pi_0(Diff(F)) \cong \pi_0(E(F)) \cong Out(\pi_1(F))$, and the inclusion of $Diff(F)$ into $E(F)$ is a homotopy equivalence. Hence $BDiff(F) \simeq BE(F) \simeq K(Out(\pi_1(F),1)$, so smooth F-bundles over B and Hurewicz fibrations with fibre F over B are classified by the (unbased) homotopy set

$$[B, K(Out(\pi_1(F),1))] = Hom(\pi_1(B), Out(\pi_1(F)))/\backsim,$$

where $\xi \backsim \xi'$ if there is an $\alpha \in Out(\pi_1(F))$ such that $\xi'(b) = \alpha\xi(b)\alpha^{-1}$ for all $b \in \pi_1(B)$.

If $\zeta\pi_1(F) \neq 1$ then $F = T$ or Kb. Left multiplication by T on itself induces homotopy equivalences from T to the identity components of $Diff(T)$ and $E(T)$. (Similarly, the standard action of S^1 on Kb induces homotopy equivalences from S^1 to the identity components of $Diff(Kb)$ and $E(Kb)$. See Theorem III.2 of [Go65].) Let $\alpha : GL(2,\mathbb{Z}) \to Aut(T) \leq Diff(T)$ be the standard linear action. Then the natural maps from the semidirect product $T \times_\alpha GL(2,\mathbb{Z})$ to $Diff(T)$ and to $E(T)$ are homotopy equivalences. Therefore $BDiff(T)$ is a $K(Z^2,2)$-fibration over $K(GL(2,\mathbb{Z}),1)$. It follows that T-bundles over B are classified by two invariants: a conjugacy class of homomorphisms $\xi : \pi_1(B) \to GL(2,\mathbb{Z})$ together with a cohomology class in $H^2(B;(Z^2)^\xi)$. A similar argument applies if $F = Kb$. □

Theorem 5.2 *Let M be a PD_4-complex and B and F aspherical closed surfaces. Then M is homotopy equivalent to the total space of an F-bundle over B if and only if $\chi(M) = \chi(B)\chi(F)$ and $\pi_1(M)$ is an extension of $\pi_1(B)$ by $\pi_1(F)$. Moreover every extension of $\pi_1(B)$ by $\pi_1(F)$ is realized by some surface bundle, which is determined up to isomorphism by the extension.*

Proof The conditions are clearly necessary. Suppose that they hold. If $\zeta\pi_1(F) = 1$ each homomorphism $\xi : \pi_1(B) \to Out(\pi_1(F))$ corresponds to an unique equivalence class of extensions of $\pi_1(B)$ by $\pi_1(F)$, by Proposition 11.4.21 of [Ro]. Hence there is an F-bundle $p : E \to B$ with $\pi_1(E) \cong \pi_1(M)$ realizing the extension, and p is unique up to bundle isomorphism. If $F = T$ then every homomorphism $\xi : \pi_1(B) \to GL(2,\mathbb{Z})$ is realizable by an extension (for instance, the semidirect product $Z^2 \times_\xi \pi_1(B)$) and the extensions realizing ξ are classified up to equivalence by $H^2(\pi_1(B);(Z^2)^\xi)$. As B is aspherical the natural map from bundles to group extensions is a bijection. Similar arguments apply if $F = Kb$. In all cases the bundle space E is aspherical, and so $\pi_1(M)$ is an FF PD_4-group. Hence $M \simeq E$, by Corollary 3.5.1. □

Such extensions (with $\chi(F) < 0$) were shown to be realizable by bundles in [Jo79].

5.2 Bundles with base and fibre aspherical surfaces

In many cases the group $\pi_1(M)$ determines the bundle up to diffeomorphism of its base. Lemma 5.3 and Theorems 5.4 and 5.5 are based on [Jo94].

Lemma 5.3 *Let G_1 and G_2 be groups with no nontrivial abelian normal subgroup. If H is a normal subgroup of $G = G_1 \times G_2$ which contains no nontrivial direct product then either $H \leq G_1 \times \{1\}$ or $H \leq \{1\} \times G_2$.*

Proof Let P_i be the projection of H onto G_i, for $i = 1, 2$. If $(h, h') \in H$, $g_1 \in G_1$ and $g_2 \in G_2$ then $([h, g_1], 1) = [(h, h'), (g_1, 1)]$ and $(1, [h', g_2])$ are in H. Hence $[P_1, P_1] \times [P_2, P_2] \leq H$. Therefore either P_1 or P_2 is abelian, and so is trivial, since P_i is normal in G_i, for $i = 1, 2$. □

Theorem 5.4 *Let π be a group with a normal subgroup K such that K and π/K are PD_2-groups with trivial centres.*

1. *If $C_\pi(K) = 1$ and K_1 is a finitely generated normal subgroup of π then $C_\pi(K_1) = 1$ also.*
2. *The index $[\pi : KC_\pi(K)]$ is finite if and only if π is virtually a direct product of PD_2-groups.*

Proof (1) Let $z \in C_\pi(K_1)$. If $K_1 \leq K$ then $[K : K_1] < \infty$ and $\zeta K_1 = 1$. Let $M = [K : K_1]!$. Then $f(k) = k^{-1} z^M k z^{-M}$ is in K_1 for all k in K. Now $f(kk_1) = k_1^{-1} f(k) k_1$ and also $f(kk_1) = f(kk_1k^{-1}k) = f(k)$ (since K_1 is a normal subgroup centralized by z), for all k in K and k_1 in K_1. Hence $f(k)$ is central in K_1, and so $f(k) = 1$ for all k in K. Thus z^M centralizes K. Since π is torsion free we must have $z = 1$. Otherwise the image of K_1 under the projection $p : \pi \to \pi/K$ is a nontrivial finitely generated normal subgroup of π/K, and so has trivial centralizer. Hence $p(z) = 1$. Now $[K, K_1] \leq K \cap K_1$ and so $K \cap K_1 \neq 1$, for otherwise $K_1 \leq C_\pi(K)$. Since z centralizes the nontrivial normal subgroup $K \cap K_1$ in K we must again have $z = 1$.

(2) Since K has trivial centre $KC_\pi(K) \cong K \times C_\pi(K)$ and so the condition is necessary. Suppose that $f : G_1 \times G_2 \to \pi$ is an isomorphism onto a subgroup of finite index, where G_1 and G_2 are PD_2-groups. Let $L = K \cap f(G_1 \times G_2)$. Then $[K : L] < \infty$ and so L is also a PD_2-group, and is normal in $f(G_1 \times G_2)$. We may assume that $L \leq f(G_1)$, by Lemma 5.3. Then $f(G_1)/L$ is finite and is isomorphic to a subgroup of $f(G_1 \times G_2)/K \leq \pi/K$, so $L = f(G_1)$. Now $f(G_2)$ normalizes K and centralizes L, and $[K : L] < \infty$. Hence $f(G_2)$ has a subgroup of finite index which centralizes K, as in part (1). Hence $[\pi : KC_\pi(K)] < \infty$. □

It follows immediately that if π and K are as in the theorem whether

(1) $C_\pi(K) \neq 1$ and $[\pi : KC_\pi(K)] = \infty$;

(2) $[\pi : KC_\pi(K)] < \infty$; or

(3) $C_\pi(K) = 1$

depends only on π and not on the subgroup K. In [Jo94] these cases are labeled as types I, II and III, respectively. (In terms of the action: if $\mathrm{Im}(\theta)$ is infinite and $\mathrm{Ker}(\theta) \neq 1$ then π is of type I, if $\mathrm{Im}(\theta)$ is finite then π is of type II, and if θ is injective then π is of type III.)

Theorem 5.5 *Let π be a group with normal subgroups K and K_1 such that K, K_1, π/K and π/K_1 are PD_2-groups with trivial centres. If $C_\pi(K) \neq 1$ but $[\pi : KC_\pi(K)] = \infty$ then $K_1 = K$ is unique. If $[\pi : KC_\pi(K)] < \infty$ then either $K_1 = K$ or $K_1 \cap K = 1$; in the latter case K and K_1 are the only such normal subgroups which are PD_2-groups with torsion free quotients.*

Proof Let $p : \pi \to \pi/K$ be the quotient epimorphism. Then $p(C_\pi(K))$ is a nontrivial normal subgroup of π/K, since $K \cap C_\pi(K) = \zeta K = 1$. Suppose that $K_1 \cap K \neq 1$. Let $\Sigma = K_1 \cap (KC_\pi(K))$. Then Σ contains $K_1 \cap K$, and $\Sigma \not\leq C_\pi(K)$, since $K_1 \cap K \cap C_\pi(K) = K_1 \cap \zeta K = 1$. Since Σ is normal in $KC_\pi(K) \cong K \times C_\pi(K)$ we must have $\Sigma \leq K_1$, by Lemma 5.3. Hence $\Sigma \leq K_1 \cap K$. Hence $p(K_1) \cap p(C_\pi(K)) = 1$, and so $p(K_1)$ centralizes the nontrivial normal subgroup $p(C_\pi(K))$ in π/K. Therefore $K_1 \leq K$ and so $[K : K_1] < \infty$. Since π/K_1 is torsion free we must have $K_1 = K$.

If $K_1 \cap K = 1$ then $[K, K_1] = 1$ (since each subgroup is normal in π) so $K_1 \leq C_\pi(K)$ and $[\pi : KC_\pi(K)] \leq [\pi/K : p(K_1)] < \infty$. Suppose K_2 is a normal subgroup of π which is a PD_2-group with $\zeta K_2 = 1$ and such that π/K_2 is torsion free and $K_2 \cap K = 1$. Then $H = K_2 \cap (KK_1)$ is normal in $KK_1 \cong K \times K_1$ and $[K_2 : H] < \infty$, so H is a PD_2-group with $\zeta H = 1$

and $H \cap K = 1$. The projection of H to K_1 is nontrivial since $H \cap K = 1$. Therefore $H \leq K_1$, by Lemma 5.3, and so $K_1 \leq K_2$. Hence $K_1 = K_2$. $\square$

Corollary 5.5.1 [Jo93] *Let α and β be automorphisms of π, and suppose that $\alpha(K) \cap K = 1$. Then $\beta(K) = K$ or $\alpha(K)$, and so $Aut(K \times K) \cong Aut(K)^2 \tilde{\times} (Z/2Z)$.*

We shall obtain a somewhat weaker result for groups of type III as a corollary of the next theorem.

Theorem 5.6 *Let π be a group with normal subgroups K and K_1 such that K, K_1 and π/K are PD_2-groups, π/K_1 is torsion free and $\chi(\pi/K) < 0$. Then either $K_1 = K$ or $K_1 \cap K = 1$ and $\pi \cong K \times K_1$ or $\chi(K_1) < \chi(\pi/K)$.*

Proof Let $p : \pi \to \pi/K$ be the quotient epimorphism. If $K_1 \leq K$ then $K_1 = K$, as in Theorem 5.5. Otherwise $p(K_1)$ has finite index in π/K and so $p(K_1)$ is also a PD_2-group. As the minimum number of generators of a PD_2-group G is $\beta_1(G; \mathbb{F}_2)$, we have $\chi(K_1) \leq \chi(p(K_1)) \leq \chi(\pi/K)$. We may assume that $\chi(K_1) \geq \chi(\pi/K)$. Hence $\chi(K_1) = \chi(\pi/K)$ and so $p|_{K_1}$ is an epimorphism. Therefore K_1 and π/K have the same orientation type, by the nondegeneracy of Poincaré duality with coefficients $\mathbb{F}_2$ and the Wu relation $w_1 \cup x = x^2$ for all $x \in H^1(G; \mathbb{F}_2)$ and PD_2-groups G. Hence $K_1 \cong \pi/K$. Since PD_2-groups are hopfian $p|_{K_1}$ is an isomorphism. Hence $[K, K_1] \leq K \cap K_1 = 1$ and so $\pi = K.K_1 \cong K \times \pi/K$. $\square$

Corollary 5.6.1 [Jo98] *The group π has only finitely many such subgroups K.*

Proof We may assume given $\chi(K) < 0$ and that π is of type III. If ρ is an epimorphism from π to $Z/\chi(\pi)Z$ such that $\rho(K) = 0$ then $\chi(\mathrm{Ker}(\rho)/K) \leq \chi(K)$. Since π is not a product K is the only such subgroup of $\mathrm{Ker}(\rho)$. Since $\chi(K)$ divides $\chi(\pi)$ and $Hom(\pi, Z/\chi(\pi)Z)$ is finite the corollary follows. $\square$

The next two corollaries follow by elementary arithmetic.

Corollary 5.6.2 *If $\chi(K) = 0$ or $\chi(K) = -1$ and π/K_1 is a PD_2-group then either $K_1 = K$ or $\pi \cong K \times K_1$.* $\square$

Corollary 5.6.3 *If K and π/K are PD_2-groups, $\chi(\pi/K) < 0$, and $\chi(K)^2 \leq \chi(\pi)$ then either K is the unique such subgroup or $\pi \cong K \times K$.* $\square$

Corollary 5.6.4 *Let M and M' be the total spaces of bundles ξ and ξ' with the same base B and fibre F, where B and F are aspherical closed surfaces such that $\chi(B) < \chi(F)$. Then M' is diffeomorphic to M via a fibre-preserving diffeomorphism if and only if $\pi_1(M') \cong \pi_1(M)$.* $\square$

Compare the statement of Melvin's Theorem on total spaces of S^2-bundles (Theorem 5.13 below.)

We can often recognise total spaces of aspherical surface bundles under weaker hypotheses on the fundamental group.

Theorem 5.7 *Let M be a PD_4-complex with fundamental group π. Then the following conditions are equivalent:*

(1) *M is homotopy equivalent to the total space of a bundle with base and fibre aspherical closed surfaces:*

(2) *π has an FP_2 normal subgroup K such that π/K is a PD_2-group and $\pi_2(M) = 0$;*

(3) *π has a normal subgroup N which is a PD_2-group, π/N is torsion free and $\pi_2(M) = 0$.*

Proof Clearly (1) implies (2) and (3). Conversely they each imply that π has one end and so M is aspherical. If K is an FP_2 normal subgroup in π and π/K is a PD_2-group then K is a PD_2-group, by Theorem 1.19. If N is a normal subgroup which is a PD_2-group then an LHSSS argument gives $H^2(\pi/N;\mathbb{Z}[\pi/N]) \cong Z$. Hence π/N is virtually a PD_2-group, by Bowditch's Theorem. Since it is torsion free it is a PD_2-group and so the theorem follows from Theorem 5.2. $\square$

If $\zeta K = 1$ we may avoid the difficult theorem of Bowditch here, for then π/K is an extension of $C_\pi(K)$ by a subgroup of $Out(K)$, so $v.c.d.\pi/K < \infty$ and thus π/K is virtually a PD_2-group, by Theorem 9.11 of [Bi].

Kapovich has given an example of an aspherical closed 4-manifold M such that $\pi_1(M)$ is an extension of a PD_2-group by a finitely generated normal subgroup which is not FP_2 [Ka98].

Theorem 5.8 *Let M be a PD_4-complex with fundamental group π and such that $\chi(M) = 0$. If π has a subnormal subgroup G of infinite index which is a PD_2-group then M is aspherical. If moreover $\zeta G = 1$ there is a subnormal chain $G < J < K \leq \pi$ such that $[\pi : K] < \infty$ and $K/J \cong J/G \cong Z$.*

Proof Let $G = G_0 < G_1 < \ldots G_n = \pi$ be a subnormal chain of minimal length. Let $j = \min\{i \mid [G_{i+1} : G] = \infty\}$. Then $[G_j : G] < \infty$, so G_j is FP. It is easily seen that the theorem holds for G if it holds for G_j. Thus we may assume that $[G_1 : G] = \infty$. A finite induction up the subnormal chain using the LHSSS gives $H^s(\pi; \mathbb{Z}[\pi]) = 0$ for $s \leq 2$. Now $\beta_1^{(2)}(G_1) = 0$, since G is finitely generated and $[G_1 : G] = \infty$ [Ga00]. (This also can be deduced from Theorem 2.2 and the fact that $Out(G)$ is virtually torsion free.) Inducting up the subnormal chain gives $\beta_1^{(2)}(\pi) = 0$ and so M is aspherical, by Theorem 3.4.

If $G < \tilde{G}$ are two normal subgroups of G_1 with cohomological dimension 2 then $\tilde{G}/G$ is locally finite, by Theorem 8.2 of [Bi]. Hence $\tilde{G}/G$ is finite, since $\chi(G) = [H : G]\chi(H)$ for any finitely generated subgroup H such that $G \leq H \leq \tilde{G}$. Moreover if $\tilde{G}$ is normal in J then $[J : N_J(G)] < \infty$, since $\tilde{G}$ has only finitely many subgroups of index $[\tilde{G} : G]$.

Therefore we may assume that G is maximal among such subgroups of G_1. Let n be an element of G_2 such that $nGn^{-1} \neq G$, and let $H = G.nGn^{-1}$. Then G is normal in H and H is normal in G_1, so $[H : G] = \infty$ and $c.d.H = 3$. Moreover H is FP and $H^s(H; \mathbb{Z}[H]) = 0$ for $s \leq 2$, so either G_1/H is locally finite or $c.d.G_1 > c.d.H$, by Theorem 8.2 of [Bi]. If G_1/H is locally finite but not finite then we again have $c.d.G_1 > c.d.H$, by Theorem 3.3 of [GS81].

If $c.d.G_1 = 4$ then $[\pi : N_\pi(G)] \leq [\pi : G_1] < \infty$. An LHSSS argument gives $H^2(N_\pi(G)/G; \mathbb{Z}[N_\pi(G)/G]) \cong Z$. Hence $N_\pi(G)/G$ is virtually a PD_2-group, by [Bo99]. Therefore π has a normal subgroup $K \leq N_\pi(G)$ such that $[\pi : K] < \infty$ and K/G is a PD_2-group of orientable type. Then $\chi(G)\chi(K/G) = [\pi : K]\chi(\pi) = 0$ and so $\chi(K/G) = 0$, since $\chi(G) < 0$. Thus $K/G \cong Z^2$, and there are clearly many possibilities for J.

If $c.d.G_1 = 3$ then G_1/H is locally finite, and hence is finite, by Theorem 3.3 of [GS81]. Therefore G_1 is FP and $H^s(G_1; \mathbb{Z}[G_1]) = 0$ for $s \leq 2$. Let $k = \min\{i \mid [G_{i+1} : G_1] = \infty\}$. Then $H^s(G_k; W) = 0$ for $s \leq 3$ and any free $\mathbb{Z}[G_k]$-module W. Hence $c.d.G_k = 4$ and so $[\pi : G_k] < \infty$, by Strebel's Theorem. An LHS spectral sequence corner argument then shows that G_k/G_{k-1} has 2 ends and $H^3(G_{k-1}); \mathbb{Z}[G_{k-1}]) \cong Z$. Thus G_{k-1} is a PD_3-group, and therefore so is G_1. By a similar argument, G_1/G has two ends also. The theorem follows easily. □

Corollary 5.8.1 *If $\zeta G = 1$ and G is normal in π then M has a finite covering space which is homotopy equivalent to the total space of a surface bundle over T.*

Proof Since G is normal in π and M is aspherical M has a finite covering which is homotopy equivalent to a $K(G,1)$-bundle over an aspherical orientable surface, as in Theorem 5.7. Since $\chi(M)=0$ the base must be T. □

If π/G is virtually Z^2 then it has a subgroup of index at most 6 which maps onto Z^2 or $Z\times_{-1}Z$.

Let G be a PD_2-group such that $\zeta G=1$. Let θ be an automorphism of G whose class in $Out(G)$ has infinite order and let $\lambda: G\to Z$ be an epimorphism. Let $\pi=(G\times Z)\times_\phi Z$ where $\phi(g,n)=(\theta(g),\lambda(g)+n)$ for all $g\in G$ and $n\in Z$. Then G is subnormal in π but this group is not virtually the group of a surface bundle over a surface.

If π has a subnormal subgroup G which is a PD_2-group with $\zeta G\neq 1$ then $\sqrt{G}\cong Z^2$ is subnormal in π and hence contained in $\sqrt{\pi}$. In this case $h(\sqrt{\pi})\geq 2$ and so either Theorem 8.1 or Theorem 9.2 applies, to show that M has a finite covering space which is homotopy equivalent to the total space of a T-bundle over an aspherical closed surface.

5.3 Bundles with aspherical base and fibre S^2 or RP^2

Let $E^+(S^2)$ denote the connected component of id_{S^2} in $E(S^2)$, i.e., the submonoid of degree 1 maps. The connected component of id_{S^2} in $E_0(S^2)$ may be identified with the double loop space Ω^2S^2.

Lemma 5.9 *Let X be a finite 2-complex. Then there are natural bijections $[X;BO(3)]\cong[X;BE(S^2)]\cong H^1(X;\mathbb{F}_2)\times H^2(X;\mathbb{F}_2)$.*

Proof As a self homotopy equivalence of a sphere is homotopic to the identity if and only if it has degree $+1$ the inclusion of $O(3)$ into $E(S^2)$ is bijective on components. Evaluation of a self map of S^2 at the basepoint determines fibrations of $SO(3)$ and $E^+(S^2)$ over S^2, with fibre $SO(2)$ and Ω^2S^2, respectively, and the map of fibres induces an isomorphism on π_1. On comparing the exact sequences of homotopy for these fibrations we see that the inclusion of $SO(3)$ in $E^+(S^2)$ also induces an isomorphism on π_1. Since the Stiefel-Whitney classes are defined for any spherical fibration and w_1 and w_2 are nontrivial on suitable S^2-bundles over S^1 and S^2, respectively, the inclusion of $BO(3)$ into $BE(S^2)$ and the map $(w_1,w_2): BE(S^2)\to K(Z/2Z,1)\times K(Z/2Z,2)$ induces isomorphisms on π_i for $i\leq 2$. The lemma follows easily. □

Thus there is a natural 1-1 correspondance between S^2-bundles and spherical fibrations over such complexes, and any such bundle ξ is determined up to isomorphism over X by its total Stiefel-Whitney class $w(\xi) = 1+w_1(\xi)+w_2(\xi)$. (From another point of view: if $w_1(\xi) = w_1(\xi')$ there is an isomorphism of the restrictions of ξ and ξ' over the 1-skeleton $X^{[1]}$. The difference $w_2(\xi) - w_2(\xi')$ is the obstruction to extending any such isomorphism over the 2-skeleton.)

Theorem 5.10 *Let M be a PD_4-complex and B an aspherical closed surface. Then the following conditions are equivalent:*

1. $\pi_1(M) \cong \pi_1(B)$ *and* $\chi(M) = 2\chi(B)$;
2. $\pi_1(M) \cong \pi_1(B)$ *and* $\widetilde{M} \simeq S^2$;
3. M *is homotopy equivalent to the total space of an S^2-bundle over B.*

Proof If (1) holds then $H_3(\widetilde{M};\mathbb{Z}) = H_4(\widetilde{M};\mathbb{Z}) = 0$, as $\pi_1(M)$ has one end, and $\pi_2(M) \cong \overline{H^2(\pi;\mathbb{Z}[\pi])} \cong Z$, by Theorem 3.12. Hence $\widetilde{M}$ is homotopy equivalent to S^2. If (2) holds we may assume that there is a Hurewicz fibration $h : M \to B$ which induces an isomorphism of fundamental groups. As the homotopy fibre of h is $\widetilde{M}$, Lemma 5.9 implies that h is fibre homotopy equivalent to the projection of an S^2-bundle over B. Clearly (3) implies the other conditions. □

We shall summarize some of the key properties of the Stiefel-Whitney classes of such bundles in the following lemma.

Lemma 5.11 *Let ξ be an S^2-bundle over a closed surface B, with total space M and projection $p : M \to B$. Then*

1. *ξ is trivial if and only if* $w(M) = p^*w(B)$;
2. $\pi_1(M) \cong \pi_1(B)$ *acts on* $\pi_2(M)$ *by multiplication by* $w_1(\xi)$;
3. *the intersection form on $H_2(M;\mathbb{F}_2)$ is even if and only if* $w_2(\xi) = 0$;
4. *if $q : B' \to B$ is a 2-fold covering map with connected domain B' then* $w_2(q^*\xi) = 0$.

Proof (1) Applying the Whitney sum formula and naturality to the tangent bundle of the B^3-bundle associated to ξ gives $w(M) = p^*w(B) \cup p^*w(\xi)$. Since p is a 2-connected map the induced homomorphism p^* is injective in degrees ≤ 2 and so $w(M) = p^*w(B)$ if and only if $w(\xi) = 1$. By Lemma 5.9 this is so if and only if ξ is trivial, since B is 2-dimensional.

(2) It is sufficient to consider the restriction of ξ over loops in B, where the result is clear.

(3) By Poincaré duality, the intersection form is even if and only if the Wu class $v_2(M) = w_2(M) + w_1(M)^2$ is 0. Now

$$\begin{aligned} v_2(M) &= p^*(w_1(B) + w_1(\xi))^2 + p^*(w_2(B) + w_1(B) \cup w_1(\xi) + w_2(\xi)) \\ &= p^*(w_2(B) + w_1(B) \cup w_1(\xi) + w_2(\xi) + w_1(B)^2 + w_1(\xi)^2) \\ &= p^*(w_2(\xi)), \end{aligned}$$

since $w_1(B) \cup \eta = \eta^2$ and $w_1(B)^2 = w_2(B)$, by the Wu relations for B. Hence $v_2(M) = 0$ if and only if $w_2(\xi) = 0$, as p^* is injective in degree 2.

(4) We have $q_*(w_2(q^*\xi) \cap [B']) = q_*((q^*w_2(\xi)) \cap [B']) = w_2(\xi) \cap q_*[B']$, by the projection formula. Since q has degree 2 this is 0, and since q_* is an isomorphism in degree 0 we find $w_2(q^*\xi) \cap [B'] = 0$. Therefore $w_2(q^*\xi) = 0$, by Poincaré duality for B'. □

Melvin has determined criteria for the total spaces of S^2-bundles over a compact surface to be diffeomorphic, in terms of their Stiefel-Whitney classes. We shall give an alternative argument for the cases with aspherical base.

Lemma 5.12 *Let B be a closed surface and w be the Poincaré dual of $w_1(B)$. If u_1 and u_2 are elements of $H_1(B;\mathbb{F}_2) - \{0, w\}$ such that $u_1.u_1 = u_2.u_2$ then there is a homeomorphism $f : B \to B$ which is a composite of Dehn twists about two-sided essential simple closed curves and such that $f_*(u_1) = u_2$.*

Proof For simplicity of notation, we shall use the same symbol for a simple closed curve u on B and its homology class in $H_1(B;\mathbb{F}_2)$. The curve u is two-sided if and only if $u.u = 0$. In that case we shall let c_u denote the automorphism of $H_1(B;\mathbb{F}_2)$ induced by a Dehn twist about u. Note also that $u.u = u.w$ and $c_v(u) = u + (u.v)v$ for all u and two-sided v in $H_1(B;\mathbb{F}_2)$.

If B is orientable it is well known that the group of isometries of the intersection form acts transitively on $H_1(B;\mathbb{F}_2)$, and is generated by the automorphisms c_u. Thus the claim is true in this case.

If $w_1(B)^2 \neq 0$ then $B \cong RP^2 \sharp T_g$, where T_g is orientable. If $u_1.u_1 = u_2.u_2 = 0$ then u_1 and u_2 are represented by simple closed curves in T_g, and so are related by a homeomorphism which is the identity on the RP^2 summand. If $u_1.u_1 = u_2.u_2 = 1$ let $v_i = u_i + w$. Then $v_i.v_i = 0$ and this case follows from the earlier one.

Suppose finally that $w_1(B) \neq 0$ but $w_1(B)^2 = 0$; equivalently, that $B \cong Kb\sharp T_g$, where T_g is orientable. Let $\{w, z\}$ be a basis for the homology of the Kb summand. In this case w is represented by a 2-sided curve. If $u_1.u_1 = u_2.u_2 = 0$ and $u_1.z = u_2.z = 0$ then u_1 and u_2 are represented by simple closed curves in T_g, and so are related by a homeomorphism which is the identity on the Kb summand. The claim then follows if $u.z = 1$ for $u = u_1$ or u_2, since we then have $c_w(u).c_w(u) = c_w(u).z = 0$. If $u.u \neq 0$ and $u.z = 0$ then $(u+z).(u+z) = 0$ and $c_{u+z}(u) = z$. If $u.u \neq 0$, $u.z \neq 0$ and $u \neq z$ then $c_{u+z+w}c_w(u) = z$. Thus if $u_1.u_1 = u_2.u_2 = 1$ both u_1 and u_2 are related to z. Thus in all cases the claim is true. □

Theorem 5.13 (Melvin) *Let ξ and ξ' be two S^2-bundles over an aspherical closed surface B. Then the following conditions are equivalent:*

(1) *there is a diffeomorphism $f : B \to B$ such that $\xi = f^*\xi'$;*

(2) *the total spaces $E(\xi)$ and $E(\xi')$ are diffeomorphic; and*

(3) *$w_1(\xi) = w_1(\xi')$ if $w_1(\xi) = 0$ or $w_1(B)$, $w_1(\xi) \cup w_1(B) = w_1(\xi') \cup w_1(B)$ and $w_2(\xi) = w_2(\xi')$.*

Proof Clearly (1) implies (2). A diffeomorphism $h : E \to E'$ induces an isomorphism on fundamental groups; hence there is a diffeomorphism $f : B \to B$ such that fp is homotopic to $p'h$. Now $h^*w(E') = w(E)$ and $f^*w(B) = w(B)$. Hence $p^*f^*w(\xi') = p^*w(\xi)$ and so $w(f^*\xi') = f^*w(\xi') = w(\xi)$. Thus $f^*\xi' = \xi$, by Theorem 5.10, and so (2) implies (1).

If (1) holds then $f^*w(\xi') = w(\xi)$. Since $w_1(B) = v_1(B)$ is the characteristic element for the cup product pairing from $H^1(B; \mathbb{F}_2)$ to $H^2(B; \mathbb{F}_2)$ and $H^2(f; \mathbb{F}_2)$ is the identity $f^*w_1(B) = w_1(B)$, $w_1(\xi) \cup w_1(B) = w_1(\xi') \cup w_1(B)$ and $w_2(\xi) = w_2(\xi')$. Hence(1) implies (3).

If $w_1(\xi) \cup w_1(B) = w_1(\xi') \cup w_1(B)$ and $w_1(\xi)$ and $w_1(\xi')$ are neither 0 nor $w_1(B)$ then there is a diffeomorphism $f : B \to B$ such that $f^*w_1(\xi') = w_1(\xi)$, by Lemma 5.12 (applied to the Poincaré dual homology classes). Hence (3) implies (1). □

Corollary 5.13.1 *There are 4 diffeomorphism classes of S^2-bundle spaces if B is orientable and $\chi(B) \leq 0$, 6 if $B = Kb$ and 8 if B is nonorientable and $\chi(B) < 0$.* □

See [Me84] for a more geometric argument, which applies also to S^2-bundles over surfaces with nonempty boundary. The theorem holds also when $B = S^2$ or RP^2; there are 2 such bundles over S^2 and 4 over RP^2. (See Chapter 12.)

Theorem 5.14 *Let M be a PD_4-complex with fundamental group π. The following are equivalent:*

(1) *M has a covering space of degree ≤ 2 which is homotopy equivalent to the total space of an S^2-bundle over an aspherical closed surface;*

(2) *the universal covering space $\widetilde{M}$ is homotopy equivalent to S^2;*

(3) *$\pi \neq 1$ and $\pi_2(M) \cong Z$.*

If these conditions hold the kernel K of the natural action of π on $\pi_2(M)$ is a PD_2-group.

Proof Clearly (1) implies (2) and (2) implies (3). Suppose that (3) holds. If π is finite and $\pi_2(M) \cong Z$ then $\widetilde{M} \simeq CP^2$, and so admits no nontrivial free group actions, by the Lefshetz fixed point theorem. Hence π must be infinite. Then $H_0(\widetilde{M};\mathbb{Z}) = Z$, $H_1(\widetilde{M};\mathbb{Z}) = 0$ and $H_2(\widetilde{M};\mathbb{Z}) = \pi_2(M)$, while $H_3(\widetilde{M};\mathbb{Z}) \cong \overline{H^1(\pi;\mathbb{Z}[\pi])}$ and $H_4(\widetilde{M};\mathbb{Z}) = 0$. Now $Hom_{\mathbb{Z}[\pi]}(\pi_2(M),\mathbb{Z}[\pi]) = 0$, since π is infinite and $\pi_2(M) \cong Z$. Therefore $H^2(\pi;\mathbb{Z}[\pi])$ is infinite cyclic, by Lemma 3.3, and so π is virtually a PD_2-group, by Bowditch's Theorem. Hence $H_3(\widetilde{M};\mathbb{Z}) = 0$ and so $\widetilde{M} \simeq S^2$. If C is a finite cyclic subgroup of K then $H_{n+3}(C;\mathbb{Z}) \cong H_n(C;H_2(\widetilde{M};\mathbb{Z}))$ for all $n \geq 2$, by Lemma 2.10. Therefore C must be trivial, so K is torsion free. Hence K is a PD_2-group and (1) now follows from Theorem 5.10. □

A straightfoward Mayer-Vietoris argument may be used to show directly that if $H^2(\pi;\mathbb{Z}[\pi]) \cong Z$ then π has one end.

Lemma 5.15 *Let X be a finite 2-complex. Then there are natural bijections $[X;BSO(3)] \cong [X;BE(RP^2)] \cong H^2(X;\mathbb{F}_2)$.*

Proof Let $(1,0,0)$ and $[1:0:0]$ be the base points for S^2 and RP^2 respectively. A based self homotopy equivalence f of RP^2 lifts to a based self homotopy equivalence F^+ of S^2. If f is based homotopic to the identity then $deg(f^+) = 1$. Conversely, any based self homotopy equivalence is based homotopic to a map which is the identity on RP^1; if moreover $deg(f^+) = 1$ then this map is the identity on the normal bundle and it quickly follows that f is based homotopic to the identity. Thus $E_0(RP^2)$ has two components. The homeomorphism g defined by $g([x:y:z]) = [x:y:-z]$ is isotopic to the identity (rotate in the (x,y)-coordinates). However $deg(g^+) = -1$. It follows that $E(RP^2)$ is connected. As every self homotopy equivalence of RP^2 is covered by a degree 1 self map of S^2, there is a natural map from $E(RP^2)$ to $E^+(S^2)$.

We may use obstruction theory to show that $\pi_1(E_0(RP^2))$ has order 2. Hence $\pi_1(E(RP^2))$ has order at most 4. Suppose that there were a homotopy f_t through self maps of RP^2 with $f_0 = f_1 = id_{RP^2}$ and such that the loop $f_t(*)$ is essential, where $*$ is a basepoint. Let F be the map from $RP^2 \times S^1$ to RP^2 determined by $F(p,t) = f_t(p)$, and let α and β be the generators of $H^1(RP^2;\mathbb{F}_2)$ and $H^1(S^1;\mathbb{F}_2)$, respectively. Then $F^*\alpha = \alpha\otimes 1 + 1\otimes\beta$ and so $(F^*\alpha)^3 = \alpha^2\otimes\beta$ which is nonzero, contradicting $\alpha^3 = 0$. Thus there can be no such homotopy, and so the homomorphism from $\pi_1(E(RP^2))$ to $\pi_1(RP^2)$ induced by the evaluation map must be trivial. It then follows from the exact sequence of homotopy for this evaluation map that the order of $\pi_1(E(RP^2))$ is at most 2. The group $SO(3) \cong O(3)/(\pm I)$ acts isometrically on RP^2. As the composite of the maps on π_1 induced by the inclusions $SO(3) \subset E(RP^2) \subset E^+(S^2)$ is an isomorphism of groups of order 2 the first map also induces an isomorphism. It follows as in Lemma 5.9 that there are natural bijections $[X;BSO(3)] \cong [X;BE(RP^2)] \cong H^2(X;\mathbb{F}_2)$. □

Thus there is a natural 1-1 correspondance between RP^2-bundles and orientable spherical fibrations over such complexes. The RP^2-bundle corresponding to an orientable S^2-bundle is the quotient by the fibrewise antipodal involution. In particular, there are two RP^2-bundles over each closed aspherical surface.

Theorem 5.16 *Let M be a PD_4-complex and B an aspherical closed surface. Then M is homotopy equivalent to the total space of an RP^2-bundle over B if and only if $\pi_1(M) \cong \pi_1(B)\times(Z/2Z)$ and $\chi(M) = \chi(B)$.*

Proof If E is the total space of an RP^2-bundle over B, with projection p, then $\chi(E) = \chi(B)$ and the long exact sequence of homotopy gives a short exact sequence $1 \to Z/2Z \to \pi_1(E) \to \pi_1(B) \to 1$. Since the fibre has a product neighbourhood, $j^*w_1(E) = w_1(RP^2)$, where $j : RP^2 \to E$ is the inclusion of the fibre over the basepoint of B, and so $w_1(E)$ considered as a homomorphism from $\pi_1(E)$ to $Z/2Z$ splits the injection j_*. Therefore $\pi_1(E) \cong \pi_1(B)\times(Z/2Z)$ and so the conditions are necessary, as they are clearly invariant under homotopy.

Suppose that they hold, and let $w : \pi_1(M) \to Z/2Z$ be the projection onto the $Z/2Z$ factor. Then the covering space associated with the kernel of w satisfies the hypotheses of Theorem 5.10 and so $\widetilde{M} \simeq S^2$. Therefore the homotopy fibre of the map h from M to B inducing the projection of $\pi_1(M)$ onto $\pi_1(B)$ is homotopy equivalent to RP^2. The map h is fibre homotopy equivalent to the projection of an RP^2-bundle over B, by Lemma 5.15. □

We may use the above results to refine some of the conclusions of Theorem 3.9 on PD_4-complexes with finitely dominated covering spaces.

Theorem 5.17 *Let M be a PD_4-complex and $p : \widehat{M} \to M$ a regular covering map, with covering group $G = Aut(p)$. If the covering space $\widehat{M}$ is finitely dominated and $H^2(G;\mathbb{Z}[G]) \cong Z$ then M has a finite covering space which is homotopy equivalent to a closed 4-manifold which fibres over an aspherical closed surface.*

Proof By Bowditch's Theorem G is virtually a PD_2-group. Therefore as $\widehat{M}$ is finitely dominated it is homotopy equivalent to a closed surface, by [Go79]. The result then follows as in Theorems 5.2, 5.10 and 5.16. □

Note that by Theorem 3.11 and the remarks in the paragraph preceding it the total spaces of such bundles with base an aspherical surface have minimal Euler characteristic for their fundamental groups (i.e. $\chi(M) = q(\pi)$).

Can the hypothesis that $\widehat{M}$ be finitely dominated be replaced by the more algebraic hypothesis that the chain complex of the universal cover $C_*(\tilde{M})$ be chain homotopy equivalent over $\mathbb{Z}[\pi_1(\widehat{M})]$ to a complex of free $\mathbb{Z}[\pi_1(\widehat{M})]$-modules which is finitely generated in degrees ≤ 2? One might hope to adapt the strategy of Theorem 4.5, by using cup-product with a generator of $H^2(G;\mathbb{Z}[G]) \cong Z$ to relate the equivariant cohomology of $\widehat{M}$ to that of M. (See also [Ba80'].)

Theorem 5.18 *A PD_4-complex M is homotopy equivalent to the total space of a surface bundle over T or Kb if and only if $\pi = \pi_1(M)$ is an extension of Z^2 or $Z \times_{-1} Z$ (respectively) by an FP_2 normal subgroup K and $\chi(M) = 0$.*

Proof The conditions are clearly necessary. If they hold then the covering space associated to the subgroup K is homotopy equivalent to a closed surface, by Corollary 4.5.3 together with Corollary 2.12.1, and so the theorem follows from Theorems 5.2, 5.10 and 5.16. □

In particular, if π is the nontrivial extension of Z^2 by $Z/2Z$ then $q(\pi) > 0$.

5.4 Bundles over S^2

Since S^2 is the union of two discs along a circle, an F-bundle over S^2 is determined by the homotopy class of the clutching function, which is an element of $\pi_1(Diff(F))$.

Theorem 5.19 *Let M be a PD_4-complex with fundamental group π and F a closed surface. Then M is homotopy equivalent to the total space of an F-bundle over S^2 if and only if $\chi(M) = 2\chi(F)$ and*

1. *(when $\chi(F) < 0$ and $w_1(F) = 0$) $\pi \cong \pi_1(F)$ and $w_1(M) = w_2(M) = 0$; or*
2. *(when $\chi(F) < 0$ and $w_1(F) \neq 0$) $\pi \cong \pi_1(F)$, $w_1(M) \neq 0$ and $w_2(M) = w_1(M)^2 = (c_M^* w_1(F))^2$; or*
3. *(when $F = T$) $\pi \cong Z^2$ and $w_1(M) = w_2(M) = 0$, or $\pi \cong Z \oplus (Z/nZ)$ for some $n > 0$ and, if $n = 1$ or 2, $w_1(M) = 0$; or*
4. *(when $F = Kb$) $\pi \cong Z \times_{-1} Z$, $w_1(M) \neq 0$ and $w_2(M) = w_1(M)^2 = 0$, or π has a presentation $\langle x, y \mid yxy^{-1} = x^{-1},\ y^{2n} = 1\rangle$ for some $n > 0$, where $w_1(M)(x) = 0$ and $w_1(M)(y) = 1$, and there is a map $p : M \to S^2$ which induces an epimorphism on π_3; or*
5. *(when $F = S^2$) $\pi = 1$ and the index $\sigma(M) = 0$; or*
6. *(when $F = RP^2$) $\pi = Z/2Z$, $w_1(M) \neq 0$ and there is a class u of infinite order in $H^2(M;\mathbb{Z})$ and such that $u^2 = 0$.*

Proof Let $p_E : E \to S^2$ be such a bundle. Then $\chi(E) = 2\chi(F)$ and $\pi_1(E) \cong \pi_1(F)/\partial\pi_2(S^2)$, where $\mathrm{Im}(\partial) \leq \zeta\pi_1(F)$ [Go68]. The characteristic classes of E restrict to the characteristic classes of the fibre, as it has a product neighbourhood. As the base is 1-connected E is orientable if and only if the fibre is orientable. Thus the conditions on χ, π and w_1 are all necessary. We shall treat the other assertions case by case.

(1) and (2) If $\chi(F) < 0$ any F-bundle over S^2 is trivial, by Lemma 5.1. Thus the conditions are necessary. Conversely, if they hold then c_M is fibre homotopy equivalent to the projection of an S^2-bundle ξ with base F, by Theorem 5.10. The conditions on the Stiefel-Whitney classes then imply that $w(\xi) = 1$ and hence that the bundle is trivial, by Lemma 5.11. Therefore M is homotopy equivalent to $S^2 \times F$.

(3) If $\partial = 0$ there is a map $q : E \to T$ which induces an isomorphism of fundamental groups, and the map $(p_E, q) : E \to S^2 \times T$ is clearly a homotopy equivalence, so $w(E) = 1$. Conversely, if $\chi(M) = 0$, $\pi \cong Z^2$ and $w(M) = 1$ then M is homotopy equivalent to $S^2 \times T$, by Theorem 5.10 and Lemma 5.11.

If $\chi(M) = 0$ and $\pi \cong Z \oplus (Z/nZ)$ for some $n > 0$ then the covering space $M_{Z/nZ}$ corresponding to the torsion subgroup Z/nZ is homotopy equivalent to a lens space L, by Corollary 4.5.3. As observed in Chapter 4 the manifold

M is homotopy equivalent to the mapping torus of a generator of the group of covering transformations $Aut(M_{Z/nZ}/M) \cong Z$. Since the generator induces the identity on $\pi_1(L) \cong Z/nZ$ it is homotopic to id_L, if $n > 2$. This is also true if $n = 1$ or 2 and M is orientable. (See Section 29 of [Co].) Therefore M is homotopy equivalent to $L \times S^1$, which fibres over S^2 via the composition of the projection to L with the Hopf fibration of L over S^2. (Hence $w(M) = 1$ in these cases also.)

(4) As in part (3), if $\pi_1(E) \cong Z \rtimes_{-1} Z = \pi_1(Kb)$ then E is homotopy equivalent to $S^2 \times Kb$ and so $w_1(E) \neq 0$ while $w_2(E) = 0$. Conversely, if $\chi(M) = 0$, $\pi \cong \pi_1(Kb)$, M is nonorientable and $w_1(M)^2 = w_2(M) = 0$ then M is homotopy equivalent to $S^2 \times Kb$. Suppose now that $\partial \neq 0$. The homomorphism $\pi_3(p_E)$ induced by the bundle projection is an epimorphism. Conversely, if M satisfies these conditions and $q : M^+ \to M$ is the orientation double cover then M^+ satisfies the hypotheses of part (3), and so $\widetilde{M} \simeq S^3$. Therefore as $\pi_3(p)$ is onto the composition of the projection of $\widetilde{M}$ onto M with p is essentially the Hopf map, and so induces isomorphisms on all higher homotopy groups. Hence the homotopy fibre of p is aspherical. As $\pi_2(M) = 0$ the fundamental group of the homotopy fibre of p is a torsion free extension of π by Z, and so the homotopy fibre must be Kb. As in Theorem 5.2 above the map p is fibre homotopy equivalent to a bundle projection.

(5) There are just two S^2-bundles over S^2, with total spaces $S^2 \times S^2$ and $S^2 \tilde{\times} S^2 = CP^2 \sharp -CP^2$, respectively. Thus the conditions are necessary. If M satisfies these conditions then $H^2(M;\mathbb{Z}) \cong Z^2$ and there is an element u in $H^2(M;\mathbb{Z})$ which generates an infinite cyclic direct summand and has square $u \cup u = 0$. Thus $u = f^* i_2$ for some map $f : M \to S^2$, where i_2 generates $H^2(S^2;\mathbb{Z})$, by Theorem 8.4.11 of [Sp]. Since u generates a direct summand there is a homology class z in $H_2(M;\mathbb{Z})$ such that $u \cap z = 1$, and therefore (by the Hurewicz theorem) there is a map $z : S^2 \to M$ such that fz is homotopic to id_{S^2}. The homotopy fibre of f is 1-connected and has $\pi_2 \cong Z$, by the long exact sequence of homotopy. It then follows easily from the spectral sequence for f that the homotopy fibre has the homology of S^2. Therefore f is fibre homotopy equivalent to the projection of an S^2-bundle over S^2.

(6) Since $\pi_1(Diff(RP^2)) = Z/2Z$ (see page 21 of [EE69]) there are two RP^2-bundles over S^2. Again the conditions are clearly necessary. If they hold then $u = g^* i_2$ for some map $g : M \to S^2$. Let $q : M^+ \to M$ be the orientation double cover and $g^+ = gq$. Since $H_2(Z/2Z;\mathbb{Z}) = 0$ the second homology of M is spherical. As we may assume u generates an infinite cyclic direct summand of $H^2(M;\mathbb{Z})$ there is a map $z = qz^+ : S^2 \to M$ such that $gz = g^+ z^+$ is homotopic to id_{S^2}. Hence the homotopy fibre of g^+ is S^2, by case (5). Since

the homotopy fibre of g has fundamental group $Z/2Z$ and is double covered by the homotopy fibre of g^+ it is homotopy equivalent to RP^2. It follows as in Theorem 5.16 that g is fibre homotopy equivalent to the projection of an RP^2-bundle over S^2. □

Theorems 5.2, 5.10 and 5.16 may each be rephrased as giving criteria for maps from M to B to be fibre homotopy equivalent to fibre bundle projections. With the hypotheses of Theorem 5.19 (and assuming also that $\partial = 0$ if $\chi(M) = 0$) we may conclude that a map $f : M \to S^2$ is fibre homotopy equivalent to a fibre bundle projection if and only if f^*i_2 generates an infinite cyclic direct summand of $H^2(M;\mathbb{Z})$.

Is there a criterion for part (4) which does not refer to π_3? The other hypotheses are not sufficient alone. (See Chapter 11.)

It follows from Theorem 5.10 that the conditions on the Stiefel-Whitney classes are independent of the other conditions when $\pi \cong \pi_1(F)$. Note also that the nonorientable S^3- and RP^3-bundles over S^1 are not T-bundles over S^2, while if $M = CP^2\sharp CP^2$ then $\pi = 1$ and $\chi(M) = 4$ but $\sigma(M) \neq 0$. See Chapter 12 for further information on parts (5) and (6).

5.5 Bundles over RP^2

Since $RP^2 = Mb \cup D^2$ is the union of a Möbius band Mb and a disc D^2, a bundle $p : E \to RP^2$ with fibre F is determined by a bundle over Mb which restricts to a trivial bundle over ∂Mb, i.e. by a conjugacy class of elements of order dividing 2 in $\pi_0(Homeo(F))$, together with the class of a gluing map over $\partial Mb = \partial D^2$ modulo those which extend across D^2 or Mb, i.e. an element of a quotient of $\pi_1(Homeo(F))$. If F is aspherical $\pi_0(Homeo(F)) \cong Out(\pi_1(F))$, while $\pi_1(Homeo(F)) \cong \zeta\pi_1(F)$ [Go65].

We may summarize the key properties of the algebraic invariants of such bundles with F an aspherical closed surface in the following lemma. Let $\tilde{Z}$ be the non-trivial infinite cyclic $Z/2Z$-module. The groups $H^1(Z/2Z;\tilde{Z})$, $H^1(Z/2Z;\mathbb{F}_2)$ and $H^1(RP^2;\tilde{Z})$ are canonically isomorphic to $Z/2Z$.

Lemma 5.20 *Let $p : E \to RP^2$ be the projection of an F-bundle, where F is an aspherical closed surface, and let x be the generator of $H^1(RP^2;\tilde{Z})$. Then*

(1) $\chi(E) = \chi(F)$;

(2) *$\partial(\pi_2(RP^2)) \leq \zeta\pi_1(F)$ and there is an exact sequence of groups*

$$0 \to \pi_2(E) \to Z \xrightarrow{\partial} \pi_1(F) \to \pi_1(E) \to Z/2Z \to 1;$$

(3) *if $\partial = 0$ then $\pi_1(E)$ has one end and acts nontrivially on $\pi_2(E) \cong Z$, and the covering space E_F with fundamental group $\pi_1(F)$ is homeomorphic to $S^2 \times F$, so $w_1(E)|_{\pi_1(F)} = w_1(E_F) = w_1(F)$ (as homomorphisms from $\pi_1(F)$ to $Z/2Z$) and $w_2(E_F) = w_1(E_F)^2$;*

(4) *if $\partial \neq 0$ then $\chi(F) = 0$, $\pi_1(E)$ has two ends, $\pi_2(E) = 0$ and $Z/2Z$ acts by inversion on $\partial(Z)$;*

(5) *$p^*x^3 = 0 \in H^3(E; p^*\tilde{Z})$.*

Proof Condition (1) holds since the Euler characteristic is multiplicative in fibrations, while (2) is part of the long exact sequence of homotopy for p. The image of ∂ is central by [Go68], and is therefore trivial unless $\chi(F) = 0$. Conditions (3) and (4) then follow as the homomorphisms in this sequence are compatible with the actions of the fundamental groups, and E_F is the total space of an F-bundle over S^2, which is a trivial bundle if $\partial = 0$, by Theorem 5.19. Condition (5) holds since $H^3(RP^2; \tilde{Z}) = 0$. □

Let π be a group which is an extension of $Z/2Z$ by a normal subgroup G, and let $t \in \pi$ be an element which maps nontrivially to $\pi/G = Z/2Z$. Then $u = t^2$ is in G and conjugation by t determines an automorphism α of G such that $\alpha(u) = u$ and α^2 is the inner automorphism given by conjugation by u.

Conversely, let α be an automorphism of G whose square is inner, say $\alpha^2(g) = ugu^{-1}$ for all $g \in G$. Let $v = \alpha(u)$. Then $\alpha^3(g) = \alpha^2(\alpha(g)) = u\alpha(g)c^{-1} = \alpha(\alpha^2(g)) = v\alpha(g)v^{-1}$ for all $g \in G$. Therefore vu^{-1} is central. In particular, if the centre of G is trivial α fixes u, and we may define an extension

$$\xi_\alpha : 1 \to G \to \Pi_\alpha \to Z/2Z \to 1$$

in which Π_α has the presentation $\langle G, t_\alpha \mid t_\alpha g t_\alpha^{-1} = \alpha(g),\ t_\alpha^2 = u\rangle$. If β is another automorphism in the same outer automorphism class then ξ_α and ξ_β are equivalent extensions. (Note that if $\beta = \alpha.c_h$, where c_h is conjugation by h, then $\beta(\alpha(h)uh) = \alpha(h)uh$ and $\beta^2(g) = \alpha(h)uh.g.(\alpha(h)uh)^{-1}$ for all $g \in G$.)

Lemma 5.21 *If $\chi(F) < 0$ or $\chi(F) = 0$ and $\partial = 0$ then an F-bundle over RP^2 is determined up to isomorphism by the corresponding extension of fundamental groups.*

Proof If $\chi(F) < 0$ such bundles and extensions are each determined by an element ξ of order 2 in $Out(\pi_1(F))$. If $\chi(F) = 0$ bundles with $\partial = 0$ are the restrictions of bundles over $RP^\infty = K(Z/2Z, 1)$ (compare Lemma 4.10). Such bundles are determined by an element ξ of order 2 in $Out(\pi_1(F))$ and a cohomology class in $H^2(Z/2Z; \zeta\pi_1(F)^\xi)$, by Lemma 5.1, and so correspond bijectively to extensions also. ∎

Lemma 5.22 *Let M be a PD_4-complex with fundamental group π. A map $f : M \to RP^2$ is fibre homotopy equivalent to the projection of a bundle over RP^2 with fibre an aspherical closed surface if $\pi_1(f)$ is an epimorphism and either*

(1) *$\chi(M) \leq 0$ and $\pi_2(f)$ is an isomorphism; or*

(2) *$\chi(M) = 0$, π has two ends and $\pi_3(f)$ is an isomorphism.*

Proof In each case π is infinite, by Lemma 3.14. In case (1) $H^2(\pi; \mathbb{Z}[\pi]) \cong Z$ (by Lemma 3.3) and so π has one end, by Bowditch's Theorem. Hence $\widetilde{M} \simeq S^2$. Moreover the homotopy fibre of f is aspherical, and its fundamental group is a surface group. (See Chapter X for details.) In case (2) $\widetilde{M} \simeq S^3$, by Corollary 4.5.3. Hence the lift $\tilde{f} : \widetilde{M} \to S^2$ is homotopic to the Hopf map, and so induces isomorphisms on all higher homotopy groups. Therefore the homotopy fibre of f is aspherical. As $\pi_2(M) = 0$ the fundamental group of the homotopy fibre is a (torsion free) infinite cyclic extension of π and so must be either Z^2 or $Z \times_{-1} Z$. Thus the homotopy fibre of f is homotopy equivalent to T or Kb. In both cases the argument of Theorem 5.2 now shows that f is fibre homotopy equivalent to a surface bundle projection. ∎

5.6 Bundles over RP^2 with $\partial = 0$

If we assume that the connecting homomorphism $\partial : \pi_2(E) \to \pi_1(F)$ is trivial then conditions (2), (3) and (5) of Lemma 5.20 simplify to conditions on E and the action of $\pi_1(E)$ on $\pi_2(E)$. These conditions almost suffice to characterize the homotopy types of such bundle spaces; there is one more necesssary condition, and for nonorientable manifolds there is a further possible obstruction, of order at most 2.

Theorem 5.23 *Let M be a PD_4-complex and let $m : M_u \to M$ be the covering associated to $\kappa = \mathrm{Ker}(u)$, where $u : \pi = \pi_1(M) \to Aut(\pi_2(M))$ is the natural action. Let x be the generator of $H^1(Z/2Z; \tilde{Z})$. If M is homotopy equivalent to the total space of a fibre bundle over RP^2 with fibre an* *aspherical closed surface and with $\partial = 0$ then $\pi_2(M) \cong Z$, u is surjective, $w_2(M_u) = w_1(M_u)^2$ and u^*x^3 has image 0 in $H^3(M;\mathbb{F}_2)$. Moreover the homomorphism from $H^2(M;Z^u)$ to $H^2(S^2;Z^u)$ induced by a generator of $\pi_2(M)$ is onto. Conversely, if M is orientable these conditions imply that M is homotopy equivalent to such a bundle space. If M is nonorientable there is a further obstruction of order at most 2.*

Proof The necessity of most of these conditions follows from Lemma 5.20. The additional condition holds since the covering projection from S^2 to RP^2 induces an isomorphism $H^2(RP^2;Z^u) \cong H^2(S^2;Z^u) = H^2(S^2;\mathbb{Z})$.

Suppose that they hold. Let $g : S^2 \to P_2(RP^2)$ and $j : S^2 \to M$ represent generators for $\pi_2(P_2(RP^2))$ and $\pi_2(M)$, respectively. After replacing M by a homotopy equivalent space if necessary, we may assume that j is the inclusion of a subcomplex. We may identify u with a map from M to $K(Z/2Z,1)$, via the isomorphism $[M,K(Z/2Z,1)] \cong Hom(\pi,Z/2Z)$. The only obstruction to the construction of a map from M to $P_2(RP^2)$ which extends g and lifts u lies in $H^3(M,S^2;Z^u)$, since $u^*\pi_2(RP^2)) \cong Z^u$. This group maps injectively to $H^3(M;Z^u)$, since restriction maps $H^2(M;Z^u)$ onto $H^2(S^2;Z^u)$, and so this obstruction is 0, since its image in $H^3(M;Z^u)$ is $u^*k_1(RP^2) = u^*x^3 = 0$. Therefore there is a map $h : M \to P_2(RP^2)$ such that $\pi_1(h) = u$ and $\pi_2(h)$ is an isomorphism. The set of such maps is parametrized by $H^2(M,S^2;Z^u)$.

As $Z/2Z$ acts trivially on $\pi_3(RP^2) \cong Z$ the second k-invariant of RP^2 lies in $H^4(P_2(RP^2);Z)$. This group is infinite cyclic, and is generated by $t = k_2(RP^2)$. (See §3.12 of [Si67].) The obstruction to lifting h to a map from M to $P_3(RP^2)$ is h^*t. Let $n : \widetilde{P}_2(RP^2) \to P_2(RP^2)$ be the universal covering, and let z be a generator of $H^2(\widetilde{P}_2(RP^2);\mathbb{Z}) \cong Z$. Then h lifts to a map $h_u : M_u \to \widetilde{P}_2(RP^2)$, so that $nh_u = hm$. (Note that h_u is determined by h_u^*z, since $\widetilde{P}_2(RP^2) \simeq K(Z,2)$.)

The covering space M_u is homotopy equivalent to the total space of an S^2-bundle $q : E \to F$, where F is an aspherical closed surface, by Theorem 5.14. Since κ acts trivially on $\pi_2(M_u)$ the bundle is orientable (i.e., $w_1(q) = 0$) and so $q^*w_2(q) = w_2(E) + w_1(E)^2$, by the Whitney sum formula. Therefore $q^*w_2(q) = 0$, since $w_2(M_u) = w_1(M_u)^2$, and so $w_2(q) = 0$, since q is 2-connected. Hence the bundle is trivial, by Lemma 5.11, and so M_u is homotopy equivalent to $S^2 \times F$. Let j_F and j_S be the inclusions of the factors. Then $h_u j_S$ generates $\pi_2(P_2)$. We may choose h so that $h_u j_F$ is null homotopic. Then h_u^*z is Poincaré dual to $j_{F*}[F]$, and so $h_u^*z^2 = 0$, since $j_{F*}[F]$ has self intersection 0. As n^*t is a multiple of z^2, it follows that $m^*h^*t = 0$.

If M is orientable $m^* = H^4(m;\mathbb{Z})$ is a monomorphism and so $h^*t = 0$. Hence h lifts to a map $f : M \to P_3(RP^2)$. As $P_3(RP^2)$ may be constructed from RP^2 by adjoining cells of dimension at least 5 we may assume that f maps M into RP^2, after a homotopy if necessary. Since $\pi_1(f) = u$ is an epimorphism and $\pi_2(f)$ is an isomorphism f is fibre homotopy equivalent to the projection of an F-bundle over RP^2, by Lemma 5.22.

In general, we may assume that h maps the 3-skeleton $M^{[3]}$ to RP^2. Let w be a generator of $H^2(P_2(RP^2);\tilde{Z}) \cong H^2(RP^2;\tilde{Z}) \cong Z$ and define a function $\mu : H^2(M;Z^u) \to H^4(M;\mathbb{Z})$ by $\mu(g) = g \cup g + g \cup h^*w$ for all $g \in H^2(M;Z^u)$. If M is nonorientable $H^4(M;\mathbb{Z}) = Z/2Z$ and μ is a homomorphism. The sole obstruction to extending $h|_{M^{[3]}}$ to a map $f : M \to RP^2$ is the image of h^*t in $\mathrm{Coker}(\mu)$, which is independent of the choice of lift h. (See §3.24 of [Si67].) $\square$

Are these hypotheses independent? A closed 4-manifold M with $\pi = \pi_1(M)$ a PD_2-group and $\pi_2(M) \cong Z$ is homotopy equivalent to the total space of an S^2-bundle $p : E \to B$, where B is an aspherical closed surface. Therefore if u is nontrivial $M_u \simeq E^+$, where $q : E^+ \to B^+$ is the bundle induced over a double cover of B. As $w_1(q) = 0$ and $q^*w_2(q) = 0$, by part (3) of Lemma 5.11, we have $w_1(E^+) = q^*w_1(B^+)$ and $w_2(E^+) = q^*w_2(B^+)$, by the Whitney sum formula. Hence $w_2(M_u) = w_1(M_u)^2$. (In particular, $w_2(M_u) = 0$ if M is orientable.) Moreover since $c.d.\pi = 2$ the condition $u^*x^3 = 0$ is automatic. (It shall follow directly from the results of Chapter 10 that any such S^2-bundle space with u nontrivial fibres over RP^2, even if it is not orientable.)

On the other hand, if $Z/2Z$ is a (semi)direct factor of π the cohomology of $Z/2Z$ is a direct summand of that of π and so the image of x^3 in $H^3(\pi;\tilde{Z})$ is nonzero.

Is the obstruction always 0 in the nonorientable cases?

Chapter 6

Simple homotopy type and surgery

The problem of determining the high-dimensional manifolds within a given homotopy type has been successfully reduced to the determination of normal invariants and surgery obstructions. This strategy applies also in dimension 4, provided that the fundamental group is in the class SA generated from groups with subexponential growth by extensions and increasing unions [FT95]. (Essentially all the groups in this class that we shall discuss in this book are in fact virtually solvable). We may often avoid this hypothesis by using 5-dimensional surgery to construct s-cobordisms.

We begin by showing that the Whitehead group of the fundamental group is trivial for surface bundles over surfaces, most circle bundles over geometric 3-manifolds and for many mapping tori. In §2 we define the modified surgery structure set, parametrizing s-cobordism classes of simply homotopy equivalences of closed 4-manifolds. This notion allows partial extensions of surgery arguments to situations where the fundamental group is not elementary amenable. Although many papers on surgery do not explicitly consider the 4-dimensional cases, their results may often be adapted to these cases. In §3 we comment briefly on approaches to the s-cobordism theorem and classification using stabilization by connected sum with copies of $S^2 \times S^2$ or by cartesian product with R.

In §4 we show that 4-manifolds M such that $\pi = \pi_1(M)$ is torsion free virtually poly-Z and $\chi(M) = 0$ are determined up to homeomorphism by their fundamental group (and Stiefel-Whitney classes, if $h(\pi) < 4$). We also characterize 4-dimensional mapping tori with torsion free, elementary amenable fundamental group and show that the structure sets for total spaces of RP^2-bundles over T or Kb are finite. In §5 we extend this finiteness to RP^2-bundle spaces over closed hyperbolic surfaces and show that total spaces of bundles with fibre S^2 or an aspherical closed surface over aspherical bases are determined up to s-cobordism by their homotopy type. (We shall consider bundles with base or fibre geometric 3-manifolds in Chapter 13).

6.1 The Whitehead group

In this section we shall rely heavily upon the work of Waldhausen in [Wd78]. The class of groups Cl is the smallest class of groups containing the trivial group and which is closed under generalised free products and HNN extensions with amalgamation over regular coherent subgroups and under filtering direct limit. This class is also closed under taking subgroups, by Proposition 19.3 of [Wd78]. If G is in Cl then $Wh(G) = 0$, by Theorem 19.4 of [Wd78]. The argument for this theorem actually shows that if $G \cong A *_C B$ and C is regular coherent then there are "Mayer-Vietoris" sequences:

$$Wh(A) \oplus Wh(B) \to Wh(G) \to \tilde{K}(\mathbb{Z}[C]) \to \tilde{K}(\mathbb{Z}[A]) \oplus \tilde{K}(\mathbb{Z}[B]) \to \tilde{K}(\mathbb{Z}[G]) \to 0,$$

and similarly if $G \cong A*_C$. (See Sections 17.1.3 and 17.2.3 of [Wd78]).

The class Cl contains all free groups and poly-Z groups and the class $\mathcal{X}$ of Chapter 2. (In particular, all the groups $Z*_m$ are in Cl). Since every PD_2-group is either poly-Z or is the generalised free product of two free groups with amalgamation over infinite cyclic subgroups it is regular coherent, and is in Cl. Hence homotopy equivalences between S^2-bundles over aspherical surfaces are simple. The following extension implies the corresponding result for quotients of such bundle spaces by free involutions.

Theorem 6.1 *Let π be a semidirect product $\rho\tilde{\times}(Z/2Z)$ where ρ is a surface group. Then $Wh(\pi) = 0$.*

Proof Assume first that $\pi \cong \rho \times (Z/2Z)$. Let $\Gamma = \mathbb{Z}[\rho]$. There is a cartesian square expressing $\Gamma[Z/2Z] = \mathbb{Z}[\rho \times (Z/2Z)]$ as the pullback of the reduction of coefficients map from Γ to $\Gamma_2 = \Gamma/2\Gamma = \mathbb{Z}/2\mathbb{Z}[\rho]$ over itself. (The two maps from $\Gamma[Z/2Z]$ to Γ send the generator of $Z/2Z$ to $+1$ and -1, respectively). The Mayer-Vietoris sequence for algebraic K-theory traps $K_1(\Gamma[Z/2Z])$ between $K_2(\Gamma_2)$ and $K_1(\Gamma)^2$ (see Theorem 6.4 of [Mi]). Now since $c.d.\rho = 2$ the higher K-theory of $R[\rho]$ can be computed in terms of the homology of ρ with coefficients in the K-theory of R (cf. the Corollary to Theorem 5 of the introduction of [Wd78]). In particular, the map from $K_2(\Gamma)$ to $K_2(\Gamma_2)$ is onto, while $K_1(\Gamma) = K_1(\mathbb{Z}) \oplus (\rho/\rho')$ and $K_1(\Gamma_2) = \rho/\rho'$. It now follows easily that $K_1(\Gamma[Z/2Z])$ is generated by the images of $K_1(\mathbb{Z}) = \{\pm 1\}$ and $\rho \times (Z/2Z)$, and so $Wh(\rho \times (Z/2Z)) = 0$.

If $\pi = \rho\tilde{\times}(Z/2Z)$ is not such a direct product it is isomorphic to a discrete subgroup of $Isom(\mathbb{X})$ which acts properly discontinuously on X, where $\mathbb{X} = \mathbb{E}^2$ or $\mathbb{H}^2$. (See [EM82], [Zi]). The singularities of the corresponding 2-orbifold

X/π are either cone points of order 2 or reflector curves; there are no corner points and no cone points of higher order. Let $|X/\pi|$ be the surface obtained by forgetting the orbifold structure of X/π, and let m be the number of cone points. Then $\chi(|X/\pi|)-(m/2)=\chi_{orb}(X/\pi)\leq 0$, by the Riemann-Hurwitz formula [Sc83'], so either $\chi(|X/\pi|)\leq 0$ or $\chi(|X/\pi|)=1$ and $m\geq 2$ or $|X/\pi|\cong S^2$ and $m\geq 4$.

We may separate X/π along embedded circles (avoiding the singularities) into pieces which are either (i) discs with at least two cone points; (ii) annuli with one cone point; (iii) annuli with one boundary a reflector curve; or (iv) surfaces other than D^2 with nonempty boundary. In each case the inclusions of the separating circles induce monomorphisms on orbifold fundamental groups, and so π is a generalized free product with amalgamation over copies of Z of groups of the form (i) $*^m(Z/2Z)$ (with $m\geq 2$); (ii) $Z*(Z/2Z)$; (iii) $Z\oplus(Z/2Z)$; or (iv) $*^m Z$, by the Van Kampen theorem for orbifolds [Sc83]. The Mayer-Vietoris sequences for algebraic K-theory now give $Wh(\pi)=0$. □

The argument for the direct product case is based on one for showing that $Wh(Z\oplus(Z/2Z))=0$ from [Kw86].

Not all such orbifold groups arise in this way. For instance, the orbifold fundamental group of a torus with one cone point of order 2 has the presentation $\langle x,y\mid [x,y]^2=1\rangle$. Hence it has torsion free abelianization, and so cannot be a semidirect product as above.

The orbifold fundamental groups of flat 2-orbifolds are the 2-dimensional crystallographic groups. Their finite subgroups are cyclic or dihedral, of order properly dividing 24, and have trivial Whitehead group. In fact $Wh(\pi)=0$ for π any such 2-dimensional crystallographic group [Pe98]. (If π is the fundamental group of an orientable hyperbolic 2-orbifold with k cone points of orders $\{n_1,\dots n_k\}$ then $Wh(\pi)\cong\oplus_{i=1}^k Wh(Z/n_iZ)$ [LS00]).

The argument for the next result is essentially due to F.T.Farrell.

Theorem 6.2 *If π is an extension of $\pi_1(B)$ by $\pi_1(F)$ where B and F are aspherical closed surfaces then $Wh(\pi)=0$.*

Proof If $\chi(B)<0$ then B admits a complete riemannian metric of constant negative curvature -1. Moreover the only virtually poly-Z subgroups of $\pi_1(B)$ are 1 and Z. If G is the preimage in π of such a subgroup then G is either $\pi_1(F)$ or is the group of a Haken 3-manifold. It follows easily that for any $n\geq 0$ the group $G\times Z^n$ is in Cl and so $Wh(G\times Z^n)=0$. Therefore any such G

is K-flat and so the bundle is admissible, in the terminology of [FJ86]. Hence $Wh(\pi) = 0$ by the main result of that paper.

If $\chi(B) = 0$ then this argument does not work, although if moreover $\chi(F) = 0$ then π is poly-Z so $Wh(\pi) = 0$ by Theorem 2.13 of [FJ]. We shall sketch an argument of Farrell for the general case. Lemma 1.4.2 and Theorem 2.1 of [FJ93] together yield a spectral sequence (with coefficients in a simplicial cosheaf) whose E^2 term is $H_i(X/\pi_1(B); Wh'_j(p^{-1}(\pi_1(B)^x)))$ and which converges to $Wh'_{i+j}(\pi)$. Here $p : \pi \to \pi_1(B)$ is the epimorphism of the extension and X is a certain universal $\pi_1(B)$-complex which is contractible and such that all the nontrivial isotropy subgroups $\pi_1(B)^x$ are infinite cyclic and the fixed point set of each infinite cyclic subgroup is a contractible (nonempty) subcomplex. The Whitehead groups with negative indices are the lower K-theory of $\mathbb{Z}[G]$ (i.e., $Wh'_n(G) = K_n(\mathbb{Z}[G])$ for all $n \leq -1$), while $Wh'_0(G) = \tilde{K}_0(\mathbb{Z}[G])$ and $Wh'_1(G) = Wh(G)$. Note that $Wh'_{-n}(G)$ is a direct summand of $Wh(G \times Z^{n+1})$. If $i+j > 1$ then $Wh'_{i+j}(\pi)$ agrees rationally with the higher Whitehead group $Wh_{i+j}(\pi)$. Since the isotropy subgroups $\pi_1(B)^x$ are infinite cyclic or trivial $Wh(p^{-1}(\pi_1(B)^x) \times Z^n) = 0$ for all $n \geq 0$, by the argument of the above paragraph, and so $Wh'_j(p^{-1}(\pi_1(B)^x)) = 0$ if $j \leq 1$. Hence the spectral sequence gives $Wh(\pi) = 0$. □

A closed 3-manifold is a *Haken manifold* if it is irreducible and contains an incompressible 2-sided surface. Every Haken 3-manifold either has solvable fundamental group or may be decomposed along a finite family of disjoint incompressible tori and Klein bottles so that the complementary components are Seifert fibred or hyperbolic. It is an open question whether every closed irreducible orientable 3-manifold with infinite fundamental group is virtually Haken (i.e., finitely covered by a Haken manifold). (Non-orientable 3-manifolds are Haken). Every virtually Haken 3-manifold is either Haken, hyperbolic or Seifert-fibred, by [CS83] and [GMT96]. A closed irreducible 3-manifold is a *graph manifold* if either it has solvable fundamental group or it may be decomposed along a finite family of disjoint incompressible tori and Klein bottles so that the complementary components are Seifert fibred. (There are several competing definitions of graph manifold in the literature).

Theorem 6.3 *Let $\pi = \nu \times_\theta Z$ where ν is torsion free and is the fundamental group of a closed 3-manifold N which is a connected sum of graph manifolds. Then ν is regular coherent and $Wh(\pi) = 0$.*

Proof The group ν is a generalized free product with amalgamation along poly-Z subgroups (1, Z^2 or $Z \times_{-1} Z$) of polycyclic groups and fundamental groups of Seifert fibred 3-manifolds (possibly with boundary). The group rings of torsion free polycyclic groups are regular noetherian, and hence regular coherent. If G is the fundamental group of a Seifert fibred 3-manifold then it has a subgroup G_o of finite index which is a central extension of the fundamental group of a surface B (possibly with boundary) by Z. We may assume that G is not solvable and hence that $\chi(B) < 0$. If ∂B is nonempty then $G_o \cong Z \times F$ and so is an iterated generalized free product of copies of Z^2, with amalgamation along infinite cyclic subgroups. Otherwise we may split B along an essential curve and represent G_o as the generalised free product of two such groups, with amalgamation along a copy of Z^2. In both cases G_o is regular coherent, and therefore so is G, since $[G : G_o] < \infty$ and $c.d.G < \infty$.

Since ν is the generalised free product with amalgamation of regular coherent groups, with amalgamation along poly-Z subgroups, it is also regular coherent. Let N_i be an irreducible summand of N and let $\nu_i = \pi_1(N_i)$. If N_i is Haken then ν_i is in *Cl*. Otherwise N_i is a Seifert fibred 3-manifold which is not sufficiently large, and the argument of [Pl80] extends easily to show that $Wh(\nu_i \times Z^s) = 0$, for any $s \geq 0$. Since $\tilde{K}(\mathbb{Z}[\nu_i])$ is a direct summand of $Wh(\nu_i \times Z)$, it follows that in all cases $\tilde{K}(\mathbb{Z}[\nu_i]) = Wh(\nu_i) = 0$. The Mayer-Vietoris sequences for algebraic K-theory now give firstly that $Wh(\nu) = \tilde{K}(\mathbb{Z}[\nu]) = 0$ and then that $Wh(\pi) = 0$ also. □

All 3-manifold groups are coherent as *groups* [Hm]. If we knew that their group *rings* were regular coherent then we could use [Wd78] instead of [FJ86] to give a purely algebraic proof of Theorem 6.2, for as surface groups are free products of free groups with amalgamation over an infinite cyclic subgroup, an extension of one surface group by another is a free product of groups with $Wh = 0$, amalgamated over the group of a surface bundle over S^1. Similarly, we could deduce from [Wd78] that $Wh(\nu \times_\theta Z) = 0$ for any torsion free group $\nu = \pi_1(N)$ where N is a closed 3-manifold whose irreducible factors are Haken, hyperbolic or Seifert fibred.

Theorem 6.4 *Let μ be a group with an infinite cyclic normal subgroup A such that $\nu = \mu/A$ is torsion free and is a free product $\nu = *_{1 \leq i \leq n} \nu_i$ where each factor is the fundamental group of an irreducible 3-manifold which is Haken, hyperbolic or Seifert fibred. Then $Wh(\mu) = Wh(\nu) = 0$.*

Proof (Note that our hypotheses allow the possibility that some of the factors ν_i are infinite cyclic). Let μ_i be the preimage of ν_i in μ, for $1 \leq i \leq n$. Then μ is the generalized free product of the μ_i's, amalgamated over infinite cyclic subgroups. For all $1 \leq i \leq n$ we have $Wh(\mu_i) = 0$, by Lemma 1.1 of [St84] if $K(\nu_i, 1)$ is Haken, by the main result of [FJ86] if it is hyperbolic, by an easy extension of the argument of [Pl80] if it is Seifert fibred but not Haken and by Theorem 19.5 of [Wd78] if ν_i is infinite cyclic. The Mayer-Vietoris sequences for algebraic K-theory now give $Wh(\mu) = Wh(\nu) = 0$ also. □

Theorem 6.4 may be used to strengthen Theorem 4.11 to give criteria for a closed 4-manifold M to be *simple* homotopy equivalent to the total space of an S^1-bundle, if the irreducible summands of the base N are all virtually Haken and $\pi_1(M)$ is torsion free.

6.2 The s-cobordism structure set

Let M be a closed 4-manifold with fundamental group π and orientation character $w : \pi \to \{\pm 1\}$, and let G/TOP have the H-space multiplication determined by its loop space structure. Then the surgery obstruction maps $\sigma_{4+i} = \sigma^M_{4+i} : [M \times D^i, \partial(M \times D^i); G/TOP, \{*\}] \to L^s_{4+i}(\pi, w)$ are homomorphisms. If π is in the class SA then $L^s_5(\pi, w)$ acts on $S_{TOP}(M)$, and the surgery sequence

$$[SM; G/TOP] \xrightarrow{\sigma_5} L^s_5(\pi, w) \xrightarrow{\omega} S_{TOP}(M) \xrightarrow{\eta} [M; G/TOP] \xrightarrow{\sigma_4} L^s_4(\pi, w)$$

is an exact sequence of groups and pointed sets, i.e., the orbits of the action ω correspond to the normal invariants $\eta(f)$ of simple homotopy equivalences [FQ, FT95]. As it is not yet known whether 5-dimensional s-cobordisms over other fundamental groups are products, we shall redefine the structure set by setting

$$S^s_{TOP}(M) = \{f : N \to M \mid N \ a \ TOP \ 4{-}manifold, \ f \ a \ simple \ h.e.\}/\approx,$$

where $f_1 \approx f_2$ if there is a map $F : W \to M$ with domain W an s-cobordism with $\partial W = N_1 \cup N_2$ and $F|_{N_i} = f_i$ for $i = 1, 2$. If the s-cobordism theorem holds over π this is the usual TOP structure set for M. We shall usually write $L_n(\pi, w)$ for $L^s_n(\pi, w)$ if $Wh(\pi) = 0$ and $L_n(\pi)$ if moreover w is trivial. When the orientation character is nontrivial and otherwise clear from the context we shall write $L_n(\pi, -)$.

The homotopy set $[M; G/TOP]$ may be identified with the set of normal maps (f, b), where $f : N \to M$ is a degree 1 map and b is a stable framing of $T_N \oplus f^*\xi$, for some TOP R^n-bundle ξ over M. (If $f : N \to M$ is a homotopy equivalence, with homotopy inverse h, we shall let $\hat{f} = (f, b)$, where $\xi = h^*\nu_N$ and b is the framing determined by a homotopy from hf to id_N). The Postnikov 4-stage of G/TOP is homotopy equivalent to $K(Z/2Z,2)\times K(Z,4)$. Let k_2 generate $H^2(G/TOP;\mathbb{F}_2)\cong Z/2Z$ and l_4 generate $H^4(G/TOP;\mathbb{Z})\cong Z$. The function from $[M;G/TOP]$ to $H^2(M;\mathbb{F}_2)\oplus H^4(M;\mathbb{Z})$ which sends $\hat{f}$ to $(\hat{f}^*(k_2),\hat{f}^*(l_4))$ is an isomorphism.

The *Kervaire-Arf invariant* of a normal map $\hat{g}:N^{2q}\to G/TOP$ is the image of the surgery obstruction in $L_{2q}(Z/2Z,-)=Z/2Z$ under the homomorphism induced by the orientation character, $c(\hat{g})=L_{2q}(w_1(N))(\sigma_{2q}(\hat{g}))$. The argument of Theorem 13.B.5 of [Wl] may be adapted to show that there are universal classes K_{4i+2} in $H^{4i+2}(G/TOP;\mathbb{F}_2)$ (for $i\geq 0$) such that

$$c(\hat{g})=(w(M)\cup\hat{g}^*((1+Sq^2+Sq^2Sq^2)\Sigma K_{4i+2}))\cap[M].$$

Moreover $K_2=k_2$, since c induces the isomorphism $\pi_2(G/TOP)=Z/2Z$. In the 4-dimensional case this expression simplifies to

$$c(\hat{g})=(w_2(M)\cup\hat{g}^*(k_2)+\hat{g}^*(Sq^2k_2))[M]=(w_1(M)^2\cup\hat{g}^*(k_2))[M].$$

The *codimension-2 Kervaire invariant* of a 4-dimensional normal map $\hat{g}$ is $kerv(\hat{g})=\hat{g}^*(k_2)$. Its value on a 2-dimensional homology class represented by an immersion $y:Y\to M$ is the Kervaire-Arf invariant of the normal map induced over the surface Y.

The structure set may overestimate the number of homeomorphism types within the homotopy type of M, if M has self homotopy equivalences which are not homotopic to homeomorphisms. Such "exotic" self homotopy equivalences may often be constructed as follows. Given $\alpha:S^2\to M$, let $\beta:S^4\to M$ be the composition $\alpha\eta S\eta$, where η is the Hopf map, and let $s:M\to M\vee S^4$ be the pinch map obtained by shrinking the boundary of a 4-disc in M. Then the composite $f_\alpha=(id_E\vee\beta)s$ is a self homotopy equivalence of M.

Lemma 6.5 [No64] *Let M be a closed 4-manifold and let $\alpha:S^2\to M$ be a map such that $\alpha_*[S^2]\neq 0$ in $H_2(M;\mathbb{F}_2)$ and $\alpha^*w_2(M)=0$. Then $kerv(\widehat{f_\alpha})\neq 0$ and so f_α is not normally cobordant to a homeomorphism.*

Proof There is a class $u\in H_2(M;\mathbb{F}_2)$ such that $\alpha_*[S^2].u=1$, since $\alpha_*[S^2]\neq 0$. As low-dimensional homology classes may be realized by singular manifolds there is a closed surface Y and a map $y:Y\to M$ transverse to f_α and such that $f_*[Y]=u$. Then $y^*kerv(\widehat{f_\alpha})[Y]$ is the Kervaire-Arf invariant of the normal map induced over Y and is nontrivial. (See Theorem 5.1 of [CH90] for details). ∎

The family of surgery obstruction maps may be identified with a natural transformation from $\mathbb{L}_0$-homology to L-theory. (In the nonorientable case we must use w-twisted $\mathbb{L}_0$-homology). In dimension 4 the cobordism invariance of surgery obstructions (as in §13B of [Wl]) leads to the following formula.

Theorem 6.6 [Da95] *There are homomorphisms $I_0 : H_0(\pi; Z^w) \to L_4(\pi, w)$ and $\kappa_2 : H_2(\pi; \mathbb{F}_2) \to L_4(\pi, w)$ such that for any $\hat{f} : M \to G/TOP$ the surgery obstruction is $\sigma_4(\hat{f}) = I_0 c_{M*}(\hat{f}^*(l_4) \cap [M]) + \kappa_2 c_{M*}(kerv(\hat{f}) \cap [M])$* □

If $w = 1$ the signature homomorphism from $L_4(\pi)$ to Z is a left inverse for $I_0 : Z \to L_4(\pi)$, but in general I_0 is not injective. This formula can be made somewhat more explicit as follows. Let $KS(M) \in H^4(M; \mathbb{F}_2)$ be the Kirby-Siebenmann obstruction to lifting the TOP normal fibration of M to a vector bundle. If M is orientable and $(f, b) : N \to M$ is a degree 1 normal map with classifying map $\hat{f}$ then

$$(KS(M) - (f^*)^{-1}KS(N) - kerv(\hat{f})^2)[M] \equiv (\sigma(M) - \sigma(N))/8 \ mod\ (2).$$

(See Lemma 15.5 of [Si71] - page 329 of [KS]).

Theorem [Da95, 6′] *If $\hat{f} = (f, b)$ where $f : N \to M$ is a degree 1 map then the surgery obstructions are given by*

$$\sigma_4(\hat{f}) = I_0((\sigma(N) - \sigma(M))/8) + \kappa_2 c_{M*}(kerv(\hat{f}) \cap [M]) \qquad \text{if } w = 1, \text{ and}$$

$$\sigma_4(\hat{f}) = I_0(KS(N) - KS(M) + kerv(\hat{f})^2) + \kappa_2 c_{M*}(kerv(\hat{f}) \cap [M]) \qquad \text{if } w \neq 1.$$

(In the latter case we identify $H^4(M; \mathbb{Z})$, $H^4(N; \mathbb{Z})$ and $H^4(M; \mathbb{F}_2)$ with $H_0(\pi; Z^w) = Z/2Z$). □

The homomorphism σ_4 is trivial on the image of η, but in general we do not know whether a 4-dimensional normal map with trivial surgery obstruction must be normally cobordant to a simple homotopy equivalence. In our applications we shall always have a simple homotopy equivalence in hand, and so if σ_4 is injective we can conclude that the homotopy equivalence is normally cobordant to the identity.

A more serious problem is that it is not clear how to define the action ω in general. We shall be able to circumvent this problem by *ad hoc* arguments in some cases. (There is always an action on the homological structure set, defined in terms of $\mathbb{Z}[\pi]$-homology equivalences [FQ]).

If we fix an isomorphism $i_Z : Z \to L_5(Z)$ we may define a function $I_\pi : \pi \to L_5^s(\pi)$ for any group π by $I_\pi(g) = g_*(i_Z(1))$, where $g_* : Z = L_5(Z) \to L_5^s(\pi)$ is induced by the homomorphism sending 1 in Z to g in π. Then $I_Z = i_Z$ and I_π is natural in the sense that if $f : \pi \to H$ is a homomorphism then $L_5(f)I_\pi = I_H f$. As abelianization and projection to the summands of Z^2 induce an isomorphism from $L_5(Z * Z)$ to $L_5(Z)^2$ [Ca73], it follows easily from naturality that I_π is a homomorphism (and so factors through π/π') [We83]. We shall extend this to the nonorientable case by defining $I_\pi^+ : \mathrm{Ker}(w) \to L_5^s(\pi; w)$ as the composite of $I_{\mathrm{Ker}(w)}$ with the homomorphism induced by inclusion.

Theorem 6.7 *Let M be a closed 4-manifold with fundamental group π and let $w = w_1(M)$. Given any $\gamma \in \mathrm{Ker}(w)$ there is a normal cobordism from id_M to itself with surgery obstruction $I_\pi^+(\gamma) \in L_5^s(\pi, w)$.*

Proof We may assume that γ is represented by a simple closed curve with a product neighbourhood $U \cong S^1 \times D^3$. Let P be the E_8 manifold [FQ] and delete the interior of a submanifold homeomorphic to $D^3 \times [0,1]$ to obtain P_o. There is a normal map $p : P_o \to D^3 \times [0,1]$ (*rel* boundary). The surgery obstruction for $p \times id_{S^1}$ in $L_5(Z) \cong L_4(1)$ is given by a codimension-1 signature (see §12B of [Wl]), and generates $L_5(Z)$. Let $Y = (M \setminus intU) \times [0,1] \cup P_o \times S^1$, where we identify $(\partial U) \times [0,1] = S^1 \times S^2 \times [0,1]$ with $S^2 \times [0,1] \times S^1$ in $\partial P_o \times S^1$. Matching together $id|_{(M \setminus intU) \times [0,1]}$ and $p \times id_{S^1}$ gives a normal cobordism Q from id_M to itself. The theorem now follows by the additivity of surgery obstructions and naturality of the homomorphisms I_π^+. □

Corollary 6.7.1 *Let $\lambda_* : L_5^s(\pi) \to L_5(Z)^d = Z^d$ be the homomorphism induced by a basis $\{\lambda_1, ..., \lambda_d\}$ for $Hom(\pi, Z)$. If M is orientable, $f : M_1 \to M$ is a simple homotopy equivalence and $\theta \in L_5(Z)^d$ there is a normal cobordism from f to itself whose surgery obstruction in $L_5(\pi)$ has image θ under λ_*.*

Proof If $\{\gamma_1, ..., \gamma_d\} \in \pi$ represents a "dual basis" for $H_1(\pi; \mathbb{Z})$ modulo torsion (so that $\lambda_i(\gamma_j) = \delta_{ij}$ for $1 \leq i, j \leq d$), then $\{\lambda_*(I_\pi(\gamma_1)), ..., \lambda_*(I_\pi(\gamma_d))\}$ is a basis for $L_5(Z)^d$. □

If π is free or is a PD_2^+-group the homomorphism λ_* is an isomorphism [Ca73]. In most of the other cases of interest to us the following corollary applies.

Corollary 6.7.2 *If M is orientable and $\mathrm{Ker}(\lambda_*)$ is finite then $S_{TOP}^s(M)$ is finite. In particular, this is so if $\mathrm{Coker}(\sigma_5)$ is finite.*

Proof The signature difference maps $[M; G/TOP] = H^4(M;\mathbb{Z}) \oplus H^2(M;\mathbb{F}_2)$ onto $L_4(1) = Z$ and so there are only finitely many normal cobordism classes of simple homotopy equivalences $f : M_1 \to M$. Moreover, $\mathrm{Ker}(\lambda_*)$ is finite if σ_5 has finite cokernel, since $[SM; G/TOP] \cong Z^d \oplus (Z/2Z)^d$. Suppose that $F : N \to M \times I$ is a normal cobordism between two simple homotopy equivalences $F_- = F|\partial_- N$ and $F_+ = F|\partial_+ N$. By Theorem 6.7 there is another normal cobordism $F' : N' \to M \times I$ from F_+ to itself with $\lambda_*(\sigma_5(F')) = \lambda_*(-\sigma_5(F))$. The union of these two normal cobordisms along $\partial_+ N = \partial_- N'$ is a normal cobordism from F_- to F_+ with surgery obstruction in $\mathrm{Ker}(\lambda_*)$. If this obstruction is 0 we may obtain an s-cobordism W by 5-dimensional surgery (rel ∂). □

The surgery obstruction groups for a semidirect product $\pi \cong G \times_\theta Z$, may be related to those of the (finitely presentable) normal subgroup G by means of Theorem 12.6 of [Wl]. If $Wh(\pi) = Wh(G) = 0$ this theorem asserts that there is an exact sequence

$$\dots L_m(G, w|_G) \xrightarrow{1-w(t)\theta_*} L_m(G, w|_G) \to L_m(\pi, w) \to L_{m-1}(G, w|_G) \dots,$$

where t generates π modulo G and $\theta_* = L_m(\theta, w|_G)$. The following lemma is adapted from Theorem 15.B.1 of [Wl].

Lemma 6.8 *Let M be the mapping torus of a self homeomorphism of an aspherical closed $(n-1)$-manifold N. Suppose that $Wh(\pi_1(M)) = 0$. If the homomorphisms σ_i^N are isomorphisms for all large i then so are the σ_i^M.*

Proof This is an application of the 5-lemma and periodicity, as in pages 229-230 of [Wl]. □

The hypotheses of this lemma are satisfied if $n = 4$ and $\pi_1(N)$ is square root closed accessible [Ca73], or N is orientable and $\beta_1(N) > 0$ [Ro00], or is hyperbolic or virtually solvable [FJ], or admits an effective S^1-action with orientable orbit space [St84, NS85]. It remains an open question whether aspherical closed manifolds with isomorphic fundamental groups must be homeomorphic. This has been verified in higher dimensions in many cases, in particular under geometric assumptions [FJ], and under assumptions on the combinatorial structure of the group [Ca73, St84, NS85]. We shall see that many aspherical 4-manifolds are determined up to s-cobordism by their groups.

There are more general "Mayer-Vietoris" sequences which lead to calculations of the surgery obstruction groups for certain generalized free products and HNN extensions in terms of those of their building blocks [Ca73, St87].

Lemma 6.9 *Let π be either the group of a finite graph of groups, all of whose vertex groups are infinite cyclic, or a square root closed accessible group of cohomological dimension 2. Then I_π^+ is an isomorphism. If M is a closed 4-manifold with fundamental group π the surgery obstruction maps $\sigma_4(M)$ and $\sigma_5(M)$ are epimorphisms.*

Proof Since π is in *Cl* we have $Wh(\pi) = 0$ and a comparison of Mayer-Vietoris sequences shows that the assembly map from $H_*(\pi;\mathbb{L}_0^w)$ to $L_*(\pi,w)$ is an isomorphism [Ca73, St87]. Since $c.d.\pi \leq 2$ and $H_1(\mathrm{Ker}(w);\mathbb{Z})$ maps onto $H_1(\pi;Z^w)$ the component of this map in degree 1 may be identified with I_π^+. In general, the surgery obstruction maps factor through the assembly map. Since $c.d.\pi \leq 2$ the homomorphism $c_{M*}: H_*(M;D) \to H_*(\pi;D)$ is onto for any local coefficient module D, and so the lemma follows. □

The class of groups considered in this lemma includes free groups, PD_2-groups and the groups $Z*_m$. Note however that if π is a PD_2-group w need not be the canonical orientation character.

6.3 Stabilization and h-cobordism

It has long been known that many results of high dimensional differential topology hold for smooth 4-manifolds after stabilizing by connected sum with copies of $S^2 \times S^2$ [CS71, FQ80, La79, Qu83]. In particular, if M and N are h-cobordant closed smooth 4-manifolds then $M\sharp(\sharp^k S^2 \times S^2)$ is diffeomorphic to $N\sharp(\sharp^k S^2 \times S^2)$ for some $k \geq 0$. In the spin case $w_2(M) = 0$ this is an elementary consequence of the existence of a well-indexed handle decomposition of the h-cobordism [Wa64]. In Chapter VII of [FQ] it is shown that 5-dimensional TOP cobordisms have handle decompositions relative to a component of their boundaries, and so a similar result holds for h-cobordant closed TOP 4-manifolds. Moreover, if M is a TOP 4-manifold then $KS(M) = 0$ if and only if $M\sharp(\sharp^k S^2 \times S^2)$ is smoothable for some $k \geq 0$ [LS71].

These results suggest the following definition. Two 4-manifolds M_1 and M_2 are *stably homeomorphic* if $M_1\sharp(\sharp^k S^2 \times S^2)$ and $M_2\sharp(\sharp^l S^2 \times S^2)$ are homeomorphic, for some k, $l \geq 0$. (Thus h-cobordant closed 4-manifolds are stably homeomorphic). Clearly $\pi_1(M)$, $w_1(M)$, the orbit of $c_{M*}[M]$ in $H_4(\pi_1(M), Z^{w_1(M)})$ under the action of $Out(\pi_1(M))$, and the parity of $\chi(M)$ are invariant under stabilization. If M is orientable $\sigma(M)$ is also invariant.

Kreck has shown that (in any dimension) classification up to stable homeomorphism (or diffeomorphism) can be reduced to bordism theory. There are three cases: If $w_2(\tilde{M}) \neq 0$ and $w_2(\tilde{N}) \neq 0$ then M and N are stably homeomorphic if and only if for some choices of orientations and identification of the fundamental groups the invariants listed above agree (in an obvious manner). If $w_2(M) = w_2(N) = 0$ then M and N are stably homeomorphic if and only if for some choices of orientations, Spin structures and identification of the fundamental group they represent the same element in $\Omega_4^{SpinTOP}(K(\pi,1))$. The most complicated case is when M and N are not Spin, but the universal covers are Spin. (See [Kr99], [Te] for expositions of Kreck's ideas).

We shall not pursue this notion of stabilization further (with one minor exception, in Chapter 14), for it is somewhat at odds with the tenor of this book. The manifolds studied here usually have minimal Euler characteristic, and often are aspherical. Each of these properties disappears after stabilization. We may however also stabilize by cartesian product with R, and there is then the following simple but satisfying result.

Lemma 6.10 *Closed* 4*-manifolds* M *and* N *are* h*-cobordant if and only if* $M \times R$ *and* $N \times R$ *are homeomorphic.*

Proof If W is an h-cobordism from M to N (with fundamental group $\pi = \pi_1(W)$) then $W \times S^1$ is an h-cobordism from $M \times S^1$ to $N \times S^1$. The torsion is 0 in $Wh(\pi \times Z)$, by Theorem 23.2 of [Co], and so there is a homeomorphism from $M \times S^1$ to $N \times S^1$ which carries $\pi_1(M)$ to $\pi_1(N)$. Hence $M \times R \cong N \times R$. Conversely, if $M \times R \cong N \times R$ then $M \times R$ contains a copy of N disjoint from $M \times \{0\}$, and the region W between $M \times \{0\}$ and N is an h-cobordism. □

6.4 Manifolds with π_1 elementary amenable and $\chi = 0$

In this section we shall show that closed manifolds satisfying the hypotheses of Theorem 3.17 and with torsion free fundamental group are determined up to homeomorphism by their homotopy type. As a consequence, closed 4-manifolds with torsion free elementary amenable fundamental group and Euler characteristic 0 are homeomorphic to mapping tori. We also estimate the structure sets for RP^2-bundles over T or Kb. In the remaining cases involving torsion computation of the surgery obstructions is much more difficult. We shall comment briefly on these cases in Chapters 10 and 11.

Theorem 6.11 *Let* M *be a closed* 4*-manifold with* $\chi(M) = 0$ *and whose fundamental group* π *is torsion free, coherent, locally virtually indicable and restrained. Then* M *is determined up to homeomorphism by its homotopy type. If moreover* $h(\pi) = 4$ *then every automorphism of* π *is realized by a self homeomorphism of* M.

Proof By Theorem 3.17 either $\pi \cong Z$ or $Z*_m$ for some $m \neq 0$, or M is aspherical, π is virtually poly-Z and $h(\pi) = 4$. Hence $Wh(\pi) = 0$, in all cases. If $\pi \cong Z$ or $Z*_m$ then the surgery obstruction homomorphisms are epimorphisms, by Lemma 6.9. We may calculate $L_4(\pi, w)$ by means of Theorem 12.6 of [Wl], or more generally §3 of [St87], and we find that if $\pi \cong Z$ or $Z*_{2n}$ then $\sigma_4(M)$ is in fact an isomorphism. If $\pi \cong Z*_{2n+1}$ then there are two normal cobordism classes of homotopy equivalences $h : X \to M$. Let ξ generate the image of $H^2(\pi; \mathbb{F}_2) \cong Z/2Z$ in $H^2(M; \mathbb{F}_2) \cong (Z/2Z)^2$, and let $j : S^2 \to M$ represent the unique nontrivial spherical class in $H_2(M; \mathbb{F}_2)$. Then $\xi^2 = 0$, since $c.d.\pi = 2$, and $\xi \cap j_*[S^2] = 0$, since $c_M j$ is nullhomotopic. It follows that $j_*[S^2]$ is Poincaré dual to ξ, and so $v_2(M) \cap j_*[S^2] = \xi^2 \cap [M] = 0$. Hence $j^* w_2(M) = j^* v_2(M) + (j^* w_1(M))^2 = 0$ and so f_j has nontrivial normal invariant, by Lemma 6.5. Therefore each of these two normal cobordism classes contains a self homotopy equivalence of M.

If M is aspherical, π is virtually poly-Z and $h(\pi) = 4$ then $S_{TOP}(M)$ has just one element, by Theorem 2.16 of [FJ]. The theorem now follows. □

Corollary 6.11.1 *Let M be a closed 4-manifold with $\chi(M) = 0$ and fundamental group $\pi \cong Z$, Z^2 or $Z \times_{-1} Z$. Then M is determined up to homeomorphism by π and $w(M)$.*

Proof If $\pi \cong Z$ then M is homotopy equivalent to the total space of an S^3-bundle over S^1, by Theorem 4.2, while if $\pi \cong Z^2$ or $Z \times_{-1} Z$ it is homotopy equivalent to the total space of an S^2-bundle over T or Kb, by Theorem 5.10. □

Is the homotopy type of M also determined by π and $w(M)$ if $\pi \cong Z*_m$ for some $|m| > 1$?

We may now give an analogue of the Farrell and Stallings fibration theorems for 4-manifolds with torsion free elementary amenable fundamental group.

Theorem 6.12 *Let M be a closed 4-manifold whose fundamental group π is torsion free and elementary amenable. A map $f : M \to S^1$ is homotopic to a fibre bundle projection if and only if $\chi(M) = 0$ and f induces an epimorphism from π to Z with almost finitely presentable kernel.*

Proof The conditions are clearly necessary. Suppose that they hold. Let $\nu = \mathrm{Ker}(\pi_1(f))$, let M_ν be the infinite cyclic covering space of M with fundamental group ν and let $t : M_\nu \to M_\nu$ be a generator of the group of covering transformations. By Corollary 4.5.3 either $\nu = 1$ (so $M_\nu \simeq S^3$) or $\nu \cong Z$ (so $M_\nu \simeq S^2 \times S^1$ or $S^2 \tilde{\times} S^1$) or M is aspherical. In the latter case π is a torsion free virtually poly-Z group, by Theorem 1.11 and Theorem 9.23 of [Bi]. Thus in all cases there is a homotopy equivalence f_ν from M_ν to a closed 3-manifold N. Moreover the self homotopy equivalence $f_\nu t f_\nu^{-1}$ of N is homotopic to a homeomorphism, g say, and so f is fibre homotopy equivalent to the canonical projection of the mapping torus $M(g)$ onto S^1. It now follows from Theorem 6.11 that any homotopy equivalence from M to $M(g)$ is homotopic to a homeomorphism. □

The structure sets of the RP^2-bundles over T or Kb are also finite.

Theorem 6.13 *Let M be the total space of an RP^2-bundle over T or Kb. Then $S_{TOP}(M)$ has order at most 32.*

Proof As M is nonorientable $H^4(M;\mathbb{Z}) = Z/2Z$ and as $\beta_1(M;\mathbb{F}_2) = 3$ and $\chi(M) = 0$ we have $H^2(M;\mathbb{F}_2) \cong (Z/2Z)^4$. Hence $[M; G/TOP]$ has order 32. Let $w = w_1(M)$. It follows from the Shaneson-Wall splitting theorem (Theorem 12.6 of [Wl]) that $L_4(\pi, w) \cong L_4(Z/2Z, -) \oplus L_2(Z/2Z, -) \cong (Z/2Z)^2$, detected by the Kervaire-Arf invariant and the codimension-2 Kervaire invariant. Similarly $L_5(\pi, w) \cong L_4(Z/2Z, -)^2$ and the projections to the factors are Kervaire-Arf invariants of normal maps induced over codimension-1 submanifolds. (In applying the splitting theorem, note that $Wh(Z \oplus (Z/2Z)) = Wh(\pi) = 0$, by Theorem 6.1 above). Hence $S_{TOP}(M)$ has order at most 128.

The Kervaire-Arf homomorphism c is onto, since $c(\hat{g}) = (w^2 \cup \hat{g}^*(k_2)) \cap [M]$, $w^2 \neq 0$ and every element of $H^2(M;\mathbb{F}_2)$ is equal to $\hat{g}^*(k_2)$ for some normal map $\hat{g} : M \to G/TOP$. Similarly there is a normal map $f_2 : X_2 \to RP^2$ with $\sigma_2(f_2) \neq 0$ in $L_2(Z/2Z, -)$. If $M = RP^2 \times B$, where $B = T$ or Kb is the base of the bundle, then $f_2 \times id_B : X_2 \times B \to RP^2 \times B$ is a normal map with surgery obstruction $(0, \sigma_2(f_2)) \in L_4(Z/2Z, -) \oplus L_2(Z/2Z, -)$. We may assume that f_2 is a homeomorphism over a disc $\Delta \subset RP^2$. As the nontrivial bundles may be obtained from the product bundles by cutting M along $RP^2 \times \partial\Delta$ and regluing via the twist map of $RP^2 \times S^1$, the normal maps for the product bundles may be compatibly modified to give normal maps with nonzero obstructions in the other cases. Hence σ_4 is onto and so $S_{TOP}(M)$ has order at most 32. □

In each case $H_2(M;\mathbb{F}_2) \cong H_2(\pi;\mathbb{F}_2)$, so the argument of Lemma 6.5 does not apply. However we can improve our estimate in the abelian case.

Theorem 6.14 *Let M be the total space of an RP^2-bundle over T. Then $L_5(\pi, w)$ acts trivially on the class of id_M in $S_{TOP}(M)$.*

Proof Let $\lambda_1, \lambda_2 : \pi \to Z$ be epimorphisms generating $Hom(\pi, Z)$ and let $t_1, t_2 \in \pi$ represent a dual basis for $\pi/(torsion)$ (i.e., $\lambda_i(t_j) = \delta_{ij}$ for $i = 1, 2$). Let u be the element of order 2 in π and let $k_i : Z \oplus (Z/2Z) \to \pi$ be the monomorphism defined by $k_i(a, b) = at_i + bu$, for $i = 1, 2$. Define splitting homomorphisms p_1, p_2 by $p_i(g) = k_i^{-1}(g - \lambda_i(g)t_i)$ for all $g \in \pi$. Then $p_i k_i = id_{Z\oplus(Z/2Z)}$ and $p_i k_{3-i}$ factors through $Z/2Z$, for $i = 1, 2$. The orientation character $w = w_1(M)$ maps the torsion subgroup of π onto $Z/2Z$, by Theorem 5.13, and t_1 and t_2 are in $\mathrm{Ker}(w)$. Therefore p_i and k_i are compatible with w, for $i = 1, 2$. As $L_5(Z/2Z, -) = 0$ it follows that $L_5(k_1)$ and $L_5(k_2)$ are inclusions of complementary summands of $L_5(\pi, w) \cong (Z/2Z)^2$, split by the projections $L_5(p_1)$ and $L_5(p_2)$.

Let γ_i be a simple closed curve in T which represents $t_i \in \pi$. Then γ_i has a product neighbourhood $N_i \cong S^1 \times [-1, 1]$ whose preimage $U_i \subset M$ is homeomorphic to $RP^2 \times S^1 \times [-1, 1]$. As in Theorem 6.13 there is a normal map $f_4 : X_4 \to RP^2 \times [-1, 1]^2$ (*rel* boundary) with $\sigma_4(f_4) \neq 0$ in $L_4(Z/2Z, -)$. Let $Y_i = (M \setminus intU_i) \times [-1, 1] \cup X_4 \times S^1$, where we identify $(\partial U_i) \times [-1, 1] = RP^2 \times S^1 \times S^0 \times [-1, 1]$ with $RP^2 \times [-1, 1] \times S^0 \times S^1$ in $\partial X_4 \times S^1$. If we match together $id_{(M\setminus intU_i)\times[-1,1]}$ and $f_4 \times id_{S^1}$ we obtain a normal cobordism Q_i from id_M to itself. The image of $\sigma_5(Q_i)$ in $L_4(\mathrm{Ker}(\lambda_i), w) \cong L_4(Z/2Z, -)$ under the splitting homomorphism is $\sigma_4(f_4)$. On the other hand its image in $L_4(\mathrm{Ker}(\lambda_{3-i}), w)$ is 0, and so it generates the image of $L_5(k_{3-i})$. Thus $L_5(\pi, w)$ is generated by $\sigma_5(Q_1)$ and $\sigma_5(Q_2)$, and so acts trivially on id_M. □

Does $L_5(\pi, w)$ act trivially on each class in $S_{TOP}(M)$ when M is an RP^2-bundle over T or Kb? If so, then $S_{TOP}(M)$ has order 8 in each case. Are these manifolds determined up to homeomorphism by their homotopy type?

6.5 Bundles over aspherical surfaces

The fundamental groups of total spaces of bundles over hyperbolic surfaces all contain nonabelian free subgroups. Nevertheless, such bundle spaces are determined up to s-cobordism by their homotopy type, except when the fibre is RP^2, in which case we can only show that the structure sets are finite.

Theorem 6.15 *Let M be a closed 4-manifold which is homotopy equivalent to the total space E of an F-bundle over B where B and F are aspherical closed surfaces. Then M is s-cobordant to E and $\widetilde{M}$ is homeomorphic to R^4.*

Proof Since $\pi_1(B)$ is either an HNN extension of Z or a generalised free product $F*_Z F'$, where F and F' are free groups, $\pi\times Z$ is a square root closed generalised free product with amalgamation of groups in *Cl*. Comparison of the Mayer-Vietoris sequences for $\mathbb{L}_0$-homology and L-theory (as in Proposition 2.6 of [St84]) shows that $S_{TOP}(E\times S^1)$ has just one element. (Note that even when $\chi(B)=0$ the groups arising in intermediate stages of the argument all have trivial Whitehead groups). Hence $M\times S^1\cong E\times S^1$, and so M is s-cobordant to E by Lemma 6.10 and Theorem 6.2. The final assertion follows from Corllary 7.3B of [FQ] since M is aspherical and π is 1-connected at ∞ [Ho77]. □

Davis has constructed aspherical 4-manifolds whose universal covering space is not 1-connected at ∞ [Da83].

Theorem 6.16 *Let M be a closed 4-manifold which is homotopy equivalent to the total space E of an S^2-bundle over an aspherical closed surface B. Then M is s-cobordant to E, and $\widetilde{M}$ is homeomorphic to $S^2\times R^2$.*

Proof Let $\pi=\pi_1(E)\cong\pi_1(B)$. Then $Wh(\pi)=0$, and $H_*(\pi;\mathbb{L}_0^w)\cong L_*(\pi,w)$, as in Lemma 6.9. Hence $L_4(\pi,w)\cong Z\oplus(Z/2Z)$ if $w=0$ and $(Z/2Z)^2$ otherwise. The surgery obstruction map $\sigma_4(E)$ is onto, by Lemma 6.9. Hence there are two normal cobordism classes of maps $h:X\to E$ with $\sigma_4(h)=0$. The kernel of the natural homomorphism from $H_2(E;\mathbb{F}_2)\cong(Z/2Z)^2$ to $H_2(\pi;\mathbb{F}_2)\cong Z/2Z$ is generated by $j_*[S^2]$, where $j:S^2\to E$ is the inclusion of a fibre. As $j_*[S^2]\neq -0$, while $w_2(E)(j_*[S^2])=j^*w_2(E)=0$ the normal invariant of f_j is nontrivial, by Lemma 6.5. Hence each of these two normal cobordism classes contains a self homotopy equivalence of E.

Let $f:M\to E$ be a homotopy equivalence (necessarily simple). Then there is a normal cobordism $F:V\to E\times[0,1]$ from f to some self homotopy equivalence of E. As I_π^+ is an isomorphism, by Lemma 6.9, there is an s-cobordism W from M to E, as in Corollary 6.7.2.

The universal covering space $\widetilde{W}$ is a proper s-cobordism from $\widetilde{M}$ to $\widetilde{E}\cong S^2\times R^2$. Since the end of $\widetilde{E}$ is tame and has fundamental group Z we may apply Corollary 7.3B of [FQ] to conclude that $\widetilde{W}$ is homeomorphic to a product. Hence $\widetilde{M}$ is homeomorphic to $S^2\times R^2$. □

Let ρ be a PD_2-group. As $\pi = \rho \times (Z/2Z)$ is square-root closed accessible from $Z/2Z$, the Mayer-Vietoris sequences of [Ca73] imply that $L_4(\pi, w) \cong L_4(Z/2Z, -) \oplus L_2(Z/2Z, -)$ and that $L_5(\pi, w) \cong L_4(Z/2Z, -)^\beta$, where $w = pr_2 : \pi \to Z/2Z$ and $\beta = \beta_1(\rho; \mathbb{F}_2)$. Since these L-groups are finite the structure sets of total spaces of RP^2-bundles over aspherical surfaces are also finite. (Moreover the arguments of Theorems 6.13 and 6.14 can be extended to show that σ_4 is an epimorphism and that most of $L_5(\pi, w)$ acts trivially on id_E, where E is such a bundle space).

Part II

4-dimensional Geometries

Chapter 7

Geometries and decompositions

Every closed connected surface is geometric, i.e., is a quotient of one of the three model 2-dimensional geometries $\mathbb{E}^2$, $\mathbb{H}^2$ or $\mathbb{S}^2$ by a free and properly discontinuous action of a discrete group of isometries. Much current research on 3-manifolds is guided by Thurston's Geometrization Conjecture, that every closed irreducible 3-manifold admits a finite decomposition into geometric pieces. In §1 we shall recall Thurston's definition of geometry, and shall describe briefly the 19 4-dimensional geometries. Our concern in the middle third of this book is not to show how this list arises (as this is properly a question of differential geometry; see [Fi], [Pa96] and [Wl85,86]), but rather to describe the geometries sufficiently well that we may subsequently characterize geometric manifolds up to homotopy equivalence or homeomorphism. In §2 we relate the notions of "geometry of solvable Lie type" and "infrasolvmanifold". The limitations of geometry in higher dimensions are illustrated in §3, where it is shown that a closed 4-manifold which admits a finite decomposition into geometric pieces is (essentially) either geometric or aspherical. The geometric viewpoint is nevertheless of considerable interest in connection with complex surfaces [Ue90,91, Wl85,86]. With the exception of the geometries $\mathbb{S}^2 \times \mathbb{H}^2$, $\mathbb{H}^2 \times \mathbb{H}^2$, $\mathbb{H}^2 \times \mathbb{E}^2$ and $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$ no closed geometric manifold has a proper geometric decomposition.

A number of the geometries support natural Seifert fibrations or compatible complex structures. In §4 we characterize the groups of aspherical 4-manifolds which are orbifold bundles over flat or hyperbolic 2-orbifolds. We outline what we need about Seifert fibrations and complex surfaces in §5 and §6.

Subsequent chapters shall consider in turn geometries whose models are contractible (Chapters 8 and 9), geometries with models diffeomorphic to $S^2 \times R^2$ (Chapter 10), the geometry $\mathbb{S}^3 \times \mathbb{E}^1$ (Chapter 11) and the geometries with compact models (Chapter 12). In Chapter 13 we shall consider geometric structures and decompositions of bundle spaces. In the final chapter of the book we shall consider knot manifolds which admit geometries.

7.1 Geometries

An *n-dimensional geometry* $\mathbb{X}$ in the sense of Thurston is represented by a pair (X, G_X) where X is a complete 1-connected n-dimensional Riemannian manifold and G_X is a Lie group which acts effectively, transitively and isometrically on X and which has discrete subgroups Γ which act freely on X so that $\Gamma\backslash X$ has finite volume. (Such subgroups are called *lattices*.) Since the stabilizer of a point in X is isomorphic to a closed subgroup of $O(n)$ it is compact, and so $\Gamma\backslash X$ is compact if and only if $\Gamma\backslash G_X$ is compact. Two such pairs (X, G) and (X', G') define the same geometry if there is a diffeomorphism $f : X \to X'$ which conjugates the action of G onto that of G'. (Thus the metric is only an adjunct to the definition.) We shall assume that G is maximal among Lie groups acting thus on X, and write $Isom(\mathbb{X}) = G$, and $Isom_o(\mathbb{X})$ for the component of the identity. A closed manifold M is an *$\mathbb{X}$-manifold* if it is a quotient $\Gamma\backslash X$ for some lattice in G_X. Under an equivalent formulation, M is an $\mathbb{X}$-manifold if it is a quotient $\Gamma\backslash X$ for some discrete group Γ of isometries acting freely on a 1-connected homogeneous space $X = G/K$, where G is a connected Lie group and K is a compact subgroup of G such that the intersection of the conjugates of K is trivial, and X has a G-invariant metric. The manifold *admits a geometry of type* $\mathbb{X}$ if it is homeomorphic to such a quotient. If G is solvable we shall say that the geometry is of *solvable Lie type*. If $\mathbb{X} = (X, G_X)$ and $\mathbb{Y} = (Y, G_Y)$ are two geometries then $X \times Y$ supports a geometry in a natural way; however the maximal group of isometries $G_{X\times Y}$ may be strictly larger than $G_X \times G_Y$.

The geometries of dimension 1 or 2 are the familiar geometries of constant curvature: $\mathbb{E}^1$, $\mathbb{E}^2$, $\mathbb{H}^2$ and $\mathbb{S}^2$. Thurston showed that there are eight maximal 3-dimensional geometries ($\mathbb{E}^3$, $\mathbb{N}il^3$, $\mathbb{S}ol^3$, $\widetilde{\mathbb{SL}}$, $\mathbb{H}^2 \times \mathbb{E}^1$, $\mathbb{H}^3$, $\mathbb{S}^2 \times \mathbb{E}^1$ and $\mathbb{S}^3$.) Manifolds with one of the first five of these geometries are aspherical Seifert fibred 3-manifolds or $\mathbb{S}ol^3$-manifolds. These are determined among irreducible 3-manifolds by their fundamental groups, which are the PD_3-groups with non-trivial Hirsch-Plotkin radical. There are just four $\mathbb{S}^2 \times \mathbb{E}^1$-manifolds. It is not yet known whether every aspherical 3-manifold whose fundamental group contains no rank 2 abelian subgroup must be hyperbolic, and the determination of the closed $\mathbb{H}^3$-manifolds remains incomplete. Nor is it known whether every 3-manifold with finite fundamental group must be spherical. For a detailed and lucid account of the 3-dimensional geometries see [Sc83'].

There are 19 maximal 4-dimensional geometries; one of these ($\mathbb{S}ol^4_{m,n}$) is in fact a family of closely related geometries, and one ($\mathbb{F}^4$) is not realizable by any closed manifold [Fi]. We shall see that the geometry is determined by the fundamental group (cf. [Wl86, Ko92]). In addition to the geometries of constant curvature and products of lower dimensional geometries there are seven "new" 4-dimensional geometries. Two of these are modeled on the irreducible Riemannian symmetric spaces $CP^2 = U(3)/U(2)$ and $H^2(C) = SU(2,1)/S(U(2)\times U(1))$. The model for the geometry $\mathbb{F}^4$ is $C\times H^2$. The component of the identity in its isometry group is the semidirect product $R^2 \times_\alpha SL(2,\mathbb{R})$, where α is the natural action of $SL(2,\mathbb{R})$ on R^2. This group acts on $C\times H^2$ as follows: if $(u,v)\in R^2$ and $A=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in SL(2,\mathbb{R})$ then $(u,v)(w,z)=(u-vz+w,z)$ and $A(w,z)=(\frac{w}{cz+d},\frac{az+b}{cz+d})$ for all $(w,z)\in C\times H^2$. The other four new geometries are of solvable Lie type, and shall be described in §2 and §3.

In most cases the model X is homeomorphic to R^4, and the corresponding geometric manifolds are aspherical. Six of these geometries ($\mathbb{E}^4$, $\mathbb{N}il^4$, $\mathbb{N}il^3\times\mathbb{E}^1$, $\mathbb{S}ol^4_{m,n}$, $\mathbb{S}ol^4_0$ and $\mathbb{S}ol^4_1$) are of solvable Lie type; in Chapter 8 we shall show manifolds admitting such geometries have Euler characteristic 0 and fundamental group a torsion free virtually poly-Z group of Hirsch length 4. Such manifolds are determined up to homeomorphism by their fundamental groups, and every such group arises in this way. In Chapter 9 we shall consider closed 4-manifolds admitting one of the other geometries of aspherical type ($\mathbb{H}^3\times\mathbb{E}^1$, $\widetilde{\mathbb{SL}}\times\mathbb{E}^1$, $\mathbb{H}^2\times\mathbb{E}^2$, $\mathbb{H}^2\times\mathbb{H}^2$, $\mathbb{H}^4$, $\mathbb{H}^2(\mathbb{C})$ and $\mathbb{F}^4$). These may be characterised up to s-cobordism by their fundamental group and Euler characteristic. However it is unknown to what extent surgery arguments apply in these cases, and we do not yet have good characterizations of the possible fundamental groups. Although no closed 4-manifold admits the geometry $\mathbb{F}^4$, there are such manifolds with proper geometric decompositions involving this geometry; we shall give examples in Chapter 13.

Three of the remaining geometries ($\mathbb{S}^2\times\mathbb{E}^2$, $\mathbb{S}^2\times\mathbb{H}^2$ and $\mathbb{S}^3\times\mathbb{E}^1$) have models homeomorphic to $S^2\times R^2$ or $S^3\times R$. (Note that we shall use $\mathbb{E}^n$ or $\mathbb{H}^n$ to refer to the *geometry* and R^n to refer to the underlying *topological space*). The final three ($\mathbb{S}^4$, $\mathbb{CP}^2$ and $\mathbb{S}^2\times\mathbb{S}^2$) have compact models, and there are only eleven such manifolds. We shall discuss these nonaspherical geometries in Chapters 10, 11 and 12.

7.2 Infranilmanifolds

The notions of "geometry of solvable Lie type" and "infrasolvmanifold" are closely related. We shall describe briefly the latter class of manifolds, from a rather utilitarian point of view. As we are only interested in closed manifolds, we shall frame our definitions accordingly. We consider the easier case of infranilmanifolds in this section, and the other infrasolvmanifolds in the next section.

A *flat n-manifold* is a quotient of R^n by a discrete torsion free subgroup of $E(n) = Isom(\mathbb{E}^n) = R^n \times_\alpha O(n)$ (where α is the natural action of $O(n)$ on R^n). A group π is a *flat n-manifold group* if it is torsion free and has a normal subgroup of finite index which is isomorphic to Z^n. (These are necessary and sufficient conditions for π to be the fundamental group of a closed flat n-manifold.) The action of π by conjugation on its *translation subgroup* $T(\pi)$ (the maximal abelian normal subgroup of π) induces a faithful action of $\pi/T(\pi)$ on $T(\pi)$. On choosing an isomorphism $T(\pi) \cong Z^n$ we may identify $\pi/T(\pi)$ with a subgroup of $GL(n,\mathbb{Z})$; this subgroup is called the *holonomy group* of π, and is well defined up to conjugacy in $GL(n,\mathbb{Z})$. We say that π is *orientable* if the holonomy group lies in $SL(n,\mathbb{Z})$. (The group is orientable if and only if the corresponding flat n-manifold is orientable.) If two discrete torsion free cocompact subgroups of $E(n)$ are isomorphic then they are conjugate in the larger group $Aff(R^n) = R^n \times_\alpha GL(n,\mathbb{R})$, and the corresponding flat n-manifolds are "affinely" diffeomorphic. There are only finitely many isomorphism classes of such flat n-manifold groups for each n.

A nilmanifold is a coset space of a 1-connected nilpotent Lie group by a discrete subgroup. More generally, an *infranilmanifold* is a quotient $\pi\backslash N$ where N is a 1-connected nilpotent Lie group and π is a discrete torsion free subgroup of the semidirect product $Aff(N) = N \times_\alpha Aut(N)$ such that $\pi \cap N$ is a lattice in N and $\pi/\pi\cap N$ is finite. Thus infranilmanifolds are finitely covered by nilmanifolds. The Lie group N is determined by $\sqrt{\pi}$, by Mal'cev's rigidity theorem, and two infranilmanifolds are diffeomorphic if and only if their fundamental groups are isomorphic. The isomorphism may then be induced by an affine diffeomorphism. The infranilmanifolds derived from the abelian Lie groups R^n are just the flat manifolds. It is not hard to see that there are just three 4-dimensional (real) nilpotent Lie algebras. (Compare the analogous argument of Theorem 1.4.) Hence there are three 1-connected 4-dimensional nilpotent Lie groups, R^4, $Nil^3 \times R$ and Nil^4.

The group Nil^3 is the subgroup of $SL(3,\mathbb{R})$ consisting of upper triangular matrices $[r,s,t] = \begin{pmatrix} 1 & r & t \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}$. It has abelianization R^2 and centre $\zeta Nil^3 = Nil^{3\prime} \cong R$. The elements $[1,0,0]$, $[0,1,0]$ and $[0,0,1/q]$ generate a discrete cocompact subgroup of Nil^3 isomorphic to Γ_q, and these are essentially the only such subgroups. (Since they act orientably on R^3 they are PD_3^+-groups.)

The coset space $N_q = Nil^3/\Gamma_q$ is the total space of the S^1-bundle over $S^1 \times S^1$ with Euler number q, and the action of ζNil^3 on Nil^3 induces a free action of $S^1 = \zeta Nil/\zeta\Gamma_q$ on N_q. The group Nil^4 is the semidirect product $R^3 \times_\theta R$, where $\theta(t) = [t, t, t^2/2]$. It has abelianization R^2 and central series $\zeta Nil^4 \cong R < \zeta^2 Nil^4 = Nil^{4\prime} \cong R^2$.

These Lie groups have natural left invariant metrics. (See [Sc83'].) The infranilmanifolds corresponding to R^4, Nil^4 and $Nil^3 \times R$ are the $\mathbb{E}^4$-, $\mathbb{N}il^4$- and $\mathbb{N}il^3 \times \mathbb{E}^1$-manifolds. (The isometry group of $\mathbb{E}^4$ is the semidirect product $R^4 \times_\alpha O(4)$; the group Nil^4 is the identity component for its isometry group, while $\mathbb{N}il^3 \times \mathbb{E}^1$ admits an additional isometric action of S^1.)

7.3 Infrasolvmanifolds

The situation for (infra)solvmanifolds is more complicated. An *infrasolvmanifold* is a quotient $M = \Gamma\backslash S$ where S is a 1-connected solvable Lie group and Γ is a closed torsion free subgroup of the semidirect product $Aff(S) = S \times_\alpha Aut(S)$ such that Γ_o (the component of the identity of Γ) is contained in the nilradical of S (the maximal connected nilpotent normal subgroup of S), $\Gamma/\Gamma \cap S$ has compact closure in $Aut(S)$ and M is compact. The pair (S, Γ) is called a presentation for M, and is discrete if Γ is a discrete subgroup of $Aff(S)$, in which case $\pi_1(M) = \Gamma$. Every infrasolvmanifold has a presentation such that $\Gamma/\Gamma \cap S$ is finite [FJ97], but we cannot assume that Γ is discrete, and S is not determined by π.

Farrell and Jones showed that in all dimensions *except* perhaps 4 infrasolvmanifolds with isomorphic fundamental groups are diffeomorphic. However an affine diffeomorphism is not always possible [FJ97]. They showed also that 4-dimensional infrasolvmanifolds are determined up to homeomorphism by their fundamental groups (see Theorem 8.2 below). Using the Mostow orbifold bundle associated to a presentation of an infrasolvmanifold (see §5 below) and standard 3-manifold theory it is possible to show that, in most cases, 4-dimensional infrasolvmanifolds are determined up to diffeomorphism by their groups ([Cb] - see Theorem 8.9 below). However there may still be a nonorientable 4-dimensional infrasolvmanifold with virtually nilpotent fundamental group which has no discrete presentation.

An important special case includes most infrasolvmanifolds of dimension ≤ 4 (and all infranilmanifolds). Let $T_n^+(\mathbb{R})$ be the subgroup of $GL(n, \mathbb{R})$ consisting of upper triangular matrices with positive diagonal entries. A Lie group S is *triangular* if is isomorphic to a closed subgroup of $T_n^+(\mathbb{R})$ for some n. The eigenvalues of the image of each element of S under the adjoint representation are then all real, and so S is *of type* R in the terminology of [Go71]. (It can be shown that a Lie group is triangular if and only if it is 1-connected and solvable of type R.) Two infrasolvmanifolds with discrete presentations (S_i, Γ_i) where each S_i is triangular (for $i = 1, 2$) are affinely diffeomorphic if and only if their fundamental groups are isomorphic, by Theorem 3.1 of [Le95]. The *translation subgroup* $S \cap \Gamma$ of a discrete pair with S triangular can be characterised intrinsically as the subgroup of Γ consisting of the elements $g \in \Gamma$ such that all the eigenvalues of the automorphisms of the abelian sections of the lower central series for $\sqrt{\Gamma}$ induced by conjugation by g are positive [De97]. Does every infrasolvmanifold with a presentation (S, Γ) where S is triangular have a discrete presentation?

Since S and Γ_o are each contractible, $X = \Gamma_o \backslash S$ is contractible also. It can be shown that $\pi = \Gamma/\Gamma_o$ acts freely on X, and so is the fundamental group of $M = \pi\backslash X$. (See Chapter III.3 of [Au73] for the solvmanifold case.) Since M is aspherical π is a PD_m group, where m is the dimension of M; since π is also virtually solvable it is thus virtually poly-Z of Hirsch length m, by Theorem 9.23 of [Bi], and $\chi(M) = \chi(\pi) = 0$. Conversely, any torsion free virtually poly-Z group is the fundamental group of a closed smooth manifold which is finitely covered by the coset space of a lattice in a 1-connected solvable Lie group [AJ76].

Let S be a connected solvable Lie group of dimension m, and let N be its nilradical. If π is a lattice in S then it is torsion free and virtually poly-Z of Hirsch length m and $\pi \cap N = \sqrt{\pi}$ is a lattice in N. If S is 1-connected then S/N is isomorphic to some vector group R^n, and $\pi/\sqrt{\pi} \cong Z^n$. A complete characterization of such lattices is not known, but a torsion free virtually poly-Z group π is a lattice in a connected solvable Lie group S if and only if $\pi/\sqrt{\pi}$ is abelian. (See Sections 4.29-31 of [Rg].)

The 4-dimensional solvable Lie geometries other than the infranil geometries are $\mathbb{S}ol^4_{m,n}$, $\mathbb{S}ol^4_0$ and $\mathbb{S}ol^4_1$, and the model spaces are solvable Lie groups with left invariant metrics. The following descriptions are based on [Wl86]. The Lie group is the identity component of the isometry group for the geometries $\mathbb{S}ol^4_{m,n}$ and $\mathbb{S}ol^4_1$; the identity component of $Isom(\mathbb{S}ol^4_0)$ is isomorphic to the semidirect product $(C \oplus R) \times_\gamma C^\times$, where $\gamma(z)(u, x) = (zu, |z|^{-2}x)$ for all (u, x) in $C \oplus R$ and z in $C^\times$, and thus $\mathbb{S}ol^4_0$ admits an additional isometric action of S^1, by rotations about an axis in $C \oplus R \cong R^3$, the radical of Sol^4_0.

$Sol^4_{m,n} = R^3 \times_{\theta_{m,n}} R$, where m and n are integers such that the polynomial $f_{m,n} = X^3 - mX^2 + nX - 1$ has distinct roots e^a, e^b and e^c (with $a < b < c$ real) and $\theta_{m,n}(t)$ is the diagonal matrix $diag[e^{at}, e^{bt}, e^{ct}]$. Since $\theta_{m,n}(t) = \theta_{n,m}(-t)$ we may assume that $m \leq n$; the condition on the roots then holds if and only if $2\sqrt{n} \leq m \leq n$. The metric given by $ds^2 = e^{-2at}dx^2 + e^{-2bt}dy^2 + e^{-2ct}dz^2 + dt^2$ (in the obvious global coordinates) is left invariant, and the automorphism of $Sol^4_{m,n}$ which sends (t, x, y, z) to (t, px, qy, rz) is an isometry if and only if $p^2 = q^2 = r^2 = 1$. Let G be the subgroup of $GL(4, \mathbb{R})$ of bordered matrices $\left(\begin{smallmatrix} D & \xi \\ 0 & 1 \end{smallmatrix}\right)$, where $D = diag[\pm e^{at}, \pm e^{bt}, \pm e^{ct}]$ and $\xi \in R^3$. Then $Sol^4_{m,n}$ is the subgroup of G with positive diagonal entries, and $G = Isom(\mathbb{S}ol^4_{m,n})$ if $m \neq n$. If $m = n$ then $b = 0$ and $\mathbb{S}ol^4_{m,m} = \mathbb{S}ol^3 \times \mathbb{E}^1$, which admits the additional isometry sending (t, x, y, z) to (t^{-1}, z, y, x), and G has index 2 in $Isom(\mathbb{S}ol^3 \times \mathbb{E}^1)$. The stabilizer of the identity in the full isometry group is $(Z/2Z)^3$ for $Sol^4_{m,n}$ if $m \neq n$ and $D_8 \times (Z/2Z)$ for $Sol^3 \times R$. In all cases $Isom(\mathbb{S}ol^4_{m,n}) \leq Aff(Sol^4_{m,n})$.

In general $\mathbb{S}ol^4_{m,n} = \mathbb{S}ol^4_{m',n'}$ if and only if $(a, b, c) = \lambda(a', b', c')$ for some $\lambda \neq 0$. Must λ be rational? (This is a case of the "problem of the four exponentials" of transcendental number theory.) If $m \neq n$ then $F_{m,n} = \mathbb{Q}[X]/(f_{m,n})$ is a totally real cubic number field, generated over $\mathbb{Q}$ by the image of X. The images of X under embeddings of $F_{m,n}$ in $\mathbb{R}$ are the roots e^a, e^b and e^c, and so it represents a unit of norm 1. The group of such units is free abelian of rank 2. Therefore if $\lambda = r/s \in \mathbb{Q}^\times$ this unit is an r^{th} power in $F_{m,n}$ (and its r^{th} root satisfies another such cubic). It can be shown that $|r| \leq \log_2(m)$, and so (modulo the problem of the four exponentials) there is a canonical "minimal" pair (m, n) representing each such geometry.

$Sol^4_0 = R^3 \times_\xi R$, where $\xi(t)$ is the diagonal matrix $diag[e^t, e^t, e^{-2t}]$. Note that if $\xi(t)$ preserves a lattice in R^3 then its characteristic polynomial has integral coefficients and constant term -1. Since it has e^t as a repeated root we must have $\xi(t) = I$. Therefore Sol^4_0 does not admit any lattices. The metric given by the expression $ds^2 = e^{-2t}(dx^2 + dy^2) + e^{4t}dz^2 + dt^2$ is left invariant, and $O(2) \times O(1)$ acts via rotations and reflections in the (x, y)-coordinates and reflection in the z-coordinate, to give the stabilizer of the identity. These actions are automorphisms of Sol^4_0, so $Isom(\mathbb{S}ol^4_0) = Sol^4_0 \rtimes (O(2) \times O(1)) \leq Aff(Sol^4_0)$. The identity component of $Isom(\mathbb{S}ol^4_0)$ is not triangular.

Sol^4_1 is the group of real matrices $\left\{\begin{pmatrix} 1 & b & c \\ 0 & \alpha & a \\ 0 & 0 & 1 \end{pmatrix} : \alpha > 0,\ a,\ b,\ c \in \mathbb{R}\right\}$. The metric given by $ds^2 = t^{-2}((1+x^2)(dt^2+dy^2)+t^2(dx^2+dz^2)-2tx(dtdx+dydz))$ is left invariant, and the stabilizer of the identity is D_8, generated by the isometries which send (t, x, y, z) to $(t, -x, y, -z)$ and to $t^{-1}(1, -y, -x, xy-tz)$. These are automorphisms. (The latter one is the restriction of the involution of $GL(3,\mathbb{R})$ which sends A to $J(A^{tr})^{-1}J$, where J reverses the order of the standard basis of $\mathbb{R}^3$.) Thus $Isom(\mathbb{S}ol_1^4) \leq Aff(Sol_1^4)$.

Closed $\mathbb{S}ol_{m,n}^4$- or $\mathbb{S}ol_1^4$-manifolds are clearly infrasolvmanifolds. The $\mathbb{S}ol_0^4$ case is more complicated. Let $\tilde{\gamma}(z)(u,x) = (e^z u, e^{-2\mathrm{Re}(z)}x)$ for all (u,x) in $C \oplus R$ and z in C. Then $\widetilde{I} = (C \oplus R) \times_{\tilde{\gamma}} C$ is the universal covering group of $Isom(\mathbb{S}ol_0^4)$. If M is a closed $\mathbb{S}ol_0^4$-manifold its fundamental group π is a semidirect product $Z^3 \times_\theta Z$, where $\theta(1) \in GL(3,\mathbb{Z})$ has two complex conjugate eigenvalues $\lambda \neq \bar{\lambda}$ with $|\lambda| \neq 0$ or 1 and one real eigenvalue ρ such that $|\rho| = |\lambda|^{-2}$ (see Chapter 8). If M is orientable (i.e., $\rho > 0$) then π is a lattice in $S_\pi = (C \oplus R) \times_{\tilde{\theta}} R < \widetilde{I}$, where $\tilde{\theta}(r) = \tilde{\gamma}(r\log(\lambda))$. In general, π is a lattice in $Aff(S_{\pi^+})$. The action of $\widetilde{I}$ on Sol_0^4 determines a diffeomorphism $S_{\pi^+}/\pi \cong M$, and so M is an infrasolvmanifold with a discrete presentation.

We shall see in Chapter 8 that every orientable 4-dimensional infrasolvmanifold is diffeomorphic to a geometric 4-manifold, but the argument uses the Mostow fibration and is differential-topological rather than differential-geometric.

7.4 Geometric decompositions

An n-manifold M admits a *geometric decomposition* if it has a finite collection of disjoint 2-sided hypersurfaces S such that each component of $M - \cup S$ is geometric of finite volume, i.e., is homeomorphic to $\Gamma\backslash X$, for some geometry $\mathbb{X}$ and lattice Γ. We shall call the hypersurfaces S *cusps* and the components of $M - \cup S$ *pieces* of M. The decomposition is *proper* if the set of cusps is nonempty.

Theorem 7.1 *If a closed 4-manifold M admits a geometric decomposition then either*

(1) *M is geometric; or*

(2) *M has a codimension-2 foliation with leaves S^2 or RP^2; or*

(3) *the components of $M - \cup S$ all have geometry $\mathbb{H}^2 \times \mathbb{H}^2$; or*

(4) *the components of $M - \cup S$ have geometry $\mathbb{H}^4$, $\mathbb{H}^3 \times \mathbb{E}^1$, $\mathbb{H}^2 \times \mathbb{E}^2$ or $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$; or*

(5) *the components of $M - \cup S$ have geometry $\mathbb{H}^2(\mathbb{C})$ or $\mathbb{F}^4$.*

In cases (3), (4) or (5) $\chi(M) \geq 0$ and in cases (4) or (5) M is aspherical.

Proof The proof consists in considering the possible ends (cusps) of complete geometric 4-manifolds of finite volume. The hypersurfaces bounding a component of $M - \cup S$ correspond to the ends of its interior. If the geometry is of solvable or compact type then there are no ends, since every lattice is then cocompact [Rg]. Thus we may concentrate on the eight geometries $\mathbb{S}^2 \times \mathbb{H}^2$, $\mathbb{H}^2 \times \mathbb{E}^2$, $\mathbb{H}^2 \times \mathbb{H}^2$, $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$, $\mathbb{H}^3 \times \mathbb{E}^1$, $\mathbb{H}^4$, $\mathbb{H}^2(\mathbb{C})$ and $\mathbb{F}^4$. The ends of a geometry of constant negative curvature $\mathbb{H}^n$ are flat [Eb80]; since any lattice in a Lie group must meet the radical in a lattice it follows easily that the ends are also flat in the mixed euclidean cases $\mathbb{H}^3 \times \mathbb{E}^1$, $\mathbb{H}^2 \times \mathbb{E}^2$ and $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$. Similarly, the ends of $\mathbb{S}^2 \times \mathbb{H}^2$-manifolds are $\mathbb{S}^2 \times \mathbb{E}^1$-manifolds. Since the elements of $PSL(2, \mathbb{C})$ corresponding to the cusps of finite area hyperbolic surfaces are parabolic, the ends of $\mathbb{F}^4$-manifolds are $\mathbb{N}il^3$-manifolds. The ends of $\mathbb{H}^2(\mathbb{C})$-manifolds are also $\mathbb{N}il^3$-manifolds [Ep87], while the ends of $\mathbb{H}^2 \times \mathbb{H}^2$-manifolds are $\mathbb{S}ol^3$-manifolds in the irreducible cases [Sh63], and graph manifolds whose fundamental groups contain nonabelian free subgroups otherwise. Clearly if two pieces are contiguous their common cusps must be homeomorphic. If the piece is not a reducible $\mathbb{H}^2 \times \mathbb{H}^2$-manifold then the inclusion of a cusp into the closure of the piece induces a monomorphism on fundamental group.

If M is a closed 4-manifold with a geometric decomposition of type (2) the inclusions of the cusps into the closures of the pieces induce isomorphisms on π_2, and a Mayer-Vietoris argument in the universal covering space $\widetilde{M}$ shows that $\widetilde{M}$ is homotopy equivalent to S^2. The natural foliation of $S^2 \times H^2$ by 2-spheres induces a codimension-2 foliation on each piece, with leaves S^2 or RP^2. The cusps bounding the closure of a piece are $\mathbb{S}^2 \times \mathbb{E}^1$-manifolds, and hence also have codimension-1 foliations, with leaves S^2 or RP^2. Together these foliations give a foliation of the closure of the piece, so that each cusp is a union of leaves. The homeomorphisms identifying cusps of contiguous pieces are isotopic to isometries of the corresponding $\mathbb{S}^2 \times \mathbb{E}^1$-manifolds. As the foliations of the cusps are preserved by isometries M admits a foliation with leaves S^2 or RP^2. (In other words, it is the total space of an orbifold bundle over a hyperbolic 2-orbifold, with general fibre S^2.)

If at least one piece has an aspherical geometry other than $\mathbb{H}^2 \times \mathbb{H}^2$ then all do and M is aspherical. Since all the pieces of type $\mathbb{H}^4$, $\mathbb{H}^2(\mathbb{C})$ or $\mathbb{H}^2 \times \mathbb{H}^2$ have strictly positive Euler characteristic while those of type $\mathbb{H}^3 \times \mathbb{E}^1$, $\mathbb{H}^2 \times \mathbb{E}^2$, $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$ or $\mathbb{F}^4$ have Euler characteristic 0 we must have $\chi(M) \geq 0$. ∎

If in case (2) M admits a foliation with all leaves homeomorphic then the projection to the leaf space is a submersion and so M is the total space of an S^2-bundle or RP^2-bundle over a hyperbolic surface. In particular, the covering space M_κ corresponding to the kernel κ of the action of $\pi_1(M)$ on $\pi_2(M) \cong Z$ is the total space of an S^2-bundle over a hyperbolic surface. In Chapter 9 we shall show that S^2-bundles and RP^2-bundles over aspherical surfaces are geometric. This surely holds also for orbifold bundles (defined in the next section) over flat or hyperbolic 2-orbifolds, with general fibre S^2.

If an aspherical closed 4-manifold has a nontrivial geometric decomposition with no pieces of type $\mathbb{H}^2 \times \mathbb{H}^2$ then its fundamental group contains nilpotent subgroups of Hirsch length 3 (corresponding to the cusps).

Is there an essentially unique minimal decomposition? Since hyperbolic surfaces are connected sums of tori, and a punctured torus admits a complete hyperbolic geometry of finite area, we cannot expect that there is an unique decomposition, even in dimension 2. Any PD_n-group satisfying *Max-c* (the maximal condition on centralizers) has an essentially unique minimal finite splitting along virtually poly-Z subgroups of Hirsch length $n-1$, by Theorem A2 of [Kr90]. Do all fundamental groups of aspherical manifolds with geometric decompositions have *Max-c*? A compact non-positively curved n-manifold ($n \geq 3$) with convex boundary is either flat or has a canonical decomposition along totally geodesic closed flat hypersurfaces into pieces which are Seifert fibred or codimension-1 atoroidal [LS00]. Which 4-manifolds with geometric decompositions admit such metrics? (Closed $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$-manifolds do not [KL96].)

Closed $\mathbb{H}^4$- or $\mathbb{H}^2(\mathbb{C})$-manifolds admit no proper geometric decompositions, since their fundamental groups have no noncyclic abelian subgroups [Pr43]. A similar argument shows that closed $\mathbb{H}^3 \times \mathbb{E}^1$-manifolds admit no proper decompositions, since they are finitely covered by cartesian products of $\mathbb{H}^3$-manifolds with S^1. Thus closed 4-manifolds with a proper geometric decomposition involving pieces of types other than $\mathbb{S}^2 \times \mathbb{H}^2$, $\mathbb{H}^2 \times \mathbb{E}^2$, $\mathbb{H}^2 \times \mathbb{H}^2$ or $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$ are never geometric.

Many $\mathbb{S}^2 \times \mathbb{H}^2$-, $\mathbb{H}^2 \times \mathbb{H}^2$-, $\mathbb{H}^2 \times \mathbb{E}^2$- and $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$-manifolds admit proper geometric decompositions. On the other hand, a manifold with a geometric decomposition into pieces of type $\mathbb{H}^2 \times \mathbb{E}^2$ need not be geometric. For instance, let $G = \langle u, v, x, y \mid [u, v] = [x, y] \rangle$ be the fundamental group of $T\sharp T$, the closed orientable surface of genus 2, and let $\theta : G \to SL(2, \mathbb{Z})$ be the epimorphism determined by $\theta(u) = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $\theta(x) = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 1 \end{smallmatrix}\right)$, Then the semidirect product $\pi = Z^2 \times_\theta G$ is the fundamental group of a torus bundle over $T\sharp T$ which has a geometric decomposition into two pieces of type $\mathbb{H}^2 \times \mathbb{E}^2$, but is not geometric, since π does not have a subgroup of finite index with centre Z^2.

It is easily seen that each $\mathbb{S}^2 \times \mathbb{E}^1$-manifold may be realized as the end of a complete $\mathbb{S}^2 \times \mathbb{H}^2$-manifold with finite volume and a single end. However, if the manifold is orientable the ends must be orientable, and if it is complex analytic then they must be $S^2 \times S^1$. Every flat 3-manifold is a cusp of some complete $\mathbb{H}^4$-manifold with finite volume [Ni98]. However if such a manifold has only one cusp the cusp cannot have holonomy $Z/3Z$ or $Z/6Z$ [LR00]. The fundamental group of a cusp of an $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$-manifold must have a chain of abelian normal subgroups $Z < Z^2 < Z^3$; not all orientable flat 3-manifold groups have such subgroups. The ends of complete, complex analytic $\mathbb{H}^2 \times \mathbb{H}^2$-manifolds with finite volume and irreducible fundamental group are orientable $\mathbb{S}ol^3$-manifolds which are mapping tori, and all such may be realized in this way [Sh63].

Let M be the double of $T_o \times T_o$, where $T_o = T - intD^2$ is the once-punctured torus. Since T_o admits a complete hyperbolic geometry of finite area M admits a geometric decomposition into two pieces of type $\mathbb{H}^2 \times \mathbb{H}^2$. However as $F(2) \times F(2)$ has cohomological dimension 2 the homomorphism of fundamental groups induced by the inclusion of the cusp into $T_o \times T_o$ has nontrivial kernel, and M is not aspherical.

7.5 Orbifold bundles

An n-dimensional orbifold B has an open covering by subspaces of the form D^n/G, where G is a finite subgroup of $O(n)$. Let F be a closed manifold. An *orbifold bundle with general fibre F over B* is a map $f : M \to B$ which is locally equivalent to a projection $G\backslash(F \times D^n) \to G\backslash D^n$, where G acts freely on F and effectively and orthogonally on D^n.

If the base B has a finite regular covering $\hat{B}$ which is a manifold, then p induces a fibre bundle projection $\hat{p} : \hat{M} \to \hat{B}$ with fibre F, and the action of the covering group maps fibres to fibres. Conversely, if $p_1 : M_1 \to B_1$ is a fibre bundle projection with fibre F_1 and G is a finite group which acts freely on M_1 and maps fibres to fibres then passing to orbit spaces gives an orbifold bundle $p : M = G\backslash M_1 \to B = H\backslash B_1$ with general fibre $F = K\backslash F_1$, where H is the induced group of homeomorphisms of B_1 and K is the kernel of the epimorphism from G to H.

Theorem 7.2 [Cb] *Let M be an infrasolvmanifold. Then there is an orbifold bundle $p : M \to B$ with general fibre an infranilmanifold and base a flat orbifold.*

Proof Let (S, Γ) be a presentation for M and let R be the nilradical of S. Then $A = S/R$ is a 1-connected abelian Lie group, and so $A \cong \mathbb{R}^d$ for some

$d \geq 0$. Since R is characteristic in S there is a natural projection $q : Aff(S) \to Aff(A)$. Let $\Gamma_S = \Gamma \cap S$ and $\Gamma_R = \Gamma \cap R$. Then the action of Γ_S on S induces an action of the discrete group $q(\Gamma_S) = R\Gamma_S/R$ on A. The Mostow fibration for $M_1 = \Gamma_S\backslash S$ is the quotient map to $B_1 = q(\Gamma_S)\backslash A$, which is a bundle projection with fibre $F_1 = \Gamma_R\backslash R$. Now Γ_o is normal in R, by Corollary 3 of Theorem 2.3 of [Rg], and Γ_R/Γ_o is a lattice in the nilpotent Lie group R/Γ_o. Therefore F_1 is a nilmanifold, while B_1 is a torus.

The finite group Γ/Γ_S acts on M_1, respecting the Mostow fibration. Let $\overline{\Gamma} = q(\Gamma)$, $K = \Gamma \cap \mathrm{Ker}(q)$ and $B = \overline{\Gamma}\backslash A$. Then the induced map $p : M \to B$ is an orbifold bundle projection with general fibre the infranilmanifold $F = K\backslash R = (K/\Gamma_o)\backslash(R/\Gamma_o)$, and base a flat orbifold. □

We shall call $p : M \to B$ the *Mostow orbifold bundle* corresponding to the presentation (S, Γ). In Theorem 8.9 we shall use this construction to show that orientable 4-dimensional infrasolvmanifolds are determined up to diffeomorphism by their fundamental groups, with the possible exception of manifolds having one of two virtually abelian groups.

7.6 Realization of virtual bundle groups

Every extension of one PD_2-group by another may be realized by some surface bundle, by Theorem 5.2. The study of Seifert fibred 4-manifolds and singular fibrations of complex surfaces lead naturally to consideration of the larger class of torsion free groups which are virtually such extensions. Johnson has asked whether such " virtual bundle groups" may be realized by aspherical 4-manifolds.

Theorem 7.3 *Let π be a torsion free group with normal subgroups $K < G < \pi$ such that K and G/K are PD_2-groups and $[\pi : G] < \infty$. Then π is the fundamental group of an aspherical closed smooth 4-manifold which is the total space of an orbifold bundle with general fibre an aspherical closed surface over a 2-dimensional orbifold.*

Proof Let $p : \pi \to \pi/K$ be the quotient homomorphism. Since π is torsion free the preimage in π of any finite subgroup of π/K is a PD_2-group. As the finite subgroups of π/K have order at most $[\pi : G]$, we may assume that π/K has no nontrivial finite normal subgroup, and so is the orbifold fundamental group of some 2-dimensional orbifold B, by the solution to the Nielsen realization problem for surfaces [Ke83]. Let F be the aspherical closed surface with

$\pi_1(F) \cong K$. If π/K is torsion free then B is a closed aspherical surface, and the result follows from Theorem 5.2. In general, B is the union of a punctured surface B_o with finitely many cone discs and regular neighborhoods of reflector curves (possibly containing corner points). The latter may be further decomposed as the union of squares with a reflector curve along one side and with at most one corner point, with two such squares meeting along sides adjacent to the reflector curve. These suborbifolds U_i (i.e., cone discs and squares) are quotients of D^2 by finite subgroups of $O(2)$. Since B is finitely covered (as an orbifold) by the aspherical surface with fundamental group G/K these finite groups embed in $\pi_1^{\mathrm{orb}}(B) \cong \pi/K$, by the Van Kampen Theorem for orbifolds.

The action of π/K on K determines an action of $\pi_1(B_o)$ on K and hence an F-bundle over B_o. Let H_i be the preimage in π of $\pi_1^{orb}(U_i)$. Then H_i is torsion free and $[H_i : K] < \infty$, so H_i acts freely and cocompactly on $\mathbb{X}^2$, where $\mathbb{X}^2 = \mathbb{R}^2$ if $\chi(K) = 0$ and $\mathbb{X}^2 = \mathbb{H}^2$ otherwise, and F is a finite covering space of $H_i\backslash\mathbb{X}^2$. The obvious action of H_i on $\mathbb{X}^2 \times D^2$ determines a bundle with general fibre F over the orbifold U_i. Since self homeomorphisms of F are determined up to isotopy by the induced element of $\mathrm{Out}(K)$, bundles over adjacent suborbifolds have isomorphic restrictions along common edges. Hence these pieces may be assembled to give a bundle with general fibre F over the orbifold B, whose total space is an aspherical closed smooth 4-manifold with fundamental group π. □

We shall verify in Theorem 9.8 that torsion free groups commensurate with products of two centreless PD_2-groups are also realizable.

We can improve upon Theorem 5.7 as follows.

Corollary 7.3.1 *Let M be a closed 4-manifold M with fundamental group π. Then the following are equivalent.*

(1) *M is homotopy equivalent to the total space of an orbifold bundle with general fibre an aspherical surface over an $\mathbb{E}^2$- or $\mathbb{H}^2$-orbifold;*

(2) *π has an FP_2 normal subgroup K such that π/K is virtually a PD_2-group and $\pi_2(M) = 0$;*

(3) *π has a normal subgroup N which is a PD_2-group and $\pi_2(M) = 0$.*

Proof Condition (1) clearly implies (2) and (3). Conversely, if they hold the argument of Theorem 5.7 shows that K is a PD_2-group and N is virtually a PD_2-group. In each case (1) now follows from Theorem 7.2. □

It follows easily from the argument of part (1) of Theorem 5.4 that if π is a group with a normal subgroup K such that K and π/K are PD_2-groups with $\zeta K = \zeta(\pi/K) = 1$, ρ is a subgroup of finite index in π and $L = K \cap \rho$ then $C_\rho(L) = 1$ if and only if $C_\pi(K) = 1$. Since ρ is virtually a product of PD_2-groups with trivial centres if and only if π is, Johnson's trichotomy extends to groups commensurate with extensions of one centreless PD_2-group by another.

Theorem 7.2 settles the realization question for groups of type I. (For suppose π has a subgroup σ of finite index with a normal subgroup ν such that ν and σ/ν are PD_2-groups with $\zeta\nu = \zeta(\sigma/\nu) = 1$. Let $G = \cap h\sigma h^{-1}$ and $K = \nu \cap G$. Then $[\pi : G] < \infty$, G is normal in π, and K and G/K are PD_2-groups. If G is of type I then K is characteristic in G, by Theorem 5.5, and so is normal in π.) Groups of type II need not have such normal PD_2-subgroups - although this is almost true. It is not known whether every type III extension of centreless PD_2-groups has a characteristic PD_2-subgroup (although this is so in many cases, by the corollaries to Theorem 5.6).

If π is an extension of Z^2 by a normal PD_2-subgroup K with $\zeta K = 1$ then $C_\pi(K) = \sqrt{\pi}$, and $[\pi : KC_\pi(K)] < \infty$ if and only if π is virtually $K \times Z^2$, so Johnson's trichotomy extends to such groups. The three types may be characterized by (I) $\sqrt{\pi} \cong Z$, (II) $\sqrt{\pi} \cong Z^2$, and (III) $\sqrt{\pi} = 1$. As these properties are shared by commensurate torsion free groups the trichotomy extends further to torsion free groups which are virtually such extensions. There is at present no uniqueness result corresponding to Theorem 5.5 for such subgroups $K < \pi$, and (excepting for groups of type II) it is not known whether every such group is realized by some aspherical closed 4-manifold. (In fact, it also appears to be unknown in how many ways a 3-dimensional mapping torus may fibre over S^1.)

The Johnson trichotomy is inappropriate if $\zeta K \neq 1$, as there are then nontrivial extensions with trivial action ($\theta = 1$). Moreover $\mathrm{Out}(K)$ is virtually free and so the action θ is never injective. However all such groups π may be realized by aspherical 4-manifolds, for either $\sqrt{\pi} \cong Z^2$ and Theorem 7.2 applies, or π is virtually poly-Z and is the fundamental group of an infrasolvmanifold. (See Chapter 8.)

7.7 Seifert fibrations

A closed 4-manifold M is *Seifert fibred* if it is the total space of an orbifold bundle with general fibre a torus or Klein bottle over a 2-orbifold. (In [Zn85], [Ue90,91] it is required that the general fibre be a torus. This is always so if the manifold is orientable.) The fundamental group π of such a 4-manifold then has a rank two free abelian normal subgroup A such that π/A is virtually a surface group. If the base orbifold is good then the manifold is finitely covered by a torus bundle over a closed surface. This is in fact so in general, by the following theorem. In particular, $\chi(M) = 0$.

Theorem (Ue) *Let S be a closed orientable 4-manifold which is Seifert fibred over the 2-orbifold B. Then*

(1) *If B is spherical or bad S has geometry $\mathbb{S}^3 \times \mathbb{E}^1$ or $\mathbb{S}^2 \times \mathbb{E}^2$;*

(2) *If B is euclidean then S has geometry $\mathbb{E}^4$, $\mathbb{N}il^4$, $\mathbb{N}il^3 \times \mathbb{E}^1$ or $\mathbb{S}ol^3 \times \mathbb{E}^1$;*

(3) *If B is orientable and hyperbolic then S is geometric if and only if it has a complex structure, in which case the geometry is either $\mathbb{H}^2 \times \mathbb{E}^2$ or $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$.*

Conversely, excepting only two flat 4-manifolds, any orientable 4-manifold admitting one of these geometries is Seifert fibred. ∎

If the base is euclidean or hyperbolic then S is determined up to diffeomorphism by $\pi_1(S)$; if moreover the base is hyperbolic or S is geometric of type $\mathbb{N}il^4$ or $\mathbb{S}ol^3 \times \mathbb{E}^1$ there is a fibre-preserving diffeomorphism. If the base is bad or spherical then S may admit many inequivalent Seifert fibrations.

Less is known about the nonorientable cases. Seifert fibred 4-manifolds with general fibre a torus and base a hyperbolic orbifold with no reflector curves are determined up to fibre preserving diffeomorphism by their fundamental groups [Zi69]. Closed 4-manifolds which fibre over S^1 with fibre a small Seifert fibred 3-manifold are determined up to diffeomorphism by their fundamental groups [Oh90]. This class includes many nonorientable Seifert fibred 4-manifolds over bad, spherical or euclidean bases, but not all. It may be true in general that a Seifert fibred 4-manifold is geometric if and only if its orientable double covering space is geometric, and that aspherical Seifert fibred 4-manifolds are determined up to diffeomorphism by their fundamental groups.

The homotopy type of a $\mathbb{S}^2 \times \mathbb{E}^2$-manifold is determined up to finite ambiguity by the fundamental group (which must be virtually Z^2), Euler characteristic (which must be 0) and Stiefel-Whitney classes. There are just nine possible fundamental groups. Six of these have infinite abelianization, and the above invariants determine the homotopy type in these cases. (See Chapter 10.) The homotopy type of a $\mathbb{S}^3 \times \mathbb{E}^1$-manifold is determined by the fundamental group (which has two ends), Euler characteristic (which is 0), orientation character w_1 and first k-invariant in $H^4(\pi;\pi_3)$. (See Chapter 11.)

Every Seifert fibred 4-manifold with base an euclidean orbifold has Euler characteristic 0 and fundamental group solvable of Hirsch length 4, and so is homeomorphic to an infrasolvmanifold, by Theorem 6.11 and [AJ76]. As no group of type $\mathbb{S}ol_0^4$, $\mathbb{S}ol_1^4$ or $\mathbb{S}ol_{m,n}^4$ (with $m \neq n$) has a rank two free abelian normal subgroup, the manifold must have one of the geometries $\mathbb{E}^4$, $\mathbb{N}il^4$, $\mathbb{N}il \times \mathbb{E}^1$ or $\mathbb{S}ol \times \mathbb{E}^1$. Conversely, excepting only three flat 4-manifolds, such manifolds are Seifert fibred. The fundamental group of a closed $\mathbb{N}il^3 \times \mathbb{E}^1$- or $\mathbb{N}il^4$-manifold has a rank two free abelian normal subgroup, by Theorem 1.5. If π is the fundamental group of a $\mathbb{S}ol^3 \times \mathbb{E}^1$-manifold then the commutator subgroup of the intersection of all index 4 subgroups is such a subgroup. (In the $\mathbb{N}il^4$ and $\mathbb{S}ol^3 \times \mathbb{E}^1$ cases there is an unique maximal such subgroup, and the general fibre must be a torus.) Case-by-case inspection of the 74 flat 4-manifold groups shows that all but three have such subgroups. The only exceptions are the semidirect products $G_6 \times_\theta Z$ where $\theta = j$, cej and $abcej$. (See Chapter 8. There is a minor oversight in [Ue90]; in fact there are *two* orientable flat four-manifolds which are not Seifert fibred.)

As $\mathbb{H}^2 \times \mathbb{E}^2$- and $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$-manifolds are aspherical, they are determined up to homotopy equivalence by their fundamental groups. See Chapter 9 for more details.

Theorem 7.3 specializes to give the following characterization of the fundamental groups of Seifert fibred 4-manifolds.

Theorem 7.4 *A group π is the fundamental group of a closed 4-manifold which is Seifert fibred over a hyperbolic base 2-orbifold with general fibre a torus if and only if it is torsion free, $\sqrt{\pi} \cong Z^2$, $\pi/\sqrt{\pi}$ has no nontrivial finite normal subgroup and $\pi/\sqrt{\pi}$ is virtually a PD_2-group.* □

If $\sqrt{\pi}$ is central ($\zeta\pi \cong Z^2$) the corresponding Seifert fibred manifold $M(\pi)$ admits an effective torus action with finite isotropy subgroups.

7.8 Complex surfaces and related structures

In this section we shall summarize what we need from [BPV], [Ue90,91], [Wl86] and [GS], and we refer to these sources for more details.

A *complex surface* shall mean a compact connected nonsingular complex analytic manifold S of complex dimension 2. It is Kähler (and thus diffeomorphic to a projective algebraic surface) if and only if $\beta_1(S)$ is even. Since the Kähler condition is local, all finite covering spaces of such a surface must also have β_1

even. If S has a complex submanifold $L \cong CP^1$ with self-intersection -1 then L may be blown down: there is a complex surface S_1 and a holomorphic map $p : S \to S_1$ such that $p(L)$ is a point and p restricts to a biholomorphic isomorphism from $S-L$ to $S_1-p(L)$. In particular, S is diffeomorphic to $S_1\sharp\overline{CP^2}$. If there is no such embedded projective line L the surface is *minimal*. Excepting only the ruled surfaces, every surface has an unique minimal representative.

For many of the 4-dimensional geometries (X, G) the identity component G_o of the isometry group preserves a natural complex structure on X, and so if π is a discrete subgroup of G_o which acts freely on X the quotient $\pi\backslash X$ is a complex surface. This is clear for the geometries $\mathbb{CP}^2$, $\mathbb{S}^2\times\mathbb{S}^2$, $\mathbb{S}^2\times\mathbb{E}^2$, $\mathbb{S}^2\times\mathbb{H}^2$, $\mathbb{H}^2\times\mathbb{E}^2$, $\mathbb{H}^2\times\mathbb{H}^2$ and $\mathbb{H}^2(\mathbb{C})$. (The corresponding model spaces may be identified with CP^2, $CP^1\times CP^1$, $CP^1\times C$, $CP^1\times H^2$, $H^2\times C$, $H^2\times H^2$ and the unit ball in C^2, respectively, where H^2 is identified with the upper half plane.) It is also true for $\mathbb{N}il^3\times\mathbb{E}^1$, $\mathbb{S}ol_0^4$, $\mathbb{S}ol_1^4$, $\widetilde{\mathbb{SL}}\times\mathbb{E}^1$ and $\mathbb{F}^4$. In addition, the subgroups $R^4\tilde{\times}U(2)$ of $E(4)$ and $U(2)\times R$ of $Isom(\mathbb{S}^3\times\mathbb{E}^1)$ act biholomorphically on C^2 and $C^2-\{0\}$, respectively, and so some $\mathbb{E}^4$- and $\mathbb{S}^3\times\mathbb{E}^1$-manifolds have complex structures. No other geometry admits a compatible complex structure. Since none of the model spaces contain an embedded S^2 with self-intersection -1 any complex surface which admits a compatible geometry must be minimal.

Complex surfaces may be coarsely classified by their Kodaira dimension κ, which may be $-\infty$, 0, 1 or 2. Within this classification, minimal surfaces may be further classified into a number of families. We have indicated in parentheses where the geometric complex surfaces appear in this classification. (The dashes signify families which include nongeometric surfaces.)

$\kappa=-\infty$: Hopf surfaces ($\mathbb{S}^3\times\mathbb{E}^1$, -); Inoue surfaces ($\mathbb{S}ol_0^4$, $\mathbb{S}ol_1^4$);

rational surfaces ($\mathbb{CP}^2$, $\mathbb{S}^2\times\mathbb{S}^2$); ruled surfaces ($\mathbb{S}^2\times\mathbb{E}^2$, $\mathbb{S}^2\times\mathbb{H}^2$, -).

$\kappa=0$: complex tori ($\mathbb{E}^4$); hyperelliptic surfaces ($\mathbb{E}^4$); Enriques surfaces (-);

K3 surfaces (-); Kodaira surfaces ($\mathbb{N}il^3\times\mathbb{E}^1$).

$\kappa=1$: minimal properly elliptic surfaces ($\widetilde{\mathbb{SL}}\times\mathbb{E}^1$, $\mathbb{H}^2\times\mathbb{E}^2$).

$\kappa=2$: minimal (algebraic) surfaces of general type ($\mathbb{H}^2\times\mathbb{H}^2$, $\mathbb{H}^2(\mathbb{C})$, -).

A *Hopf surface* is a complex surface whose universal covering space is homeomorphic to $S^3\times R\cong C^2-\{0\}$. Some Hopf surfaces admit no compatible geometry, and there are $\mathbb{S}^3\times\mathbb{E}^1$-manifolds that admit no complex structure. The *Inoue* surfaces are exactly the complex surfaces admitting one of the geometries $\mathbb{S}ol_0^4$ or $\mathbb{S}ol_1^4$.

A *rational surface* is a complex surface birationally equivalent to CP^2. Minimal rational surfaces are diffeomorphic to CP^2 or to $CP^1 \times CP^1$. A *ruled surface* is a complex surface which is holomorphically fibred over a smooth complex curve (closed orientable 2-manifold) of genus $g > 0$ with fibre CP^1. Rational and ruled surfaces may be characterized as the complex surfaces S with $\kappa(S) = -\infty$ and $\beta_1(S)$ even. Not all ruled surfaces admit geometries compatible with their complex structures.

A *complex torus* is a quotient of C^2 by a lattice, and a *hyperelliptic surface* is one properly covered by a complex torus. If S is a complex surface which is homeomorphic to a flat 4-manifold then S is a complex torus or is hyperelliptic, since it is finitely covered by a complex torus. Since S is orientable and $\beta_1(S)$ is even $\pi = \pi_1(S)$ must be one of the eight flat 4-manifold groups of orientable type and with $\pi \cong Z^4$ or $I(\pi) \cong Z^2$. In each case the holonomy group is cyclic, and so is conjugate (in $GL^+(4,\mathbb{R})$) to a subgroup of $GL(2,\mathbb{C})$. (See Chapter 8.) Thus all of these groups may be realized by complex surfaces. A *Kodaira surface* is finitely covered by a surface which fibres holomorphically over an elliptic curve with fibres of genus 1.

An *elliptic surface* S is a complex surface which admits a holomorphic map p to a complex curve such that the generic fibres of p are diffeomorphic to the torus T. If the elliptic surface S has no singular fibres it is Seifert fibred, and it then has a geometric structure if and only if the base is a good orbifold. An orientable Seifert fibred 4-manifold over a hyperbolic base has a geometric structure if and only if it is an elliptic surface without singular fibres [Ue90]. The elliptic surfaces S with $\kappa(S) = -\infty$ and $\beta_1(S)$ odd are the geometric Hopf surfaces. The elliptic surfaces S with $\kappa(S) = -\infty$ and $\beta_1(S)$ even are the cartesian products of elliptic curves with CP^1.

All rational, ruled and hyperelliptic surfaces are projective algebraic surfaces, as are all surfaces with $\kappa = 2$. Complex tori and surfaces with geometry $\mathbb{H}^2 \times \mathbb{E}^2$ are diffeomorphic to projective algebraic surfaces. Hopf, Inoue and Kodaira surfaces and surfaces with geometry $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$ all have β_1 odd, and so are not Kähler, let alone projective algebraic.

An *almost complex structure* on a smooth $2n$-manifold M is a reduction of the structure group of its tangent bundle to $GL(n,\mathbb{C}) < GL^+(2n,\mathbb{R})$. Such a structure determines an orientation on M. If M is a closed oriented 4-manifold and $c \in H^2(M;\mathbb{Z})$ then there is an almost complex structure on M with first Chern class c and inducing the given orientation if and only if $c \equiv w_2(M)$ *mod* (2) and $c^2 \cap [M] = 3\sigma(M) + 2\chi(M)$, by a theorem of Wu. (See the Appendix to Chapter I of [GS] for a recent account.)

A *symplectic structure* on a closed smooth manifold M is a closed nondegenerate 2-form ω. Nondegenerate means that for all $x \in M$ and all $u \in T_xM$ there is a $v \in T_xM$ such that $\omega(u,v) \neq 0$. Manifolds admitting symplectic structures are even-dimensional and orientable. A condition equivalent to nondegeneracy is that the n-fold wedge $\omega^{\wedge n}$ is nowhere 0, where $2n$ is the dimension of M. The n^{th} cup power of the corresponding cohomology class $[\omega]$ is then a nonzero element of $H^{2n}(M;\mathbb{R})$. Any two of a riemannian metric, a symplectic structure and an almost complex structure together determine a third, if the given two are compatible. In dimension 4, this is essentially equivalent to the fact that $SO(4)\cap Sp(4) = SO(4)\cap GL(2,\mathbb{C}) = Sp(4)\cap GL(2,\mathbb{C}) = U(2)$, as subgroups of $GL(4,\mathbb{R})$. (See [GS] for a discussion of relations between these structures.) In particular, Kähler surfaces have natural symplectic structures, and symplectic 4-manifolds admit compatible almost complex tangential structures. However orientable $\mathbb{S}ol^3 \times \mathbb{E}^1$-manifolds which fibre over T are symplectic [Ge92] but have no complex structure (by the classification of surfaces) and Hopf surfaces are complex manifolds with no symplectic structure (since $\beta_2 = 0$).

Chapter 8

Solvable Lie geometries

The main result of this chapter is the characterization of 4-dimensional infrasolvmanifolds up to homeomorphism, given in §1. All such manifolds are either mapping tori of self homeomorphisms of 3-dimensional infrasolvmanifolds or are unions of two twisted I-bundles over such 3-manifolds. In the rest of the chapter we consider each of the possible 4-dimensional geometries of solvable Lie type.

In §2 we determine the automorphism groups of the flat 3-manifold groups, while in §3 and §4 we determine *ab initio* the 74 flat 4-manifold groups. There have been several independent computations of these groups; the consensus reported on page 126 of [Wo] is that there are 27 orientable groups and 48 nonorientable groups. However the tables of 4-dimensional crystallographic groups in [B-Z] list only 74 torsion free groups. As these computer-generated tables give little insight into how these groups arise, and as the earlier computations were never published in detail, we shall give a direct and elementary computation, motivated by Lemma 3.14. Our conclusions as to the numbers of groups with abelianization of given rank, isomorphism type of holonomy group and orientation type agree with those of [B-Z]. (We have not attempted to make the lists correspond.)

There are infinitely many examples for each of the other geometries. In §5 we show how these geometries may be distinguished, in terms of the group theoretic properties of their lattices. In §6, §7 and §8 we consider mapping tori of self homeomorphisms of $\mathbb{E}^3$-, $\mathbb{N}il^3$- and $\mathbb{S}ol^3$-manifolds, respectively. In §9 we show directly that "most" groups allowed by Theorem 8.1 are realized geometrically and outline classifications for them, while in §10 we show that "most" 4-dimensional infrasolvmanifolds are determined up to diffeomorphism by their fundamental groups.

8.1 The characterization

In this section we show that 4-dimensional infrasolvmanifolds may be characterized up to homeomorphism in terms of the fundamental group and Euler characteristic.

Theorem 8.1 *Let M be a closed 4-manifold with fundamental group π and such that $\chi(M) = 0$. The following conditions are equivalent:*

(1) *π is torsion free and virtually poly-Z and $h(\pi) = 4$;*

(2) *$h(\sqrt{\pi}) \geq 3$;*

(3) *π has an elementary amenable normal subgroup ρ with $h(\rho) \geq 3$, and $H^2(\pi; Z[\pi]) = 0$; and*

(4) *π is restrained, every finitely generated subgroup of π is FP_3 and π maps onto a virtually poly-Z group Q with $h(Q) \geq 3$.*

Moreover if these conditions hold M is aspherical, and is determined up to homeomorphism by π, and every automorphism of π may be realized by a self homeomorphism of M.

Proof If (1) holds then $h(\sqrt{\pi}) \geq 3$, by Theorem 1.6, and so (2) holds. This in turn implies (3), by Theorem 1.17. If (3) holds then π has one end, by Lemma 1.15, and $\beta_1^{(2)}(\pi) = 0$, by Corollary 2.3.1. Hence M is aspherical, by Corollary 3.5.2. Hence π is a PD_4-group and $3 \leq h(\rho) \leq c.d.\rho \leq 4$. In particular, ρ is virtually solvable, by Theorem 1.11. If $c.d.\rho = 4$ then $[\pi : \rho]$ is finite, by Strebel's Theorem, and so π is virtually solvable also. If $c.d.\rho = 3$ then $c.d.\rho = h(\rho)$ and so ρ is a duality group and is FP [Kr86]. Therefore $H^q(\rho; \mathbb{Q}[\pi]) \cong H^q(\rho; \mathbb{Q}[\rho]) \otimes \mathbb{Q}[\pi/\rho]$ and is 0 unless $q = 3$. It then follows from the LHSSS for π as an extension of π/ρ by ρ (with coefficients $\mathbb{Q}[\pi]$) that $H^4(\pi; \mathbb{Q}[\pi]) \cong H^1(\pi/\rho; \mathbb{Q}[\pi/\rho]) \otimes H^3(\rho; \mathbb{Q}[\rho])$. Therefore $H^1(\pi/\rho; \mathbb{Q}[\pi/\rho]) \cong Q$, so π/ρ has two ends and we again find that π is virtually solvable. In all cases π is torsion free and virtually poly-Z, by Theorem 9.23 of [Bi], and $h(\pi) = 4$.

If (4) holds then π is an ascending HNN extension $\pi \cong B*_\phi$ with base FP_3 and so M is aspherical, by Theorem 3.16. As in Theorem 2.13 we may deduce from [BG85] that B must be a PD_3-group and ϕ an isomorphism, and hence B and π are virtually poly-Z. Conversely (1) clearly implies (4).

The final assertions follow from Theorem 2.16 of [FJ], as in Theorem 6.11 above. □

Does the hypothesis $h(\rho) \geq 3$ in (3) imply $H^2(\pi; \mathbb{Z}[\pi]) = 0$? The examples $F \times S^1 \times S^1$ where $F = S^2$ or is a closed hyperbolic surface show that the condition that $h(\rho) > 2$ is necessary. (See also §1 of Chapter 9.)

Corollary 8.1.1 *The 4-manifold M is homeomorphic to an infrasolvmanifold if and only if the equivalent conditions of Theorem 8.1 hold.*

Proof If M is homeomorphic to an infrasolvmanifold then $\chi(M)=0$, π is torsion free and virtually poly-Z and $h(\pi)=4$ (see Chapter 7). Conversely, if these conditions hold then π is the fundamental group of an infrasolvmanifold, by [AJ76]. □

It is easy to see that all such groups are realizable by closed smooth 4-manifolds with Euler characteristic 0.

Theorem 8.2 *If π is torsion free and virtually poly-Z of Hirsch length 4 then it is the fundamental group of a closed smooth 4-manifold M which is either a mapping torus of a self homeomorphism of a closed 3-dimensional infrasolvmanifold or is the union of two twisted I-bundles over such a 3-manifold. Moreover, the 4-manifold M is determined up to homeomorphism by the group.*

Proof The Eilenberg-Mac Lane space $K(\pi,1)$ is a PD_4-complex with Euler characteristic 0. By Lemma 3.14, either there is an epimorphism $\phi:\pi\to Z$, in which case π is a semidirect product $G\times_\theta Z$ where $G=\mathrm{Ker}(\phi)$, or $\pi\cong G_1*_G G_2$ where $[G_1:G]=[G_2:G]=2$. The subgroups G, G_1 and G_2 are torsion free and virtually poly-Z. Since in each case π/G has Hirsch length 1 these subgroups have Hirsch length 3 and so are fundamental groups of closed 3-dimensional infrasolvmanifolds. The existence of such a manifold now follows by standard 3-manifold topology, while its uniqueness up to homeomorphism was proven in Theorem 6.11. □

The first part of this theorem may be stated and proven in purely algebraic terms, since torsion free virtually poly-Z groups are Poincaré duality groups. (See Chapter III of [Bi].) If π is such a group then either it is virtually nilpotent or $\sqrt{\pi}\cong Z^3$ or Γ_q for some q, by Theorems 1.5 and 1.6. In the following sections we shall consider how such groups may be realized geometrically. The geometry is largely determined by $\sqrt{\pi}$. We shall consider first the virtually abelian cases.

8.2 Flat 3-manifold groups and their automorphisms

The flat n-manifold groups for $n\leq 2$ are Z, Z^2 and $K=Z\times_{-1}Z$, the Klein bottle group. There are six orientable and four nonorientable flat 3-manifold groups. The first of the orientable flat 3-manifold groups G_1 - G_6 is $G_1=Z^3$. The next four have $I(G_i)\cong Z^2$ and are semidirect products $Z^2\times_T Z$ where $T=-I$, $\left(\begin{smallmatrix}0&-1\\1&-1\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix}0&-1\\1&1\end{smallmatrix}\right)$, respectively, is an element of finite order in $SL(2,\mathbb{Z})$. These groups all have cyclic holonomy groups, of orders 2, 3, 4 and 6, respectively. The group G_6 is the group of the Hantzsche-Wendt flat 3-manifold, and has a presentation

$$\langle x, y \mid xy^2x^{-1} = y^{-2},\ yx^2y^{-1} = y^{-2}\rangle.$$

Its maximal abelian normal subgroup is generated by x^2, y^2 and $(xy)^2$ and its holonomy group is the diagonal subgroup of $SL(3,\mathbb{Z})$, which is isomorphic to $(Z/2Z)^2$. (This group is the generalized free product of two copies of K, amalgamated over their maximal abelian subgroups, and so maps onto D.)

The nonorientable flat 3-manifold groups B_1 - B_4 are semidirect products $K \times_\theta Z$, corresponding to the classes in $Out(K) \cong (Z/2Z)^2$. In terms of the presentation $\langle x, y \mid xyx^{-1} = y^{-1}\rangle$ for K these classes are represented by the automorphisms θ which fix y and send x to x, xy, x^{-1} and $x^{-1}y$, respectively. The groups B_1 and B_2 are also semidirect products $Z^2 \times_T Z$, where $T = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ has determinant -1 and $T^2 = I$. They have holonomy groups of order 2, while the holonomy groups of B_3 and B_4 are isomorphic to $(Z/2Z)^2$.

All the flat 3-manifold groups either map onto Z or map onto D. The methods of this chapter may be easily adapted to find all such groups. Assuming these are all known we may use Sylow theory and a little topology to show that there are no others. We sketch here such an argument. Suppose that π is a flat 3-manifold group with finite abelianization. Then $0 = \chi(\pi) = 1+\beta_2(\pi)-\beta_3(\pi)$, so $\beta_3(\pi) \neq 0$ and π must be orientable. Hence the holonomy group $F = \pi/T(\pi)$ is a subgroup of $SL(3,\mathbb{Z})$. Let f be a nontrivial element of F. Then f has order 2, 3, 4 or 6, and has a $+1$-eigenspace of rank 1, since it is orientation preserving. This eigenspace is invariant under the action of the normalizer $N_F(\langle f\rangle)$, and the induced action of $N_F(\langle f\rangle)$ on the quotient space is faithful. Thus $N_F(\langle f\rangle)$ is isomorphic to a subgroup of $GL(2,\mathbb{Z})$ and so is cyclic or dihedral of order dividing 24. This estimate applies to the Sylow subgroups of F, since p-groups have nontrivial centres, and so the order of F divides 24. If F has a nontrivial cyclic normal subgroup then π has a normal subgroup isomorphic to Z^2 and hence maps onto Z or D. Otherwise F has a nontrivial Sylow 3-subgroup C which is not normal in F. The number of Sylow 3-subgroups is congruent to $1\ mod\ (3)$ and divides the order of F. The action of F by conjugation on the set of such subgroups is transitive. It must also be faithful. (For otherwise $\cap_{g\in F} gN_F(C)g^{-1} \neq 1$. As $N_F(C)$ is cyclic or dihedral it would follow that F must have a nontrivial cyclic normal subgroup, contrary to hypothesis.) Hence F must be A_4 or S_4, and so contains $V \cong (Z/2Z)^2$ as a normal subgroup. But any orientable flat 3-manifold group with holonomy V must have finite abelianization. As $Z/3Z$ cannot act freely on a $\mathbb{Q}$-homology 3-sphere (by the

Lefshetz fixed point theorem) it follows that A_4 cannot be the holonomy group of a flat 3-manifold. Hence we may exclude S_4 also.

We shall now determine the (outer) automorphism groups of each of the flat 3-manifold groups. Clearly $Out(G_1) = Aut(G_1) = GL(3,\mathbb{Z})$. If $2 \leq i \leq 5$ let $t \in G_i$ represent a generator of the quotient $G_i/I(G_i) \cong Z$. The automorphisms of G_i must preserve the characteristic subgroup $I(G_i)$ and so may be identified with triples $(v, A, \epsilon) \in Z^2 \times GL(2,\mathbb{Z}) \times \{\pm 1\}$ such that $ATA^{-1} = T^\epsilon$ and which act via A on $I(G_i) = Z^2$ and send t to $t^\epsilon v$. Such an automorphism is orientation preserving if and only if $\epsilon = det(A)$. The multiplication is given by $(v, A, \epsilon)(w, B, \eta) = (\Xi v + Aw, AB, \epsilon\eta)$, where $\Xi = I$ if $\eta = 1$ and $\Xi = -T^\epsilon$ if $\eta = -1$. The inner automorphisms are generated by $(0, T, 1)$ and $((T - I)Z^2, I, 1)$.

In particular, $Aut(G_2) \cong (Z^2 \times_\alpha GL(2,\mathbb{Z})) \times \{\pm 1\}$, where α is the natural action of $GL(2,\mathbb{Z})$ on Z^2, for Ξ is always I if $T = -I$. The involution $(0, I, -1)$ is central in $Aut(G_2)$, and is orientation reversing. Hence $Out(G_2)$ is isomorphic to $((Z/2Z)^2 \times_{P\alpha} PGL(2,\mathbb{Z})) \times (Z/2Z)$, where $P\alpha$ is the induced action of $PGL(2,\mathbb{Z})$ on $(Z/2Z)^2$.

If $n = 3$, 4 or 5 the normal subgroup $I(G_i)$ may be viewed as a module over the ring $R = \mathbb{Z}[t]/(\phi(t))$, where $\phi(t) = t^2+t+1$, t^2+1 or t^2-t+1, respectively. As these rings are principal ideal domains and $I(G_i)$ is torsion free of rank 2 as an abelian group, in each case it is free of rank 1 as an R-module. Thus matrices A such that $AT = TA$ correspond to units of R. Hence automorphisms of G_i which induce the identity on $G_i/I(G_i)$ have the form $(v, \pm T^m, 1)$, for some $m \in Z$ and $v \in Z^2$. There is also an involution $(0, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), -1)$ which sends t to t^{-1}. In all cases $\epsilon = det(A)$. It follows that $Out(G_3) \cong S_3 \times (Z/2Z)$, $Out(G_4) \cong (Z/2Z)^2$ and $Out(G_5) = Z/2Z$. All these automorphisms are orientation preserving.

The subgroup A of G_6 generated by $\{x^2, y^2, (xy)^2\}$ is the maximal abelian normal subgroup of G_6, and $G_6/A \cong (Z/2Z)^2$. Let a, b, c, d, e, f, i and j be the automorphisms of G_6 which send x to $x^{-1}, x, x, x, y^2x, (xy)^2x, y, xy$ and y to $y, y^{-1}, (xy)^2y, x^2y, y, (xy)^2y, x, x$, respectively. The natural homomorphism from $Aut(G_6)$ to $Aut(G_6/A) \cong GL(2,\mathbb{F}_2)$ is onto, as the images of i and j generate $GL(2,\mathbb{F}_2)$, and its kernel E is generated by $\{a, b, c, d, e, f\}$. (For an automorphism which induces the identity on G_6/A must send x to $x^{2p}y^{2q}(xy)^{2r}x$, and y to $x^{2s}y^{2t}(xy)^{2u}y$. The images of x^2, y^2 and $(xy)^2$ are then x^{4p+2}, y^{4t+2} and $(xy)^{4(r-u)+2}$, which generate A if and only if $p = 0$ or -1, $t = 0$ or -1 and $r = u - 1$ or u. Composing such an automorphism appropriately with a, b and c we may acheive $p = t = 0$ and $r = u$. Then by composing with powers of d, e and f we may obtain the identity automorphism.) The inner automorphisms are generated by bcd (conjugation by x) and $acef$ (conjugation by y). Then $Out(G_6)$ has a presentation

$$\langle a,b,c,e,i,j \mid a^2=b^2=c^2=e^2=i^2=j^6=1,\ a,b,c,e \text{ commute},\ iai=b,$$
$$ici=ae,\ jaj^{-1}=c,\ jbj^{-1}=abc,\ jcj^{-1}=be,\ j^3=abce,\ (ji)^2=bc\rangle.$$

The generators $a,b,c,$ and j represent orientation reversing automorphisms. (Note that $jej^{-1}=bc$ follows from the other relations. See [Zn90] for an alternative description.)

The group $B_1=Z\times K$ has a presentation

$$\langle t,x,y \mid tx=xt,\ ty=yt,\ xyx^{-1}=y^{-1}\rangle.$$

An automorphism of B_1 must preserve the centre ζB_1 (which has basis t,x^2) and $I(B_1)$ (which is generated by y). Thus the automorphisms of B_1 may be identified with triples $(A,m,\epsilon)\in\Upsilon_2\times Z\times\{\pm1\}$, where Υ_2 is the subgroup of $GL(2,\mathbb{Z})$ consisting of matrices congruent $mod\,(2)$ to upper triangular matrices. Such an automorphism sends t to t^ax^b, x to $t^cx^dy^m$ and y to y^ϵ, and induces multiplication by A on $B_1/I(B_1)\cong Z^2$. Composition of automorphisms is given by $(A,m,\epsilon)(B,n,\eta)=(AB,m+\epsilon n,\epsilon\eta)$. The inner automorphisms are generated by $(I,1,-1)$ and $(I,2,1)$, and so $Out(B_1)\cong\Upsilon_2\times(Z/2Z)$.

The group B_2 has a presentation

$$\langle t,x,y \mid txt^{-1}=xy,\ ty=yt,\ xyx^{-1}=y^{-1}\rangle.$$

Automorphisms of B_2 may be identified with triples $(A,(m,n),\epsilon)$, where $A\in\Gamma_2$, $m,n\in Z$, $\epsilon=\pm1$ and $m=(A_{11}-\epsilon)/2$. Such an automorphism sends t to $t^ax^by^m$, x to $t^cx^dy^n$ and y to y^ϵ, and induces multiplication by A on $B_2/I(B_2)\cong Z^2$. The automorphisms which induce the identity on $B_2/I(B_2)$ are all inner, and so $Out(B_2)\cong\Upsilon_2$.

The group B_3 has a presentation

$$\langle t,x,y \mid txt^{-1}=x^{-1},\ ty=yt,\ xyx^{-1}=y^{-1}\rangle.$$

An automorphism of B_3 must preserve $I(B_3)\cong K$ (which is generated by x,y) and $I(I(B_3))$ (which is generated by y). It follows easily that $Out(B_3)\cong(Z/2Z)^3$, and is generated by the classes of the automorphisms which fix y and send t to t^{-1},t,tx^2 and x to x,xy,x, respectively.

A similar argument using the presentation

$$\langle t,x,y \mid txt^{-1}=x^{-1}y,\ ty=yt,\ xyx^{-1}=y^{-1}\rangle$$

for B_4 shows that $Out(B_4)\cong(Z/2Z)^3$, and is generated by the classes of the automorphisms which fix y and send t to $t^{-1}y^{-1},t,tx^2$ and x to x,x^{-1},x, respectively.

8.3 Flat 4-manifold groups with infinite abelianization

We shall organize our determination of the flat 4-manifold groups π in terms of $I(\pi)$. Let π be a flat 4-manifold group, $\beta = \beta_1(\pi)$ and $h = h(I(\pi))$. Then $\pi/I(\pi) \cong Z^\beta$ and $h + \beta = 4$. If $I(\pi)$ is abelian then $C_\pi(I(\pi))$ is a nilpotent normal subgroup of π and so is a subgroup of the Hirsch-Plotkin radical $\sqrt{\pi}$, which is here the maximal abelian normal subgroup $T(\pi)$. Hence $C_\pi(I(\pi)) = T(\pi)$ and the holonomy group is isomorphic to $\pi/C_\pi(I(\pi))$.

$h = 0$ In this case $I(\pi) = 1$, so $\pi \cong Z^4$ and is orientable.

$h = 1$ In this case $I(\pi) \cong Z$ and π is nonabelian, so $\pi/C_\pi(I(\pi)) = Z/2Z$. Hence π has a presentation of the form

$$\langle t, x, y, z \mid txt^{-1} = xz^a,\ tyt^{-1} = yz^b,\ tzt^{-1} = z^{-1},\ x, y, z \ \mathit{commute}\rangle,$$

for some integers a, b. On replacing x by xy or interchanging x and y if necessary we may assume that a is even. On then replacing x by $xz^{a/2}$ and y by $yz^{[b/2]}$ we may assume that $a = 0$ and $b = 0$ or 1. Thus π is a semidirect product $Z^3 \times_T Z$, where the normal subgroup Z^3 is generated by the images of x, y and z, and the action of t is determined by a matrix $T = \left(\begin{smallmatrix} I_2 & 0 \\ (0,b) & -1 \end{smallmatrix}\right)$ in $GL(3,\mathbb{Z})$. Hence $\pi \cong Z \times B_1 = Z^2 \times K$ or $Z \times B_2$. Both of these groups are nonorientable.

$h = 2$ If $I(\pi) \cong Z^2$ and $\pi/C_\pi(I(\pi))$ is cyclic then we may again assume that π is a semidirect product $Z^3 \times_T Z$, where $T = \left(\begin{smallmatrix} 1 & 0 \\ \mu & U \end{smallmatrix}\right)$, with $\mu = \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)$ and $U \in GL(2,\mathbb{Z})$ is of order 2, 3, 4 or 6 and does not have 1 as an eigenvalue. Thus $U = -I_2$, $\left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)$. Conjugating T by $\left(\begin{smallmatrix} 1 & 0 \\ \nu & I_2 \end{smallmatrix}\right)$ replaces μ by $\mu + (I_2 - U)\nu$. In each case the choice $a = b = 0$ leads to a group of the form $\pi \cong Z \times G$, where G is an orientable flat 3-manifold group with $\beta_1(G) = 1$. For each of the first three of these matrices there is one other possible group. However if $U = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)$ then $I_2 - U$ is invertible and so $Z \times G_5$ is the only possibility. All seven of these groups are orientable.

If $I(\pi) \cong Z^2$ and $\pi/C_\pi(I(\pi))$ is not cyclic then $\pi/C_\pi(I(\pi)) \cong (Z/2Z)^2$. There are two conjugacy classes of embeddings of $(Z/2Z)^2$ in $GL(2,\mathbb{Z})$. One has image the subgroup of diagonal matrices. The corresponding groups π have presentations of the form

$$\langle t, u, x, y \mid tx = xt,\ tyt^{-1} = y^{-1},\ uxu^{-1} = x^{-1},\ uyu^{-1} = y^{-1},\ xy = yx,\\ tut^{-1}u^{-1} = x^m y^n\rangle,$$

for some integers m, n. On replacing t by $tx^{-[m/2]}y^{[n/2]}$ if necessary we may assume that $0 \leq m, n \leq 1$. On then replacing t by tu and interchanging x and y if necessary we may assume that $m \leq n$. The only infinite cyclic subgroups of $I(\pi)$ which are normal in π are the subgroups $\langle x \rangle$ and $\langle y \rangle$. On comparing the quotients of these groups π by such subgroups we see that the three possibilities are distinct. The other embedding of $(Z/2Z)^2$ in $GL(2,\mathbb{Z})$ has image generated by $-I$ and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. The corresponding groups π have presentations of the form

$$\langle t, u, x, y \mid txt^{-1} = y,\ tyt^{-1} = x,\ uxu^{-1} = x^{-1},\ uyu^{-1} = y^{-1},\ xy = yx,\\ tut^{-1}u^{-1} = x^m y^n \rangle,$$

for some integers m, n. On replacing t by $tx^{[(m-n)/2]}$ and u by ux^{-m} if necessary we may assume that $m = 0$ and $n = 0$ or 1. Thus there two such groups. All five of these groups are nonorientable.

Otherwise, $I(\pi) \cong K$, $I(I(\pi)) \cong Z$ and $G = \pi/I(I(\pi))$ is a flat 3-manifold group with $\beta_1(G) = 2$, but with $I(G) = I(\pi)/I(I(\pi))$ not contained in G' (since it acts nontrivially on $I(I(\pi))$). Therefore $G \cong B_1 = Z \times K$, and so has a presentation

$$\langle t, x, y \mid tx = xt,\ ty = yt,\ xyx^{-1} = y^{-1} \rangle.$$

If $w : G \to Aut(Z)$ is a homomorphism which restricts nontrivially to $I(G)$ then we may assume (up to isomorphism of G) that $w(x) = 1$ and $w(y) = -1$. Groups π which are extensions of $Z \times K$ by Z corresponding to the action with $w(t) = w$ $(= \pm 1)$ have presentations of the form

$$\langle t, x, y, z \mid txt^{-1} = xz^a,\ tyt^{-1} = yz^b,\ tzt^{-1} = z^w,\ xyx^{-1} = y^{-1}z^c,\ xz = zx,\\ yzy^{-1} = z^{-1} \rangle,$$

for some integers a, b. Any group with such a presentation is easily seen to be an extension of $Z \times K$ by a cyclic normal subgroup. However conjugating the fourth relation leads to the equation

$$txt^{-1}tyt^{-1}(txt^{-1})^{-1} = txyx^{-1}t^{-1} = ty^{-1}z^c t^{-1} = tyt^{-1}(tzt^{-1})^c$$

which simplifies to $xz^a yz^b z^{-a}x^{-1} = (yz^b)^{-1}z^{wc}$ and hence to $z^{c-2a} = z^{wc}$. Hence this cyclic normal subgroup is finite unless $2a = (1-w)c$.

Suppose first that $w = 1$. Then $z^{2a} = 1$ and so we must have $a = 0$. On replacing t by $tz^{[b/2]}$ and x by $xz^{[c/2]}$, if necessary, we may assume that $0 \leq b, c \leq 1$. If $b = 0$ then $\pi \cong Z \times B_3$ or $Z \times B_4$. Otherwise, after further replacing x by txz if necessary we may assume that $c = 0$. The three remaining possibilities may be distinguished by their abelianizations, and so there are three such groups. In each case the subgroup generated by $\{t, x^2, y^2, z\}$ is maximal abelian, and the holonomy group is isomorphic to $(Z/2Z)^2$.

If instead $w = -1$ then $z^{2(c-a)} = 1$ and so we must have $a = c$. On replacing y by $yz^{[b/2]}$ and x by $xz^{[c/2]}$ if necessary we may assume that $0 \leq b, c \leq 1$. If $b = 1$ then after replacing x by txy, if necessary, we may assume that $a = 0$. If $a = b = 0$ then $\pi/\pi' \cong Z^2 \oplus (Z/2Z)^2$. The remaining two possibilities both have abelianization $Z^2 \oplus (Z/2Z)$, but one has centre of rank 2 and the other has centre of rank 1. Thus there are three such groups. The subgroup generated by $\{ty, x^2, y^2, z\}$ is maximal abelian, and the holonomy group is isomorphic to $(Z/2Z)^2$. All of these groups π with $I(\pi) \cong K$ are nonorientable.

$h = 3$ In this case π is uniquely a semidirect product $\pi \cong I(\pi) \times_\theta Z$, where $I(\pi)$ is a flat 3-manifold group and θ is an automorphism of $I(\pi)$ such that the induced automorphism of $I(\pi)/I(I(\pi))$ has no eigenvalue 1, and whose image in $Out(I(\pi))$ has finite order. (The conjugacy class of the image of θ in $Out(I(\pi))$ is determined up to inversion by π.)

Since $T(I(\pi))$ is the maximal abelian normal subgroup of $I(\pi)$ it is normal in π. It follows easily that $T(\pi) \cap I(\pi) = T(I(\pi))$. Hence the holonomy group of $I(\pi)$ is isomorphic to a normal subgroup of the holonomy subgroup of π, with quotient cyclic of order dividing the order of θ in $Out(I(\pi))$. (The order of the quotient can be strictly smaller.)

If $I(\pi) \cong Z^3$ then $Out(I(\pi)) \cong GL(3, \mathbb{Z})$. If $T \in GL(3, \mathbb{Z})$ has finite order n and $\beta_1(Z^3 \times_T Z) = 1$ then either $T = -I$ or $n = 4$ or 6 and the characteristic polynomial of T is $(t+1)\phi(t)$ with $\phi(t) = t^2+1$, t^2+t+1 or t^2-t+1. In the latter cases T is conjugate to a matrix of the form $\left(\begin{smallmatrix} -1 & \mu \\ 0 & A \end{smallmatrix}\right)$, where $A = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)$, respectively. The row vector $\mu = (m_1, m_2)$ is well defined $mod\ Z^2(A+I)$. Thus there are seven such conjugacy classes. All but one pair (corresponding to $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)$ and $\mu \notin Z^2(A+I)$) are self-inverse, and so there are six such groups. The holonomy group is cyclic, of order equal to the order of T. As such matrices all have determinant -1 all of these groups are nonorientable.

If $I(\pi) \cong G_i$ for $2 \leq i \leq 5$ the automorphism $\theta = (v, A, \epsilon)$ must have $\epsilon = -1$, for otherwise $\beta_1(\pi) = 2$. We have $Out(G_2) \cong ((Z/2Z)^2 \tilde{\times} PGL(2, \mathbb{Z})) \times (Z/2Z)$. The five conjugacy classes of finite order in $PGL(2, \mathbb{Z})$ are represented by the matrices I, $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 1 \end{smallmatrix}\right)$. The numbers of conjugacy classes in $Out(G_2)$ with $\epsilon = -1$ corresponding to these matrices are two, two, two, three and one, respectively. All of these conjugacy classes are self-inverse. Of these, only the two conjugacy classes corresponding to $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and the three conjugacy classes corresponding to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ give rise to orientable groups. The holonomy groups are all isomorphic to $(Z/2Z)^2$, except when $A = \left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix}0 & 1\\ -1 & 1\end{smallmatrix}\right)$, when they are isomorphic to $Z/4Z$ or $Z/6Z \oplus Z/2Z$, respectively. There are five orientable groups and five nonorientable groups.

As $Out(G_3) \cong S_3 \times (Z/2Z)$, $Out(G_4) \cong (Z/2Z)^2$ and $Out(G_5) = Z/2Z$, there are three, two and one conjugacy classes corresponding to automorphisms with $\epsilon = -1$, respectively, and all these conjugacy classes are closed under inversion. The holonomy groups are dihedral of order 6, 8 and 12, respectively. The six such groups are all orientable.

The centre of $Out(G_6)$ is generated by the image of ab, and the image of ce in the quotient $Out(G_6)/\langle ab\rangle$ generates a central $Z/2Z$ direct factor. The quotient $Out(G_6)/\langle ab, ce\rangle$ is isomorphic to the semidirect product of a normal subgroup $(Z/2Z)^2$ (generated by the images of a and c) with S_3 (generated by the images of ia and j), and has five conjugacy classes, represented by $1, a, i, j$ and ci. Hence $Out(G_6)/\langle ab\rangle$ has ten conjugacy classes, represented by $1, ce, a, ace, i, cei, j, cej, ci$ and $cice = ei$. Thus $Out(G_6)$ itself has between 10 and 20 conjugacy classes. In fact $Out(G_6)$ has 14 conjugacy classes, of which those represented by $1, ab, ace, bce, i, cej$, $abcej$ and ei are orientation preserving, and those represented by a, ce, cei, j, abj and ci are orientation reversing. All of these classes are self inverse, except for j and abj, which are mutually inverse ($j^{-1} = ai(abj)ia$). The holonomy groups corresponding to the classes $1, ab, ace$ and bce are isomorphic to $(Z/2Z)^2$, those corresponding to a and ce are isomorphic to $(Z/2Z)^3$, those corresponding to i, ei, cei and ci are dihedral of order 8, those corresponding to cej and $abcej$ are isomorphic to A_4 and the one corresponding to j has order 24. There are eight orientable groups and five nonorientable groups.

All the remaining cases give rise to nonorientable groups.

$I(\pi) \cong Z \times K$. If a matrix A in Υ_2 has finite order then as its trace is even the order must be 1, 2 or 4. If moreover A does not have 1 as an eigenvalue then either $A = -I$ or A has order 4 and is conjugate (in Υ_2) to $\left(\begin{smallmatrix}-1 & 1\\ -2 & 1\end{smallmatrix}\right)$. Each of the four corresponding conjugacy classes in $\Upsilon_2 \times \{\pm 1\}$ is self inverse, and so there are four such groups. The holonomy groups are isomorphic to $Z/nZ \oplus Z/2Z$, where $n = 2$ or 4 is the order of A.

$I(\pi) \cong B_2$. As $Out(B_2) \cong \Upsilon_2$ there are two relevant conjugacy classes and hence two such groups. The holonomy groups are again isomorphic to $Z/nZ \oplus Z/2Z$, where $n = 2$ or 4 is the order of A.

$I(\pi) \cong B_3$ or B_4. In each case $Out(H) \cong (Z/2Z)^3$, and there are four outer automorphism classes determining semidirect products with $\beta = 1$. (Note that here conjugacy classes are singletons and are self-inverse.) The holonomy groups are all isomorphic to $(Z/2Z)^3$.

8.4 Flat 4-manifold groups with finite abelianization

There remains the case when π/π' is finite (equivalently, $h = 4$). By Lemma 3.14 if π is such a flat 4-manifold group it is nonorientable and is isomorphic to a generalized free product $J *_\phi \tilde{J}$, where ϕ is an isomorphism from $G < J$ to $\tilde{G} < \tilde{J}$ and $[J : G] = [\tilde{J} : \tilde{G}] = 2$. The groups G, J and $\tilde{J}$ are then flat 3-manifold groups. If λ and $\tilde{\lambda}$ are automorphisms of G and $\tilde{G}$ which extend to J and $\tilde{J}$, respectively, then $J *_\phi \tilde{J}$ and $J *_{\tilde{\lambda}\phi\lambda} \tilde{J}$ are isomorphic, and so we shall say that ϕ and $\tilde{\lambda}\phi\lambda$ are equivalent isomorphisms. The major difficulty in handling these cases is that some such flat 4-manifold groups split as a generalised free product in several essentially distinct ways.

It follows from the Mayer-Vietoris sequence for $\pi \cong J *_\phi \tilde{J}$ that $H_1(G;\mathbb{Q})$ maps onto $H_1(J;\mathbb{Q}) \oplus H_1(\tilde{J};\mathbb{Q})$, and hence that $\beta_1(J) + \beta_1(\tilde{J}) \leq \beta_1(G)$. Since G_3, G_4, B_3 and B_4 are only subgroups of other flat 3-manifold groups via maps inducing isomorphisms on $H_1(-;\mathbb{Q})$ and G_5 and G_6 are not index 2 subgroups of any flat 3-manifold group we may assume that $G \cong Z^3$, G_2, B_1 or B_2. If j and $\tilde{j}$ are the automorphisms of $T(J)$ and $T(\tilde{J})$ determined by conjugation in J and $\tilde{J}$, respectively, then π is a flat 4-manifold group if and only if $\Phi = jT(\phi)^{-1}\tilde{j}T(\phi)$ has finite order. In particular, the trace of Φ must have absolute value at most 3. At this point detailed computation seems unavoidable. (We note in passing that any generalised free product $J *_G \tilde{J}$ with $G \cong G_3$, G_4, B_3 or B_4, J and $\tilde{J}$ torsion free and $[J : G] = [\tilde{J} : G] = 2$ is a flat 4-manifold group, since $Out(G)$ is then finite. However all such groups have infinite abelianization.)

Suppose first that $G \cong Z^3$, with basis $\{x, y, z\}$. Then J and $\tilde{J}$ must have holonomy of order ≤ 2, and $\beta_1(J) + \beta_1(\tilde{J}) \leq 3$. Hence we may assume that $J \cong G_2$ and $\tilde{J} \cong G_2$, B_1 or B_2. In each case we have $G = T(J)$ and $\tilde{G} = T(\tilde{J})$. We may assume that J and $\tilde{J}$ are generated by G and elements s and t, respectively, such that $s^2 = x$ and $t^2 \in \tilde{G}$. We may also assume that the action of s on G has matrix $j = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -I \end{smallmatrix}\right)$ with respect to the basis $\{x, y, z\}$. Fix an isomorphism $\phi : G \to \tilde{G}$ and let $T = T(\phi)^{-1}\tilde{j}T(\phi) = \left(\begin{smallmatrix} a & \delta \\ \gamma & D \end{smallmatrix}\right)$ be the matrix corresponding to the action of t on $\tilde{G}$. (Here γ is a 2×1 column vector, δ is a 1×2 row vector and D is a 2×2 matrix, possibly singular.) Then $T^2 = I$ and so the trace of T is odd. Since $j \equiv I \ mod\ (2)$ the trace of $\Phi = jT$ is also odd, and so Φ cannot have order 3 or 6. Therefore $\Phi^4 = I$. If $\Phi = I$ then

π/π' is infinite. If Φ has order 2 then $jT = Tj$ and so $\gamma = 0$, $\delta = 0$ and $D^2 = I_2$. Moreover we must have $a = -1$ for otherwise π/π' is infinite. After conjugating T by a matrix commuting with j if necessary we may assume that $D = I_2$ or $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. (Since $\tilde{J}$ must be torsion free we cannot have $D = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.) These two matrices correspond to the generalized free products $G_2 *_\phi B_1$ and $G_2 *_\phi G_2$, with presentations

$$\begin{aligned} &\langle s,t,z \mid st^2s^{-1} = t^{-2},\ szs^{-1} = z^{-1},\ ts^2t^{-1} = s^{-2},\ tz = zt\rangle \\ \text{and}\quad &\langle s,t,z \mid st^2s^{-1} = t^{-2},\ szs^{-1} = z^{-1},\ ts^2t^{-1} = s^{-2},\ tzt^{-1} = z^{-1}\rangle, \end{aligned}$$

respectively. These groups each have holonomy group isomorphic to $(Z/2Z)^2$. If Φ has order 4 then we must have $(jT)^2 = (jT)^{-2} = (Tj)^2$ and so $(jT)^2$ commutes with j. It can then be shown that after conjugating T by a matrix commuting with j if necessary we may assume that T is the elementary matrix which interchanges the first and third rows. The corresponding group $G_2 *_\phi B_2$ has a presentation

$$\langle s,t,z \mid st^2s^{-1} = t^{-2},\ szs^{-1} = z^{-1},\ ts^2t^{-1} = z,\ tzt^{-1} = s^2\rangle.$$

Its holonomy group is isomorphic to the dihedral group of order 8.

If $G \cong B_1$ or B_2 then J and $\tilde{J}$ are nonorientable and $\beta_1(J) + \beta_1(\tilde{J}) \leq 2$. Hence J and $\tilde{J}$ are B_3 or B_4. Since neither of these groups contains B_2 as an index 2 subgroup we must have $G \cong B_1$. In each case there are two essentially different embeddings of B_1 as an index 2 subgroup of B_3 or B_4. (The image of one contains $I(B_i)$ while the other does not.) In all cases we find that j and $\tilde{j}$ are diagonal matrices with determinant -1, and that $T(\phi) = \left(\begin{smallmatrix} M & 0 \\ 0 & \pm 1 \end{smallmatrix}\right)$ for some $M \in \Gamma_2$. Calculation now shows that if Φ has finite order then M is diagonal and hence $\beta_1(J *_\phi \tilde{J}) > 0$. Thus there are no flat 4-manifold groups (with finite abelianization) which are generalized free products with amalgamation over copies of B_1 or B_2.

If $G \cong G_2$ then $\beta_1(J) + \beta_1(\tilde{J}) \leq 1$, so we may assume that $J \cong G_6$. The other factor $\tilde{J}$ must then be one of G_2, G_4, G_6, B_3 or B_4, and then every amalgamation has finite abelianization. In each case the images of any two embeddings of G_2 in one of these groups are equivalent up to composition with an automorphism of the larger group. In all cases the matrices for j and $\tilde{j}$ have the form $\left(\begin{smallmatrix} \pm 1 & 0 \\ 0 & N \end{smallmatrix}\right)$ where $N^4 = I \in GL(2,\mathbb{Z})$, and $T(\phi) = \left(\begin{smallmatrix} \epsilon & 0 \\ 0 & M \end{smallmatrix}\right)$ for some $M \in GL(2,\mathbb{Z})$. Calculation shows that Φ has finite order if and only if M is in the dihedral subgroup D_8 of $GL(2,\mathbb{Z})$ generated by the diagonal matrices and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. (In other words, either M is diagonal or both diagonal elements of M are 0.) Now the subgroup of $Aut(G_2)$ consisting of automorphisms which extend to G_6 is $(Z^2 \times_\alpha D_8) \times \{\pm 1\}$. Hence any two such isomorphisms ϕ from

G to $\tilde{G}$ are equivalent, and so there is an unique such flat 4-manifold group $G_6 *_\phi \tilde{J}$ for each of these choices of $\tilde{J}$. The corresponding presentations are

$$\langle u,x,y \mid xux^{-1}=u^{-1},\ y^2=u^2,\ yx^2y^{-1}=x^{-2},\ u(xy)^2=(xy)^2u\rangle,$$
$$\langle u,x,y \mid yx^2y^{-1}=x^{-2},\ uy^2u^{-1}=(xy)^2,\ u(xy)^2u^{-1}=y^{-2},\ x=u^2\rangle,$$
$$\langle u,x,y \mid xy^2x^{-1}=y^{-2},\ yx^2y^{-1}=ux^2u^{-1}=x^{-2},\ y^2=u^2,\ yxy=uxu\rangle,$$
$$\langle t,x,y \mid xy^2x^{-1}=y^{-2},\ yx^2y^{-1}=x^{-2},\ x^2=t^2,\ y^2=(t^{-1}x)^2,\ t(xy)^2=(xy)^2t\rangle$$
$$\text{and}\quad \langle t,x,y \mid xy^2x^{-1}=y^{-2},\ yx^2y^{-1}=x^{-2},\ x^2=t^2(xy)^2,\ y^2=(t^{-1}x)^2,\ t(xy)^2=(xy)^2t\rangle,$$

respectively. The corresponding holonomy groups are isomorphic to $(Z/2Z)^3$, D_8, $(Z/2Z)^2$, $(Z/2Z)^3$ and $(Z/2Z)^3$, respectively.

Thus we have found eight generalized free products $J *_G \tilde{J}$ which are flat 4-manifold groups with $\beta=0$. The groups $G_2 *_\phi B_1$, $G_2 *_\phi G_2$ and $G_6 *_\phi G_6$ are all easily seen to be semidirect products of G_6 with an infinite cyclic normal subgroup, on which G_6 acts nontrivially. It follows easily that these three groups are in fact isomorphic, and so there is just one flat 4-manifold group with finite abelianization and holonomy isomorphic to $(Z/2Z)^2$.

The above presentations of $G_2 *_\phi B_2$ and $G_6 *_\phi G_4$ are in fact equivalent; the function sending s to y, t to yu^{-1} and z to uy^2u^{-1} determines an isomorphism between these groups. Thus there is just one flat 4-manifold group with finite abelianization and holonomy isomorphic to D_8.

The above presentations of $G_6 *_\phi G_2$ and $G_6 *_\phi B_4$ are also equivalent; the function sending x to xt^{-1}, y to yt and u to $xy^{-1}t$ determines an isomorphism between these groups (with inverse sending x to $uy^{-1}x^{-2}$, y to ux^{-1} and t to xuy^{-1}). (This isomorphism and the one in the paragraph above were found by Derek Holt, using the program described in [HR92].) The translation subgroups of $G_6 *_\phi B_3$ and $G_6 *_\phi B_4$ are generated by the images of $U=(ty)^2$, $X=x^2$, $Y=y^2$ and $Z=(xy)^2$ (with respect to the above presentations). In each case the images of t, x and y act diagonally, via the matrices $diag[-1,1,-1,1]$, $diag[1,1,-1,-1]$ and $diag[-1,-1,1,-1]$, respectively. However the maximal orientable subgroups have abelianization $Z\oplus(Z/2)^3$ and $Z\oplus(Z/4Z)\oplus(Z/2Z)$, respectively, and so $G_6 *_\phi B_3$ is not isomorphic to $G_6 *_\phi B_4$. Thus there are two flat 4-manifold groups with finite abelianization and holonomy isomorphic to $(Z/2Z)^3$.

In summary, there are 27 orientable flat 4-manifold groups (all with $\beta>0$), 43 nonorientable flat 4-manifold groups with $\beta>0$ and 4 (nonorientable) flat 4-manifold groups with $\beta=0$. (We suspect that the discrepancy with the results reported in [Wo] may be explained by an unnoticed isomorphism between two examples with finite abelianization.)

8.5 Distinguishing between the geometries

Let M be a closed 4-manifold with fundamental group π and with a geometry of solvable Lie type. We shall show that the geometry is largely determined by the structure of $\sqrt{\pi}$. (See also Proposition 10.4 of [Wl86].) As a geometric structure on a manifold lifts to each covering space of the manifold it shall suffice to show that the geometries on suitable finite covering spaces (corresponding to subgroups of finite index in π) can be recognized.

If M is an infranilmanifold then $[\pi : \sqrt{\pi}] < \infty$. If it is flat then $\sqrt{\pi} \cong Z^4$, while if it has the geometry $\mathbb{N}il^3 \times \mathbb{E}^1$ or $\mathbb{N}il^4$ then $\sqrt{\pi}$ is nilpotent of class 2 or 3 respectively. (These cases may also be distinguished by the rank of $\zeta\sqrt{\pi}$.) All such groups have been classified, and may be realized geometrically. (See [De] for explicit representations of the $\mathbb{N}il^3 \times \mathbb{E}^1$- and $\mathbb{N}il^4$-groups as lattices in $Aff(Nil^3 \times R)$ and $Aff(Nil^4)$, respectively.)

If M is a $\mathbb{S}ol_0^4$- or $\mathbb{S}ol_{m,n}^4$-manifold then $\sqrt{\pi} \cong Z^3$. Hence $h(\pi/\sqrt{\pi}) = 1$ and so π has a normal subgroup σ of finite index which is a semidirect product $\sqrt{\pi} \times_\theta Z$, where the action of a generator t of Z by conjugation on $\sqrt{\pi}$ is given by a matrix θ in $GL(3,\mathbb{Z})$. We may further assume that θ is in $SL(3,\mathbb{Z})$ and has no negative eigenvalues, and that σ is maximal among such normal subgroups. The characteristic polynomial of θ is $X^3 - mX^2 + nX - 1$, where $m = trace(\theta)$ and $n = trace(\theta^{-1})$. The matrix θ has infinite order, for otherwise the subgroup generated by $\sqrt{\pi}$ and a suitable power of t would be abelian of rank 4. Moreover the eigenvalues must be distinct. For otherwise they would be all 1, so $(\theta - I)^3 = 0$ and π would be virtually nilpotent.

If M is a $\mathbb{S}ol_0^4$-manifold two of the eigenvalues are complex conjugates. They cannot be roots of unity, since θ has infinite order, and so the real eigenvalue is not 1. If M is a $\mathbb{S}ol_{m,n}^4$-manifold the eigenvalues of θ are distinct and real. The geometry is $\mathbb{S}ol^3 \times \mathbb{E}^1 (= \mathbb{S}ol_{m,m}^4$ for any $m \geq 4)$ if and only if θ has 1 as a simple eigenvalue.

The groups of $\mathbb{E}^4$-, $\mathbb{N}il^3 \times \mathbb{E}^1$- and $\mathbb{N}il^4$-manifolds also have finite index subgroups $\sigma \cong Z^3 \times_\theta Z$. We may assume that all the eigenvalues of θ are 1, so $N = \theta - I$ is nilpotent. If the geometry is $\mathbb{E}^4$ then $N = 0$; if it is $\mathbb{N}il^3 \times \mathbb{E}^1$ then $N \neq 0$ but $N^2 = 0$, while if it is $\mathbb{N}il^4$ then $N^2 \neq 0$ but $N^3 = 0$. (Conversely, it is easy to see that such semidirect products may be realized by lattices in the corresponding Lie groups.)

Finally, if M is a $\mathbb{S}ol_1^4$-manifold then $\sqrt{\pi} \cong \Gamma_q$ for some $q \geq 1$ (and so is nonabelian, of Hirsch length 3).

If $h(\sqrt{\pi}) = 3$ then π is an extension of Z or D by a normal subgroup ν which contains $\sqrt{\pi}$ as a subgroup of finite index. Hence either M is the mapping torus of a self homeomorphism of a flat 3-manifold or a $\mathbb{N}il^3$-manifold, or it is the union of two twisted I-bundles over such 3-manifolds and is doubly covered by such a mapping torus. (Compare Theorem 8.2.)

We shall consider the converse question of realizing geometrically such torsion free virtually poly-Z groups π (with $h(\pi) = 4$ and $h(\sqrt{\pi}) = 3$) in §9.

8.6 Mapping tori of self homeomorphisms of $\mathbb{E}^3$-manifolds

It follows from the above that a 4-dimensional infrasolvmanifold M admits one of the product geometries of type $\mathbb{E}^4$, $\mathbb{N}il^3 \times \mathbb{E}^1$ or $\mathbb{S}ol^3 \times \mathbb{E}^1$ if and only if $\pi_1(M)$ has a subgroup of finite index of the form $\nu \times Z$, where ν is abelian, nilpotent of class 2 or solvable but not virtually nilpotent, respectively. In the next two sections we shall examine when M is the mapping torus of a self homeomorphism of a 3-dimensional infrasolvmanifold. (Note that if M is orientable then it must be a mapping torus, by Lemma 3.14 and Theorem 6.11.)

Theorem 8.3 *Let ν be the fundamental group of a flat 3-manifold, and let θ be an automorphism of ν. Then*

(1) *$\sqrt{\nu}$ is the maximal abelian subgroup of ν and $\nu/\sqrt{\nu}$ embeds in $Aut(\sqrt{\nu})$;*
(2) *$Out(\nu)$ is finite if and only if $[\nu : \sqrt{\nu}] > 2$;*
(3) *the kernel of the restriction homomorphism from $Out(\nu)$ to $Aut(\sqrt{\nu})$ is finite;*
(4) *if $[\nu : \sqrt{\nu}] = 2$ then $(\theta|_{\sqrt{\nu}})^2$ has 1 as an eigenvalue;*
(5) *if $[\nu : \sqrt{\nu}] = 2$ and $\theta|_{\sqrt{\nu}}$ has infinite order but all of its eigenvalues are roots of unity then $((\theta|_{\sqrt{\nu}})^2 - I)^2 = 0$.*

Proof It follows immediately from Theorem 1.5 that $\sqrt{\nu} \cong Z^3$ and is thus the maximal abelian subgroup of ν. The kernel of the homomorphism from ν to $Aut(\sqrt{\nu})$ determined by conjugation is the centralizer $C = C_\nu(\sqrt{\nu})$. As $\sqrt{\nu}$ is central in C and $[C : \sqrt{\nu}]$ is finite, C has finite commutator subgroup, by Schur's Theorem (Proposition 10.1.4 of [Ro]). Since C is torsion free it must be abelian and so $C = \sqrt{\nu}$. Hence $H = \nu/\sqrt{\nu}$ embeds in $Aut(\sqrt{\nu}) \cong GL(3, \mathbb{Z})$. (This is just the holonomy representation.)

If H has order 2 then θ induces the identity on H; if H has order greater than 2 then some power of θ induces the identity on H, since $\sqrt{\nu}$ is a characteristic subgroup of finite index. The matrix $\theta|_{\sqrt{\nu}}$ then commutes with each element of the image of H in $GL(3,\mathbb{Z})$, and the remaining assertions follow from simple calculations, on considering the possibilities for π and H listed in §3 above. □

Corollary 8.3.1 *The mapping torus $M(\phi) = N \times_\phi S^1$ of a self homeomorphism ϕ of a flat 3-manifold N is flat if and only if the outer automorphism $[\phi_*]$ induced by ϕ has finite order.* □

If N is flat and $[\phi_*]$ has infinite order then $M(\phi)$ may admit one of the other product geometries $\mathbb{S}ol^3 \times \mathbb{E}^1$ or $\mathbb{N}il^3 \times \mathbb{E}^1$; otherwise it must be a $\mathbb{S}ol^4_{m,n}$-, $\mathbb{S}ol^4_0$- or $\mathbb{N}il^4$-manifold. (The latter can only happen if $N = R^3/Z^3$, by part (v) of the theorem.)

Theorem 8.4 *Let M be an infrasolvmanifold with fundamental group π such that $\sqrt{\pi} \cong Z^3$ and $\pi/\sqrt{\pi}$ is an extension of D by a finite normal subgroup. Then M is a $\mathbb{S}ol^3 \times \mathbb{E}^1$-manifold.*

Proof Let $p : \pi \to D$ be an epimorphism with kernel K containing $\sqrt{\pi}$ as a subgroup of finite index, and let t and u be elements of π whose images under p generate D and such that $p(t)$ generates an infinite cyclic subgroup of index 2 in D. Then there is an $N > 0$ such that the image of $s = t^N$ in $\pi/\sqrt{\pi}$ generates a normal subgroup. In particular, the subgroup generated by s and $\sqrt{\pi}$ is normal in π and usu^{-1} and s^{-1} have the same image in $\pi/\sqrt{\pi}$. Let θ be the matrix of the action of s on $\sqrt{\pi}$, with respect to some basis $\sqrt{\pi} \cong Z^3$. Then θ is conjugate to its inverse, since usu^{-1} and s^{-1} agree modulo $\sqrt{\pi}$. Hence one of the eigenvalues of θ is ± 1. Since π is not virtually nilpotent the eigenvalues of θ must be distinct, and so the geometry must be of type $\mathbb{S}ol^3 \times \mathbb{E}^1$. □

Corollary 8.4.1 *If M admits one of the geometries $\mathbb{S}ol^4_0$ or $\mathbb{S}ol^4_{m,n}$ with $m \neq n$ then it is the mapping torus of a self homeomorphism of R^3/Z^3, and so $\pi \cong Z^3 \times_\theta Z$ for some θ in $GL(3,\mathbb{Z})$ and is a metabelian poly-Z group.*

Proof This follows immediately from Theorems 8.3 and 8.4. □

We may use the idea of Theorem 8.2 to give examples of $\mathbb{E}^4$-, $\mathbb{N}il^4$-, $\mathbb{N}il^3 \times \mathbb{E}^1$- and $\mathbb{S}ol^3 \times \mathbb{E}^1$-manifolds which are not mapping tori. For instance, the groups with presentations

$$\langle u,v,x,y,z \mid xy=yx,\ xz=zx,\ yz=zy,\ uxu^{-1}=x^{-1},\ u^2=y,\ uzu^{-1}=z^{-1},\\ v^2=z,\ vxv^{-1}=x^{-1},\ vyv^{-1}=y^{-1}\rangle,$$

$$\langle u,v,x,y,z \mid xy=yx,\ xz=zx,\ yz=zy,\ u^2=x,\ uyu^{-1}=y^{-1},\ uzu^{-1}=z^{-1},\\ v^2=x,\ vyv^{-1}=v^{-4}y^{-1},\ vzv^{-1}=z^{-1}\rangle$$

and $$\langle u,v,x,y,z \mid xy=yx,\ xz=zx,\ yz=zy,\ u^2=x,\ v^2=y,\\ uyu^{-1}=x^4y^{-1},\ vxv^{-1}=x^{-1}y^2,\ uzu^{-1}=vzv^{-1}=z^{-1}\rangle$$

are each generalised free products of two copies of $Z^2\times_{-I}Z$ amalgamated over their maximal abelian subgroups. The Hirsch-Plotkin radicals of these groups are isomorphic to Z^4 (generated by $\{(uv)^2,x,y,z\}$), $\Gamma_2\times Z$ (generated by $\{uv,x,y,z\}$) and Z^3 (generated by $\{x,y,z\}$), respectively. The group with presentation

$$\langle u,v,x,y,z \mid xy=yx,\ xz=zx,\ yz=zy,\ u^2=x,\ uz=zu,\ uyu^{-1}=x^2y^{-1},\\ v^2=y,\ vxv^{-1}=x^{-1},\ vzv^{-1}=v^4z^{-1}\rangle$$

is a generalised free product of copies of $(Z\times_{-1}Z)\times Z$ (generated by $\{u,y,z\}$) and $Z^2\times_{-I}Z$ (generated by $\{v,x,z,\}$) amalgamated over their maximal abelian subgroups. Its Hirsch-Plotkin radical is the subgroup of index 4 generated by $\{(uv)^2,x,y,z\}$, and is nilpotent of class 3. The manifolds corresponding to these groups admit the geometries $\mathbb{E}^4$, $\mathbb{N}il^3\times\mathbb{E}^1$, $\mathbb{S}ol^3\times\mathbb{E}^1$ and $\mathbb{N}il^4$, respectively. However they cannot be mapping tori, as these groups each have finite abelianization.

8.7 Mapping tori of self homeomorphisms of $\mathbb{N}il^3$-manifolds

Let ϕ be an automorphism of Γ_q, sending x to $x^ay^bz^m$ and y to $x^cy^dz^n$ for some $a\dots n$ in Z. Then $A=\left(\begin{smallmatrix} a & c\\ b & d\end{smallmatrix}\right)$ is in $GL(2,\mathbb{Z})$ and $\phi(z)=z^{det(A)}$. (In particular, the PD_3-group Γ_q is orientable, as already observed in §2 of Chapter 7, and ϕ is orientation preserving, by the criterion of page 177 of [Bi], or by the argument of §3 of Chapter 18 below.) Every pair (A,μ) in the set $GL(2,\mathbb{Z})\times Z^2$ determines an automorphism (with $\mu=(m,n)$). However $Aut(\Gamma_q)$ is not the direct product of $GL(2,\mathbb{Z})$ and Z^2, as

$$(A,\mu)(B,\nu)=(AB,\mu B+det(A)\nu+q\omega(A,B)),$$

where $\omega(A,B)$ is biquadratic in the entries of A and B. The natural map $p: Aut(\Gamma_q)\to Aut(\Gamma_q/\zeta\Gamma_q)=GL(2,\mathbb{Z})$ sends (A,μ) to A and is an epimorphism, with $\mathrm{Ker}(p)\cong Z^2$. The inner automorphisms are represented by $q\mathrm{Ker}(p)$, and $Out(\Gamma_q)$ is the semidirect product of $GL(2,\mathbb{Z})$ with the normal subgroup $(Z/qZ)^2$. (Let $[A,\mu]$ be the image of (A,μ) in $Out(\Gamma_q)$. Then $[A,\mu][B,\nu] = [AB, \mu B + det(A)\nu]$.) In particular, $Out(\Gamma_1) = GL(2,\mathbb{Z})$.

Theorem 8.5 *Let ν be the fundamental group of a $\mathbb{N}il^3$-manifold N. Then*

(1) *$\nu/\sqrt{\nu}$ embeds in $Aut(\sqrt{\nu}/\zeta\sqrt{\nu}) \cong GL(2,\mathbb{Z})$;*

(2) *$\bar{\nu} = \nu/\zeta\sqrt{\nu}$ is a 2-dimensional crystallographic group;*

(3) *the images of elements of $\bar{\nu}$ of finite order under the holonomy representation in $Aut(\sqrt{\bar{\nu}}) \cong GL(2,\mathbb{Z})$ have determinant 1;*

(4) *$Out(\bar{\nu})$ is infinite if and only if $\bar{\nu} \cong Z^2$ or $Z^2 \times_{-I} (Z/2Z)$;*

(5) *the kernel of the natural homomorphism from $Out(\nu)$ to $Out(\bar{\nu})$ is finite.*

(6) *ν is orientable and every automorphism of ν is orientation preserving.*

Proof Let $h : \nu \to Aut(\sqrt{\nu}/\zeta\sqrt{\nu})$ be the homomorphism determined by conjugation, and let $C = \mathrm{Ker}(h)$. Then $\sqrt{\nu}/\zeta\sqrt{\nu}$ is central in $C/\zeta\sqrt{\nu}$ and $[C/\zeta\sqrt{\nu} : \sqrt{\nu}/\zeta\sqrt{\nu}]$ is finite, so $C/\zeta\sqrt{\nu}$ has finite commutator subgroup, by Schur's Theorem (Proposition 10.1.4 of [Ro].) Since C is torsion free it follows easily that C is nilpotent and hence that $C = \sqrt{\nu}$. This proves (1) and (2). In particular, h factors through the holonomy representation for $\bar{\nu}$, and $gzg^{-1} = z^{d(g)}$ for all $g \in \nu$ and $z \in \zeta\sqrt{\nu}$, where $d(g) = det(h(g))$. If $g \in \nu$ is such that $g \neq 1$ and $g^k \in \zeta\sqrt{\nu}$ for some $k > 0$ then $g^k \neq 1$ and so g must commute with elements of $\zeta\sqrt{\nu}$, i.e., the determinant of the image of g is 1. Condition (4) follows as in Theorem 8.3, on considering the possible finite subgroups of $GL(2,\mathbb{Z})$. (See Theorem 1.3.)

If $\zeta\nu \neq 1$ then $\zeta\nu = \zeta\sqrt{\nu} \cong Z$ and so the kernel of the natural homomorphism from $Aut(\nu)$ to $Aut(\bar{\nu})$ is isomorphic to $Hom(\nu/\nu', Z)$. If ν/ν' is finite this kernel is trivial. If $\bar{\nu} \cong Z^2$ then $\nu = \sqrt{\nu} \cong \Gamma_q$, for some $q \geq 1$, and the kernel is isomorphic to $(Z/qZ)^2$. Otherwise $\bar{\nu} \cong Z \times_{-1} Z$, $Z \times D$ or $D \times_\tau Z$ (where τ is the automorphism of $D = (Z/2Z) * (Z/2Z)$ which interchanges the factors). But then $H^2(\bar{\nu};\mathbb{Z})$ is finite and so any central extension of such a group by Z is virtually abelian, and thus not a $\mathbb{N}il^3$-manifold group.

If $\zeta\nu = 1$ then $\nu/\sqrt{\nu} < GL(2,\mathbb{Z})$ has an element of order 2 with determinant -1. No such element can be conjugate to $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, for otherwise ν would not be torsion free. Hence the image of $\nu/\sqrt{\nu}$ in $GL(2,\mathbb{Z})$ is conjugate to a subgroup of the group of diagonal matrices $\left(\begin{smallmatrix} \epsilon & 0 \\ 0 & \epsilon' \end{smallmatrix}\right)$, with $|\epsilon| = |\epsilon'| = 1$. If $\nu/\sqrt{\nu}$ is generated by $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ then $\nu/\zeta\sqrt{\nu} \cong Z \times_{-1} Z$ and $\nu \cong Z^2 \times_\theta Z$, where $\theta = \left(\begin{smallmatrix} -1 & r \\ 0 & -1 \end{smallmatrix}\right)$ for some nonzero integer r, and N is a circle bundle over the Klein bottle. If $\nu/\sqrt{\nu} \cong (Z/2Z)^2$ then ν has a presentation

$$\langle t, u, z \mid u^2 = z, tzt^{-1} = z^{-1},\ ut^2u^{-1} = t^{-2}z^s\rangle,$$

and N is a Seifert bundle over the orbifold $P(22)$. It may be verified in each case that the kernel of the natural homomorphism from $Out(\nu)$ to $Out(\bar{\nu})$ is finite. Therefore (5) holds.

Since $\sqrt{\nu} \cong \Gamma_q$ is a PD_3^+-group, $[\nu : \sqrt{\nu}] < \infty$ and every automorphism of Γ_q is orientation preserving ν must also be orientable. Since $\sqrt{\nu}$ is characteristic in ν and the image of $H_3(\sqrt{\nu};\mathbb{Z})$ in $H_3(\nu;\mathbb{Z})$ has index $[\nu : \sqrt{\nu}]$ it follows easily that any automorphism of ν must be orientation preserving. □

In fact every $\mathbb{N}il^3$-manifold is a Seifert bundle over a 2-dimensional euclidean orbifold [Sc83']. The base orbifold must be one of the seven such with no reflector curves, by (3).

Theorem 8.6 *The mapping torus $M(\phi) = N \times_\phi S^1$ of a self homeomorphism ϕ of a $\mathbb{N}il^3$-manifold N is orientable, and is a $\mathbb{N}il^3 \times \mathbb{E}^1$-manifold if and only if the outer automorphism $[\phi_*]$ induced by ϕ has finite order.*

Proof Since N is orientable and ϕ is orientation preserving (by part (6) of Theorem 8.5) $M(\phi)$ must be orientable.

The subgroup $\zeta\sqrt{\nu}$ is characteristic in ν and hence normal in π, and $\nu/\zeta\sqrt{\nu}$ is virtually Z^2. If $M(\phi)$ is a $\mathbb{N}il^3 \times \mathbb{E}^1$-manifold then $\pi/\zeta\sqrt{\nu}$ is also virtually abelian. It follows easily that that the image of ϕ_* in $Aut(\nu/\zeta\sqrt{\nu})$ has finite order. Hence $[\phi_*]$ has finite order also, by Theorem 8.5. Conversely, if $[\phi_*]$ has finite order in $Out(\nu)$ then π has a subgroup of finite index which is isomorphic to $\nu \times Z$, and so $M(\phi)$ has the product geometry, by the discussion above. □

Theorem 4.2 of [KLR83] (which extends Bieberbach's theorem to the virtually nilpotent case) may be used to show directly that every outer automorphism class of finite order of the fundamental group of an $\mathbb{E}^3$- or $\mathbb{N}il^3$-manifold is realizable by an isometry of an affinely equivalent manifold.

The image of an automorphism θ of Γ_q in $Out(\Gamma_q)$ has finite order if and only if the induced automorphism $\bar{\theta}$ of $\bar{\Gamma}_q = \Gamma_q/\zeta\Gamma_q \cong Z^2$ has finite order in $Aut(\bar{\Gamma}_q) \cong GL(2,\mathbb{Z})$. If $\bar{\theta}$ has infinite order but has trace ± 2 (i.e., if $\bar{\theta}^2 - I$ is a nonzero nilpotent matrix) then $\pi = \Gamma_q \times_\theta Z$ is virtually nilpotent of class 3. If the trace of $\bar{\theta}$ has absolute value greater than 2 then $h(\sqrt{\pi}) = 3$.

Theorem 8.7 *Let M be a closed 4-manifold which admits one of the geometries $\mathbb{N}il^4$ or $\mathbb{S}ol_1^4$. Then M is the mapping torus of a self homeomorphism of a $\mathbb{N}il^3$-manifold if and only if it is orientable.*

Proof If M is such a mapping torus then it is orientable, by Theorem 8.6. Conversely, if M is orientable then $\pi = \pi_1(M)$ has infinite abelianization, by Lemma 3.14. Let $p : \pi \to Z$ be an epimorphism with kernel K, and let t be an element of π such that $p(t)$ generates Z. If K is virtually nilpotent of class 2 we are done, by Theorem 6.12. (Note that this must be the case if M is a $\mathbb{S}ol_1^4$-manifold.) If K is virtually abelian then $K \cong Z^3$, by part (5) of Theorem 8.3. The matrix corresponding to the action of t on K by conjugation must be orientation preserving, since M is orientable. It follows easily that π is nilpotent. Hence there is another epimorphism with kernel nilpotent of class 2, and so the theorem is proven. □

Corollary 8.7.1 *Let M be a closed $\mathbb{S}ol_1^4$-manifold with fundamental group π. Then $\beta_1(M) \leq 1$ and M is orientable if and only if $\beta_1(M) = 1$.*

Proof The first assertion is clear if π is a semidirect product $\Gamma_q \times_\theta Z$, and then follows in general. Hence if there is an epimorphism $p\colon \pi \to Z$ with kernel K then K must be virtually nilpotent of class 2 and the result follows from the theorem. □

If M is a $\mathbb{N}il^3 \times \mathbb{E}^1$- or $\mathbb{N}il^4$-manifold then $\beta_1(\pi) \leq 3$ or 2, respectively, with equality if and only if π is nilpotent. In the latter case M is orientable, and is a mapping torus, both of a self homeomorphism of R^3/Z^3 and also of a self homeomorphism of a $\mathbb{N}il^3$-manifold. We have already seen that $\mathbb{N}il^3 \times \mathbb{E}^1$- and $\mathbb{N}il^4$-manifolds need not be mapping tori at all. We shall round out this discussion with examples illustrating the remaining combinations of mapping torus structure and orientation compatible with Lemma 3.14 and Theorem 8.7. As the groups have abelianization of rank 1 the corresponding manifolds are mapping tori in an essentially unique way. The groups with presentations

$$\langle t, x, y, z \mid xz = zx,\ yz = zy,\ txt^{-1} = x^{-1},\ tyt^{-1} = y^{-1},\ tzt^{-1} = yz^{-1}\rangle,$$
$$\langle t, x, y, z \mid xyx^{-1}y^{-1} = z,\ xz = zx,\ yz = zy,\ txt^{-1} = x^{-1},\ tyt^{-1} = y^{-1}\rangle$$
$$\text{and}\quad \langle t, x, y, z \mid xy = yx,\ zxz^{-1} = x^{-1},\ zyz^{-1} = y^{-1},\ txt^{-1} = x^{-1},\ ty = yt,\ tzt^{-1} = z^{-1}\rangle$$

are each virtually nilpotent of class 2. The corresponding $\mathbb{N}il^3 \times \mathbb{E}^1$-manifolds are mapping tori of self homeomorphisms of R^3/Z^3, a $\mathbb{N}il^3$-manifold and a flat manifold, respectively. The latter two of these manifolds are orientable. The groups with presentations

$$\langle t,x,y,z \mid xz=zx,\ yz=zy,\ txt^{-1}=x^{-1},\ tyt^{-1}=xy^{-1},\ tzt^{-1}=yz^{-1}\rangle$$

and $\langle t,x,y,z \mid xyx^{-1}y^{-1}=z,\ xz=zx,\ yz=zy,\ txt^{-1}=x^{-1},\ tyt^{-1}=xy^{-1}\rangle$

are each virtually nilpotent of class 3. The corresponding $\mathbb{N}il^4$-manifolds are mapping tori of self homeomorphisms of R^3/Z^3 and of a $\mathbb{N}il^3$-manifold, respectively.

The group with presentation

$$\langle t,u,x,y,z \mid xyx^{-1}y^{-1}=z^2,\ xz=zx,\ yz=zy,\ txt^{-1}=x^2y,\ tyt^{-1}=xy,$$
$$tz=zt,\ u^4=z,\ uxu^{-1}=y^{-1},\ uyu^{-1}=x,\ utu^{-1}=t^{-1}\rangle$$

has Hirsch-Plotkin radical isomorphic to Γ_2 (generated by $\{x,y,z\}$), and has finite abelianization. The corresponding $\mathbb{S}ol_1^4$-manifold is nonorientable and is not a mapping torus.

8.8 Mapping tori of self homeomorphisms of $\mathbb{S}ol^3$-manifolds

The arguments in this section are again analogous to those of §6.

Theorem 8.8 *Let σ be the fundamental group of a $\mathbb{S}ol^3$-manifold. Then*

(1) $\sqrt{\sigma}\cong Z^2$ *and* $\sigma/\sqrt{\sigma}\cong Z$ *or* D*;*

(2) $Out(\sigma)$ *is finite.*

Proof The argument of Theorem 1.6 implies that $h(\sqrt{\sigma})>1$. Since σ is not virtually nilpotent $h(\sqrt{\sigma})<3$. Hence $\sqrt{\sigma}\cong Z^2$, by Theorem 1.5. Let $\tilde{F}$ be the preimage in σ of the maximal finite normal subgroup of $\sigma/\sqrt{\nu}$, let t be an element of σ whose image generates the maximal abelian subgroup of $\sigma/\tilde{F}$ and let τ be the automorphism of $\tilde{F}$ determined by conjugation by t. Let σ_1 be the subgroup of σ generated by $\tilde{F}$ and t. Then $\sigma_1\cong\tilde{F}\times_\tau Z$, $[\sigma:\sigma_1]\leq 2$, $\tilde{F}$ is torsion free and $h(\tilde{F})=2$. If $\tilde{F}\neq\sqrt{\sigma}$ then $\tilde{F}\cong Z\times_{-1}Z$. But extensions of Z by $Z\times_{-1}Z$ are virtually abelian, since $Out(Z\times_{-1}Z)$ is finite. Hence $\tilde{F}=\sqrt{\sigma}$ and so $\sigma/\sqrt{\sigma}\cong Z$ or D.

Every automorphism of σ induces automorphisms of $\sqrt{\sigma}$ and of $\sigma/\sqrt{\sigma}$. Let $Out^+(\sigma)$ be the subgroup of $Out(\sigma)$ represented by automorphisms which induce the identity on $\sigma/\sqrt{\sigma}$. The restriction of any such automorphism to $\sqrt{\sigma}$ commutes with τ. We may view $\sqrt{\sigma}$ as a module over the ring $R=$

$\mathbb{Z}[X]/(\lambda(X))$, where $\lambda(X) = X^2 - tr(\tau)X + det(\tau)$ is the characteristic polynomial of τ. The polynomial λ is irreducible and has real roots which are not roots of unity, for otherwise $\sqrt{\sigma} \times_\tau Z$ would be virtually nilpotent. Therefore R is a domain and its field of fractions $\mathbb{Q}[X]/(\lambda(X))$ is a real quadratic number field. The R-module $\sqrt{\sigma}$ is clearly finitely generated, R-torsion free and of rank 1. Hence the endomorphism ring $End_R(\sqrt{\sigma})$ is a subring of $\tilde{R}$, the integral closure of R. Since $\tilde{R}$ is the ring of integers in $\mathbb{Q}[X]/(\lambda(X))$ the group of units $\tilde{R}^\times$ is isomorphic to $\{\pm 1\} \times Z$. Since τ determines a unit of infinite order in $R^\times$ the index $[\tilde{R}^\times : \tau^Z]$ is finite.

Suppose now that $\sigma/\sqrt{\sigma} \cong Z$. If f is an automorphism which induces the identity on $\sqrt{\sigma}$ and on $\sigma/\sqrt{\sigma}$ then $f(t) = tw$ for some w in $\sqrt{\sigma}$. If w is in the image of $\tau - 1$ then f is an inner automorphism. Now $\sqrt{\sigma}/(\tau-1)\sqrt{\sigma}$ is finite, of order $det(\tau - 1)$. Since τ is the image of an inner automorphism of σ it follows that $Out^+(\sigma)$ is an extension of a subgroup of $\tilde{R}^\times/\tau^Z$ by $\sqrt{\sigma}/(\tau-1)\sqrt{\sigma}$. Hence $Out(\sigma)$ has order dividing $2[\tilde{R}^\times : \tau^Z]det(\tau - 1)$.

If $\sigma/\sqrt{\sigma} \cong D$ then σ has a characteristic subgroup σ_1 such that $[\sigma : \sigma_1] = 2$, $\sqrt{\sigma} < \sigma_1$ and $\sigma_1/\sqrt{\sigma} \cong Z = \sqrt{D}$. Every automorphism of σ restricts to an automorphism of σ_1. It is easily verified that the restriction from $Aut(\sigma)$ to $Aut(\sigma_1)$ is a monomorphism. Since $Out(\sigma_1)$ is finite it follows that $Out(\sigma)$ is also finite. □

Corollary 8.8.1 *The mapping torus of a self homeomorphism of a $\mathbb{S}ol^3$-manifold is a $\mathbb{S}ol^3 \times \mathbb{E}^1$-manifold.* □

The group with presentation

$$\langle x, y, t \mid xy = yx,\ txt^{-1} = x^3y^2,\ tyt^{-1} = x^2y\rangle$$

is the fundamental group of a nonorientable $\mathbb{S}ol^3$-manifold Σ. The nonorientable $\mathbb{S}ol^3 \times \mathbb{E}^1$-manifold $\Sigma \times S^1$ is the mapping torus of id_Σ and is also the mapping torus of a self homeomorphism of R^3/Z^3.

The groups with presentations

$$\langle t, x, y, z \mid xy = yx,\ zxz^{-1} = x^{-1},\ zyz^{-1} = y^{-1},\ txt^{-1} = xy,\ tyt^{-1} = x,\ tzt^{-1} = z^{-1}\rangle,$$

$$\langle t, x, y, z \mid xy = yx,\ zxz^{-1} = x^2y,\ zyz^{-1} = xy,\ tx = xt,\ tyt^{-1} = x^{-1}y^{-1},\ tzt^{-1} = z^{-1}\rangle,$$

$$\langle t, x, y, z \mid xy = yx,\ xz = zx,\ yz = zy,\ txt^{-1} = x^2y,\ tyt^{-1} = xy,\ tzt^{-1} = z^{-1}\rangle$$

and $$\langle t, u, x, y \mid xy = yx,\ txt^{-1} = x^2y,\ tyt^{-1} = xy,\ uxu^{-1} = y^{-1},\ uyu^{-1} = x,\ utu^{-1} = t^{-1}\rangle$$

have Hirsch-Plotkin radical Z^3 and abelianization of rank 1. The corresponding $\mathbb{S}ol^3 \times \mathbb{E}^1$-manifolds are mapping tori in an essentially unique way. The first two are orientable, and are mapping tori of self homeomorphisms of the orientable flat 3-manifold with holonomy of order 2 and of an orientable $\mathbb{S}ol^3$-manifold, respectively. The latter two are nonorientable, and are mapping tori of orientation reversing self homeomorphisms of R^3/Z^3 and of the same orientable $\mathbb{S}ol^3$-manifold, respectively.

8.9 Realization and classification

Let π be a torsion free virtually poly-Z group of Hirsch length 4. If π is virtually abelian then it is the fundamental group of a flat 4-manifold, by the work of Bieberbach, and such groups are listed in §2-§4 above.

If π is virtually nilpotent but not virtually abelian then $\sqrt{\pi}$ is nilpotent of class 2 or 3. In the first case it has a characteristic chain $\sqrt{\pi}' \cong Z < C = \zeta\sqrt{\pi} \cong Z^2$. Let $\theta : \pi \to Aut(C) \cong GL(2,\mathbb{Z})$ be the homomorphism induced by conjugation in π. Then $\mathrm{Im}(\theta)$ is finite and triangular, and so is 1, $Z/2Z$ or $(Z/2Z)^2$. Let $K = C_\pi(C) = \mathrm{Ker}(\theta)$. Then K is torsion free and $\zeta K = C$, so K/C is a flat 2-orbifold group. Moreover as $K/\sqrt{K}$ acts trivially on $\sqrt{\pi}'$ it must act orientably on $\sqrt{K}/C$, and so $K/\sqrt{K}$ is cyclic of order 1, 2, 3, 4 or 6. As $\sqrt{\pi}$ is the preimage of $\sqrt{K}$ in π we see that $[\pi : \sqrt{\pi}] \leq 24$. (In fact $\pi/\sqrt{\pi} \cong F$ or $F \oplus (Z/2Z)$, where F is a finite subgroup of $GL(2,\mathbb{Z})$, excepting only direct sums of the dihedral groups of order 6, 8 or 12 with $(Z/2Z)$ [De].) Otherwise (if $\sqrt{\pi}' \not\leq \zeta\sqrt{\pi}$) it has a subgroup of index ≤ 2 which is a semidirect product $Z^3 \times_\theta Z$, by part (5) of Theorem 8.3. Since $(\theta^2 - I)$ is nilpotent it follows that $\pi/\sqrt{\pi} = 1$, $Z/2Z$ or $(Z/2Z)^2$. All these possibilities occur.

Such virtually nilpotent groups are fundamental groups of $\mathbb{N}il^3 \times \mathbb{E}^1$- and $\mathbb{N}il^4$-manifolds (respectively), and are classified in [De]. Dekimpe observes that π has a characteristic subgroup Z such that $Q = \pi/Z$ is a $\mathbb{N}il^3$- or $\mathbb{E}^3$-orbifold group and classifies the torsion free extensions of such Q by Z. There are 61 families of $\mathbb{N}il^3 \times \mathbb{E}^1$-groups and 7 families of $\mathbb{N}il^4$-groups. He also gives a faithful affine representation for each such group.

We shall sketch an alternative approach for the geometry $\mathbb{N}il^4$, which applies also to $\mathbb{S}ol^4_{m,n}$, $\mathbb{S}ol^4_0$ and $\mathbb{S}ol^4_1$. Each such group π has a characteristic subgroup ν of Hirsch length 3, and such that $\pi/\nu \cong Z$ or D. The preimage in π of $\sqrt{\pi/\nu}$ is characteristic, and is a semidirect product $\nu \times_\theta Z$. Hence it is determined up to isomorphism by the union of the conjugacy classes of θ and θ^{-1} in $Out(\nu)$, by Lemma 1.1. All such semidirect products may be realized as lattices and have faithful affine representations.

If the geometry is $\mathbb{N}il^4$ then $\nu = C_{\sqrt{\pi}}(\zeta_2\sqrt{\pi}) \cong Z^3$, by Theorem 1.5 and part (5) of Theorem 8.3. Moreover ν has a basis x, y, z such that $\langle z \rangle = \zeta\sqrt{\pi}$ and $\langle y, z \rangle = \zeta_2\sqrt{\pi}$. As these subgroups are characteristic the matrix of θ with respect to such a basis is $\pm(I + N)$, where N is strictly lower triangular and $n_{21}n_{32} \neq 0$. (See §5 above.) The conjugacy class of θ is determined by $(det(\theta), |n_{21}|, |n_{32}|, [n_{31}\ mod\ (n_{32})])$. (Thus θ is conjugate to θ^{-1} if and only if n_{32} divides $2n_{31}$.) The classification is more complicated if $\pi/\nu \cong D$.

If the geometry is $\mathbb{S}ol^4_{m,n}$ for some $m \neq n$ then $\pi \cong Z^3 \times_\theta Z$, where the eigenvalues of θ are distinct and real, and not ± 1, by the Corollary to Theorem 8.4. The translation subgroup $\pi \cap Sol^4_{m,n}$ is $Z^3 \times_A Z$, where $A = \theta$ or θ^2 is the least nontrivial power of θ with all eigenvalues positive, and has index ≤ 2 in π. Conversely, it is clear from the description of the isometries of $Sol^4_{m,n}$ in §3 of Chapter 7 that every such group is a lattice in $Isom(\mathbb{S}ol^4_{m,n})$. The conjugacy class of θ is determined by its characteristic polynomial $\Delta_\theta(t)$ and the ideal class of $\nu \cong Z^3$, considered as a rank 1 module over the order $\Lambda/(\Delta_\theta(t))$, by Theorem 1.4. (No such θ is conjugate to its inverse, as neither 1 nor -1 is an eigenvalue.)

A similar argument applies for $\mathbb{S}ol^4_0$. Although Sol^4_0 has no lattice subgroups, any semidirect product $Z^3 \times_\theta Z$ where θ has a pair of complex conjugate roots which are not roots of unity is a lattice in $Isom(\mathbb{S}ol^4_0)$. Such groups are again classified by the characteristic polynomial and an ideal class.

If the geometry is $\mathbb{S}ol^4_1$ then $\sqrt{\pi} \cong \Gamma_q$ for some $q \geq 1$, and either $\nu = \sqrt{\pi}$ or $\nu/\sqrt{\pi} = Z/2Z$ and $\nu/\zeta\sqrt{\pi} \cong Z^2 \times_{-I} (Z/2Z)$. (In the latter case ν is uniquely determined by q.) Moreover π is orientable if and only if $\beta_1(\pi) = 1$. In particular, $\mathrm{Ker}(w_1(\pi)) \cong \nu \times_\theta Z$ for some $\theta \in Aut(\nu)$. Let $A = \theta|_{\sqrt{\pi}}$ and let $\overline{A}$ be its image in $Aut(\sqrt{\pi}/\zeta\sqrt{\pi}) \cong GL(2, \mathbb{Z})$. If $\nu = \sqrt{\pi}$ the translation subgroup $\pi \cap Sol^4_1$ is $T = \Gamma_q \times_B Z$, where $B = A$ or A^2 is the least nontrivial power of A such that both eigenvalues of $\overline{A}$ are positive. If $\nu \neq \sqrt{\pi}$ the conjugacy class of $\overline{A}$ is only well-defined up to sign. If moreover $\pi/\nu \cong D$ then $\overline{A}$ is conjugate to its inverse, and so $det(\overline{A}) = 1$, since $\overline{A}$ has infinite order. We can then choose θ and hence A so that $T = \sqrt{\pi} \times_A Z$. In all cases we find that $[\pi : T]$ divides 4. (Note that $Isom(\mathbb{S}ol^4_1)$ has 8 components.)

Conversely, it is fairly easy to verify that a torsion free semidirect product $\nu \times_\theta Z$ (with $[\nu : \Gamma_q] \leq 2$ and ν as above) which is not virtually nilpotent is a lattice in the group of upper triangular matrices generated by Sol^4_1 and the diagonal matrix $diag[\pm 1, 1, \pm 1]$, which is contained in $Isom(\mathbb{S}ol^4_1)$. The conjugacy class of θ is determined up to a finite ambiguity by the characteristic polynomial of $\overline{A}$. Realization and classification of the nonorientable groups seems more difficult.

In the remaining case $\mathbb{S}ol^3 \times \mathbb{E}^1$ the subgroup ν is one of the four flat 3-manifold groups Z^3, $Z^2 \times_{-I} Z$, B_1 or B_2, and $\theta|\sqrt{\nu}$ has distinct real eigenvalues, one being ± 1. The index of the translation subgroup $\pi \cap (Sol^3 \times R)$ in π divides 8. (Note that $Isom(\mathbb{S}ol^3 \times \mathbb{E}^1)$ has 16 components.) Conversely any such semidirect product $\nu \times_\theta Z$ can be realized as a lattice in the index 2 subgroup $G < Isom(Sol^3 \times \mathbb{E}^1)$ defined in §3 of Chapter 7. Realization and classification of the groups with $\pi/\nu \cong D$ seems more difficult. (The number of subcases to be considered makes any classification an uninviting task. See however [Cb].)

8.10 Diffeomorphism

In all dimensions $n \neq 4$ it is known that infrasolvmanifolds with isomorphic fundamental group are diffeomorphic [FJ97]. In general one cannot expect to find affine diffeomorphisms, and the argument of Farrell and Jones uses differential topology rather than Lie theory for the cases $n \geq 5$. The cases with $n \leq 3$ follow from standard results of low dimensional topology. We shall show that related arguments also cover most 4-dimensional infrasolvmanifolds. The following theorem extends the main result of [Cb] (in which it was assumed that π is not virtually nilpotent).

Theorem 8.9 *Let M and M' be 4-manifolds which are total spaces of orbifold bundles $p : M \to B$ and $p' : M' \to B'$ with flat orbifold bases and infranilmanifold fibres, and suppose that $\pi_1(M) \cong \pi_1(M') \cong \pi$. Suppose that either π is orientable or $\beta_1(\pi) = 3$ or $\beta_1(\pi) = 2$ and $(\sqrt{\pi})' \cong Z$. Then M and M' are diffeomorphic.*

Proof We may assume that $d = dim(B) \leq d' = dim(B')$. Clearly $d' \leq 4 - \beta_1(\sqrt{\pi})$. Suppose first that π is not virtually abelian or virtually nilpotent of class 2 (i.e., suppose that $(\sqrt{\pi})' \not\leq \zeta\sqrt{\pi}$). Then all subgroups of finite index in π have $\beta_1 \leq 2$, and so $1 \leq d \leq d' \leq 2$. Moreover π has a characteristic nilpotent subgroup $\tilde{\nu}$ such that $h(\pi/\tilde{\nu}) = 1$, by Theorems 1.5 and 1.6. Let ν be the preimage in π of the maximal finite normal subgroup of $\pi/\tilde{\nu}$. Then ν is a characteristic virtually nilpotent subgroup (with $\sqrt{\nu} = \tilde{\nu}$) and $\pi/\nu \cong Z$ or D. If $d = 1$ then $\pi_1(F) = \nu$ and $p : M \to B$ induces this isomorphism. If $d = 2$ the image of ν in $\pi_1^{orb}(B)$ is normal. Hence there is an orbifold map q from B to the circle S^1 or the reflector interval $\mathbb{I}$ such that qp is an orbifold bundle projection. A similar analysis applies to M'. In either case, M and M' are canonically mapping tori or the unions of two twisted I-bundles, and the theorem follows via standard 3-manifold theory.

If π is virtually nilpotent it is realized by an infranilmanifold M_0 [DeK]. Hence we may assume that $M = M_0$, $d = 0$ or 4 and $(\sqrt{\pi})' \leq \zeta\sqrt{\pi}$. If $d' = 0$ or 4 then M' is also an infranilmanifold and the result is clear. If $d' = 1$ or if $\beta_1(\pi) + d' > 4$ then M' is a mapping torus or the union of twisted I-bundles, and π is a semidirect product $\kappa \rtimes Z$ or a generalized free product with amalgamation $G *_J H$ where $[G : J] = [H : J] = 2$. Hence the model M_0 is also a mapping torus or the union of twisted I-bundles, and we may argue as before.

Therefore we may assume that either $d' = 2$ and $\beta_1(\pi) \leq 2$ or $d' = 3$ and $\beta_1(\pi) \leq 1$. If $d' = 2$ then M and M' are Seifert fibred. If moreover π is orientable then M is diffeomorphic to M', by [Ue90]. If $\beta_1(\pi) = 2$ then either $\pi_1^{orb}(B')$ maps onto Z or π is virtually abelian.

If π is orientable then $\beta_1(\pi) > 0$, by Lemma 3.14. Therefore the remaining possibility is that $d' = 3$ and $\beta_1(\pi) = 1$. If $\pi_1^{orb}(B')$ maps onto Z then we may argue as before. Otherwise $\pi_1(F) \cap \pi' = 1$, so π is virtually abelian and the kernel of the induced homomorphism from π to $\pi_1^{orb}(B)$ is infinite cyclic and central. Hence the orbifold projection is the orbit map of an S^1-action on M. If M is orientable it is determined up to diffeomorphism by the orbifold data and an Euler class corresponding to the central extension of $\pi_1^{orb}(B)$ by Z [Fi78]. Thus M and M' are diffeomorphic. □

It is highly probable that the arguments of Ue and of Fintushel can be extended to all 4-manifolds which are Seifert fibred or admit smooth S^1-actions, and the theorem is surely true without any restrictions on π. (If $d' = 3$ and $\beta_1(\pi) = 0$ then π maps onto D, by Lemma 3.14, and $\pi_1(F) \cong Z$. It is not difficult to determine the maximal infinite cyclic normal subgroups of the flat 4-manifold groups π with $\beta_1(\pi) = 0$, and to verify that in each case the quotient maps onto D. Otherwise $\pi_1(F) = (\sqrt{\pi})'$, since $d' = 3$, and any epimorphism from π to D must factor through $\pi_1^{orb}(B') \cong \pi/(\sqrt{\pi})'$.)

We may now compare the following notions for M a closed smooth 4-manifold:

(1) *M is geometric of solvable Lie type;*

(2) *M is an infrasolvmanifold;*

(3) *M is the total space of an orbifold bundle with infranilmanifold fibre and flat base.*

Geometric 4-manifolds of solvable Lie type are infrasolvmanifolds, by the observations in §3 of Chapter 7, and the Mostow orbifold bundle of an infrasolvmanifold is as in (3), by Theorem 7.2. If π is orientable then it is realized geometrically and determines the total space of such an orbifold bundle up to diffeomorphism. Hence orientable smooth 4-manifolds admitting such orbifold fibrations are diffeomorphic to geometric 4-manifolds of solvable Lie type.

Are these three notions equivalent in general?

Chapter 9

The other aspherical geometries

The aspherical geometries of nonsolvable type which are realizable by closed 4-manifolds are the "mixed" geometries $\mathbb{H}^2 \times \mathbb{E}^2$, $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$, $\mathbb{H}^3 \times \mathbb{E}^1$ and the "semisimple" geometries $\mathbb{H}^2 \times \mathbb{H}^2$, $\mathbb{H}^4$ and $\mathbb{H}^2(\mathbb{C})$. (We shall consider the geometry $\mathbb{F}^4$ briefly in Chapter 13.) Closed $\mathbb{H}^2 \times \mathbb{E}^2$- or $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$-manifolds are Seifert fibred, have Euler characteristic 0 and their fundamental groups have Hirsch-Plotkin radical Z^2. In §1 and §2 we examine to what extent these properties characterize such manifolds and their fundamental groups. Closed $\mathbb{H}^3 \times \mathbb{E}^1$-manifolds also have Euler characteristic 0, but we have only a conjectural characterization of their fundamental groups (§3). In §4 we determine the mapping tori of self homeomorphisms of geometric 3-manifolds which admit one of these mixed geometries. (We return to this topic in Chapter 13.) In §5 we consider the three semisimple geometries. All closed 4-manifolds with product geometries other than $\mathbb{H}^2 \times \mathbb{H}^2$ are finitely covered by cartesian products. We characterize the fundamental groups of $\mathbb{H}^2 \times \mathbb{H}^2$-manifolds with this property; there are also "irreducible" $\mathbb{H}^2 \times \mathbb{H}^2$-manifolds which are not virtually products. Little is known about manifolds admitting one of the two hyperbolic geometries.

Although it is not yet known whether the disk embedding theorem holds over lattices for such geometries, we can show that the fundamental group and Euler characteristic determine the manifold up to s-cobordism (§6). Moreover an aspherical orientable closed 4-manifold which is finitely covered by a geometric manifold is homotopy equivalent to a geometric manifold (excepting perhaps if the geometry is $\mathbb{H}^2 \times \mathbb{E}^2$ or $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$).

9.1 Aspherical Seifert fibred 4-manifolds

In Chapter 8 we saw that if M is a closed 4-manifold with fundamental group π such that $\chi(M) = 0$ and $h(\sqrt{\pi}) \geq 3$ then M is homeomorphic to an infrasolvmanifold. Here we shall show that if $\chi(M) = 0$, $h(\sqrt{\pi}) = 2$ and $[\pi : \sqrt{\pi}] = \infty$ then M is homotopy equivalent to a 4-manifold which is Seifert fibred over a hyperbolic 2-orbifold. (We shall consider the case when $\chi(M) = 0$, $h(\sqrt{\pi}) = 2$ and $[\pi : \sqrt{\pi}] < \infty$ in Chapter 10.)

Theorem 9.1 *Let M be a closed 4-manifold with fundamental group π. If $\chi(M) = 0$ and π has an elementary amenable normal subgroup ρ with $h(\rho) = 2$ and such that either $H^2(\pi; \mathbb{Z}[\pi]) = 0$ or ρ is torsion free and $[\pi : \rho] = \infty$ then M is aspherical and ρ is virtually abelian.*

Proof Since π has one end, by Corollary 1.16.1, and $\beta_1^{(2)}(\pi) = 0$, by Theorem 2.3, M is aspherical if also $H^2(\pi; \mathbb{Z}[\pi]) = 0$, by Corollary 3.5.2. In this case ρ is torsion free and of infinite index in π, and so we may assume this henceforth. Since ρ is torsion free elementary amenable and $h(\rho) = 2$ it is virtually solvable, by Theorem 1.11. Therefore $A = \sqrt{\rho}$ is nontrivial, and as it is characteristic in ρ it is normal in π. Since A is torsion free and $h(A) \leq 2$ it is abelian, by Theorem 1.5.

Suppose first that $h(A) = 1$. Then A is isomorphic to a subgroup of Q and the homomorphism from $B = \rho/A$ to $Aut(A)$ induced by conjugation in ρ is injective. Since $Aut(A)$ is isomorphic to a subgroup of $\mathbb{Q}^\times$ and $h(B) = 1$ either $B \cong Z$ or $B \cong Z \oplus (Z/2Z)$. We must in fact have $B \cong Z$, since ρ is torsion free. Moreover A is not finitely generated and the centre of ρ is trivial. The quotient group π/A has one end as the image of ρ is an infinite cyclic normal subgroup of infinite index. Therefore π is 1-connected at ∞, by Theorem 1 of [Mi87], and so $H^s(\pi; \mathbb{Z}[\pi]) = 0$ for $s \leq 2$ [GM86]. Hence M is aspherical and π is a PD_4-group.

As A is a characteristic subgroup every automorphism of ρ restricts to an automorphism of A. This restriction from $Aut(\rho)$ to $Aut(A)$ is an epimorphism, with kernel isomorphic to A, and so $Aut(\rho)$ is solvable. Let $C = C_\pi(\rho)$ be the centralizer of ρ in π. Then C is nontrivial, for otherwise π would be isomorphic to a subgroup of $Aut(\rho)$ and hence would be virtually poly-Z. But then A would be finitely generated, ρ would be virtually abelian and $h(A) = 2$. Moreover $C \cap \rho = \zeta\rho = 1$, so $C\rho \cong C \times \rho$ and $c.d.C + c.d.\rho = c.d.C\rho \leq c.d.\pi = 4$. The quotient group $\pi/C\rho$ is isomorphic to a subgroup of $Out(\rho)$.

If $c.d.C\rho \leq 3$ then as C is nontrivial and $h(\rho) = 2$ we must have $c.d.C = 1$ and $c.d.\rho = h(\rho) = 2$. Therefore C is free and ρ is of type FP [Kr86]. By Theorem 1.13 ρ is an ascending HNN group with base a finitely generated subgroup of A and so has a presentation $\langle a, t \mid tat^{-1} = a^n \rangle$ for some nonzero integer n. We may assume $|n| > 1$, as ρ is not virtually abelian. The subgroup of $Aut(\rho)$ represented by $(n-1)A$ consists of inner automorphisms. Since $n > 1$ the quotient $A/(n-1)A \cong Z/(n-1)Z$ is finite, and as $Aut(A) \cong \mathbb{Z}[1/n]^\times$ it follows that $Out(\rho)$ is virtually abelian. Therefore π has a subgroup σ of finite index which contains $C\rho$ and such that $\sigma/C\rho$ is a finitely generated free abelian group, and in particular $c.d.\sigma/C\rho$ is finite. As σ is a PD_4-group it follows from Theorem 9.11 of [Bi] that $C\rho$ is a PD_3-group and hence that ρ is a PD_2-group. We reach the same conclusion if $c.d.C\rho = 4$, for then $[\pi : C\rho]$ is finite, by Strebel's Theorem, and so $C\rho$ is a PD_4-group. As a solvable PD_2-group is virtually Z^2 our original assumption must have been wrong.

Therefore $h(A) = 2$. As π/A is finitely generated and infinite π is not elementary amenable of Hirsch length 2. Hence $H^s(\pi;\mathbb{Z}[\pi]) = 0$ for $s \leq 2$, by Theorem 1.17, and so M is aspherical. Moreover as every finitely generated subgroup of ρ is either isomorphic to $Z \times_{-1} Z$ or is abelian $[\rho : A] \leq 2$. □

The group $Z*_n$ (with presentation $\langle a,t \mid tat^{-1} = a^n\rangle$) is torsion free and solvable of Hirsch length 2, and is the fundamental group of a closed orientable 4-manifold M with $\chi(M) = 0$. (See Chapter 3.) Thus the hypothesis that the subgroup ρ have infinite index in π is necessary for the above theorem. Do the other hypotheses imply that ρ must be torsion free?

Theorem 9.2 *Let M be a closed 4-manifold with fundamental group π. If $h(\sqrt{\pi}) = 2$, $[\pi : \sqrt{\pi}] = \infty$ and $\chi(M) = 0$ then M is aspherical and $\sqrt{\pi} \cong Z^2$.*

Proof As $H^s(\pi;\mathbb{Z}[\pi]) = 0$ for $s \leq 2$, by Theorem 1.17, M is aspherical, by Theorem 9.1. We may assume henceforth that $\sqrt{\pi}$ is a torsion free abelian group of rank 2 which is not finitely generated.

Suppose first that $[\pi : C] = \infty$, where $C = C_\pi(\sqrt{\pi})$. Then $c.d.C \leq 3$, by Strebel's Theorem. Since $\sqrt{\pi}$ is not finitely generated $c.d.\sqrt{\pi} = h(\sqrt{\pi}) + 1 = 3$, by Theorem 7.14 of [Bi]. Hence $C = \sqrt{\pi}$, by Theorem 8.8 of [Bi], so the homomorphism from $\pi/\sqrt{\pi}$ to $Aut(\sqrt{\pi})$ determined by conjugation in π is a monomorphism. Since $\sqrt{\pi}$ is torsion free abelian of rank 2 $Aut(\sqrt{\pi})$ is isomorphic to a subgroup of $GL(2,\mathbb{Q})$ and therefore any torsion subgroup of $Aut(\sqrt{\pi})$ is finite, by Corollary 1.3.1. Thus if $\pi'\sqrt{\pi}/\sqrt{\pi}$ is a torsion group $\pi'\sqrt{\pi}$ is elementary amenable and so π is itself elementary amenable, contradicting our assumption. Hence we may suppose that there is an element g in π' which has infinite order modulo $\sqrt{\pi}$. The subgroup $\langle\sqrt{\pi}, g\rangle$ generated by $\sqrt{\pi}$ and g is an extension of Z by $\sqrt{\pi}$ and has infinite index in π, for otherwise π would be virtually solvable. Hence $c.d.\langle\sqrt{\pi}, g\rangle = 3 = h(\langle\sqrt{\pi}, g\rangle)$, by Strebel's Theorem. By Theorem 7.15 of [Bi], $L = H_2(\sqrt{\pi};\mathbb{Z})$ is the underlying abelian group of a subring $\mathbb{Z}[m^{-1}]$ of Q, and the action of g on L is multiplication by a rational number a/b, where a and b are relatively prime and ab and m have the same prime divisors. But g acts on $\sqrt{\pi}$ as an element of $GL(2,\mathbb{Q})' \leq SL(2,\mathbb{Q})$. Since $L = \sqrt{\pi} \wedge \sqrt{\pi}$, by Proposition 11.4.16 of [Ro], g acts on L via $det(g) = 1$.

Therefore $m = 1$ and so L must be finitely generated. But then $\sqrt{\pi}$ must also be finitely generated, again contradicting our assumption.

Thus we may assume that C has finite index in π. Let $A < \sqrt{\pi}$ be a subgroup of $\sqrt{\pi}$ which is free abelian of rank 2. Then A_1 is central in C and C/A is finitely presentable. Since $[\pi : C]$ is finite A has only finitely many distinct conjugates in π, and they are all subgroups of ζC. Let N be their product. Then N is a finitely generated torsion free abelian normal subgroup of π and $2 \leq h(N) \leq h(\sqrt{C}) \leq h(\sqrt{\pi}) = 2$. An LHSSS argument gives $H^2(\pi/N; \mathbb{Z}[\pi/N]) \cong Z$, and so π/N is virtually a PD_2-group, by Bowditch's Theorem. Since $\sqrt{\pi}/N$ is a torsion group it must be finite, and so $\sqrt{\pi} \cong Z^2$. □

Corollary 9.2.1 *The manifold M is homotopy equivalent to one which is Seifert fibred with general fibre T or Kb over a hyperbolic 2-orbifold if and only if $h(\sqrt{\pi}) = 2$, $[\pi : \sqrt{\pi}] = \infty$ and $\chi(M) = 0$.*

Proof This follows from the theorem together with Theorem 7.3. □

9.2 The Seifert geometries: $\mathbb{H}^2 \times \mathbb{E}^2$ and $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$

A manifold with geometry $\mathbb{H}^2 \times \mathbb{E}^2$ or $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$ is Seifert fibred with base a hyperbolic orbifold. However not all such Seifert fibred 4-manifolds are geometric. An orientable Seifert fibred 4-manifold over an orientable hyperbolic base is geometric if and only if it is an elliptic surface; the relevant geometries are then $\mathbb{H}^2 \times \mathbb{E}^2$ and $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$ [Ue90,91].

In this section we shall show that such manifolds may be characterized up to homotopy equivalence in terms of their fundamental groups.

Theorem 9.3 *Let M be a closed $\mathbb{H}^3 \times \mathbb{E}^1$-, $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$- or $\mathbb{H}^2 \times \mathbb{E}^2$-manifold. Then M has a finite covering space which is diffeomorphic to a product $N \times S^1$.*

Proof If M is an $\mathbb{H}^3 \times \mathbb{E}^1$-manifold then $\pi = \pi_1(M)$ is a discrete cocompact subgroup of $G = Isom(\mathbb{H}^3 \times \mathbb{E}^1)$. The radical of this group is $Rad(G) \cong R$, and $G_o/Rad(G) \cong PSL(2, \mathbb{C})$, where G_o is the component of the identity in G. Therefore $A = \pi \cap Rad(G)$ is a lattice subgroup, by Proposition 8.27 of [Rg]. Since R/A is compact the image of π/A in $Isom(\mathbb{H}^3)$ is again a discrete cocompact subgroup. Hence $\sqrt{\pi} = A \cong Z$. Moreover π preserves the foliation of the model space by euclidean lines, so M is an orbifold bundle with general fibre S^1 over an $\mathbb{H}^3$-orbifold with orbifold fundamental group $\pi/\sqrt{\pi}$.

On passing to a 2-fold covering space, if necessary, we may assume that $\pi \leq Isom(\mathbb{H}^3) \times R$ and (hence) $\zeta\pi = \sqrt{\pi}$. Projection to the second factor maps $\sqrt{\pi}$ monomorphically to R. Hence on passing to a further finite covering space, if necessary, we may assume that $\pi \cong \nu \times Z$, where $\nu = \pi/\sqrt{\pi} \cong \pi_1(N)$ for some closed orientable $\mathbb{H}^3$-manifold N. (Note that we do *not* claim that $\pi = \nu \times Z$ as a subgroup of $PSL(2,\mathbb{R}) \times R$.) The foliation of $H^3 \times R$ by lines induces an S^1-bundle structure on M, with base N. As such bundles (with aspherical base) are determined by their fundamental groups, M is diffeomorphic to $N \times S^1$.

Similar arguments apply in the other two cases. If $G = Isom(\mathbb{X})$ where $\mathbb{X} = \mathbb{H}^2 \times \mathbb{E}^2$ or $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$, then $Rad(G) \cong R^2$, and $G_o/R^2 \cong PSL(2,\mathbb{R})$. The intersection $A = \pi \cap Rad(G)$ is again a lattice subgroup, and the image of π/A in $PSL(2,\mathbb{R})$ is a discrete cocompact subgroup. Hence $\sqrt{\pi} = A \cong Z^2$ and $\pi/\sqrt{\pi}$ is virtually a PD_2-group. If $\mathbb{X} = \widetilde{\mathbb{SL}} \times \mathbb{E}^1$ then (after passing to a 2-fold covering space, if necessary) we may assume that $\pi \leq Isom(\widetilde{\mathbb{SL}}) \times R$. If $\mathbb{X} = \mathbb{H}^2 \times \mathbb{E}^2$ then $PSL(2,\mathbb{R}) \times R^2$ is a cocompact subgroup of $Isom(\mathbb{X})$. Hence $\pi \cap PSL(2,\mathbb{R}) \times R^2$ has finite index in π. In each case projection to the second factor maps $\sqrt{\pi}$ monomorphically. Moreover π preserves the foliation of the model space by copies of the euclidean factor. As before, M is virtually a product. □

In general, there may not be such a covering which is geometrically a cartesian product. Let ν be a discrete cocompact subgroup of $Isom(\mathbb{X})$ where $\mathbb{X} = \mathbb{H}^3$ or $\widetilde{\mathbb{SL}}$ which admits an epimorphism $\alpha : \nu \to Z$. Define a homomorphism $\theta : \nu \times Z \to Isom(\mathbb{X} \times \mathbb{E}^1)$ by $\theta(g,n)(x,r) = (g(x), r + n + \alpha(g)\sqrt{2})$ for all $g \in \nu$, $n \in Z$, $x \in X$ and $r \in R$. Then θ is a monomorphism onto a discrete subgroup which acts freely and cocompactly on $X \times R$, but the image of $\theta(\nu \times Z)$ in $E(1)$ has rank 2.

Orientable $\mathbb{H}^2 \times \mathbb{E}^2$- and $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$-manifolds are determined up to diffeomorphism (among such geometric manifolds) by their fundamental groups [Ue91]. However we do not yet have a complete characterization of the possible groups.

Corollary 9.3.1 *Let M be a closed 4-manifold with fundamental group π. Then M has a covering space of degree dividing 4 which is homotopy equivalent to a $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$- or $\mathbb{H}^2 \times \mathbb{E}^2$-manifold if and only if $\sqrt{\pi} \cong Z^2$, $[\pi : \sqrt{\pi}] = \infty$, $[\pi : C_\pi(\sqrt{\pi})] < \infty$ and $\chi(M) = 0$.*

Proof The necessity of most of these conditions is clear from the proof of the theorem. If $\mathbb{X} = \mathbb{H}^2 \times \mathbb{E}^2$ then π has a subgroup of finite index which is isomorphic to $\tau \times Z^2$, where $\zeta\tau = 1$. If $\mathbb{X} = \widetilde{\mathbb{SL}} \times \mathbb{E}^1$ then π has a normal subgroup of finite index which is isomorphic to a product $Z \times \sigma$, and $\sqrt{\pi}$ has a characteristic infinite cyclic subgroup. Hence $\pi/C_\pi(\sqrt{\pi})$ is isomorphic to a finite upper triangular subgroup of $GL(2,\mathbb{Z})$. Since M is aspherical and $\sqrt{\pi}$ is infinite $\chi(M) = 0$.

If these conditions hold $\beta_1^{(2)}(\pi) = 0$ and $H^s(\pi;\mathbb{Z}[\pi]) = 0$ for $s \leq 2$, and so M is aspherical, by Corollary 3.5.2. Hence M is homotopy equivalent to a manifold $M(\pi)$ which is Seifert fibred over a hyperbolic base orbifold, by Theorem 7.3. On passing to a covering space of degree dividing 4, if necessary, we may assume that M and the base orbifold are each orientable. Since π must then act on $\sqrt{\pi}$ through a finite subgroup of $SL(2,\mathbb{Z})$ (which is upper triangular if $\sqrt{\pi}$ is not a direct factor of a subgroup of finite index in π) the result follows from Theorem B of §5 of [Ue91]. □

Corollary 9.3.2 *A group π is the fundamental group of a closed orientable $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$- or $\mathbb{H}^2 \times \mathbb{E}^2$-manifold with orientable base orbifold if and only if it is a PD_4^+-group, $\sqrt{\pi} \cong Z^2$, $[\pi : \sqrt{\pi}] = \infty$ and π acts on $\sqrt{\pi}$ through a finite cyclic subgroup of $SL(2,\mathbb{Z})$.* □

The geometry is $\mathbb{H}^2 \times \mathbb{E}^2$ if and only if $\sqrt{\pi}$ is virtually a direct factor in π. This case may also be distinguished as follows.

Theorem 9.4 *Let M be a closed 4-manifold with fundamental group π. Then M has a covering space of degree dividing 4 which is homotopy equivalent to a $\mathbb{H}^2 \times \mathbb{E}^2$-manifold if and only if π has a finitely generated infinite subgroup ρ such that $[\pi : N_\pi(\rho)] < \infty$, $\sqrt{\rho} = 1$, $\zeta C_\pi(\rho) \cong Z^2$ and $\chi(M) = 0$.*

Proof The necessity of the conditions follows from Theorem 9.3. Suppose that they hold. Then M is aspherical and so π is a PD_4-group. Let $C = C_\pi(\rho)$. Then C is also normal in $\nu = N_\pi(\rho)$, and $C \cap \rho = 1$, since $\sqrt{\rho} = 1$. Hence $\rho \times C \cong \rho.C \leq \pi$. Now ρ is nontrivial. If ρ were free then an argument using the LHSSS for $H^*(\pi;\mathbb{Q}[\pi])$ would imply that ρ has two ends, and hence that $\sqrt{\rho} = \rho \cong Z$. Hence $c.d.\rho \geq 2$. Since moreover $Z^2 \leq C$ we must have $c.d.\rho = c.d.C = 2$ and $[\pi : \rho.C] < \infty$. It follows easily that $\sqrt{\pi} \cong Z^2$ and that $[\pi : C_\pi(\sqrt{\pi})] < \infty$. Hence we may apply Corollary 9.3.1. Since π is virtually a product it must be of type $\mathbb{H}^2 \times \mathbb{E}^2$. □

Is it possible to give a more self-contained argument for this case? It is not hard to see that $\pi/\sqrt{\pi}$ acts discretely, cocompactly and isometrically on H^2. However it is more difficult to find a suitable homomorphism from π to $E(2)$.

Theorems 9.1 and 9.2 suggest that there should be a characterization of closed $\mathbb{H}^2 \times \mathbb{E}^2$- and $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$-manifolds parallel to Theorem 8.1, i.e., in terms of the conditions "$\chi(M) = 0$" and "π has an elementary amenable normal subgroup of Hirsch length 2 and infinite index".

9.3 $\mathbb{H}^3 \times \mathbb{E}^1$-manifolds

We have only conjectural characterizations of manifolds homotopy equivalent to $\mathbb{H}^3 \times \mathbb{E}^1$-manifolds and of their fundamental groups. An argument similar to that of Corollary 9.3.1 shows that a 4-manifold M with fundamental group π is virtually simple homotopy equivalent to an $\mathbb{H}^3 \times \mathbb{E}^1$-manifold if and only if $\chi(M) = 0$, $\sqrt{\pi} = Z$ and π has a normal subgroup of finite index which is isomorphic to $\rho \times Z$ where ρ is a discrete cocompact subgroup of $PSL(2, \mathbb{C})$. If every PD_3-group is the fundamental group of an aspherical closed 3-manifold and if every atoroidal aspherical closed 3-manifold is hyperbolic we could replace the last assertion by the more intrinsic conditions that ρ have one end (which would suffice with the other conditions to imply that M is aspherical and hence that ρ is a PD_3-group), no noncyclic abelian subgroups and $\sqrt{\rho} = 1$ (which would imply that any irreducible 3-manifold with fundamental group ρ is atoroidal). Similarly, a group G should be the fundamental group of an $\mathbb{H}^3 \times \mathbb{E}^1$-manifold if and only if it is torsion free and has a normal subgroup of finite index isomorphic to $\rho \times Z$ where ρ is a PD_3-group with $\sqrt{\rho} = 1$ and no noncyclic abelian subgroups.

Lemma 9.5 *Let π be a finitely generated group with $\sqrt{\pi} \cong Z$, and which has a subgroup G of finite index such that $\sqrt{\pi} \cap G' = 1$. Then there is a homomorphism $\lambda : \pi \to D$ which is injective on $\sqrt{\pi}$.*

Proof We may assume that G is normal in π and that $G < C_\pi(\sqrt{\pi})$. Let $H = \pi/I(G)$ and let A be the image of $\sqrt{\pi}$ in H. Then H is an extension of the finite group π/G by the finitely generated free abelian group $G/I(G)$, and $A \cong Z$. Conjugation in H determines a homomorphism w from π/G to $Aut(A) = \{\pm 1\}$. Since the rational group ring $\mathbb{Q}[\pi/G]$ is semisimple $\mathbb{Q} \otimes A$ is a direct summand of $\mathbb{Q} \otimes (G/I(G))$, and so there is a $\mathbb{Z}[\pi/G]$-linear homomorphism $p : G/I(G) \to Z^w$ which is injective on A. The kernel is a normal subgroup of H, and $H/\mathrm{Ker}(p)$ has two ends. The lemma now follows easily. □

The foliation of $H^3 \times R$ by copies of H^3 induces a codimension 1 foliation of any closed $\mathbb{H}^3 \times \mathbb{E}^1$-manifold. If all the leaves are compact, then it is either a mapping torus or the union of two twisted I-bundles.

Theorem 9.6 *Let M be a closed $\mathbb{H}^3 \times \mathbb{E}^1$-manifold. If $\zeta\pi \cong Z$ then M is homotopy equivalent to a mapping torus of a self homeomorphism of an $\mathbb{H}^3$-manifold; otherwise M is homotopy equivalent to the union of two twisted I-bundles over $\mathbb{H}^3$-manifold bases.*

Proof Let $\lambda : \pi \to D$ be a homomorphism as in Lemma 9.5 and let $K = \mathrm{Ker}(\lambda)$. Then $K \cap \sqrt{\pi} = 1$, so K is isomorphic to a subgroup of finite index in $\pi/\sqrt{\pi}$. Therefore $K \cong \pi_1(N)$ for some closed $\mathbb{H}^3$-manifold, since it is torsion free. If $\zeta\pi = Z$ then $\mathrm{Im}(\lambda) \cong Z$ (since $\zeta D = 1$); if $\zeta\pi = 1$ then $w \neq 1$ and so $\mathrm{Im}(\lambda) \cong D$. The theorem now follows easily. □

Is M itself such a mapping torus or union of I-bundles?

9.4 Mapping tori

In this section we shall use 3-manifold theory to characterize mapping tori with one of the geometries $\mathbb{H}^3 \times \mathbb{E}^1$, $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$ or $\mathbb{H}^2 \times \mathbb{E}^2$.

Theorem 9.7 *Let ϕ be a self homeomorphism of a closed 3-manifold N which admits the geometry $\mathbb{H}^2 \times \mathbb{E}^1$ or $\widetilde{\mathbb{SL}}$. Then the mapping torus $M(\phi) = N \times_\phi S^1$ admits the corresponding product geometry if and only if the outer automorphism $[\phi_*]$ induced by ϕ has finite order. The mapping torus of a self homeomorphism ϕ of an $\mathbb{H}^3$-manifold N admits the geometry $\mathbb{H}^3 \times \mathbb{E}^1$.*

Proof Let $\nu = \pi_1(N)$ and let t be an element of $\pi = \pi_1(M(\phi))$ which projects to a generator of $\pi_1(S^1)$. If $M(\phi)$ has geometry $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$ then after passing to the 2-fold covering space $M(\phi^2)$, if necessary, we may assume that π is a discrete cocompact subgroup of $Isom(\widetilde{\mathbb{SL}}) \times R$. As in Theorem 9.3 the intersection of π with the centre of this group is a lattice subgroup $L \cong Z^2$. Since the centre of ν is Z the image of L in π/ν is nontrivial, and so π has a subgroup σ of finite index which is isomorphic to $\nu \times Z$. In particular, conjugation by $t^{[\pi:\sigma]}$ induces an inner automorphism of ν.

If $M(\phi)$ has geometry $\mathbb{H}^2 \times \mathbb{E}^2$ a similar argument implies that π has a subgroup σ of finite index which is isomorphic to $\rho \times Z^2$, where ρ is a discrete cocompact subgroup of $PSL(2,\mathbb{R})$, and is a subgroup of ν. It again follows that $t^{[\pi:\sigma]}$ induces an inner automorphism of ν.

Conversely, suppose that N has a geometry of type $\mathbb{H}^2 \times \mathbb{E}^1$ or $\widetilde{\mathbb{SL}}$ and that $[\phi_*]$ has finite order in $Out(\nu)$. Then ϕ is homotopic to a self homeomorphism of (perhaps larger) finite order [Zn80] and is therefore isotopic to such a self homeomorphism [Sc85,BO91], which may be assumed to preserve the geometric structure [MS86]. Thus we may assume that ϕ is an isometry. The self homeomorphism of $N \times R$ sending (n, r) to $(\phi(n), r+1)$ is then an isometry for the product geometry and the mapping torus has the product geometry.

If N is hyperbolic then ϕ is homotopic to an isometry of finite order, by Mostow rigidity [Ms68], and is therefore isotopic to such an isometry [GMT96], so the mapping torus again has the product geometry. □

A closed 4-manifold M which admits an effective T-action with hyperbolic base orbifold is homotopy equivalent to such a mapping torus. For then $\zeta\pi = \sqrt{\pi}$ and the LHSSS for homology gives an exact sequence

$$H_2(\pi/\zeta\pi; \mathbb{Q}) \to H_1(\zeta\pi; \mathbb{Q}) \to H_1(\pi; \mathbb{Q}).$$

As $\pi/\zeta\pi$ is virtually a PD_2-group $H_2(\pi/\zeta\pi; \mathbb{Q}) \cong Q$ or 0, so $\zeta\pi/\zeta\pi \cap \pi'$ has rank at least 1. Hence $\pi \cong \nu \times_\theta Z$ where $\zeta\nu \cong Z$, $\nu/\zeta\nu$ is virtually a PD_2-group and $[\theta]$ has finite order in $Out(\nu)$. If moreover M is orientable then it is geometric ([Ue90,91] - see also §7 of Chapter 7). Note also that if M is a $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$-manifold then $\zeta\pi = \sqrt{\pi}$ if and only if $\pi \leq Isom_o(\widetilde{\mathbb{SL}} \times \mathbb{E}^1)$.

Let F be a closed hyperbolic surface and $\alpha : F \to F$ a pseudo-Anasov homeomorphism. Let $\Theta(f, z) = (\alpha(f), \bar{z})$ for all (f, z) in $N = F \times S^1$. Then N is an $\mathbb{H}^2 \times \mathbb{E}^1$-manifold. The mapping torus of Θ is homeomorphic to an $\mathbb{H}^3 \times \mathbb{E}^1$-manifold which is not a mapping torus of any self-homeomorphism of an $\mathbb{H}^3$-manifold. In this case $[\Theta_*]$ has infinite order. However if N is a $\widetilde{\mathbb{SL}}$-manifold and $[\phi_*]$ has infinite order then $M(\phi)$ admits no geometric structure, for then $\sqrt{\pi} \cong Z$ but is not a direct factor of any subgroup of finite index.

If $\zeta\nu \cong Z$ and $\zeta(\nu/\zeta\nu) = 1$ then $Hom(\nu/\nu', \zeta\nu)$ embeds in $Out(\nu)$, and thus ν has outer automorphisms of infinite order, in most cases [CR77].

Let N be an aspherical closed $\mathbb{X}^3$-manifold where $\mathbb{X}^3 = \mathbb{H}^3$, $\widetilde{\mathbb{SL}}$ or $\mathbb{H}^2 \times \mathbb{E}^1$, and suppose that $\beta_1(N) > 0$ but N is not a mapping torus. Choose an epimorphism $\lambda : \pi_1(N) \to Z$ and let $\widehat{N}$ be the 2-fold covering space associated to the subgroup $\lambda^{-1}(2Z)$. If $\nu : \widehat{N} \to \widehat{N}$ is the covering involution then $\mu(n, z) = (\nu(n), \bar{z})$ defines a free involution on $N \times S^1$, and the orbit space M is an $\mathbb{X}^3 \times \mathbb{E}^1$-manifold with $\beta_1(M) > 0$ which is not a mapping torus.

9.5 The semisimple geometries: $\mathbb{H}^2 \times \mathbb{H}^2$, $\mathbb{H}^4$ and $\mathbb{H}^2(\mathbb{C})$

In this section we shall consider the remaining three geometries realizable by closed 4-manifolds. (Not much is known about $\mathbb{H}^4$ or $\mathbb{H}^2(\mathbb{C})$.)

Let $P = PSL(2,\mathbb{R})$ be the group of orientation preserving isometries of $\mathbb{H}^2$. Then $Isom(\mathbb{H}^2 \times \mathbb{H}^2)$ contains $P \times P$ as a normal subgroup of index 8. If M is a closed $\mathbb{H}^2 \times \mathbb{H}^2$-manifold then $\sigma(M) = 0$ and $\chi(M) > 0$. It is *reducible* if it has a finite cover isometric to a product of closed surfaces. The model space for $\mathbb{H}^2 \times \mathbb{H}^2$ may be taken as the unit polydisc

$$\{(w,z) \in C^2 : |w| < 1, |z| < 1\}.$$

Thus M is a complex surface if (and only if) $\pi_1(M)$ is a subgroup of $P \times P$.

We have the following characterizations of the fundamental groups of reducible $\mathbb{H}^2 \times \mathbb{H}^2$-manifolds.

Theorem 9.8 *A group π is the fundamental group of a reducible $\mathbb{H}^2 \times \mathbb{H}^2$-manifold if and only if it is torsion free, $\sqrt{\pi} = 1$ and π has a subgroup of finite index which is isomorphic to a product of PD_2-groups.*

Proof The conditions are clearly necessary. Suppose that they hold. Then π is a PD_4-group and has a subgroup of finite index which is a direct product $\alpha.\beta \cong \alpha \times \beta$, where α and β are PD_2-groups. Let N be the intersection of the conjugates of $\alpha.\beta$ in π. Then N is normal in π, so $\sqrt{N} = 1$ also, and $[\pi : N] < \infty$. Let $K = \alpha \cap N$ and $L = \beta \cap N$. Then K and L are PD_2-groups with trivial centre, and $K.L \cong K \times L$ is normal in N and has finite index in π. Moreover N/K and N/L are isomorphic to subgroups of finite index in β and α, respectively, and so are also PD_2-groups. Since any automorphism of N must either fix these subgroups or interchange them, by Theorem 5.6, $K.L$ is normal in π and $[\pi : N_\pi(K)] \leq 2$.

Let $\nu = N_\pi(K)$. Then $L \leq C_\pi(K) \leq \nu$ and $\nu = N_\pi(L)$ also. After enlarging K and L, if necessary, we may assume that $L = C_\pi(K)$ and $K = C_\pi(L)$. Hence ν/K and ν/L have no nontrivial finite normal subgroup. (For if K_1 is normal in ν and contains K as a subgroup of finite index then $K_1 \cap L$ is finite, hence trivial, and so $K_1 \leq C_\pi(L)$.) The action of ν/L by conjugation on K has finite image in $Out(K)$, and so ν/L embeds as a discrete cocompact subgroup of $Isom(\mathbb{H}^2)$, by the Nielsen conjecture [Ke83]. Together with a similar embedding for ν/K we obtain a homomorphism from ν to a discrete cocompact subgroup of $Isom(\mathbb{H}^2 \times \mathbb{H}^2)$.

If $[\pi : \nu] = 2$ let t be an element of $\pi - \nu$, and let $j : \nu/K \to Isom(\mathbb{H}^2)$ be an embedding onto a discrete cocompact subgroup S. Then $tKt^{-1} = L$ and conjugation by t induces an isomorphism $f : \nu/K \to \nu/L$. The homomorphisms j and $j \circ f^{-1}$ determine an embedding $J : \nu \to Isom(\mathbb{H}^2 \times \mathbb{H}^2)$ onto a discrete cocompact subgroup of finite index in $S\times S$. Now $t^2\in\nu$ and $J(t^2)=(s,s)$, where $s=j(t^2K)$. We may extend J to an embedding of π in $Isom(\mathbb{H}^2\times\mathbb{H}^2)$ by defining $J(t)$ to be the isometry sending (x,y) to $(y,s.x)$. Thus (in either case) π acts isometrically and properly discontinuously on $H^2\times H^2$. Since π is torsion free the action is free, and so $\pi=\pi_1(M)$, where $M=\pi\backslash(H^2\times H^2)$. □

Corollary 9.8.1 *Let M be a $\mathbb{H}^2\times\mathbb{H}^2$-manifold. Then M is reducible if and only if it has a 2-fold covering space which is homotopy equivalent to the total space of an orbifold bundle over a hyperbolic 2-orbifold.*

Proof That reducible manifolds have such coverings was proven in the theorem. Conversely, an irreducible lattice in $P\times P$ cannot have any nontrivial normal subgroups of infinite index, by Theorem IX.6.14 of [Ma]. Hence an $\mathbb{H}^2\times\mathbb{H}^2$-manifold which is finitely covered by the total space of a surface bundle is virtually a cartesian product. □

Is the 2-fold covering space itself such a bundle space over a 2-orbifold?

In general, we cannot assume that M is itself fibred over a 2-orbifold. Let G be a PD_2-group with $\zeta G=1$ and let x be a nontrivial element of G. A cocompact free action of G on H^2 determines a cocompact free action of

$$\pi=\langle G\times G,t\mid t(g_1,g_2)t^{-1}=(xg_2x^{-1},g_1)\text{ for all }(g_1,g_2)\in G\times G,t^2=(x,x)\rangle$$

on $H^2\times H^2$, by $(g_1,g_2).(h_1,h_2)=(g_1.h_1,g_2.h_2)$ and $t.(h_1,h_2)=(x.h_2,h_1)$, for all $(g_1,g_2)\in G\times G$ and $(h_1,h_2)\in H^2\times H^2$. The group π has no normal subgroup which is a PD_2-group. (Note also that if G is orientable $\pi\backslash(H^2\times H^2)$ is a compact complex surface.)

We may use Theorem 9.8 to give several characterizations of the homotopy types of such manifolds.

Theorem 9.9 *Let M be a closed 4-manifold with fundamental group π. Then the following are equivalent:*

1. *M is homotopy equivalent to a reducible $\mathbb{H}^2\times\mathbb{H}^2$-manifold;*
2. *π has a subnormal subgroup G which is FP_2, has one end and such that $C_\pi(G)$ is not a free group, $\pi_2(M)=0$ and $\chi(M)\neq 0$;*
3. *π has a subgroup ρ of finite index which is isomorphic to a product of two PD_2-groups and $\chi(M)[\pi:\rho]=\chi(\rho)\neq 0$.*

(4) *π is virtually a PD_4-group, $\sqrt{\pi}=1$ and π has a torsion free subgroup of finite index which is isomorphic to a nontrivial product $\sigma\times\tau$ where $\chi(M)[\pi:\sigma\times\tau]=(2-\beta_1(\sigma))(2-\beta_1(\tau))$.*

Proof If (1) holds then M is aspherical and so (2) holds, by Theorem 9.8 and its Corollary.

Suppose now that (2) holds. Then π has one end, by an iterated LHSSS argument, since G does. Hence M is aspherical and π is a PD_4-group, since $\pi_2(M)=0$. Since $\chi(M)\neq 0$ we must have $\sqrt{\pi}=1$. (For otherwise $\beta_i^{(2)}(\pi)=0$ for all i, by Theorem 2.3, and so $\chi(M)=0$.) In particular, every subnormal subgroup of π has trivial centre. Therefore $G\cap C_\pi(G)=\zeta G=1$ and so $G\times C_\pi(G)\cong\rho=G.C_\pi(G)\leq\pi$. Hence $c.d.C_\pi(G)\leq 2$. Since $C_\pi(G)$ is not free $c.d.G\times C_\pi(G)=4$ and so ρ has finite index in π. (In particular, $[C_\pi(C_\pi(G)):G]$ is finite.) Hence ρ is a PD_4-group and G and $C_\pi(G)$ are PD_2-groups, so π is virtually a product. Thus (2) implies (1), by Theorem 9.8.

It is clear that (1) implies (3). If (3) holds then on applying Theorems 2.2 and 3.5 to the finite covering space associated to ρ we see that M is aspherical, so π is a PD_4-group and (4) holds. Similarly, M is asperical if (4) holds. In particular, π is a PD_4-group and so is torsion free. Since $\sqrt{\pi}=1$ neither σ nor τ can be infinite cyclic, and so they are each PD_2-groups. Therefore π is the fundamental group of a reducible $\mathbb{H}^2\times\mathbb{H}^2$-manifold, by Theorem 9.8, and $M\simeq\pi\backslash H^2\times H^2$, by asphericity. □

The asphericity of M could be ensured by assuming that π be PD_4 and $\chi(M)=\chi(\pi)$, instead of assuming that $\pi_2(M)=0$.

For $\mathbb{H}^2\times\mathbb{H}^2$-manifolds we can give more precise criteria for reducibility.

Theorem 9.10 *Let M be a closed $\mathbb{H}^2\times\mathbb{H}^2$-manifold with fundamental group π. Then the following are equivalent:*

(1) *π has a subgroup of finite index which is a nontrivial direct product;*

(2) *$Z^2<\pi$;*

(3) *π has a nontrivial element with nonabelian centralizer;*

(4) *$\pi\cap(\{1\}\times P)\neq 1$;*

(5) *$\pi\cap(P\times\{1\})\neq 1$;*

(6) *M is reducible.*

Proof Since π is torsion free each of the above conditions is invariant under passage to subgroups of finite index, and so we may assume without loss of generality that $\pi \leq P \times P$. Suppose that σ is a subgroup of finite index in π which is a nontrivial direct product. Since $\chi(\sigma) \neq 0$ neither factor can be infinite cyclic, and so the factors must be PD_2-groups. In particular, $Z^2 < \sigma$ and the centraliser of any element of either direct factor is nonabelian. Thus (1) implies (2) and (3).

Suppose that (a,b) and (a',b') generate a subgroup of π isomorphic to Z^2. Since centralizers of elements of infinite order in P are cyclic the subgroup of P generated by $\{a,a'\}$ is infinite cyclic or is finite. Hence we may assume without loss of generality that $a' = 1$, and so (2) implies (4). Similarly, (2) implies (5).

Let $g = (g_1, g_2) \in P \times P$ be nontrivial. Since centralizers of elements of infinite order in P are infinite cyclic and $C_{P\times P}(\langle g\rangle) = C_P(\langle g_1\rangle) \times C_P(\langle g_2\rangle)$ it follows that if $C_\pi(\langle g\rangle)$ is nonabelian then either g_1 or g_2 has finite order. Thus (3) implies (4) and (5).

Let $K_1 = \pi \cap (\{1\} \times P)$ and $K_2 = \pi \cap (P \times \{1\})$. Then K_i is normal in π, and there are exact sequences

$$1 \to K_i \to \pi \to L_i \to 1,$$

where $L_i = pr_i(\pi)$ is the image of π under projection to the i^{th} factor of $P \times P$, for $i = 1$ and 2. Moreover K_i is normalised by L_{3-i}, for $i = 1$ and 2. Suppose that $K_1 \neq 1$. Then K_1 is non abelian, since it is normal in π and $\chi(\pi) \neq 0$. If L_2 were not discrete then elements of L_2 sufficiently close to the identity would centralize K_1. As centralizers of nonidentity elements of P are abelian, this would imply that K_1 is abelian. Hence L_2 is discrete. Now $L_2\backslash H^2$ is a quotient of $\pi\backslash H \times H$ and so is compact. Therefore L_2 is virtually a PD_2-group. Now $c.d.K_2 + v.c.d.L_2 \geq c.d.\pi = 4$, so $c.d.K_2 \geq 2$. In particular, $K_2 \neq 1$ and so a similar argument now shows that $c.d.K_1 \geq 2$. Hence $c.d.K_1 \times K_2 \geq 4$. Since $K_1 \times K_2 \cong K_1.K_2 \leq \pi$ it follows that π is virtually a product, and M is finitely covered by $(K_1\backslash H^2) \times (K_2\backslash H^2)$. Thus (4) and (5) are equivalent, and imply (6). Clearly (6) implies (1). ∎

The idea used in showing that (4) implies (5) and (6) derives from one used in the proof of Theorem 6.3 of [Wl85].

If Γ is a discrete cocompact subgroup of $P \times P$ such that $M = \Gamma\backslash H^2 \times H^2$ is irreducible then $\Gamma \cap P \times \{1\} = \Gamma \cap \{1\} \times P = 1$, by the theorem. Hence the natural foliations of $H^2 \times H^2$ descend to give a pair of transverse foliations of M by copies of H^2. (Conversely, if M is a closed Riemannian 4-manifold with a codimension 2 metric foliation by totally geodesic surfaces then M has a finite cover which either admits the geometry $\mathbb{H}^2 \times \mathbb{E}^2$ or $\mathbb{H}^2 \times \mathbb{H}^2$ or is the total space of an S^2 or T-bundle over a closed surface or is the mapping torus of a self homeomorphism of R^3/Z^3, $S^2 \times S^1$ or a lens space [Ca90]).

An irreducible $\mathbb{H}^2 \times \mathbb{H}^2$-lattice is an arithmetic subgroup of $Isom(\mathbb{H}^2 \times \mathbb{H}^2)$, and has no nontrivial normal subgroups of infinite index, by Theorems IX.6.5 and 14 of [Ma]. Such irreducible lattices are rigid, and so the argument of Theorem 8.1 of [Wa72] implies that there are only finitely many irreducible $\mathbb{H}^2 \times \mathbb{H}^2$-manifolds with given Euler characteristic. What values of χ are realized by such manifolds?

Examples of irreducible $\mathbb{H}^2 \times \mathbb{H}^2$-manifolds may be constructed as follows. Let F be a totally real number field, with ring of integers O_F. Let H be a skew field which is a quaternion algebra over F such that $H \otimes_\sigma \mathbb{R} \cong M_2(\mathbb{R})$ for exactly two embeddings σ of F in $\mathbb{R}$. If A is an order in H (a subring which is also a finitely generated O_F-submodule and such that $F.A = H$) then the quotient of the group of units $A^\times$ by ± 1 embeds as a discrete cocompact subgroup of $P \times P$, and the corresponding $\mathbb{H}^2 \times \mathbb{H}^2$-manifold is irreducible. (See Chapter IV of [Vi].) It can be shown that every irreducible, cocompact $\mathbb{H}^2 \times \mathbb{H}^2$-lattice is commensurable with such a subgroup.

Much less is known about $\mathbb{H}^4$- or $\mathbb{H}^2(\mathbb{C})$-manifolds. If M is a closed orientable $\mathbb{H}^4$-manifold then $\sigma(M) = 0$ and $\chi(M) > 0$ [Ko92]. If M is a closed $\mathbb{H}^2(\mathbb{C})$-manifold it is orientable and $\chi(M) = 3\sigma(M) > 0$ [Wl86]. The isometry group of $\mathbb{H}^2(\mathbb{C})$ has two components; the identity component is $SU(2,1)$ and acts via holomorphic isomorphisms on the unit ball

$$\{(w, z) \in C^2 : |w|^2 + |z|^2 < 1\}.$$

(No closed $\mathbb{H}^4$-manifold admits a complex structure.) There are only finitely many closed $\mathbb{H}^4$- or $\mathbb{H}^2(\mathbb{C})$-manifolds with a given Euler characteristic (see Theorem 8.1 of [Wa72]). The 120-*cell space* of Davis is a closed orientable $\mathbb{H}^4$-manifold with $\chi = 26$ and $\beta_1 = 24 > 0$ [Da85, TS01], so all positive multiples of 26 are realized. Examples of $\mathbb{H}^2(\mathbb{C})$-manifolds due to Mumford and Hirzebruch have the homology of CP^2 (so $\chi = 3$), and $\chi = 15$ and $\beta_1 > 0$, respectively [HP96]. It is not known whether all positive multiples of 3 are realized. Since H^4 and $H^2(\mathbb{C})$ are rank 1 symmetric spaces the fundamental groups can contain no noncyclic abelian subgroups [Pr43]. In each case there are cocompact lattices which are not arithmetic. At present there are not even conjectural intrinsic characterizations of such groups. (See also [Rt] for the geometries $\mathbb{H}^n$ and [Go] for the geometries $\mathbb{H}^n(\mathbb{C})$.)

Each of the geometries $\mathbb{H}^2 \times \mathbb{H}^2$, $\mathbb{H}^4$ and $\mathbb{H}^2(\mathbb{C})$ admits cocompact lattices which are not almost coherent (see §1 of Chapter 4 above, [BM94] and [Ka98], respectively). Is this true of every such lattice for one of these geometries? (Lattices for the other geometries are coherent.)

9.6 Miscellany

A homotopy equivalence between two closed $\mathbb{H}^n$- or $\mathbb{H}^n(\mathbb{C})$-manifolds of dimension ≥ 3 is homotopic to an isometry, by *Mostow rigidity* [Ms68]. Farrell and Jones have established "topological" analogues of Mostow rigidity, which apply when the model manifold has a geometry of nonpositive curvature and dimension ≥ 5. By taking cartesian products with S^1, we can use their work in dimension 4 also.

Theorem 9.11 *Let M be a closed 4-manifold M with fundamental group π. Then M is s-cobordant to an $\mathbb{X}^4$-manifold where $\mathbb{X}^4 = \mathbb{H}^2 \times \mathbb{H}^2$, $\mathbb{H}^4$, $\mathbb{H}^2(\mathbb{C})$, $\mathbb{H}^3 \times \mathbb{E}^1$ or $\mathbb{H}^2 \times \mathbb{E}^2$ if and only if π is isomorphic to a cocompact lattice in $Isom(\mathbb{X}^4)$ and $\chi(M) = \chi(\pi)$.*

Proof The conditions are clearly necessary. If they hold M is aspherical and so $c_M : M \to \pi\backslash X$ is a homotopy equivalence, by Theorem 3.5. In all cases the geometry has nonpositive sectional curvatures, so $Wh(\pi) = Wh(\pi \times Z) = 0$ and $M \times S^1$ is homeomorphic to $(\pi\backslash X) \times S^1$ [FJ93']. Hence M and $\pi\backslash X$ are s-cobordant, by Lemma 6.10. □

A similar result holds for $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$-manifolds, provided that $\pi \leq Isom_o(\widetilde{\mathbb{SL}} \times \mathbb{E}^1)$. This is equivalent to the condition "$\zeta\pi = \sqrt{\pi}$". Although closed $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$-manifolds do not admit metrics of nonpositive curvature [KL96], they do admit effective T-actions if $\zeta\pi = \sqrt{\pi}$, and we then may appeal to [NS85] instead of [FJ93']. (See also Theorem 13.2 below.) The hypothesis that the Seifert structure derive from a toral group action may well be unnecessary.

Does a similar result hold for aspherical closed 4-manifolds with a geometric decomposition? Let M be such a manifold and let $\pi = \pi_1(M)$. Then π is built from the fundamental groups of the pieces by amalgamation along torsion free virtually poly-Z subgroups. As the Whitehead groups of the geometric pieces are trivial (by the argument of [FJ86]) and the amalgamated subgroups are regular noetherian it follows from the K-theoretic Mayer-Vietoris sequence of Waldhausen that $Wh(\pi) = 0$. Is there a corresponding argument in L-theory?

For the semisimple geometries we may avoid the appeal to L^2-methods to establish asphericity as follows. Since $\chi(M) > 0$ and π is infinite and residually finite there is a subgroup σ of finite index such that the associated covering spaces M_σ and $\sigma\backslash X$ are orientable and $\chi(M_\sigma) = \chi(\sigma) > 2$. In particular, $H^2(M_\sigma;\mathbb{Z})$ has elements of infinite order. Since the classifying map $c_{M_\sigma} : M_\sigma \to \sigma\backslash X$ is 2-connected it induces an isomorphism on H^2 and hence is a degree-1 map, by Poincaré duality. Therefore it is a homotopy equivalence, by Theorem 3.2.

Theorem 9.12 *If M is an aspherical closed 4-manifold which is finitely covered by a manifold with a geometry other than $\mathbb{H}^2 \times \mathbb{E}^2$ or $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$ then M is homotopy equivalent to a geometric 4-manifold.*

Proof The result is clear for infrasolvmanifolds, and follows from Theorem 9.8 if M is finitely covered by a reducible $\mathbb{H}^2 \times \mathbb{H}^2$-manifold. It holds for the other closed $\mathbb{H}^2 \times \mathbb{H}^2$-manifolds and for the geometries $\mathbb{H}^4$ and $\mathbb{H}^2(\mathbb{C})$ by Mostow rigidity.

If the geometry is $\mathbb{H}^3 \times \mathbb{E}^1$ then $\sqrt{\pi} \cong Z$ and $\pi/\sqrt{\pi}$ is virtually the group of a $\mathbb{H}^3$-manifold. Hence $\pi/\sqrt{\pi}$ acts isometrically and properly discontinuously on $\mathbb{H}^3$, by Mostow rigidity. Moreover as the hypotheses of Lemma 9.5 are satisfied, by Theorem 9.3, there is a homomorphism $\lambda : \pi \to D < Isom(\mathbb{E}^1)$ which maps $\sqrt{\pi}$ injectively. Together these actions determine a discrete and cocompact action of π by isometries on $H^3 \times R$. Since π is torsion free this action is free, and so M is homotopy equivalent to an $\mathbb{H}^3 \times \mathbb{E}^1$-manifold. □

The result is not yet clear for $\mathbb{H}^2\times\mathbb{E}^2$, $\widetilde{\mathbb{SL}}\times\mathbb{E}^1$, $\mathbb{S}^2\times\mathbb{E}^2$ or $\mathbb{S}^2\times\mathbb{H}^2$. The theorem holds also for $\mathbb{S}^4$ and $\mathbb{CP}^2$, but fails for $\mathbb{S}^3 \times \mathbb{E}^1$ or $\mathbb{S}^2 \times \mathbb{S}^2$. In particular, there is a closed nonorientable 4-manifold which is doubly covered by $S^2 \times S^2$ but is not homotopy equivalent to an $\mathbb{S}^2 \times \mathbb{S}^2$-manifold. (See Chapters 11 and 12.)

If π is the fundamental group of an aspherical closed geometric 4-manifold then $\beta_s^{(2)}(\pi) = 0$ for $s = 0$ or 1, and so $\beta_2^{(2)}(\pi) = \chi(\pi)$, by Theorem 1.35 of [Lü]. Therefore $\mathrm{def}(\pi) \leq \min\{0, 1 - \chi(\pi)\}$, by Theorems 2.4 and 2.5. If π is orientable this gives $\mathrm{def}(\pi) \leq 2\beta_1(\pi) - \beta_2(\pi) - 1$. When $\beta_1(\pi) = 0$ this is an improvement on the estimate $\mathrm{def}(\pi) \leq \beta_1(\pi) - \beta_2(\pi)$ derived from the ordinary homology of a 2-complex with fundamental group π.

Chapter 10

Manifolds covered by $S^2 \times R^2$

If the universal covering space of a closed 4-manifold with infinite fundamental group is homotopy equivalent to a finite complex then it is either contractible or homotopy equivalent to S^2 or S^3, by Theorem 3.9. The cases when M is aspherical have been considered in Chapters 8 and 9. In this chapter and the next we shall consider the spherical cases. We show first that if $\widetilde{M} \simeq S^2$ then M has a finite covering space which is s-cobordant to a product $S^2 \times B$, where B is an aspherical surface, and π is the group of a $\mathbb{S}^2 \times \mathbb{E}^2$- or $\mathbb{S}^2 \times \mathbb{H}^2$-manifold. In §2 we show that there are only finitely many homotopy types of such manifolds for each such group π. In §3 we show that all S^2- and RP^2-bundles over aspherical closed surfaces are geometric. We shall then determine the nine possible elementary amenable groups (corresponding to the geometry $\mathbb{S}^2 \times \mathbb{E}^2$). Six of these groups have infinite abelianization, and in §5 we show that for these groups the homotopy types may be distinguished by their Stiefel-Whitney classes. We conclude with some remarks on the homeomorphism classification.

For brevity, we shall let $\mathbb{X}^2$ denote both $\mathbb{E}^2$ and $\mathbb{H}^2$.

10.1 Fundamental groups

The determination of the closed 4-manifolds with universal covering space homotopy equivalent to S^2 rests on Bowditch's Theorem, via Theorem 5.14.

Theorem 10.1 *Let M be a closed 4-manifold with fundamental group π. Then the following conditions are equivalent:*

(1) *π is virtually a PD_2-group and $\chi(M) = 2\chi(\pi)$;*

(2) *$\pi \neq 1$ and $\pi_2(M) \cong Z$;*

(3) *M has a covering space of degree dividing 4 which is s-cobordant to $S^2 \times B$, where B is an aspherical closed orientable surface;*

(4) *M is virtually s-cobordant to an $\mathbb{S}^2 \times \mathbb{X}^2$-manifold.*

If these conditions hold then $\widetilde{M}$ is homeomorphic to $S^2 \times R^2$.

Proof If (1) holds then $\pi_2(M) \cong Z$, by Theorem 5.10, and so (2) holds. If (2) holds then the covering space associated to the kernel of the natural action of π on $\pi_2(M)$ is homotopy equivalent to the total space of an S^2-bundle ξ over an aspherical closed surface with $w_1(\xi) = 0$, by Lemma 5.11 and Theorem 5.14. On passing to a 2-fold covering space, if necessary, we may assume that $w_2(\xi) = w_1(M) = 0$ also. Hence ξ is trivial and so the corresponding covering space of M is s-cobordant to a product $S^2 \times B$ with B orientable. Moreover $\widetilde{M} \cong S^2 \times R^2$, by Theorem 6.16. It is clear that (3) implies (4) and (4) implies (1). □

This follows also from [Fa74] instead of [Bo99] if we know also that $\chi(M) \leq 0$. If π is infinite and $\pi_2(M) \cong Z$ then π may be realized geometrically.

Theorem 10.2 *Let M be a closed 4-manifold with fundamental group π and such that $\pi_2(M) \cong Z$. Then π is the fundamental group of a closed manifold admitting the geometry $\mathbb{S}^2 \times \mathbb{E}^2$, if π is virtually Z^2, or $\mathbb{S}^2 \times \mathbb{H}^2$ otherwise.*

Proof If π is torsion free then it is itself a surface group. If π has a nontrivial finite normal subgroup then it is a direct product $\mathrm{Ker}(u) \times (Z/2Z)$, where $u : \pi \to \{\pm 1\} = Aut(\pi_2(M))$ is the natural homomorphism. (See Theorem 5.14). In either case π is the fundamental group of a corresponding product of surfaces. Otherwise π is a semidirect product $\mathrm{Ker}(u) \tilde{\times} (Z/2Z)$ and is a plane motion group, by a theorem of Nielsen ([Zi]; see also Theorem A of [EM82]). This means that there is a monomorphism $f : \pi \to Isom(\mathbb{X}^2)$ with image a discrete subgroup which acts cocompactly on X, where X is the Euclidean or hyperbolic plane, according as π is virtually abelian or not. The homomorphism $(u, f) : \pi \to \{\pm I\} \times Isom(\mathbb{X}^2) \leq Isom(\mathbb{S}^2 \times \mathbb{X}^2)$ is then a monomorphism onto a discrete subgroup which acts freely and cocompactly on $S^2 \times R^2$. In all cases such a group may be realised geometrically. □

The orbit space of the geometric action of π described above is a cartesian product with S^2 if u is trivial and fibres over RP^2 otherwise.

10.2 Homotopy type

In this section we shall extend an argument of Hambleton and Kreck to show that there are only finitely many homotopy types of manifolds with universal cover $S^2 \times R^2$ and given fundamental group.

We shall first show that the orientation character and the action of π on π_2 determine each other.

Lemma 10.3 *Let M be a closed 4-manifold with fundamental group $\pi \neq 1$ and such that $\pi_2(M) \cong Z$. Then $H^2(\pi;\mathbb{Z}[\pi]) \cong Z$ and $u = w_1(M) + v$, where $u : \pi \to Aut(\pi_2(M)) = Z/2Z$ and $v : \pi \to Aut(H^2(\pi;\mathbb{Z}[\pi])) = Z/2Z$ are the natural actions.*

Proof Since π is infinite $Hom_{\mathbb{Z}[\pi]}(\pi_2(M),\mathbb{Z}[\pi]) = 0$ and so $\overline{H^2(\pi;\mathbb{Z}[\pi])} \cong \pi_2(M)$, by Lemma 3.3. Now $\overline{H^2(\pi;\mathbb{Z}[\pi])} \cong H^2(\pi;\mathbb{Z}[\pi]) \otimes Z^{w_1(M)}$, (where the tensor product is over Z and has the diagonal π-action). Therefore $Z^u = Z^v \otimes Z^{w_1(M)}$ and so $u = w_1(M) + v$. □

Note that u and $w_1(M)$ are constrained by the further conditions that $K = \mathrm{Ker}(u)$ is torsion free and $\mathrm{Ker}(w_1(M))$ has infinite abelianization if $\chi(M) \leq 0$. If $\pi < Isom(\mathbb{X}^2)$ is a plane motion group then $v(g)$ detects whether $g \in \pi$ preserves the orientation of X^2. If π is torsion free then M is homotopy equivalent to the total space of an S^2-bundle ξ over an aspherical closed surface B, and the equation $u = w_1(M) + v$ follows from Lemma 5.11.

Let β^u be the Bockstein operator associated with the exact sequence of coefficients

$$0 \to Z^u \to Z^u \to \mathbb{F}_2 \to 0,$$

and let $\overline{\beta^u}$ be the composition with reduction modulo (2). In general β^u is NOT the Bockstein operator for the untwisted sequence $0 \to Z \to Z \to \mathbb{F}_2 \to 0$, and $\overline{\beta^u}$ is not Sq^1, as can be seen already for cohomology of the group $Z/2Z$ acting nontrivially on Z.

Lemma 10.4 *Let M be a closed 4-manifold with fundamental group π and such that $\pi_2(M) \cong Z$. If π has nontrivial torsion $H^s(M;\mathbb{F}_2) \cong H^s(\pi;\mathbb{F}_2)$ for $s \leq 2$. The Bockstein operator $\beta^u : H^2(\pi;\mathbb{F}_2) \to H^3(\pi;Z^u)$ is onto, and reduction mod 2 from $H^3(\pi;Z^u)$ to $H^3(\pi;\mathbb{F}_2)$ is a monomorphism. The restriction of $k_1(M)$ to each subgroup of order 2 is nontrivial. Its image in $H^3(M;Z^u)$ is 0.*

Proof Most of these assertions hold vacuously if π is torsion free, so we may assume that π has an element of order 2. Then M has a covering space $\hat{M}$ homotopy equivalent to RP^2, and so the mod-2 Hurewicz homomorphism from $\pi_2(M)$ to $H_2(M;\mathbb{F}_2)$ is trivial, since it factors through $H_2(\hat{M};\mathbb{F}_2)$. Since we may construct $K(\pi,1)$ from M by adjoining cells to kill the higher homotopy of M the first assertion follows easily.

The group $H^3(\pi;Z^u)$ has exponent dividing 2, since the composition of restriction to $H^3(K;\mathbb{Z}) = 0$ with the corestriction back to $H^3(\pi;Z^u)$ is multiplication by the index $[\pi : K]$. Consideration of the long exact sequence associated to the coefficient sequence shows that β^u is onto. If $f : Z/2Z \to \pi$ is a monomorphism then $f^*k_1(M)$ is the first k-invariant of $\widetilde{M}/f(Z/2Z) \simeq RP^2$, which generates $H^3(Z/2Z; \pi_2(M)) = Z/2Z$. The final assertion is clear. □

Theorem 10.5 *Let M be a closed 4-manifold such that $\pi_2(M) \cong Z$. Then there are only finitely many homotopy types of such manifolds with fundamental group π and orientation character $w_1(M)$. If $w_1(M) \neq 0$ there are at most two such homotopy types with given first k-invariant.*

Proof By the lemma, the action of π on $\pi_2(M)$ is determined by $w_1(M)$. As $c.d.\rho = 2$, an LHSSS calculation shows that $H^3(\pi; \pi_2(M))$ is finite, so there are only finitely many possible k-invariants. The action and the first k-invariant $k_1(M)$ determine $P = P_2(M)$, the second stage of the Postnikov tower for M. Let $\widetilde{P} \simeq K(Z, 2)$ denote the universal covering space of P.

As $f_M : M \to P$ is 3-connected we may define a class w in $H^1(P; \mathbb{Z}/2\mathbb{Z})$ by $f_M^* w = w_1(M)$. Let $S_4^{PD}(P)$ be the set of "polarized" PD_4-complexes (X, f) where $f : X \to P$ is 3-connected and $w_1(X) = f^*w$, modulo homotopy equivalence over P. (Note that as π is one-ended the universal cover of X is homotopy equivalent to S^2). Let $[X]$ be the fundamental class of X in $H_4(X; Z^w)$. It follows as in Lemma 1.3 of [HK88] that given two such polarized complexes (X, f) and (Y, g) there is a map $h : X \to Y$ with $gh = f$ if and only if $f_*[X] = g_*[Y]$ in $H_4(P; Z^w)$. Since $\widetilde{X} \simeq \widetilde{Y} \simeq S^2$ and f and g are 3-connected such a map h must be a homotopy equivalence.

From the Cartan-Leray homology spectral sequence for the classifying map $c_P : P \to K = K(\pi, 1)$ we see that there is an exact sequence

$$0 \to H_2(\pi; H_2(\widetilde{P}) \otimes Z^w)/im(d^2_{5,0}) \to H_4(P; Z^w)/J \to H_4(\pi; Z^w),$$

where $J = H_0(\pi; H_4(\widetilde{P}; \mathbb{Z}) \otimes Z^w)/im(d^2_{3,2} + d^4_{5,0})$ is the image of $H_4(\widetilde{P}; \mathbb{Z}) \otimes Z^w$ in $H_4(P; Z^w)$. On comparing this spectral sequence with that for c_X we see that f induces an isomorphism from $H_4(X; Z^w)$ to $H_4(P; Z^w)/J$. We also see that $H_3(f; Z^w)$ is an isomorphism. Hence the cokernel of $H_4(f; Z^w)$ is $H_4(P, X; Z^w) \cong H_0(\pi; H_4(\widetilde{P}, \widetilde{X}; \mathbb{Z}) \otimes Z^w)$, by the exact sequence of homology with coefficients Z^w for the pair (P, X). Since $H_4(\widetilde{P}, \widetilde{X}; \mathbb{Z}) \cong Z$ as a π-module this cokernel is Z if $w = 0$ and $Z/2Z$ otherwise. Hence $J \cong \mathrm{Coker}(H_4(f; Z^w))$. (Note that $H_2(\pi; H_2(\widetilde{P}) \otimes Z^w)/(torsion) \cong Z$ and the groups $H_p(\pi; Z^w)$ are finite if $p > 2$). Thus if $w \neq 0$ there are at most two possible values for $f_*[X]$, up to sign. If $w = 0$ we shall show that there are only finitely many orbits of fundamental classes of such polarized complexes under the action of the group

G of (based) self homotopy equivalences of P which induce the identity on π and $\pi_2(P)$.

The cohomology spectral sequence for c_P gives rise to an exact sequence

$$0 \to H^2(\pi; Z^u) \to H^2(P; Z^u) \to H^0(\pi; H^2(\widetilde{P}; \mathbb{Z}) \otimes Z^u) \cong Z \to H^3(\pi; Z^u).$$

Note that $H^2(\pi; Z^u) \cong Z$ modulo 2-torsion (since $w = 0$), $H^2(\widetilde{P}; \mathbb{Z}) \cong Z^u$ and $Z^u \otimes Z^u \cong Z$ as π-modules. Moreover the right hand map is the transgression, with image generated by $k_1(M)$. There is a parallel exact sequence with rational coefficients

$$0 \to H^2(\pi; Q^u) \cong Q \to H^2(P; Q^u) \to H^0(\pi; H^2(\widetilde{P}; \mathbb{Z}) \otimes Q^u) \cong Q \to 0.$$

Thus $H^2(P; Q^u)$ has a $\mathbb{Q}$-basis t, z in the image of $H^2(P; Z^u)$ such that t is the image of a generator of $H^2(\pi; Z^u)/(torsion)$ and z has nonzero restriction to $H^2(\widetilde{P}; \mathbb{Z})$. The spectral sequence also gives an exact sequence

$$0 \to H^2(\pi; H^2(\widetilde{P}; \mathbb{Q})) \to H^4(P; \mathbb{Q}) \to H^0(\pi; H^4(\widetilde{P}; \mathbb{Q})) \cong Q \to 0.$$

(Note that $H^2(\widetilde{P}; \mathbb{Q}) \cong Q^u$ as a $\mathbb{Q}[\pi]$-module). Since $cd_{\mathbb{Q}}\pi = 2$ we have $t^2 = 0$ in $H^4(P; Q^u \otimes Q^u) = H^4(P; \mathbb{Q})$; since $\widetilde{P} \simeq K(Z, 2)$ we have $z^2 \neq 0$. Thus tz, z^2 is a $\mathbb{Q}$-basis for $H^4(P; \mathbb{Q})$. A self homotopy h in G induces the identity on π, and its lift to a self map of $\widetilde{P}$ is homotopic to the identity. Hence $h^*t = t$ and $h^*z \equiv z$ modulo $\mathbb{Q}t$. Nevertheless we shall see that the action of G on $H^2(P; Q^u)$ is nontrivial.

Suppose first that $u = 0$, so π is an orientable surface group and $k_1(M) = 0$. Then $P \simeq K(\pi, 1) \times K(Z, 2)$ and $G \cong [K(\pi, 1), K(Z, 2)]$. Let $f : K(\pi, 1) \to K(Z, 2)$ be a map which induces an isomorphism on H^2 and fix a generator ζ for $H^2(K(Z, 2); \mathbb{Z})$. Then $t = pr_1^* f^* \zeta$ and $z = pr_2^* \zeta$ freely generate $H^2(P; \mathbb{Z})$, and so tz, z^2 freely generate $H^4(P; \mathbb{Z})$. Each $g \in [K(\pi, 1), K(Z, 2)]$ determines a self homotopy equivalence $\tilde{g} : P \to P$ by $\tilde{g}(k, n) = (k, g(k).n)$, where $K(Z, 2) = \Omega K(Z, 3)$ has the natural loop multiplication. Clearly $\tilde{g}$ is in G, and all elements of G are of this form [Ts80]. Let $d : G \to Z$ be the isomorphism determined by the equation $g^*\zeta = d(g) f^*\zeta$. Then $\tilde{g}^*t = (f pr_1 \tilde{g})^*\zeta = t$ and $\tilde{g}^*z = (pr_2\tilde{g})^*\zeta = (g pr_1)^*\zeta + pr_2^*\zeta = pr_1^*(g^*\zeta) + z = z + d(g)t$. On taking cup products we have $h^*(tz) = tz$ and $h^*(z^2) = z^2 + 2d(g)tz$. On passing to homology we see that there are two G-orbits of elements in $H_4(P; \mathbb{Z})$ whose images generate $H_4(P; \mathbb{Z})/J$.

In general let P_K denote the covering space corresponding to the subgroup K, and let G_K be the image of G in the group of self homotopy equivalences of P_K. Then lifting self homotopy equivalences defines a homomorphism from G to G_K, which by [Ts80] may be identified with the restriction from $H^2(\pi; Z^u)$

to $H^2(K;\mathbb{Z}) \cong Z$, which has image of index ≤ 2. Moreover the projection induces an isomorphism from $H^4(P;\mathbb{Q})$ to $H^4(P_K;\mathbb{Q})$. Hence the action of G on $H_4(P;\mathbb{Z})/(torsion) \cong Z^2$ is nontrivial, and so there are only finitely many G-orbits of elements whose images generate $H_4(P;\mathbb{Z})/J$. This proves the theorem. □

As a consequence of Lemma 10.4 we may assume that the cohomology class z in the above theorem restricts to 2 times a generator of $H^2(\widetilde{P};\mathbb{Z})$, if $k_1(M) \neq 0$. A closer study of the action of G on $H^2(P;Z^u)$ suggests that in general there are at most 4 homotopy types with given π, w_1 and k-invariant. However we have not succeeded in proving this.

Significant features of the duality pairing such as $w_2(M)$ are not reflected in the Postnikov 2-stage. If π is torsion free $k_1(M) = 0$ and w_2 is the only other invariant needed. For then π is a surface group and each such manifold is homotopy equivalent to the total space of an S^2-bundle. There are two such bundle spaces for each group and orientation character, distinguished by the value of $w_2(M)$.

For RP^2-bundles $u = w_1$ and $\pi \cong K \times (Z/2Z)$. The element of order 2 in π is unique, and the splitting is unique up to composition with an automorphism of π. There are two such bundle spaces for each surface group K, again distinguished by $w_2(M)$. Can it be seen *a priori* that the k-invariant must be standard?

If $w_1(M) = 0$, $w_2(M)$ restricts to 0 in $H^2(K;\mathbb{F}_2)$, $u \neq 0$ and $H^3(u;Z^u)$ is 0 then M is homotopy equivalent to the total space of a surface bundle over RP^2, by Theorem 5.23.

In general, we may view the classifying map $c_M : M \to K(\pi,1)$ as a fibration with fibre S^2. Fix a homotopy equivalence $\widetilde{M} \simeq S^2$. Then the action of π on $\widetilde{M}$ determines a homomorphism $j : \pi \to Homeo(\widetilde{M}) \to E(S^2)$, and the fibration c_M is induced from the universal S^2-fibration over $BE(S^2)$ by the map $Bj : K(\pi,1) \to BE(S^2)$. The orientation character of this fibration is $w_1(c_M) = u$, and is induced by the composite $c_{BE(S^2)}Bj : K(\pi,1) \to K(\pi_0(E(S^2)),1)$. The (twisted) Euler class is the first obstruction to a cross-section of c_M, and so equals $k_1(M)$. Hence the reduction modulo (2) of $k_1(M)$ is $w_3(c_M) \in H^3(\pi;\mathbb{F}_2)$. Calculation show that $\beta^u : H^2(BE(S^2);\mathbb{F}_2) \to H^3(BE(S^2);Z^u)$ is an isomorphism, and so $w_3(c_M)$ also determines $k_1(M)$. In particular, if j factors through $\{\pm I\} < O(3)$ then $k_1(M) = \beta^u(U^2)$, where $U \in H^1(\pi;\mathbb{F}_2)$ is the cohomology class determined by u. (This is so when M is a $\mathbb{S}^2 \times \mathbb{X}^2$-manifold and π is generated by elements of order 2, by Lemma 10.6 below).

As M is finitely covered by a cartesian product $S^2 \times B$, where B is a closed orientable surface, $w_2(M)$ restricts to 0 in $H^2(\widetilde{M};\mathbb{F}_2)$ and so is induced from π. The Wu formulae for M then imply that the total Stiefel-Whitney class $w(M)$ is induced from π. It can be shown that $c_M^*(w(c_M))$ is determined by $w(M)$ and π; unfortunately as $c_M^*(w_3(c_M)) = 0$ (by exactness of the Gysin sequence for c_M) we do not know whether $k_1(M)$ is also determined by these invariants.

Is the homotopy type of M determined by $\pi_1(M)$, $w(M)$ and $k_1(M)$? What is the role of the exotic class in $H^3(BE(S^2);\mathbb{F}_2)$? Are there any PD_4-complexes M with $\widetilde{M} \simeq S^2$ and such that the image of this class under $(Bj)^*$ is nonzero?

10.3 Bundle spaces are geometric

All $\mathbb{S}^2\times\mathbb{X}^2$-manifolds are total spaces of orbifold bundles over $\mathbb{X}^2$-orbifolds. We shall determine the S^2- and RP^2-bundle spaces among them in terms of their fundamental groups, and then show that all such bundle spaces are geometric.

Lemma 10.6 *Let $J = (A,\theta) \in O(3) \times Isom(\mathbb{X}^2)$ be an isometry of order 2 which is fixed point free. Then $A = -I$. If moreover J is orientation reversing then $\theta = id_X$ or has a single fixed point.*

Proof Since any involution of R^2 (such as θ) must fix a point, a line or be the identity, $A \in O(3)$ must be a fixed point free involution, and so $A = -I$. If J is orientation reversing then θ is orientation preserving, and so must fix a point or be the identity. □

Theorem 10.7 *Let M be a closed $\mathbb{S}^2\times\mathbb{X}^2$-manifold with fundamental group π. Then*

(1) *M is the total space of an orbifold bundle with base an $\mathbb{X}^2$-orbifold and general fibre S^2 or RP^2;*

(2) *M is the total space of an S^2-bundle over a closed aspherical surface if and only if π is torsion free;*

(3) *M is the total space of an RP^2-bundle over a closed aspherical surface if and only if $\pi \cong (Z/2Z) \times K$, where K is torsion free.*

Proof (1) The group π is a discrete subgroup of the isometry group $Isom(\mathbb{S}^2 \times\mathbb{X}^2) = O(3) \times Isom(\mathbb{X}^2)$ which acts freely and cocompactly on $S^2 \times R^2$. In particular, $N = \pi \cap (O(3) \times \{1\})$ is finite and acts freely on S^2, so has order ≤ 2. Let p_1 and p_2 be the projections of $Isom(\mathbb{S}^2 \times \mathbb{X}^2)$ onto $O(3)$ and $Isom(\mathbb{X}^2)$, respectively. Then $p_2(\pi)$ is a discrete subgroup of $Isom(\mathbb{X}^2)$ which acts cocompactly on R^2, and so has no nontrivial finite normal subgroup. Hence N is the maximal finite normal subgroup of π. Projection of $S^2 \times R^2$ onto R^2 induces an orbifold bundle projection of M onto $p_2(\pi)\backslash R^2$ and general fibre $N\backslash S^2$. If $N \neq 1$ then $N \cong Z/2Z$ and $\pi \cong (Z/2Z) \times K$, where K is a PD_2-group, by Theorem 5.14.

(2) The condition is clearly necessary. (See Theorem 5.10). The kernel of the projection of π onto its image in $Isom(\mathbb{X}^2)$ is the subgroup N. Therefore if π is torsion free it is isomorphic to its image in $Isom(\mathbb{X}^2)$, which acts freely on R^2. The projection $\rho : S^2 \times R^2 \to R^2$ induces a map $r : M \to \pi\backslash R^2$, and we have a commutative diagram:

$$\begin{array}{ccc} S^2 \times R^2 & \xrightarrow{\rho} & R^2 \\ \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle \bar{f}} \\ M = \pi\backslash(S^2 \times R^2) & \xrightarrow{r} & \pi\backslash R^2 \end{array}$$

where f and $\bar{f}$ are covering projections. It is easily seen that r is an S^2-bundle projection.

(3) The condition is necessary, by Theorem 5.16. Suppose that it holds. Then K acts freely and properly discontinuously on R^2, with compact quotient. Let g generate the torsion subgroup of π. Then $p_1(g) = -I$, by Lemma 10.6. Since $p_2(g)^2 = \mathrm{id}_{R^2}$ the fixed point set $F = \{x \in R^2 \mid p_2(g)(x) = x\}$ is nonempty, and is either a point, a line, or the whole of R^2. Since $p_2(g)$ commutes with the action of K on R^2 we have $KF = F$, and so K acts freely and properly discontinuously on F. But K is neither trivial nor infinite cyclic, and so we must have $F = R^2$. Hence $p_2(g) = \mathrm{id}_{R^2}$. The result now follows, as $K\backslash(S^2 \times R^2)$ is the total space of an S^2-bundle over $K\backslash R^2$, by part (1), and g acts as the antipodal involution on the fibres. $\square$

If the $\mathbb{S}^2 \times \mathbb{X}^2$-manifold M is the total space of an S^2-bundle ξ then $w_1(\xi)$ is detected by the determinant: $\det(p_1(g)) = (-1)^{w_1(\xi)(g)}$ for all $g \in \pi$.

The total space of an RP^2-bundle over B is the quotient of its orientation double cover (which is an S^2-bundle over B) by the fibrewise antipodal involution and so there is a bijective correspondance between orientable S^2-bundles over B and RP^2-bundles over B.

Let $(A, \beta, C) \in O(3) \times E(2) = O(3) \times (R^2 \tilde{\times} O(2))$ be the $\mathbb{S}^2 \times \mathbb{E}^2$-isometry which sends $(v, x) \in S^2 \times R^2$ to $(Av, Cx + \beta)$.

Theorem 10.8 *Let M be the total space of an S^2- or RP^2-bundle over T or Kb. Then M admits the geometry $\mathbb{S}^2 \times \mathbb{E}^2$.*

Proof Let $R_i \in O(3)$ be the reflection of R^3 which changes the sign of the i^{th} coordinate, for $i = 1, 2, 3$. If A and B are products of such reflections then the subgroups of $Isom(\mathbb{S}^2 \times \mathbb{E}^2)$ generated by $\alpha = (A, \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), I)$ and $\beta = (B, \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right), I)$ are discrete, isomorphic to Z^2 and act freely and cocompactly on $S^2 \times R^2$. Taking

(1) $A = B = I$;

(2) $A = R_1R_2, B = R_1R_3$;

(3) $A = R_1, B = I$; and

(4) $A = R_1, B = R_1R_2$

gives four S^2-bundles η_i over the torus. If instead we use the isometries $\alpha = (A, \left(\begin{smallmatrix}\frac{1}{2}\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}1 & 0\\0 & -1\end{smallmatrix}\right))$ and $\beta = (B, \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right), I)$ we obtain discrete subgroups isomorphic to $Z \times_{-1} Z$ which act freely and cocompactly. Taking

(1) $A = R_1, B = I$;

(2) $A = R_1, B = R_2R_3$;

(3) $A = I, B = R_1$;

(4) $A = R_1R_2, B = R_1$;

(5) $A = B = I$; and

(6) $A = I, B = R_1R_2$

gives six S^2-bundles ξ_i over the Klein bottle.

To see that these are genuinely distinct, we check first the fundamental groups, then the orientation character of the total space; consecutive pairs of generators determine bundles with the same orientation character, and we distinguish these by means of the second Stiefel-Whitney classes, by computing the self-intersections of cross-sections. (See Lemma 5.11.(2). We shall use the stereographic projection of $S^2 \subset R^3 = C \times R$ onto $\widehat{C} = C \cup \{\infty\}$, to identify the reflections $R_i : S^2 \to S^2$ with the antiholomorphic involutions:

$$z \xmapsto{R_1} \overline{z}, \quad z \xmapsto{R_2} -\overline{z}, \quad z \xmapsto{R_3} \overline{z}^{-1}.$$

Let $\mathfrak{T} = \{(s,t) \in R^2 | 0 \leq s, t \leq 1\}$ be the fundamental domain for the standard action of Z^2 on R^2. A section $\sigma : \mathfrak{T} \to S^2 \times R^2$ of the projection to R^2 over $\mathfrak{T}$ such that $\sigma(1,t) = \alpha\sigma(0,t)$ and $\sigma(s,1) = \beta\sigma(s,0)$ induces a section of the bundle ξ_i.

As the orientable cases (η_1, η_2, ξ_1 and ξ_2) have been treated in [Ue90] we may concentrate on the nonorientable cases. In the case of η_3 each fixed point P of A determines a section σ_P with $\sigma_P(s,t) = (P,s,t)$. Since A fixes a circle on S^2 it follows that sections determined by distinct fixed points are isotopic and disjoint. Therefore $\sigma \cdot \sigma = 0$, so $v_2(M) = 0$ and hence $w_2(\eta_3) = 0$.

We may define a 1-parameter family of sections for η_4 by

$$\sigma_\lambda(s,t) = ((1-\lambda)(2t-1) + \lambda(4t^2-2))e^{\pi i\lambda(s-\frac{1}{2})}.$$

Now σ_0 and σ_1 intersect transversely in a single point, corresponding to $s = 1/2$ and $t = (1+\sqrt{5})/4$. Hence $\sigma \cdot \sigma = 1$, so $v_2(M) \neq 0$ and $w_2(\eta_4) \neq 0$.

The remaining cases correspond to S^2-bundles over Kb with nonorientable total space. We now take $\mathfrak{K} = \{(s,t) \in R^2 | 0 \leq s \leq 1, |t| \leq \frac{1}{2}\}$ as the fundamental domain for the action of $Z \times_{-1} Z$ on R^2. In this case it suffices to find $\sigma : \mathfrak{K} \to S^2 \times R^2$ such that $\sigma(1,t) = \alpha\sigma(0,-t)$ and $\sigma(s,\frac{1}{2}) = \beta\sigma(s,-\frac{1}{2})$.

The cases of ξ_3 and ξ_5 are similar to that of η_3: there are obvious one-parameter families of disjoint sections, and so $w_2(\xi_3) = w_2(\xi_5) = 0$. However $w_1(\xi_3) \neq w_1(\xi_5)$. (In fact ξ_5 is the product bundle).

The functions $\sigma_\lambda(s,t) = \lambda(2s-1+it)$ define a 1-parameter family of sections for ξ_4 such that σ_0 and σ_1 intersect transversely in one point, so that $\sigma \cdot \sigma = 1$. Hence $v_2(M) \neq 0$ and so $w_2(\xi_4) \neq 0$.

For ξ_6 the functions $\sigma_\lambda(s,t) = \lambda(2s-1)t + i(1-\lambda)(4t^2-1)$ define a 1-parameter family of sections such that σ_0 and $\sigma_1(s,t)$ intersect transversely in one point, so that $\sigma \cdot \sigma = 1$. Hence $v_2(M) \neq 0$ and so $w_2(\xi_6) \neq 0$.

Thus these bundles are all distinct, and so all S^2-bundles over T or Kb are geometric of type $\mathbb{S}^2 \times \mathbb{E}^2$.

Adjoining the fixed point free involution $(-I,0,I)$ to any one of the above ten sets of generators for the S^2-bundle groups amounts to dividing out the S^2 fibres by the antipodal map and so we obtain the corresponding RP^2-bundles. (Note that there are just four such RP^2-bundles - but each has several distinct double covers which are S^2-bundles). □

Theorem 10.9 *Let M be the total space of an S^2- or RP^2-bundle over a closed hyperbolic surface. Then M admits the geometry $\mathbb{S}^2 \times \mathbb{H}^2$.*

Proof Let T_g be the closed orientable surface of genus g, and let $\mathfrak{T}^g \subset \mathbb{H}^2$ be a $2g$-gon representing the fundamental domain of T_g. The map $\Omega : \mathfrak{T}^g \to \mathfrak{T}$

that collapses $2g-4$ sides of $\mathfrak{T}^g$ to a single vertex in the rectangle $\mathfrak{T}$ induces a degree 1 map $\widehat{\Omega}$ from T_g to T that collapses $g-1$ handles on T^g to a single point on T. (We may assume the induced epimorphism from

$$\pi_1(T_g) = \langle a_1, b_1, \dots, a_g, b_g \mid \Pi_{i=1}^g [a_i, b_i] = 1 \rangle$$

to Z^2 kills the generators a_j, b_j for $j > 1$). Hence given an S^2-bundle ξ over T with total space $M_\xi = \Gamma_\xi \backslash (S^2 \times E^2)$, where

$$\Gamma_\xi = \{(\xi(h), h) \mid h \in \pi_1(T)\} \leq Isom(S^2 \times E^2)$$

and $\xi : \mathbb{Z}^2 \to O(3)$ is as in Theorem 10.8, the pullback $\widehat{\Omega}^*(\xi)$ is an S^2-bundle over T_g, with total space $M_{\xi\Omega} = \Gamma_{\xi\Omega} \backslash (S^2 \times \mathbb{H}^2)$, where $\Gamma_{\xi\Omega} = \{(\xi\Omega(h), h) \mid h \in \Pi_1(T^g)\} \leq Isom(S^2 \times \mathbb{H}^2)$. As $\widehat{\Omega}$ is of degree 1 it induces monomorphisms in cohomology, so $w(\xi)$ is nontrivial if and only if $w(\widehat{\Omega}^*(\xi)) = \widehat{\Omega}^* w(\xi)$ is nontrivial. Hence all S^2-bundles over T^g for $g \geq 2$ are geometric of type $S^2 \times \mathbb{H}^2$.

Suppose now that B is the closed surface $\#^3 RP^2 = T \# RP^2 = Kb \# RP^2$. Then there is a map $\widehat{\Omega} : T\#RP^2 \to RP^2$ that collapses the torus summand to a single point. This map $\widehat{\Omega}$ again has degree 1 and so induces monomorphisms in cohomology. In particular $\widehat{\Omega}^*$ preserves the orientation character, that is $w_1(\widehat{\Omega}^*(\xi)) = \widehat{\Omega}^* w_1(RP^2) = w_1(B)$, and is an isomorphism on H^2. We may pull back the four S^2-bundles ξ over RP^2 along $\widehat{\Omega}$ to obtain the four bundles over B with first Stiefel-Whitney class $w_1(\widehat{\Omega}^*\xi)$ either 0 or $w_1(B)$.

Similarly there is a map $\widehat{\Upsilon} : Kb\#RP^2 \to RP^2$ that collapses the Klein bottle summand to a single point. This map $\widehat{\Upsilon}$ has degree 1 mod 2 so that $\widehat{\Upsilon}^* w_1(RP^2)$ has nonzero square since $w_1(RP^2)^2 \neq 0$. Note that in this case $\widehat{\Upsilon}^* w_1(RP^2) \neq w_1(B)$. Hence we may pull back the two S^2-bundles ξ over RP^2 with $w_1(\xi) = w_1(RP^2)$ to obtain a further two bundles over B with $w_1(\widehat{\Upsilon}^*(\xi))^2 = \widehat{\Upsilon}^* w_1(\xi)^2 \neq 0$, as $\widehat{\Upsilon}$ is a ring monomorphism.

There is again a map $\widehat{\Theta} : Kb\#RP^2 \to Kb$ that collapses the Klein bottle summand to a single point. Once again $\widehat{\Theta}$ is of degree 1 mod 2 so that we may pull back the two S^2-bundles ξ over Kb with $w_1(\xi) = w_1(Kb)$ along $\widehat{\Theta}$ to obtain the remaining two S^2-bundles over B. These two bundles $\widehat{\Theta}^*(\xi)$ have $w_1(\widehat{\Theta}^*(\xi)) \neq 0$ but $w_1(\widehat{\Theta}^*(\xi))^2 = 0$; as $w_1(Kb) \neq 0$ but $w_1(Kb)^2 = 0$ and $\widehat{\Theta}^*$ is a monomorphism.

Similar arguments apply to bundles over $\#^n RP^2$ where $n > 3$.

Thus all S^2-bundles over all closed aspherical surfaces are geometric. Furthermore since the antipodal involution of a geometric S^2-bundle is induced by an isometry $(-I, id_{\mathbb{H}^2}) \in O(3) \times Isom(\mathbb{H}^2)$ we have that all RP^2-bundles over closed aspherical surfaces are geometric. $\square$

An alternative route to Theorems 10.8 and 10.9 would be to first show that orientable 4-manifolds which are total spaces of S^2-bundles are geometric, then deduce that RP^2-bundles are geometric (as above); and finally observe that every S^2-bundle space double covers an RP^2-bundle space.

The other $\mathbb{S}^2 \times \mathbb{X}^2$-manifolds are orbifold bundles over flat or hyperbolic orbifolds, with general fibre S^2. In other words, they have codimension-2 foliation whose leaves are homeomorphic to S^2 or RP^2. Is every such closed 4-manifold geometric?

If $\chi(F) < 0$ or $\chi(F) = 0$ and $\partial = 0$ then every F-bundle over RP^2 is geometric, by Lemma 5.21 and the remark following Theorem 10.2.

However it is not generally true that the projection of $S^2 \times X$ onto S^2 induces an orbifold bundle projection from M to an $\mathbb{S}^2$-orbifold. For instance, if ρ and ρ' are rotations of S^2 about a common axis which generate a rank 2 abelian subgroup of $SO(3)$ then $(\rho,(1,0))$ and $(\rho',(0,1))$ generate a discrete subgroup of $SO(3)\times R^2$ which acts freely, cocompactly and isometrically on $S^2\times R^2$. The orbit space is homeomorphic to $S^2\times T$. (It is an orientable S^2-bundle over the torus, with disjoint sections, detemined by the ends of the axis of the rotations). Thus it is Seifert fibred over S^2, but the fibration is not canonically associated to the metric structure, for $\langle\rho,\rho'\rangle$ does not act properly discontinuously on S^2.

10.4 Fundamental groups of $\mathbb{S}^2 \times \mathbb{E}^2$-manifolds

We shall show first that if M is a closed 4-manifold any two of the conditions "$\chi(M) = 0$", "$\pi_1(M)$ is virtually Z^2" and "$\pi_2(M) \cong Z$" imply the third, and then determine the possible fundamental groups.

Theorem 10.10 *Let M be a closed 4-manifold with fundamental group π. Then the following conditions are equivalent:*

(1) *π is virtually Z^2 and $\chi(M) = 0$;*

(2) *π has an infinite restrained normal subgroup and $\pi_2(M) \cong Z$;*

(3) *$\chi(M) = 0$ and $\pi_2(M) \cong Z$; and*

(4) *M has a covering space of degree dividing 4 which is homeomorphic to $S^2 \times T$.*

(5) *M is virtually homeomorphic to an $\mathbb{S}^2 \times \mathbb{E}^2$-manifold.*

Proof If π is virtually a PD_2-group and either $\chi(\pi) = 0$ or π has an infinite restrained normal subgroup then π is virtually Z^2. Hence the equivalence of these conditions follows from Theorem 10.1, with the exception of the assertions regarding homeomorphisms, which then follow from Theorem 6.11. □

We shall assume henceforth that the conditions of Theorem 10.10 hold, and shall show next that there are nine possible groups. Seven of them are 2-dimensional crystallographic groups, and we shall give also the name of the corresponding $\mathbb{E}^2$-orbifold, following Appendix A of [Mo]. (The restriction on finite subgroups eliminates the remaining ten $\mathbb{E}^2$-orbifold groups from consideration).

Theorem 10.11 *Let M be a closed 4-manifold such that $\pi = \pi_1(M)$ is virtually Z^2 and $\chi(M) = 0$. Let A and F be the maximal abelian and maximal finite normal subgroups (respectively) of π. If π is torsion free then either*

(1) $\pi = A \cong Z^2$ *(the torus); or*

(2) $\pi \cong Z \times_{-1} Z$ *(the Klein bottle).*
If $F = 1$ but π has nontrivial torsion and $[\pi : A] = 2$ then either

(3) $\pi \cong D \times Z \cong (Z \oplus (Z/2Z)) *_Z (Z \oplus (Z/2Z))$, *with the presentation* $\langle s, x, y \mid x^2 = y^2 = 1,\ sx = xs,\ sy = ys\rangle$ *(the silvered annulus); or*

(4) $\pi \cong D \tilde{\times} Z \cong Z *_Z (Z \oplus (Z/2Z))$, *with the presentation* $\langle t, x \mid x^2 = 1,\ t^2x = xt^2\rangle$ *(the silvered Möbius band); or*

(5) $\pi \cong (Z^2) \times_{-I} (Z/2Z) \cong D *_Z D$, *with the presentations* $\langle s, t, x \mid x^2 = 1,\ xsx = s^{-1},\ xtx = t^{-1},\ st = ts\rangle$ *and (setting $y = xt$)* $\langle s, x, y \mid x^2 = y^2 = 1,\ xsx = ysy = s^{-1}\rangle$ *(the pillowcase $S(2222)$).*
If $F = 1$ and $[\pi : A] = 4$ then either

(6) $\pi \cong D *_Z (Z \oplus (Z/2Z))$, *with the presentations* $\langle s, t, x \mid x^2 = 1,\ xsx = s^{-1},\ xtx = t^{-1},\ tst^{-1} = s^{-1}\rangle$ *and (setting $y = xt$)* $\langle s, x, y \mid x^2 = y^2 = 1,\ xsx = s^{-1},\ ys = sy\rangle$ *($D(22)$); or*

(7) $\pi \cong Z *_Z D$, *with the presentations* $\langle r, s, x \mid x^2 = 1,\ xrx = r^{-1},\ xsx = rs^{-1},\ srs^{-1} = r^{-1}\rangle$ *and (setting $t = xs$)* $\langle t, x \mid x^2 = 1,\ xt^2x = t^{-2}\rangle$ *($P(22)$).*
If F is nontrivial then either

(8) $\pi \cong Z^2 \oplus (Z/2Z)$*; or*

(9) $\pi \cong (Z \times_{-1} Z) \times (Z/2Z)$.

Proof Let $u : \pi \to \{\pm 1\} = Aut(\pi_2(M))$ be the natural homomorphism. Since $\mathrm{Ker}(u)$ is torsion free it is either Z^2 or $Z \times_{-1} Z$; since it has index at most 2 it follows that $[\pi : A]$ divides 4 and that F has order at most 2. If $F = 1$ then $A \cong Z^2$ and π/A acts effectively on A, so π is a 2-dimensional crystallographic group. If $F \neq 1$ then it is central in π and u maps F isomorphically to $Z/2Z$, so $\pi \cong (Z/2Z) \times \mathrm{Ker}(u)$. □

Each of these groups may be realised geometrically, by Theorem 10.2. It is easy to see that any $\mathbb{S}^2 \times E^2$-manifold whose fundamental group has infinite abelianization is a mapping torus, and hence is determined up to diffeomorphism by its homotopy type. (See Theorems 10.8 and 10.12). We shall show next that there are four affine diffeomorphism classes of $\mathbb{S}^2 \times \mathbb{E}^2$-manifolds whose fundamental groups have finite abelianization.

Let Ω be a discrete subgroup of $Isom(\mathbb{S}^2 \times \mathbb{E}^2) = O(3) \times E(2)$ which acts freely and cocompactly on $S^2 \times R^2$. If $\Omega \cong D *_Z D$ or $D *_Z (Z \oplus (Z/2Z))$ it is generated by elements of order 2, and so $p_1(\Omega) = \{\pm I\}$, by Lemma 10.6. Since $p_2(\Omega) < E(2)$ is a 2-dimensional crystallographic group it is determined up to conjugacy in $Aff(2) = R^2 \tilde{\times} GL(2, \mathbb{R})$ by its isomorphism type, Ω is determined up to conjugacy in $O(3) \times Aff(2)$ and the corresponding geometric 4-manifold is determined up to affine diffeomorphism.

Although $Z *_Z D$ is not generated by involutions, a similar argument applies. The isometries $T = (\tau, \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix})$ and $X = (-I, \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, -I)$ generate a discrete subgroup of $Isom(\mathbb{S}^2 \times \mathbb{E}^2)$ isomorphic to $Z *_Z D$ and which acts freely and cocompactly on $S^2 \times R^2$, provided $\tau^2 = I$. Since $x^2 = (xt^2)^2 = 1$ this condition is necessary, by Lemma 10.6. We shall see below that we may assume that T is orientation preserving, i.e., that $\det(\tau) = -1$. (The isometries T^2 and XT generate $\mathrm{Ker}(u)$). Thus there are two affine diffeomorphism classes of such manifolds, corresponding to the choices $\tau = -I$ or R_3.

None of these manifolds fibre over S^1, since in each case π/π' is finite. However if Ω is a $\mathbb{S}^2 \times \mathbb{E}^2$-lattice such that $p_1(\Omega) \leq \{\pm I\}$ then $\Omega \backslash (S^2 \times R^2)$ fibres over RP^2, since the map sending $(v, x) \in S^2 \times R^2$ to $[\pm v] \in RP^2$ is compatible with the action of Ω. If $p_1(\Omega) = \{\pm I\}$ the fibre is $\omega \backslash R^2$, where $\omega = \Omega \cap (\{1\} \times E(2))$; otherwise it has two components. Thus three of these four manifolds fibre over RP^2 (excepting perhaps only the case $\Omega \cong Z *_Z D$ and $R_3 \in p_1(\Omega)$).

10.5 Homotopy types of $\mathbb{S}^2 \times \mathbb{E}^2$-manifolds

Our next result shows that if M satisfies the conditions of Theorem 10.10 and its fundamental group has infinite abelianization then it is determined up to homotopy by $\pi_1(M)$ and its Stiefel-Whitney classes.

Theorem 10.12 *Let M be a closed 4-manifold with $\chi(M) = 0$ and such that $\pi = \pi_1(M)$ is virtually Z^2. If π/π' is infinite then M is homotopy equivalent to an $\mathbb{S}^2 \times \mathbb{E}^2$-manifold which fibres over S^1.*

Proof The infinite cyclic covering space of M determined by an epimorphism $\lambda : \pi \to Z$ is a PD_3-complex, by Theorem 4.5, and therefore is homotopy equivalent to

(1) $S^2 \times S^1$ (if $\mathrm{Ker}(\lambda) \cong Z$ is torsion free and $w_1(M)|_{\mathrm{Ker}(\lambda)} = 0$),
(2) $S^2 \tilde{\times} S^1$ (if $\mathrm{Ker}(\lambda) \cong Z$ and $w_1(M)|_{\mathrm{Ker}(\lambda)} \neq 0$),
(3) $RP^2 \times S^1$ (if $\mathrm{Ker}(\lambda) \cong Z \oplus (Z/2Z)$) or
(4) $RP^3 \sharp RP^3$ (if $\mathrm{Ker}(\lambda) \cong D$).

Therefore M is homotopy equivalent to the mapping torus $M(\phi)$ of a self homotopy equivalence of one of these spaces.

The group of free homotopy classes of self homotopy equivalences $E(S^2 \times S^1)$ is generated by the reflections in each factor and the twist map, and has order 8. The group $E(S^2 \tilde{\times} S^1)$ has order 4 [KR90]. Two of the corresponding mapping tori also arise from self homeomorphisms of $S^2 \times S^1$. The other two have nonintegral w_1. The group $E(RP^2 \times S^1)$ is generated by the reflection in the second factor and by a twist map, and has order 4. As all these mapping tori are also S^2- or RP^2-bundles over the torus or Klein bottle, they are geometric by Theorem 10.8.

The group $E(RP^3 \sharp RP^3)$ is generated by the reflection interchanging the summands and the fixed point free involution (cf. page 251 of [Ba']), and has order 4. Let $\alpha = (-I, 0, \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right))$, $\beta = (I, \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), I)$ $\gamma = (I, \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right), I)$ and $\delta = (-I, \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right), I)$ Then the subgroups generated by $\{\alpha, \beta, \gamma\}$, $\{\alpha, \beta, \delta\}$, $\{\alpha, \beta\gamma\}$ and $\{\alpha, \beta\delta\}$, respectively, give the four $RP^3 \sharp RP^3$-bundles. (Note that these may be distinguished by their groups and orientation characters). □

A T-bundle over RP^2 which does not also fibre over S^1 has fundamental group $D *_Z D$, while the group of a Kb-bundle over RP^2 which does not also fibre over S^1 is $D *_Z (Z \oplus (Z/2Z))$ or $Z *_Z D$ (assuming throughout that π is virtually Z^2).

When π is torsion free every homomorphism from π to $Z/2Z$ arises as the orientation character for some M with fundamental group π. However if $\pi \cong D \times Z$ or $D \tilde{\times} Z$ the orientation character must be trivial on all elements of order 2, while if $F \neq 1$ the orientation character is determined up to composition with an automorphism of π.

Theorem 10.13 *Let M be a closed 4-manifold with $\chi(M) = 0$ and such that $\pi = \pi_1(M)$ is an extension of Z by an almost finitely presentable infinite normal subgroup N with a nontrivial finite normal subgroup F. Then M is homotopy equivalent to the mapping torus of a self homeomorphism of $RP^2 \times S^1$.*

Proof Let $\widetilde{M}$ be the universal covering space of M. Since N is infinite and finitely generated π has one end, and so $H_i(\widetilde{M};\mathbb{Z}) = 0$ for $i \neq 0$ or 2. Let $\Pi = \pi_2(M) = H_2(\widetilde{M};\mathbb{Z})$. We wish to show that $\Pi \cong Z$, and that $w = w_1(M)$ maps F isomorphically onto $\{\pm 1\}$. Since $\beta_1^{(2)}(\pi) = 0$ by Lemma 2.1, there is an isomorphism of left $\mathbb{Z}[\pi]$-modules $\Pi \cong \overline{H^2(\pi;\mathbb{Z}[\pi])}$, by Theorem 3.4. An LHSSS argument then gives $\Pi \cong \overline{H^1(N;\mathbb{Z}[N])}$, which is a free abelian group.

The normal closure of F in π is the product of the conjugates of F, which are finite normal subgroups of N, and so is locally finite. If it is infinite then N has one end and so $H^s(\pi;\mathbb{Z}[\pi]) = 0$ for $s \leq 2$, by an LHSSS argument. Since locally finite groups are amenable $\beta_1^{(2)}(\pi) = 0$, by Theorem 2.3, and so M must be aspherical, by Corollary 3.5.2, contradicting the hypothesis that π has nontrivial torsion. Hence we may assume that F is normal in π.

Let f be a nontrivial element of F. Since F is normal in π the centralizer $C_\pi(f)$ of f has finite index in π, and we may assume without loss of generality that F is generated by f and is central in π. It follows from the spectral sequence for the projection of $\widetilde{M}$ onto $F\backslash\widetilde{M}$ that there are isomorphisms $H_{s+3}(F;\mathbb{Z}) \cong H_s(F;\Pi)$ for all $s \geq 4$, since $F\backslash\widetilde{M}$ is a 4-dimensional complex. Here F acts trivially on Z, but we must determine its action on Π.

Now central elements n of N act trivially on $H^1(N;\mathbb{Z}[N])$ and hence via $w(n)$ on Π. (See Theorem 2.11). Thus if $w(f) = 1$ the sequence

$$0 \to Z/|f|Z \to \Pi \to \Pi \to 0$$

is exact, where the right hand homomorphism is multiplication by $|f|$. As Π is torsion free this contradicts $f \neq 1$. Therefore if f is nontrivial it has order 2 and $w(f) = -1$. Hence $w : F \to \{\pm 1\}$ is an isomorphism and there is an exact sequence

$$0 \to \Pi \to \Pi \to Z/2Z \to 0,$$

where the left hand homomorphism is multiplication by 2. Since Π is a free abelian group it must be infinite cyclic, and so $\widetilde{M} \simeq S^2$. The theorem now follows from Theorems 10.10 and 10.12. ∎

The possible orientation characters for the groups with finite abelianization are restricted by Lemma 3.14, which implies that $\mathrm{Ker}(w_1)$ must have infinite abelianization. For $D *_Z D$ we must have $w_1(x) = w_1(y) = 1$ and $w_1(s) = 0$. For $D *_Z (Z \oplus (Z/2Z))$ we must have $w_1(s) = 0$ and $w_1(x) = 1$; since the subgroup generated by the commutator subgroup and y is isomorphic to $D \times Z$ we must also have $w_1(y) = 0$. Thus the orientation characters are uniquely determined for these groups. For $Z *_Z D$ we must have $w_1(x) = 1$, but $w_1(t)$ may be either 0 or 1. As there is an automorphism ϕ of $Z *_Z D$ determined by $\phi(t) = xt$ and $\phi(x) = x$ we may assume that $w_1(t) = 0$ in this case.

In all cases, to each choice of orientation character there corresponds a unique action of π on $\pi_2(M)$, by Lemma 10.3. However the homomorphism from π to $Z/2Z$ determining the action may differ from $w_1(M)$. (Note also that elements of order 2 must act nontrivially, by Theorem 10.1).

We shall need the following lemma about plane bundles over RP^2 in order to calculate self intersections here and in Chapter 12.

Lemma 10.14 *The total space of the* R^2*-bundle* p *over* RP^2 *with* $w_1(p) = 0$ *and* $w_2(p) \neq 0$ *is* $S^2 \times R^2/\langle g \rangle$*, where* $g(s,v) = (-s,-v)$ *for all* $(s,v) \in S^2 \times R^2$*.*

Proof Let $[s]$ and $[s,v]$ be the images of s in RP^2 and of (s,v) in $N = S^2 \times R^2/\langle g \rangle$, respectively, and let $p([s,v]) = [s]$, for $s \in S^2$ and $v \in R^2$. Then $p : N \to RP^2$ is an R^2-bundle projection, and $w_1(N) = p^* w_1(RP^2)$, so $w_1(p) = 0$. Let $\sigma_t([s]) = [s, t(x,y)]$, where $s = (x,y,z) \in S^2$ and $t \in R$. The embedding $\sigma_t : RP^2 \to N$ is isotopic to the 0-section σ_0, and $\sigma_t(RP^2)$ meets $\sigma_0(RP^2)$ transversally in one point, if $t > 0$. Hence $w_2(p) \neq 0$. ∎

Lemma 10.15 *Let* M *be the* $\mathbb{S}^2 \times \mathbb{E}^2$*-manifold with* $\pi_1(M) \cong Z *_Z D$ *generated by the isometries* $(-I, \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix})$ *and* $(-I, \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, -I)$*. Then* $v_2(M) = U^2$ *and* $U^4 = 0$ *in* $H^4(M; \mathbb{F}_2)$*.*

Proof This manifold is fibred over RP^2 with fibre Kb. As $\begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \end{pmatrix}$ is a fixed point of the involution $(\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, -I)$ of R^2 there is a cross-section given by $\sigma([s]) = [s, \begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \end{pmatrix}]$. Hence $H_2(M; \mathbb{F}_2)$ has a basis represented by embedded copies of Kb and RP^2, with self-intersection numbers 0 and 1, respectively. (See Lemma 10.14). Thus the characteristic element for the intersection pairing is $[Kb]$, and $v_2(M)$ is the Poincaré dual to $[Kb]$. The cohomology class $U \in H^1(M; \mathbb{F}_2)$ is induced from the generator of $H^1(RP^2; \mathbb{F}_2)$. The projection formula gives $p_*(U^2 \cap \sigma_*[RP^2]) = 1$ and $p_*(U^2 \cap [Kb]) = 0$. Hence we have also $v_2(M) = U^2$ and so $U^4 = 0$. □

This lemma is used below to compute some products in $H^*(Z *_Z D; \mathbb{F}_2)$. Ideally, we would have a purely algebraic argument.

Theorem 10.16 *Let M be a closed 4-manifold such that $\pi_2(M) \cong Z$ and $\beta_1(M) = \chi(M) = 0$, and let $\pi = \pi_1(M)$. Let U be the cohomology class in $H^1(\pi; \mathbb{F}_2)$ corresponding to the action $u : \pi \to Aut(\pi_2(M))$. Then*

(1) *if $\pi \cong D *_Z D$ or $D *_Z (Z \oplus (Z/2Z))$ then*
$$H^*(M; \mathbb{F}_2) \cong \mathbb{F}_2[S, T, U]/(S^2 + SU, T^2 + TU, U^3),$$
where S, T and U have degree 1;

(2) *if $\pi \cong Z *_Z D$ then*
$$H^*(M; \mathbb{F}_2) \cong \mathbb{F}_2[S, U, V, W]/I,$$
where S, U have degree 1, V has degree 2 and W has degree 3, and $I = (S^2, SU, SV, U^3, UV, UW, VW, V^2 + U^2V, V^2 + SW, W^2)$;

(3) *$v_2(M) = U^2$ and $k_1(M) = \beta^u(U^2) \in H^3(\pi; Z^u)$, in all cases.*

Proof We shall consider the three possible fundamental groups in turn.

$D *_Z D$ **:** Since x, y and xs have order 2 in $D *_Z D$ they act nontrivially, and so $K = \langle s, t \rangle \cong Z^2$. Let S, T, U be the basis for $H^1(\pi; \mathbb{F}_2)$ determined by the equations $S(t) = S(x) = T(s) = T(x) = U(s) = U(t) = 0$. It follows easily from the LHSSS for π as an extension of $Z/2Z$ by K that $H^2(\pi; \mathbb{F}_2)$ has dimension ≤ 4. We may check that the classes $\{U^2, US, UT, ST\}$ are linearly independent, by restriction to the cyclic subgroups generated by x, xs, xt and xst. Therefore they form a basis of $H^2(\pi; \mathbb{F}_2)$. The squares S^2 and T^2 must be linear combinations of the above basis elements. On restricting such linear combinations to subgroups as before we find that $S^2 = US$ and $T^2 = UT$. Now $H^s(\pi; \mathbb{F}_2) \cong H^s(M; \mathbb{F}_2)$ for $s \leq 2$, by Lemma 10.4. It follows easily from the nondegeneracy of Poincaré duality that $U^2ST \neq 0$ in $H^4(M; \mathbb{F}_2)$, while $U^3S = U^3T = U^4 = 0$, so $U^3 = 0$. Hence the cohomology ring $H^*(M; \mathbb{F}_2)$ is isomorphic to the ring $\mathbb{F}_2[S, T, U]/(S^2 + SU, T^2 + TU, U^3)$. Moreover $v_2(M) = U^2$, since $(US)^2 = USU^2$, $(UT)^2 = UTU^2$ and $(ST)^2 = STU^2$. An element of π has order 2 if and only if it is of the form xs^mt^n for some $(m, n) \in Z^2$. It is easy to check that the only linear combination $aU^2 + bUS + cUT + dST$ which has nonzero restriction to all subgroups of order 2 is U^2. Hence $k_1(M) = \beta^u(U^2)$.

$D *_Z (Z \oplus (Z/2Z))$ **:** Since x, y and xs have order 2 in $D *_Z (Z \oplus (Z/2Z))$ they act nontrivially, and so $K = \langle s, t\rangle \cong Z \times_{-1} Z$. Let S, T, U be the basis for $H^1(\pi; \mathbb{F}_2)$ determined by the equations $S(t) = S(x) = T(s) = T(x) = U(s) = U(t) = 0$. We again find that $\{U^2, US, UT, ST\}$ forms a basis for $H^2(\pi; \mathbb{F}_2) \cong H^2(M; \mathbb{F}_2)$, and may check that $S^2 = US$ and $T^2 = UT$, by restriction to the subgroups generated by $\{x, xs\}$, $\{x, xt\}$ and K. As before, the nondegeneracy of Poincaré duality implies that $H^*(M; \mathbb{F}_2)$ is isomorphic to the ring $\mathbb{F}_2[S, T, U]/(S^2 + SU, T^2 + TU, U^3)$, while $v_2(M) = U^2$. An element of π has order 2 if and only if it is of the form xs^mt^n for some $(m, n) \in Z^2$, with either $m = 0$ or n even. Hence U^2 and $U^2 + ST$ are the only elements of $H^2(\pi; \mathbb{F}_2)$ with nonzero restriction to all subgroups of order 2. Now $H^1(\pi; Z^u) \cong Z \oplus (Z/2Z)$ and $H^1(\pi; \mathbb{F}_2) \cong (Z/2Z)^3$. Since $\pi/K = Z/2Z$ acts nontrivially on $H^1(K; \mathbb{Z})$ it follows from the LHSSS with coefficients Z^u that $H^2(\pi; Z^u) \leq E_2^{0,2} = Z/2Z$. As the functions $f(x^a s^m t^n) = (-1)^a n$ and $g(x^a s^m t^n) = (1 - (-1)^a)/2$ define crossed homomorphisms from G to Z^u (i.e., $f(wz) = u(w)f(z) + f(w)$ for all w, z in G) which reduce modulo (2) to T and U, respectively, $H^2(\pi; Z^u)$ is generated by $\beta^u(S)$ and has order 2. We may check that $\overline{\beta^u}(S) = ST$, by restriction to the subgroups generated by $\{x, xs\}$, $\{x, xt\}$ and K. Hence $k_1(M) = \beta^u(U^2) = \beta^u(U^2 + ST)$.

$Z *_Z D$ **:** If $\pi \cong Z *_Z D$ then $\pi/\pi' \cong (Z/4Z) \oplus (Z/2Z)$ and we may assume that $K \cong Z \times_{-1} Z$ is generated by r and s. Let S, U be the basis for $H^1(\pi; \mathbb{F}_2)$ determined by the equations $S(x) = U(s) = 0$. Note that S is in fact the *mod*-2 reduction of the homomorphism $\tilde{S} : \pi \to Z/4Z$ given by $\tilde{S}(s) = 1$ and $\tilde{S}(x) = 0$. Therefore $S^2 = Sq^1 S = 0$. Let $f : \pi \to \mathbb{F}_2$ be the function defined by $f(k) = f(rsk) = f(xrk) = f(xrsk) = 0$ and $f(rk) = f(sk) = f(xk) = f(xsk) = 1$ for all $k \in K'$. Then $U(g)S(h) = f(g) + f(h) + f(gh) = \delta f(g, h)$ for all $g, h \in \pi$, and so $US = 0$ in $H^2(\pi; \mathbb{F}_2)$. In the LHSSS all differentials ending on the bottom row must be 0, since π is a semidirect product of $Z/2Z$ with the normal subgroup K. Since $H^p(Z/2Z; H^1(K; \mathbb{F}_2)) = 0$ for all $p > 0$, it follows that $H^n(\pi; \mathbb{F}_2)$ has dimension 2, for all $n \geq 1$.

In particular, $H^2(M; \mathbb{F}_2) \cong H^2(\pi; \mathbb{F}_2)$ has a basis $\{U^2, V\}$, where $V|_K$ generates $H^2(K; \mathbb{F}_2)$. Moreover $H^4(M; \mathbb{F}_2)$ is a quotient of $H^4(\pi; \mathbb{F}_2)$. It follows from Lemma 10.15 that $\{U^4, U^2V\}$ is a basis for $H^4(\pi; \mathbb{F}_2)$ and $V^2 = U^2V + mU^4$ in $H^4(\pi; \mathbb{F}_2)$, for some $m = 0$ or 1. Let $\sigma : Z/2Z \to \pi$ be the inclusion of the subgroup $\langle x\rangle$, which splits the projection onto π/K. Then $\sigma^*(V) = \sigma^*(U^2)$ or 0, while $\sigma^* U^4 \neq 0$ in $H^4(\pi; \mathbb{F}_2)$. Hence $m\sigma^*(U^4) = \sigma^*(V^2 + U^2V) = 0$ and so $V^2 = U^2V$ in $H^4(\pi; \mathbb{F}_2)$. Therefore if M is *any* closed 4-manifold with $\pi_1(M) \cong Z *_Z D$ and $\chi(M) = 0$ the image of U^4 in $H^4(M; \mathbb{F}_2)$ must be 0, and hence $v_2(M) = U^2$, by Poincaré duality. Moreover $H^*(M;\mathbb{F}_2)$ is generated by S, U (in degree 1), V (in degree 2) and an element W in degree 3 such that $SW \neq 0$ and $UW = 0$. Hence $H^*(M;\mathbb{F}_2)$ is isomorphic to the quotient of $\mathbb{F}_2[S,U,V,W]$ by the ideal $(S^2, SU, SV, U^3, UV, UW, VW, V^2+U^2V, V^2+SW, W^2)$.

Since $U^3U = U^3S = 0$ in $H^4(M;\mathbb{F}_2)$ the image of U^3 in $H^3(M;\mathbb{F}_2)$ must also be 0, by Poincaré duality. Now $k_1(M)$ has image 0 in $H^3(M;\mathbb{F}_2)$ and nonzero restriction to subgroups of order 2. Therefore $k_1(M) = \beta^u(U^2)$, as reduction modulo (2) is injective, by Lemma 10.4. ◻

The example $M = RP^2 \times T$ has $v_2(M) = 0$ and $k_1(M) \neq 0$, and so in general $k_1(M)$ need not equal $\beta^u(v_2(M))$. Is it always $\beta^u(U^2)$?

Corollary 10.16.1 *The covering space associated to $K = \mathrm{Ker}(u)$ is homeomorphic to $S^2 \times T$ if $\pi \cong D *_Z D$ and to $S^2 \times Kb$ if $\pi \cong D *_Z (Z \oplus (Z/2Z))$ or $Z *_Z D$.*

Proof Since ρ is Z^2 or $Z \times_{-1} Z$ these assertions follow from Theorem 6.11, on computing the Stiefel-Whitney classes of the double cover. Since $D *_Z D$ acts orientably on the euclidean plane R^2 we have $w_1(M) = U$, by Lemma 10.3, and so $w_2(M) = v_2(M) + w_1(M)^2 = 0$. Hence the double cover is $S^2 \times T$. If $\pi \cong D *_Z (Z \oplus (Z/2Z))$ or $Z *_Z D$ then $w_1(M)|_K = w_1(K)$, while $w_2(M)|_K = 0$, so the double cover is $S^2 \times Kb$, in both cases. ◻

The $\mathbb{S}^2 \times \mathbb{E}^2$-manifolds with groups $D *_Z D$ and $D *_Z (Z \oplus (Z/2Z))$ are unique up to affine diffeomorphism. In each case there is at most one other homotopy type of closed 4-manifold with this fundamental group and Euler characteristic 0, by Theorems 10.5 and 10.16 and the remark following Theorem 10.13. Are the two affine diffeomorphism classes of $\mathbb{S}^2 \times \mathbb{E}^2$-manifolds with $\pi \cong Z *_Z D$ homotopy equivalent? There are again at most 2 homotopy types. In summary, there are 22 affine diffeomorphism classes of closed $\mathbb{S}^2 \times \mathbb{E}^2$-manifolds (representing at least 21 homotopy types) and between 21 and 24 homotopy types of closed 4-manifolds covered by $S^2 \times R^2$ and with Euler characteristic 0.

10.6 Some remarks on the homeomorphism types

In Chapter 6 we showed that if π is Z^2 or $Z \times_{-1} Z$ then M must be homeomorphic to the total space of an S^2-bundle over the torus or Klein bottle, and we were able to estimate the size of the structure sets when π has $Z/2Z$ as a direct factor. The other groups of Theorem 10.11 are not "square-root closed accessible" and we have not been able to compute the surgery obstruction groups completely. However the Mayer-Vietoris sequences of [Ca73] are exact modulo 2-torsion, and we may compare the ranks of $[SM; G/TOP]$ and $L_5(\pi, w_1)$. This is sufficient in some cases to show that the structure set is infinite. For instance, the rank of $L_5(D \times Z)$ is 3 and that of $L_5(D \tilde{\times} Z)$ is 2, while the rank of $L_5(D *_Z (Z \oplus (Z/2Z)), w_1)$ is 2. (The groups $L_*(\pi) \otimes \mathbb{Z}[\frac{1}{2}]$ have been computed for all cocompact planar groups π [LS00]). If M is orientable and $\pi \cong D \times Z$ or $D \times_\tau Z$ then $[SM; G/TOP] \cong H^3(M; \mathbb{Z}) \oplus H^1(M; \mathbb{F}_2) \cong H_1(M; \mathbb{Z}) \oplus H^1(M; \mathbb{F}_2)$ has rank 1. Therefore $S_{TOP}(M)$ is infinite. If $\pi \cong D *_Z (Z \oplus (Z/2Z))$ then $H_1(M; \mathbb{Q}) = 0$, $H_2(M; \mathbb{Q}) = H_2(\pi; \mathbb{Q}) = 0$ and $H_4(M; \mathbb{Q}) = 0$, since M is nonorientable. Hence $H^3(M; \mathbb{Q}) \cong Q$, since $\chi(M) = 0$. Therefore $[SM; G/TOP]$ again has rank 1 and $S_{TOP}(M)$ is infinite. These estimates do not suffice to decide whether there are infinitely many homeomorphism classes in the homotopy type of M. To decide this we need to study the action of the group $E(M)$ on $S_{TOP}(M)$. A scheme for analyzing $E(M)$ as a tower of extensions involving actions of cohomology groups with coefficients determined by Whitehead's Γ-functors is outlined on page 52 of [Ba'].

Chapter 11

Manifolds covered by $S^3 \times R$

In this chapter we shall show that a closed 4-manifold M is covered by $S^3 \times R$ if and only if $\pi = \pi_1(M)$ has two ends and $\chi(M) = 0$. Its homotopy type is then determined by π and the first k-invariant $k_1(M)$. The maximal finite normal subgroup of π is either the group of a $\mathbb{S}^3$-manifold or one of the groups $Q(8a, b, c) \times Z/dZ$ with a, b, c and d odd. (There are examples of the latter type, and no such M is homotopy equivalent to a $\mathbb{S}^3 \times \mathbb{E}^1$-manifold.) The possibilities for π are not yet known even when F is a $\mathbb{S}^3$-manifold group and $\pi/F \cong Z$. Solving this problem may involve first determining which k-invariants are realizable when F is cyclic; this is also not yet known.

Manifolds which fibre over RP^2 with fibre T or Kb and $\partial \neq 0$ have universal cover $S^3 \times R$. In §6 we determine the possible fundamental groups, and show that an orientable 4-manifold M with such a group and with $\chi(M) = 0$ must be homotopy equivalent to a $\mathbb{S}^3 \times \mathbb{E}^1$-manifold which fibres over RP^2.

As groups with two ends are virtually solvable, surgery techniques may be used to study manifolds covered by $S^3 \times R$. However computing $Wh(\pi)$ and $L_*(\pi; w_1)$ is a major task. Simple estimates suggest that there are usually infinitely many nonhomeomorphic manifolds within a given homotopy type.

11.1 Invariants for the homotopy type

The determination of the closed 4-manifolds with universal covering space homotopy equivalent to S^3 is based on the structure of groups with two ends.

Theorem 11.1 *Let M be a closed 4-manifold with fundamental group π. Then $\widetilde{M} \simeq S^3$ if and only if π has two ends and $\chi(M) = 0$. If so*

(1) *M is finitely covered by $S^3 \times S^1$ and so $\widetilde{M} \cong S^3 \times R \cong R^4 \backslash \{0\}$;*

(2) *the maximal finite normal subgroup F of π has cohomological period dividing 4, acts trivially on $\pi_3(M) \cong Z$ and the corresponding covering space M_F has the homotopy type of an orientable finite PD_3-complex;*

(3) *the homotopy type of M is determined by π and the orbit of the first nontrivial k-invariant $k(M) \in H^4(\pi; Z^w)$ under $Out(\pi) \times \{\pm 1\}$; and*

(4) *the restriction of $k(M)$ to $H^4(F;Z)$ is a generator.*

Proof If $\widetilde{M} \simeq S^3$ then $H^1(\pi;\mathbb{Z}[\pi]) \cong Z$ and so π has two ends. Hence π is virtually Z. The covering space M_A corresponding to an infinite cyclic subgroup A is homotopy equivalent to the mapping torus of a self homotopy equivalence of $S^3 \simeq \widetilde{M}$, and so $\chi(M_A) = 0$. As $[\pi : A] < \infty$ it follows that $\chi(M) = 0$ also.

Suppose conversely that $\chi(M) = 0$ and π is virtually Z. Then $H_3(\widetilde{M};\mathbb{Z}) \cong Z$ and $H_4(\widetilde{M};\mathbb{Z}) = 0$. Let M_Z be an orientable finite covering space with fundamental group Z. Then $\chi(M_Z) = 0$ and so $H_2(M_Z;\mathbb{Z}) = 0$. The homology groups of $\widetilde{M} = \widetilde{M}_Z$ may be regarded as modules over $\mathbb{Z}[t,t^{-1}] \cong \mathbb{Z}[Z]$. Multiplication by $t-1$ maps $H_2(\widetilde{M};\mathbb{Z})$ onto itself, by the Wang sequence for the projection of $\widetilde{M}$ onto M_Z. Therefore $Hom_{\mathbb{Z}[Z]}(H_2(\widetilde{M};\mathbb{Z}),\mathbb{Z}[Z]) = 0$ and so $\pi_2(M) = \pi_2(M_Z) = 0$, by Lemma 3.3. Therefore the map from S^3 to $\widetilde{M}$ representing a generator of $\pi_3(M)$ is a homotopy equivalence. Since M_Z is orientable the generator of the group of covering translations $Aut(\widetilde{M}/M_Z) \cong Z$ is homotopic to the identity, and so $M_Z \simeq \widetilde{M} \times S^1 \simeq S^3 \times S^1$. Therefore $M_Z \cong S^3 \times S^1$, by surgery over Z. Hence $\widetilde{M} \cong S^3 \times R$.

Let F be the maximal finite normal subgroup of π. Since F acts freely on $\widetilde{M} \simeq S^3$ it has cohomological period dividing 4 and $M_F = \widetilde{M}/F$ is a PD_3-complex. In particular, M_F is orientable and F acts trivially on $\pi_3(M)$. The image of the finiteness obstruction for M_F under the "geometrically significant injection" of $K_0(\mathbb{Z}[F])$ into $Wh(F \times Z)$ of [Rn86] is the obstruction to $M_F \times S^1$ being a simple PD-complex. If $f : M_F \to M_F$ is a self homotopy equivalence which induces the identity on $\pi_1(M_F) \cong F$ and on $\pi_3(M_F) \cong Z$ then f is homotopic to the identity, by obstruction theory. (See [Pl82].) Therefore $\pi_0(E(M_F))$ is finite and so M has a finite cover which is homotopy equivalent to $M_F \times S^1$. Since manifolds are simple PD_n-complexes M_F must be finite.

The first nonzero k-invariant lies in $H^4(\pi;Z^w)$, since $\pi_2(M) = 0$ and π acts on $\pi_3(M) \cong Z$ via the orientation character. As it restricts to the k-invariant for M_F in $H^4(F;Z^w)$ it generates this group, and (4) follows as in Theorem 2.9. □

The list of finite groups with cohomological period dividing 4 is well known (see [DM85]). There are the generalized quaternionic groups $Q(2^n a,b,c)$ (with $n \geq 3$ and a,b,c odd), the extended binary tetrahedral groups T_k^*, the extended binary octahedral groups O_k^*, the binary icosahedral group I^*, the dihedral groups $A(m,e)$ (with m odd >1), and the direct products of any one of these with a cyclic group Z/dZ of relatively prime order. (In particular, a p-group with periodic cohomology is cyclic if p is odd and cyclic or quaternionic if $p=2$.) We shall give presentations for these groups in §2.

Each such group F is the fundamental group of some PD_3-complex [Sw60]. Such *Swan complexes* for F are orientable, and are determined up to homotopy equivalence by their k-invariants, which are generators of $H^3(F;\mathbb{Z})\cong Z/|F|Z$, by Theorem 2.9. Thus they are parametrized up to homotopy by the quotient of $(Z/|F|Z)^\times$ under the action of $Out(F)\times\{\pm 1\}$. The set of finiteness obstructions for all such complexes forms a coset of the "Swan subgroup" of $\tilde{K}_0(\mathbb{Z}[F])$ and there is a finite complex of this type if and only if the coset contains 0. (This condition fails if F has a subgroup isomorphic to $Q(16,3,1)$ and hence if $F\cong O_k^*\times(Z/dZ)$ for some $k>1$, by Corollary 3.16 of [DM85].) If X is a Swan complex for F then $X\times S^1$ is a finite PD_4^+-complex with $\pi_1(X\times S^1)\cong F\times Z$ and $\chi(X\times S^1)=0$.

If $\pi/F\cong Z$ then $k(M)$ is a generator of $H^4(\pi;\pi_3(M))\cong H^4(F;\mathbb{Z})\cong Z/|F|Z$. If $\pi/F\cong D$ then $\pi\cong G*_F H$, where $[G:F]=[H:F]=2$, and $H^4(\pi;\mathbb{Z})\cong\{(\zeta,\xi)\in(Z/|G|Z)\oplus(Z/|H|Z)\mid \zeta\equiv\xi \bmod (|F|)\}$, which is isomorphic to $(Z/2|F|Z)\oplus(Z/2Z)$, and the k-invariant restricts to a generator of each of the groups $H^4(G;\mathbb{Z})$ and $H^4(H;\mathbb{Z})$. In particular, if $\pi\cong D$ the k-invariant is unique, and so any closed 4-manifold M with $\pi_1(M)\cong D$ and $\chi(M)=0$ is homotopy equivalent to $RP^4\sharp RP^4$.

Theorem 11.2 *Let M be a closed 4-manifold such that $\pi=\pi_1(M)$ has two ends and with $\chi(M)=0$. Then the group of unbased homotopy classes of self homotopy equivalences of M is finite.*

Proof We may assume that M has a finite cell structure with a single 4-cell. Suppose that $f:M\to M$ is a self homotopy equivalence which fixes a base point and induces the identity on π and on $\pi_3(M)\cong Z$. Then there are no obstructions to constructing a homotopy from f to $id_{\widetilde{M}}$ on the 3-skeleton $M_0=M\backslash intD^4$, and since $\pi_4(M)=\pi_4(S^3)=Z/2Z$ there are just two possibilities for f. It is easily seen that $Out(\pi)$ is finite. Since every self map is homotopic to one which fixes a basepoint the group of unbased homotopy classes of self homotopy equivalences of M is finite. □

If π is a semidirect product $F\times_\theta Z$ then $Aut(\pi)$ is finite and the group of *based* homotopy classes of based self homotopy equivalences is also finite.

11.2 The action of π/F on F

Let F be a finite group with cohomological period dividing 4. Automorphisms of F act on $H_*(F;\mathbb{Z})$ and $H^*(F;\mathbb{Z})$ through $Out(F)$, since inner automorphisms induce the identity on (co)homology. Let $J_+(F)$ be the kernel of the action on $H_3(F;\mathbb{Z})$, and let $J(F)$ be the subgroup of $Out(F)$ which acts by ± 1.

An outer automorphism class induces a well defined action on $H^4(S;\mathbb{Z})$ for each Sylow subgroup S of F, since all p-Sylow subgroups are conjugate in F and the inclusion of such a subgroup induces an isomorphism from the p-torsion of $H^4(F;\mathbb{Z}) \cong Z/|F|Z$ to $H^4(S;\mathbb{Z}) \cong Z/|S|Z$, by Shapiro's Lemma. Therefore an outer automorphism class of F induces multiplication by r on $H^4(F;\mathbb{Z})$ if and only if it does so for each Sylow subgroup of F, by the Chinese Remainder Theorem.

The map sending a self homotopy equivalence h of a Swan complex X_F for F to the induced outer automorphism class determines a homomorphism from the group of (unbased) homotopy classes of self homotopy equivalences $E(X_F)$ to $Out(F)$. The image of this homomorphism is $J(F)$, and it is a monomorphism if $|F| > 2$, by Corollary 1.3 of [Pl82]. (Note that [Pl82] works with *based* homotopies.) If $F = 1$ or $Z/2Z$ the orientation reversing involution of X_F ($\simeq S^3$ or RP^3, respectively) induces the identity on F.

Lemma 11.3 *Let M be a closed 4-manifold with universal cover $S^3 \times R$, and let F be the maximal finite normal subgroup of $\pi = \pi_1(M)$. The quotient π/F acts on $\pi_3(M)$ and $H^4(F;\mathbb{Z})$ through multiplication by ± 1. It acts trivially if the order of F is divisible by 4 or by any prime congruent to 3 mod (4).*

Proof The group π/F must act through ± 1 on the infinite cyclic groups $\pi_3(M)$ and $H_3(M_F;\mathbb{Z})$. By the universal coefficient theorem $H^4(F;\mathbb{Z})$ is isomorphic to $H_3(F;\mathbb{Z})$, which is the cokernel of the Hurewicz homomorphism from $\pi_3(M)$ to $H_3(M_F;\mathbb{Z})$. This implies the first assertion.

To prove the second assertion we may pass to the Sylow subgroups of F, by Shapiro's Lemma. Since the p-Sylow subgroups of F also have cohomological period 4 they are cyclic if p is an odd prime and are cyclic or quaternionic ($Q(2^n)$) if $p = 2$. In all cases an automorphism induces multiplication by a square on the third homology [Sw60]. But -1 is not a square modulo 4 nor modulo any prime $p = 4n + 3$. $\square$

Thus the groups $\pi \cong F \rtimes Z$ realized by such 4-manifolds correspond to outer automorphisms in $J(F)$ or $J_+(F)$. We shall next determine these subgroups of $Out(F)$ for F a group of cohomological period dividing 4. If m is an integer let $l(m)$ be the number of odd prime divisors of m.

$Z/dZ = \langle x \mid x^d = 1\rangle$.

$Out(Z/dZ) = Aut(Z/dZ) = (Z/dZ)^\times$.

Hence $J(Z/dZ) = \{s \in (Z/dZ)^\times \mid s^2 = \pm 1\}$. $J_+(Z/dZ) = (Z/2Z)^{l(d)}$ if $d \not\equiv 0\,(4)$, $(Z/2Z)^{l(d)+1}$ if $d \equiv 4\,(8)$, and $(Z/2Z)^{l(d)+2}$ if $d \equiv 0\,(8)$.

$Q(8) = \langle x, y \mid x^2 = y^2 = (xy)^2\rangle$.

An automorphism of $Q = Q(8)$ induces the identity on Q/Q' if and only if it is inner, and every automorphism of Q/Q' lifts to one of Q. In fact $Aut(Q)$ is the semidirect product of $Out(Q) \cong Aut(Q/Q') \cong SL(2, \mathbb{F}_2)$ with the normal subgroup $Inn(Q) = Q/Q' \cong (Z/2Z)^2$. Moreover $J(Q) = Out(Q)$, generated by the images of the automorphisms σ and τ, where σ sends x and y to y and xy, respectively, and τ interchanges x and y.

$Q(8k) = \langle x, y \mid x^{4k} = 1,\ x^{2k} = y^2,\ yxy^{-1} = x^{-1}\rangle$, where $k > 1$.

All automorphisms of $Q(8k)$ are of the form $[i, s]$, where $(s, 2k) = 1$, $[i, s](x) = x^s$ and $[i, s](y) = x^i y$, and $Aut(Q(8k))$ is the semidirect product of $(Z/4kZ)^\times$ with the normal subgroup $\langle[1, 1]\rangle \cong Z/4kZ$.

$Out(Q(8k)) = (Z/2Z) \oplus ((Z/4kZ)^\times/(\pm 1))$, generated by the images of the $[0, s]$ and [1,1]. The automorphism $[i, s]$ induces multiplication by s^2 on $H^4(Q(2^n);\mathbb{Z})$ [Sw60]. Hence $J(Q(8k)) = (Z/2Z)^{l(k)+1}$ if k is odd and $(Z/2Z)^{l(k)+2}$ if k is even.

$T_k^* = \langle Q(8), z \mid z^{3^k} = 1,\ zxz^{-1} = y,\ zyz^{-1} = xy\rangle$, where $k \geq 1$.

Let ρ be the automorphism which sends x, y and z to y^{-1}, x^{-1} and z^2 respectively. Let ξ, η and ζ be the inner automorphisms determined by conjugation by x, y and z, respectively (i.e., $\xi(g) = xgx^{-1}$, and so on). Then $Aut(T_k^*)$ has the presentation

$$\langle \rho, \xi, \eta, \zeta \mid \rho^{2.3^{k-1}} = \eta^2 = \zeta^3 = (\eta\zeta)^3 = 1,\ \rho\zeta\rho^{-1} = \zeta^2,\ \rho\eta\rho^{-1} = \zeta^{-1}\eta\zeta = \xi\rangle.$$

An induction on k gives $4^{3^k} = 1 + m3^{k+1}$ for some $m \equiv 1$ mod (3). Hence the image of ρ generates $Aut(T_k^*/T_k^{*\prime}) \cong (Z/3^kZ)^\times$, and so $Out(T_k^*) \cong (Z/3^kZ)^\times$. The 3-Sylow subgroup generated by z is preserved by ρ, and it follows that $J(T_k^*) = Z/2Z$ (generated by the image of $\rho^{3^{k-1}}$).

$O_k^* = \langle T_k^*, w \mid w^2 = x^2,\ wxw^{-1} = yx,\ wzw^{-1} = z^{-1}\rangle$, where $k \geq 1$.

(Note that the relations imply $wyw^{-1} = y^{-1}$.) As we may extend ρ to an automorphism of O_k^* via $\rho(w) = w^{-1}z^2$ the restriction from $Aut(O_k^*)$ to $Aut(T_k^*)$ is onto. An automorphism in the kernel sends w to wv for some $v \in T_k^*$, and the relations for O_k^* imply that v must be central in T_k^*. Hence the kernel is generated by the involution α which sends w, x, y, z to $w^{-1} = wx^2, x, y, z$, respectively. Now $\rho^{3^{k-1}} = \sigma\alpha$, where σ is conjugation by wz in O_k^*, and so the image of ρ generates $Out(O_k^*)$. The subgroup $\langle u, x\rangle$ generated by $u = xw$ and x is isomorphic to $Q(16)$, and is a 2-Sylow subgroup. As $\alpha(u) = u^5$ and $\alpha(x) = x$ it is preserved by α, and $H^4(\alpha|_{\langle u,x\rangle}; \mathbb{Z})$ is multiplication by 25. As $H^4(\rho|_{\langle z\rangle}; \mathbb{Z})$ is multiplication by 4 it follows that $J(O_k^*) = 1$.

$I^* = \langle x, y \mid x^2 = y^3 = (xy)^5\rangle$.

The map sending the generators x, y to $\left(\begin{smallmatrix} 2 & 0 \\ 1 & 3 \end{smallmatrix}\right)$ and $y = \left(\begin{smallmatrix} 2 & 2 \\ 1 & 4 \end{smallmatrix}\right)$, respectively, induces an isomorphism from I^* to $SL(2, \mathbb{F}_5)$. Conjugation in $GL(2, \mathbb{F}_5)$ induces a monomorphism from $PGL(2, \mathbb{F}_5)$ to $Aut(I^*)$. The natural map from $Aut(I^*)$ to $Aut(I^*/\zeta I^*)$ is injective, since I^* is perfect. Now $I^*/\zeta I^* \cong PSL(2, \mathbb{F}_5) \cong A_5$. The alternating group A_5 is generated by 3-cycles, and has ten 3-Sylow subgroups, each of order 3. It has five subgroups isomorphic to A_4 generated by pairs of such 3-Sylow subgroups. The intersection of any two of them has order 3, and is invariant under any automorphism of A_5 which leaves invariant each of these subgroups. It is not hard to see that such an automorphism must fix the 3-cycles. Thus $Aut(A_5)$ embeds in the group S_5 of permutations of these subgroups. Since $|PGL(2, \mathbb{F}_5)| = |S_5| = 120$ it follows that $Aut(I^*) \cong S_5$ and $Out(I^*) = Z/2Z$. The outer automorphism class is represented by the matrix $\omega = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ in $GL(2, \mathbb{F}_5)$.

Lemma 11.4 [Pl83] $J(I^*) = 1$.

Proof The element $\gamma = x^3y = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ generates a 5-Sylow subgroup of I^*. It is easily seen that $\omega\gamma\omega^{-1} = \gamma^2$, and so ω induces multiplication by 2 on $H^2(Z/5Z; \mathbb{Z}) \cong H_1(Z/5Z; \mathbb{Z}) = Z/5Z$. Since $H^4(Z/5Z; \mathbb{Z}) \cong Z/5Z$ is generated by the square of a generator for $H^2(Z/5Z; \mathbb{Z})$ we see that $H^4(\omega; \mathbb{Z})$ is multiplication by $4 = -1$ on 5-torsion. Hence $J(I^*) = 1$. □

In fact $H^4(\omega; \mathbb{Z})$ is multiplication by 49 [Pl83].

$A(m, e) = \langle x, y \mid x^m = y^{2^e} = 1,\ yxy^{-1} = x^{-1}\rangle$, where $e \geq 1$ and $m > 1$ is odd.

All automorphisms of $A(m,e)$ are of the form $[s,t,u]$, where $(s,m)=(t,2)=1$, $[s,t,u](x)=x^s$ and $[s,t,u](y)=x^uy^t$. $Out(A(m,e))$ is generated by the images of $[s,1,0]$ and $[1,t,0]$ and is isomorphic to $(Z/2^e)^\times\oplus((Z/mZ)^\times/(\pm1))$. $J(A(m,1))=\{s\in(Z/mZ)^\times\mid s^2=\pm1\}/(\pm1)$,

$J(A(m,2))=(Z/2Z)^{l(m)}$, $J(A(m,e))=(Z/2Z)^{l(m)+1}$ if $e>2$.

$Q(2^na,b,c)=\langle Q(2^n),u\mid u^{abc}=1,\ xu^{ab}=u^{ab}x,\ xu^cx^{-1}=u^{-c},\ yu^{ac}=u^{ac}y,\ yu^by^{-1}=u^{-b}\rangle$, where a, b and c are odd and relatively prime, and either $n=3$ and at most one of a, b and c is 1 or $n>3$ and $bc>1$.

An automorphism of $G=Q(2^na,b,c)$ must induce the identity on G/G'. If it induces the identity on the characteristic subgroup $\langle u\rangle\cong Z/abcZ$ and on $G/\langle u\rangle\cong Q(2^n)$ it is inner, and so $Out(Q(2^na,b,c))$ is a subquotient of $Out(Q(2^n))\times(Z/abcZ)^\times$. In particular, $Out(Q(8a,b,c))\cong(Z/abcZ)^\times$, and $J(Q(8a,b,c))\cong(Z/2Z)^{l(abc)}$. (We need only consider $n=3$, by §5 below.)

As $Aut(G\times H)=Aut(G)\times Aut(H)$ and $Out(G\times H)=Out(G)\times Out(H)$ if G and H are finite groups of relatively prime order, we have $J_+(G\times Z/dZ)=J_+(G)\times J_+(Z/dZ)$. In particular, if G is not cyclic or dihedral $J(G\times Z/dZ)=J_+(G\times Z/dZ)=J(G)\times J_+(Z/dZ)$. In all cases except when F is cyclic or $Q(8)\times Z/dZ$ the group $J(F)$ has exponent 2 and hence π has a subgroup of index at most 4 which is isomorphic to $F\times Z$.

11.3 Extensions of D

We shall now assume that $\pi/F\cong D$. Let $u,v\in D$ be a pair of involutions which generate D and let $s=uv$. Then $s^{-n}us^n=us^{2n}$, and any involution in D is conjugate to u or to $v=us$. Hence any pair of involutions $\{u',v'\}$ which generates D is conjugate to the pair $\{u,v\}$, up to change of order.

Theorem 11.5 *Let M be a closed 4-manifold with $\chi(M)=0$, and such that there is an epimorphism $p:\pi=\pi_1(M)\to D$ with finite kernel F. Let $\hat u$ and $\hat v$ be a pair of elements of π whose images $u=p(\hat u)$ and $v=p(\hat v)$ in D are involutions which together generate D. Then*

(1) *M is nonorientable and $\hat u,\hat v$ each represent orientation reversing loops;*

(2) *the subgroups G and H generated by F and $\hat u$ and by F and $\hat v$, respectively, each have cohomological period dividing 4, and the unordered pair $\{G,H\}$ of groups is determined up to isomorphisms by π alone;*

(3) *conversely, π is determined up to isomorphism by the unordered pair $\{G, H\}$ of groups with index 2 subgroups isomorphic to F as the free product with amalgamation $\pi = G *_F H$;*

(4) *π acts trivially on $\pi_3(M)$.*

Proof Let $\hat{s} = \hat{u}\hat{v}$. Suppose that $\hat{u}$ is orientation preserving. Then the subgroup σ generated by $\hat{u}$ and $\hat{s}^2$ is orientation preserving so the corresponding covering space M_σ is orientable. As σ has finite index in π and σ/σ' is finite this contradicts Lemma 3.14. Similarly, $\hat{v}$ must be orientation reversing.

By assumption, $\hat{u}^2$ and $\hat{v}^2$ are in F, and $[G : F] = [H : F] = 2$. If F is not isomorphic to $Q \times Z/dZ$ then $J(F)$ is abelian and so the (normal) subgroup generated by F and $\hat{s}^2$ is isomorphic to $F \times Z$. In any case the subgroup generated by F and $\hat{s}^k$ is normal, and is isomorphic to $F \times Z$ if k is a nonzero multiple of 12. The uniqueness up to isomorphisms of the pair $\{G, H\}$ follows from the uniqueness up to conjugation and order of the pair of generating involutions for D. Since G and H act freely on $\widetilde{M}$ they also have cohomological period dividing 4. On examining the list above we see that F must be cyclic or the product of $Q(8k)$, $T(v)$ or $A(m, e)$ with a cyclic group of relatively prime order, as it is the kernel of a map from G to $Z/2Z$. It is easily verified that in all such cases every automorphism of F is the restriction of automorphisms of G and H. Hence π is determined up to isomorphism as the amalgamated free product $G *_F H$ by the unordered pair $\{G, H\}$ of groups with index 2 subgroups isomorphic to F (i.e., it is unnecessary to specify the identifications of F with these subgroups).

The final assertion follows because each of the spaces $M_G = \widetilde{M}/G$ and $M_H = \widetilde{M}/H$ are PD_3-complexes with finite fundamental group and therefore are orientable, and π is generated by G and H. ∎

Must the spaces M_G and M_H be homotopy equivalent to finite complexes?

11.4 $\mathbb{S}^3 \times \mathbb{E}^1$-manifolds

With the exception of O_k^* (with $k > 1$), $A(m, 1)$ and $Q(2^n a, b, c)$ (with either $n = 3$ and at most one of a, b and c is 1 or $n > 3$ and $bc > 1$) and their products with cyclic groups, all of the groups listed in §2 have fixed point free representations in $SO(4)$ and so act linearly on S^3. (Cyclic groups, the binary dihedral groups $D_{4m}^* = A(m, 2)$, with m odd, and $D_{8k}^* = Q(8k, 1, 1)$, with $k \geq 1$ and the three binary polyhedral groups T_1^*, O_1^* and I^* are subgroups of S^3.) We shall call such groups $\mathbb{S}^3$-*groups*. If F is cyclic then every Swan complex for F is homotopy equivalent to a lens space. If $F = Q(2^k)$ or T_k^* for some $k > 1$ then S^3/F is the unique finite Swan complex for F [Th80]. For the other noncyclic $\mathbb{S}^3$-groups the corresponding $\mathbb{S}^3$-manifold is unique, but in general there may be other finite Swan complexes. (In particular, there are exotic finite Swan complexes for T_1.)

Let N be a $\mathbb{S}^3$-manifold with $\pi_1(N) = F$. Then the projection of $Isom(N)$ onto its group of path components splits, and the inclusion of $Isom(N)$ into $Diff(N)$ induces an isomorphism on path components. Moreover if $|F| > 2$ then an isometry which induces the identity outer automorphism is isotopic to the identity, and so $\pi_0(Isom(M))$ maps injectively to $Out(F)$. (See [Mc02].)

Theorem 11.6 *Let M be a closed 4-manifold with $\chi(M) = 0$ and $\pi = \pi_1(M) \cong F \times_\theta Z$, where F is finite. Then M is homeomorphic to a $\mathbb{S}^3 \times \mathbb{E}^1$-manifold if and only if M is the mapping torus of a self homeomorphism of a $\mathbb{S}^3$-manifold with fundamental group F, and such manifolds are determined up to homeomorphism by their homotopy type.*

Proof Let p_1 and p_2 be the projections of $Isom(\mathbb{S}^3 \times \mathbb{E}^1) = O(4) \times E(1)$ onto $O(4)$ and $E(1)$ respectively. If π is a discrete subgroup of $Isom(\mathbb{S}^3 \times \mathbb{E}^1)$ which acts freely on $S^3 \times R$ then p_1 maps F monomorphically and $p_1(F)$ acts freely on S^3, since every isometry of R of finite order has nonempty fixed point set. Moreover $p_2(\pi)$ is a discrete subgroup of $E(1)$ which acts cocompactly on R, and so has no nontrivial finite normal subgroup. Hence $F = \pi \cap (O(4) \times \{1\})$. If $\pi/F \cong Z$ and $t \in \pi$ represents a generator of π/F then conjugation by t induces an isometry θ of S^3/F, and $M \cong M(\theta)$. Conversely any self homeomorphism of a $\mathbb{S}^3$-manifold is isotopic to an isometry of finite order, and so the mapping torus is homeomorphic to a $\mathbb{S}^3 \times \mathbb{E}^1$-manifold. The final assertion follows from Theorem 3 of [Oh90]. □

If s is an integer such that $s^2 \equiv \pm 1$ modulo (d) then there is an isometry of the lens space $L(d, s)$ inducing multiplication by s, and the mapping torus has fundamental group $(Z/dZ) \times_s Z$. (This group may also be realized by mapping tori of self homotopy equivalences of other lens spaces.) If $d > 2$ a closed 4-manifold with this group and with Euler characteristic 0 is orientable if and only if $s^2 \equiv 1\,(d)$.

If F is a noncyclic $\mathbb{S}^3$-group there is a unique linear k-invariant, and so for each $\theta \in Aut(F)$ there is at most one homeomorphism class of $\mathbb{S}^3 \times \mathbb{E}^1$-manifolds with fundamental group $\pi = F \times_\theta Z$. Every class in $J(F)$ is realizable by an orientation preserving isometry of S^3/F, if $F = Q(8)$, T_k^*, O_1^*, I^*, $A(p^i, e)$, $Q(8) \times Z/q^jZ$ or $A(p^i, 2) \times Z/q^jZ$, where p and q are odd primes and $e > 1$. For the other $\mathbb{S}^3$-groups the subgroup of $J(F)$ realizable by homeomorphisms of S^3/F is usually quite small. (See [Mc02].)

Suppose now that G and H are $\mathbb{S}^3$-groups with index 2 subgroups isomorphic to F. If F, G and H are each noncyclic then the corresponding $\mathbb{S}^3$-manifolds are uniquely determined, and we may construct a nonorientable $\mathbb{S}^3 \times \mathbb{E}^1$-manifold with fundamental group $\pi = G *_F H$ as follows. Let u and $v : S^3/F \to S^3/F$ be the covering involutions with quotient spaces S^3/G and S^3/H, respectively, and let $\phi = uv$. (Note that u and v are isometries of S^3/F.) Then $U([x, t]) = [u(x), 1 - t]$ defines a fixed point free involution on the mapping torus $M(\phi)$ and the quotient space has fundamental group π. A similar construction works if F is cyclic and $G \cong H$ or if G is cyclic.

11.5 Realization of the groups

Let F be a finite group with cohomological period dividing 4, and let X_F denote a finite Swan complex for F. If θ is an automorphism of F which induces ± 1 on $H_3(F; \mathbb{Z})$ there is a self homotopy equivalence h of X_F which induces $[\theta] \in J(F)$. The mapping torus $M(h)$ is a finite PD_4-complex with $\pi_1(M) \cong F \times_\theta Z$ and $\chi(M(h)) = 0$. Conversely, every PD_4-complex M with $\chi(M) = 0$ and such that $\pi_1(M)$ is an extension of Z by a finite normal subgroup F is homotopy equivalent to such a mapping torus. Moreover, if $\pi \cong F \times Z$ and $|F| > 2$ then h is homotopic to the identity and so $M(h)$ is homotopy equivalent to $X_F \times S^1$.

Since every PD_n-complex may be obtained by attaching an n-cell to a complex which is homologically of dimension $< n$, the exotic characteristic class of the Spivak normal fibration of a PD_3-complex X in $H^3(X; \mathbb{F}_2)$ is trivial. Hence every 3-dimensional Swan complex X_F has a TOP reduction, i.e., there are normal maps $(f, b) : N^3 \to X_F$. Such a map has a "proper surgery" obstruction $\lambda^p(f, b)$ in $L_3^p(F)$, which is 0 if and only if $(f, b) \times id_{S^1}$ is normally cobordant to a simple homotopy equivalence. In particular, a surgery semicharacteristic must be 0. Hence all subgroups of F of order $2p$ (with p prime) are cyclic, and $Q(2^n a, b, c)$ (with $n > 3$ and b or $c > 1$) cannot occur [HM86]. As the $2p$ condition excludes groups with subgroups isomorphic to $A(m, 1)$ (with $m > 1$) the cases remaining to be decided are when $F \cong Q(8a, b, c) \times Z/dZ$, where a, b and c are odd and at most one of them is 1. The main result of [HM86] is that in such a case $F \times Z$ acts freely and properly "with almost linear k-invariant" if and only if some arithmetical conditions depending on subgroups of F of the form $Q(8a,b,1)$ hold. (Here "almost linear" means that all covering spaces corresponding to subgroups isomorphic to $A(m,e)\times Z/dZ$ or $Q(8k)\times Z/dZ$ must be homotopy equivalent to $\mathbb{S}^3$-manifolds. The constructive part of the argument may be extended to the 4-dimensional case by reference to [FQ].)

The following more direct argument for the existence of a free proper action of $F\times Z$ on $S^3\times R$ was outlined in [KS88], for the cases when F acts freely on an homology 3-sphere Σ. Let Σ and its universal covering space $\widetilde{\Sigma}$ have equivariant cellular decompositions lifted from a cellular decomposition of Σ/F, and let $\Pi=\pi_1(\Sigma/F)$. Then $C_*(\Sigma)=\mathbb{Z}[F]\otimes_\Pi C_*(\widetilde{\Sigma})$ is a finitely generated free $\mathbb{Z}[F]$-complex, and may be realized by a finite Swan complex X. The chain map (over the epimorphism : $\Pi\to F$) from $C_*(\widetilde{\Sigma})$ to $C_*(\widetilde{X})$ may be realized by a map $h:\Sigma/F\to X$, since these spaces are 3-dimensional. As $h\times id_{S^1}$ is a simple $\mathbb{Z}[F\times Z]$-homology equivalence it has surgery obstruction 0 in $L_4^s(F\times Z)$, and so is normally cobordant to a simple homotopy equivalence. For example, the group $Q(24,313,1)$ acts freely on an homology 3-sphere (see §6 of [DM85]). Is there an explicit action on some Brieskorn homology 3-sphere? Is $Q(24,313,1)$ a 3-manifold group? (This seems unlikely.)

Although $Q(24,13,1)$ cannot act freely on any homology 3-sphere [DM85], there is a closed orientable 4-manifold with fundamental group $Q(24,13,1)\times Z$, by the argument of [HM86]. No such 4-manifold can fibre over S^1, since $Q(24,13,1)$ is not a 3-manifold group. Thus such a manifold is a counter example to a 4-dimensional analogue of the Farrell fibration theorem (of a different kind from that of [We87]), and is not geometric.

If $F=T_k^*$, $Q(8k)$ or $A(m,2)$ then $F\times Z$ can only act freely and properly on $R^4\backslash\{0\}$ with the k-invariant corresponding to the free linear action of F on S^3. (For the group $A(m,2)$, this follows from Corollary C of [HM86'], which also implies that the restriction of the k-invariant to $A(k,r+1)$ and hence to the odd-Sylow subgroup of $Q(2^nk)$ is linear. The nonlinear k-invariants for $Q(2^n)$ have nonzero finiteness obstruction. As the k-invariants of free linear representations of $Q(2^nk)$ are given by elements in $H^4(Q(2^nk);\mathbb{Z})$ whose restrictions to Z/kZ are squares and whose restrictions to $Q(2^n)$ are squares times the basic generator (see page 120 of [Wl78], only the linear k-invariant is realizable in this case also). However in general it is not known which k-invariants are realizable. Every group of the form $Q(8a,b,c)\times Z/dZ\times Z$ admits an "almost linear" k-invariant, but there may be other actions. (See [HM86, 86'] for more on this issue.)

In considering the realization of more general extensions of Z or D by finite normal subgroups the following question seems central. If M is a closed 4-manifold with $\pi=\pi_1(M)\cong(Z/dZ)\times_s Z$ where $s^2\equiv 1$ but $s\not\equiv\pm1\,(d)$

and $\chi(M) = 0$ is M homotopy equivalent to the $\mathbb{S}^3 \times \mathbb{E}^1$-manifold with this fundamental group? Since M is homotopy equivalent to the mapping torus of a self homotopy equivalence $[s] : L(d,r) \to L(d,r)$ (for some r determined by $k(M)$), it would suffice to show that if $r \neq \pm s$ or $\pm s^{-1}$ the Whitehead torsion of the duality homomorphism of $M([s])$ is nonzero. Proposition 4.1 of [Rn86] gives a formula for the Whitehead torsion of such mapping tori. Unfortunately the associated Reidemeister-Franz torsion appears to be 0 in all cases. For other groups F can one use the fact that a closed 4-manifold is a *simple* PD_4-complex to bound the realizable subgroup of $J(F)$?

A positive answer to this question would enhance the interest of the following subsidiary question. If F is a noncyclic $\mathbb{S}^3$-group must an automorphism of F whose restrictions to (characteristic) cyclic subgroups $C < F$ are realized by isometries of the corresponding covering spaces of S^3/F be realized by an isometry of S^3/F? (In particular, is this so for $F = Q(2^t)$ or $A(m,2)$ with m composite?.)

If F is cyclic but neither G nor H is cyclic there may be no geometric manifold with fundamental group $\pi = G *_F H$. If the double covers of $G\backslash S^3$ and $H\backslash S^3$ are homotopy equivalent then π is realised by the union of two twisted I-bundles via a homotopy equivalence, which is a finite (but possibly nonsimple?) PD_4-complex with $\chi = 0$. For instance, the spherical space forms corresponding to $G = Q(40)$ and $H = Q(8) \times (Z/5Z)$ are doubly covered by forms doubly covered by $L(20,1)$ and $L(20,9)$, respectively, which are homotopy equivalent but not homeomorphic. The spherical space forms corresponding to $G = Q(24)$ and $H = Q(8) \times (Z/3Z)$ are doubly covered by $L(12,1)$ and $L(12,5)$, respectively, which are not homotopy equivalent.

11.6 T- and Kb-bundles over RP^2 with $\partial \neq 0$

Let $p : E \to RP^2$ be a bundle with fibre T or Kb. Then $\pi = \pi_1(E)$ is an extension of $Z/2Z$ by $G/\partial Z$, where G is the fundamental group of the fibre and ∂ is the connecting homomorphism. If $\partial \neq 0$ then π has two ends, F is cyclic and central in $G/\partial Z$ and π acts on it by inversion, since π acts nontrivially on $Z = \pi_2(RP^2)$.

If the fibre is T then π has a presentation of the form

$$\langle t, u, v \mid uv = vu,\ u^n = 1,\ tut^{-1} = u^{-1},\ tvt^{-1} = u^a v^\epsilon,\ t^2 = u^b v^c\rangle,$$

where $n > 0$ and $\epsilon = \pm 1$. Either

(1) F is cyclic, $\pi \cong (Z/nZ) \times_{-1} Z$ and $\pi/F \cong Z$; or

(2) $F = \langle s, u \mid s^2 = u^m,\ sus^{-1} = u^{-1}\rangle$; or (if $\epsilon = -1$)

(3) F is cyclic, $\pi = \langle s, t, u \mid s^2 = t^2 = u^b,\ sus^{-1} = tut^{-1} = u^{-1}\rangle$ and $\pi/F \cong D$.

In case (2) F cannot be dihedral. If m is odd $F \cong A(m, 2)$ while if $m = 2^r k$ with $r \geq 1$ and k odd $F \cong Q(2^{r+2}k)$. On replacing v by $u^{[a/2]}v$, if necessary, we may arrange that $a = 0$, in which case $\pi \cong F \times Z$, or $a = 1$, in which case

$$\pi = \langle t, u, v \mid t^2 = u^m,\ tut^{-1} = u^{-1},\ vtv^{-1} = tu,\ uv = vu\rangle,$$

so $\pi/F \cong Z$.

If the fibre is Kb then π has a presentation of the form

$$\langle t, u, w \mid uwu^{-1} = w^{-1},\ u^n = 1,\ tut^{-1} = u^{-1},\ twt^{-1} = u^a w^\epsilon,\ t^2 = u^b w^c\rangle,$$

where $n > 0$ is even (since $\mathrm{Im}(\partial) \leq \zeta\pi_1(Kb)$) and $\epsilon = \pm 1$. On replacing t by ut, if necessary, we may assume that $\epsilon = 1$. Moreover, $tw^2t^{-1} = w^{\pm 2}$ since w^2 generates the commutator subgroup of $G/\partial Z$, so a is even and $2a \equiv 0\ mod\ (n)$, $t^2u = ut^2$ implies that $c = 0$, and $t.t^2.t^{-1} = t^2$ implies that $2b \equiv 0\ mod\ (n)$. As F is generated by t and u^2, and cannot be dihedral, we must have $n = 2b$. Moreover b must be even, as w has infinite order and $t^2w = wt^2$. Therefore

(4) $F \cong Q(8k)$, $\pi/F \cong D$ and

$\pi = \langle t, u, w \mid uwu^{-1} = w^{-1},\ tut^{-1} = u^{-1},\ tw = u^a wt,\ t^2 = u^{2k}\rangle$.

In all cases π has a subgroup of index at most 2 which is isomorphic to $F \times Z$.

Each of these groups is the fundamental group of such a bundle space. (This may be seen by using the description of such bundle spaces given in §5 of Chapter 5.) Orientable 4-manifolds which fibre over RP^2 with fibre T and $\partial \neq 0$ are mapping tori of involutions of $\mathbb{S}^3$-manifolds, and if F is not cyclic two such bundle spaces with the same group are diffeomorphic [Ue91].

Theorem 11.7 *Let M be a closed orientable 4-manifold with fundamental group π. Then M is homotopy equivalent to an $\mathbb{S}^3 \times \mathbb{E}^1$-manifold which fibres over RP^2 if and only $\chi(M) = 0$ and π is of type (1) or (2) above.*

Proof If M is an orientable $\mathbb{S}^3 \times \mathbb{E}^1$-manifold then $\chi(M) = 0$ and $\pi/F \cong Z$, by Theorem 11.1 and Lemma 3.14. Moreover π must be of type (1) or (2) if M fibres over RP^2, and so the conditions are necessary.

Suppose that they hold. Then $\widetilde{M} \cong R^4\backslash\{0\}$ and the homotopy type of M is determined by π and $k(M)$, by Theorem 11.1. If $F \cong Z/nZ$ then $M_F = \widetilde{M}/F$

is homotopy equivalent to some lens space $L(n,s)$. As the involution of Z/nZ which inverts a generator can be realized by an isometry of $L(n,s)$, M is homotopy equivalent to an $\mathbb{S}^3 \times \mathbb{E}^1$-manifold which fibres over S^1.

If $F \cong Q(2^{r+2}k)$ or $A(m,2)$ then $F \times Z$ can only act freely and properly on $R^4 \setminus \{0\}$ with the "linear" k-invariant [HM86]. Therefore M_F is homotopy equivalent to a spherical space form S^3/F. The class in $Out(Q(2^{r+2}k))$ represented by the automorphism which sends the generator t to tu and fixes u is induced by conjugation in $Q(2^{r+3}k)$ and so can be realized by a (fixed point free) isometry θ of $S^3/Q(2^{r+2}k)$. Hence M is homotopy equivalent to a bundle space $(S^3/Q(2^{r+2}k)) \times S^1$ or $(S^3/Q(2^{r+2}k)) \times_\theta S^1$ if $F \cong Q(2^{r+2}k)$. A similar conclusion holds when $F \cong A(m,2)$ as the corresponding automorphism is induced by conjugation in $Q(2^3 d)$.

With the results of [Ue91] it follows in all cases that M is homotopy equivalent to the total space of a torus bundle over RP^2. □

Theorem 11.7 makes no assumption that there be a homomorphism $u : \pi \to Z/2Z$ such that $u^*(x)^3 = 0$ (as in §5 of Chapter 5). If F is cyclic or $A(m,2)$ this condition is a purely algebraic consequence of the other hypotheses. For let C be a cyclic normal subgroup of maximal order in F. (There is an unique such subgroup, except when $F = Q(8)$.) The centralizer $C_\pi(C)$ has index 2 in π and so there is a homomorphism $u : \pi \to Z/2Z$ with kernel $C_\pi(C)$.

When F is cyclic u factors through Z and so the induced map on cohomology factors through $H^3(Z;\tilde{Z}) = 0$.

When $F \cong A(m,2)$ the 2-Sylow subgroup is cyclic of order 4, and the inclusion of $Z/4Z$ into τ induces isomorphisms on cohomology with 2-local coefficients. In particular, $H^q(F;\tilde{Z}_{(2)}) = 0$ or $Z/2Z$ according as q is even or odd. It follows easily that the restriction from $H^3(\pi;\tilde{Z}_{(2)})$ to $H^3(Z/4Z;\tilde{Z}_{(2)})$ is an isomorphism. Let y be the image of $u^*(x)$ in $H^1(Z/4Z;\tilde{Z}_{(2)}) = Z/2Z$. Then y^2 is an element of order 2 in $H^2(Z/4Z;\tilde{Z}_{(2)} \otimes \tilde{Z}_{(2)}) = H^2(Z/4Z;Z_{(2)}) \cong Z/4Z$, and so $y^2 = 2z$ for some $z \in H^2(Z/4Z;Z_{(2)})$. But then $y^3 = 2yz = 0$ in $H^3(Z/4Z;\tilde{Z}_{(2)}) = Z/2Z$, and so $u^*(x)^3$ has image 0 in $H^3(\pi;\tilde{Z}_{(2)}) = Z/2Z$. Since x is a 2-torsion class this implies that $u^*(x)^3 = 0$.

Is there a similar argument when F is a generalized quaternionic group?

If M is nonorientable, $\chi(M) = 0$ and has fundamental group π of type (1) or (2) then M is homotopy equivalent to the mapping torus of the orientation reversing self homeomorphism of S^3 or of RP^3, and does not fibre over RP^2.

If π is of type (3) or (4) then the 2-fold covering space with fundamental group $F \times Z$ is homotopy equivalent to a product $L(n,s) \times S^1$. However we do not know which k-invariants give total spaces of bundles over RP^2.

11.7 Some remarks on the homeomorphism types

In this brief section we shall assume that M is orientable and that $\pi \cong F \times_\theta Z$. In contrast to the situation for the other geometries, the Whitehead groups of fundamental groups of $\mathbb{S}^3 \times \mathbb{E}^1$-manifolds are usually nontrivial. Computation of $Wh(\pi)$ is difficult as the Nil groups occuring in the Waldhausen exact sequence relating $Wh(\pi)$ to the algebraic K-theory of F seem intractable.

We can however compute the relevant surgery obstruction groups modulo 2-torsion and show that the structure sets are usually infinite. There is a Mayer-Vietoris sequence $L_5^s(F) \to L_5^s(\pi) \to L_4^u(F) \to L_4^s(F)$, where the superscript u signifies that the torsion must lie in a certain subgroup of $Wh(F)$ [Ca73]. The right hand map is (essentially) $\theta_* - 1$. Now $L_5^s(F)$ is a finite 2-group and $L_4^u(F) \sim L_4^s(F) \sim Z^R$ modulo 2-torsion, where R is the set of irreducible real representations of F (see Chapter 13A of [Wl]). The latter correspond to the conjugacy classes of F, up to inversion. (See §12.4 of [Se].) In particular, if $\pi \cong F \times Z$ then $L_5^s(\pi) \sim Z^R$ modulo 2-torsion, and so has rank at least 2 if $F \neq 1$. As $[\Sigma M, G/TOP] \cong Z$ modulo 2-torsion and the group of self homotopy equivalences of such a manifold is finite, by Theorem 11.3, there are infinitely many distinct topological 4-manifolds simple homotopy equivalent to M. For instance, as $Wh(Z \oplus (Z/2Z)) = 0$ [Kw86] and $L_5(Z \oplus (Z/2Z), +) \cong Z^2$, by Theorem 13A.8 of [Wl], the set $S_{TOP}(RP^3 \times S^1)$ is infinite. Although all of the manifolds in this homotopy type are doubly covered by $S^3 \times S^1$ only $RP^3 \times S^1$ is itself geometric. Similar estimates hold for the other manifolds covered by $S^3 \times R$ (if $\pi \neq Z$).

Chapter 12

Geometries with compact models

There are three geometries with compact models, namely $\mathbb{S}^4$, $\mathbb{CP}^2$ and $\mathbb{S}^2 \times \mathbb{S}^2$. The first two of these are easily dealt with, as there is only one other geometric manifold, namely RP^4, and for each of the two projective spaces there is one other (nonsmoothable) manifold of the same homotopy type. With the geometry $\mathbb{S}^2 \times \mathbb{S}^2$ we shall consider also the bundle space $S^2 \tilde{\times} S^2$. There are eight $\mathbb{S}^2 \times \mathbb{S}^2$-manifolds, seven of which are total spaces of bundles with base and fibre each S^2 or RP^2, and there are two other such bundle spaces covered by $S^2 \tilde{\times} S^2$.

The universal covering space $\widetilde{M}$ of a closed 4-manifold M is homeomorphic to $S^2 \times S^2$ if and only if $\pi = \pi_1(M)$ is finite, $\chi(M)|\pi| = 4$ and $w_2(\widetilde{M}) = 0$. (The condition $w_2(\widetilde{M}) = 0$ may be restated entirely in terms of M, but at somewhat greater length.) If these conditions hold and π is cyclic then M is homotopy equivalent to an $\mathbb{S}^2 \times \mathbb{S}^2$-manifold, except when $\pi = Z/2Z$ and M is nonorientable, in which case there is one other homotopy type. The $\mathbb{F}_2$-cohomology ring, Stiefel-Whitney classes and k-invariants must agree with those of bundle spaces when $\pi \cong (Z/2Z)^2$. However there remains an ambiguity of order at most 4 in determining the homotopy type. If $\chi(M)|\pi| = 4$ and $w_2(\widetilde{M}) \neq 0$ then either $\pi = 1$, in which case $M \simeq S^2 \tilde{\times} S^2$ or $CP^2 \sharp CP^2$, or M is nonorientable and $\pi = Z/2Z$; in the latter case $M \simeq RP^4 \sharp CP^2$, the nontrivial RP^2-bundle over S^2, and $\widetilde{M} \simeq S^2 \tilde{\times} S^2$.

The number of homeomorphism classes within each homotopy type is at most two if $\pi = Z/2Z$ and M is orientable, two if $\pi = Z/2Z$, M is nonorientable and $w_2(\widetilde{M}) = 0$, four if $\pi = Z/2Z$ and $w_2(\widetilde{M}) \neq 0$, at most four if $\pi \cong Z/4Z$, and at most eight if $\pi \cong (Z/2Z)^2$. We do not know whether there are enough exotic self homotopy equivalences to account for all the normal invariants with trivial surgery obstruction. However a PL 4-manifold with the same homotopy type as a geometric manifold or $S^2 \tilde{\times} S^2$ is homeomorphic to it, in (at least) nine of the 13 cases. (In seven of these cases the homotopy type is determined by the Euler characteristic, fundamental group and Stiefel-Whitney classes.)

For the full details of some of the arguments in the cases $\pi \cong Z/2Z$ we refer to the papers [KKR92], [HKT94] and [Te95].

12.1 The geometries $\mathbb{S}^4$ and $\mathbb{CP}^2$

The unique element of $Isom(\mathbb{S}^4) = O(5)$ of order 2 which acts freely on S^4 is $-I$. Therefore S^4 and RP^4 are the only $\mathbb{S}^4$-manifolds. The manifold S^4 is determined up to homeomorphism by the conditions $\chi(S^4) = 2$ and $\pi_1(S^4) = 1$ [FQ].

Lemma 12.1 *A closed 4-manifold M is homotopy equivalent to RP^4 if and only if $\chi(M) = 1$ and $\pi_1(M) = Z/2Z$.*

Proof The conditions are clearly necessary. Suppose that they hold. Then $\widetilde{M} \simeq S^4$ and $w_1(M) = w_1(RP^4) = w$, say, since any orientation preserving self homeomorphism of $\widetilde{M}$ has Lefshetz number 2. Since $RP^\infty = K(Z/2Z, 1)$ may be obtained from RP^4 by adjoining cells of dimension at least 5 we may assume $c_M = c_{RP^4} f$, where $f : M \to RP^4$. Since c_{RP^4} and c_M are each 4-connected f induces isomorphisms on homology with coefficients $Z/2Z$. Considering the exact sequence of homology corresponding to the short exact sequence of coefficients

$$0 \to Z^w \to Z^w \to Z/2Z \to 0,$$

we see that f has odd degree. By modifying f on a 4-cell $D^4 \subset M$ we may arrange that f has degree 1, and the lemma then follows from Theorem 3.2. □

This lemma may also be proven by comparison of the k-invariants of M and RP^4, as in Theorem 4.3 of [Wl67].

By Theorems 13.A.1 and 13.B.5 of [Wl] the surgery obstruction homomorphism is determined by an Arf invariant and maps $[RP^4; G/TOP]$ onto $Z/2Z$, and hence the structure set $S_{TOP}(RP^4)$ has two elements. (See the discussion of nonorientable manifolds with fundamental group $Z/2Z$ in Section 6 below for more details.) As every self homotopy equivalence of RP^4 is homotopic to the identity [Ol53] there is one fake RP^4. The fake RP^4 is denoted $*RP^4$ and is not smoothable [Ru84].

There is a similar characterization of the homotopy type of the complex projective plane.

Lemma 12.2 *A closed 4-manifold M is homotopy equivalent to CP^2 if and only if $\chi(M) = 3$ and $\pi_1(M) = 1$.*

Proof The conditions are clearly necessary. Suppose that they hold. Then $H^2(M;\mathbb{Z})$ is infinite cyclic and so there is a map $f_M : M \to CP^\infty = K(Z,2)$ which induces an isomorphism on H^2. Since CP^∞ may be obtained from CP^2 by adjoining cells of dimension at least 6 we may assume $f_M = f_{CP^2}g$, where $g : M \to CP^2$ and $f_{CP^2} : CP^2 \to CP^\infty$ is the natural inclusion. As $H^4(M;\mathbb{Z})$ is generated by $H^2(M;\mathbb{Z})$, by Poincaré duality, g induces an isomorphism on cohomology and hence is a homotopy equivalence. □

In this case the surgery obstruction homomorphism is determined by the difference of signatures and maps $[CP^2; G/TOP]$ onto Z. The structure set $S_{TOP}(CP^2)$ again has two elements. Since $[CP^2, CP^2] \cong [CP^2, CP^\infty] \cong H^2(CP^2;\mathbb{Z})$, by obstruction theory, there are two homotopy classes of self homotopy equivalences, represented by the identity and by complex conjugation. Thus every self homotopy equivalence of CP^2 is homotopic to a homeomorphism, and so there is one fake CP^2. The fake CP^2 is also known as the Chern manifold Ch or $*CP^2$, and is not smoothable [FQ]. Neither of these manifolds admits a nontrivial fixed point free action, as any self map of CP^2 or $*CP^2$ has nonzero Lefshetz number, and so CP^2 is the only $\mathbb{CP}^2$-manifold.

12.2 The geometry $\mathbb{S}^2 \times \mathbb{S}^2$

The manifold $S^2 \times S^2$ is determined up to homotopy equivalence by the conditions $\chi(S^2 \times S^2) = 4$, $\pi_1(S^2 \times S^2) = 1$ and $w_2(S^2 \times S^2) = 0$, by Theorem 5.19. These conditions in fact determine $S^2 \times S^2$ up to homeomorphism [FQ]. Hence if M is an $\mathbb{S}^2 \times \mathbb{S}^2$-manifold its fundamental group π is finite, $\chi(M)|\pi| = 4$ and $w_2(\widetilde{M}) = 0$.

The isometry group of $\mathbb{S}^2 \times \mathbb{S}^2$ is a semidirect product $(O(3) \times O(3))\tilde{\times}(Z/2Z)$. The $Z/2Z$ subgroup is generated by the involution τ which switches the factors ($\tau(x,y) = (y,x)$), and acts on $O(3) \times O(3)$ by $\tau(A,B)\tau = (B,A)$ for $A,B \in O(3)$. In particular, $(\tau(A,B))^2 = id$ if and only if $AB = I$, and so such an involution fixes (x, Ax), for any $x \in S^2$. Thus there are no free $Z/2Z$-actions in which the factors are switched. The element (A,B) generates a free $Z/2Z$-action if and only if $A^2 = B^2 = I$ and at least one of A, B acts freely, i.e. if A or $B = -I$. After conjugation with τ if necessary we may assume that $B = -I$, and so there are four conjugacy classes in $Isom(\mathbb{S}^2 \times \mathbb{S}^2)$ of free $Z/2Z$-actions. (The conjugacy classes may be distinguished by the multiplicity (0, 1, 2 or 3) of 1 as an eigenvalue of A.) In each case the projection onto the second factor gives rise to a fibre bundle projection from the orbit space to RP^2, with fibre S^2.

If the involutions (A, B) and (C, D) generate a free $(Z/2Z)^2$-action (AC, BD) is also a free involution. By the above paragraph, one element of each of these ordered pairs must be $-I$. It follows easily that (after conjugation with τ if necessary) the $(Z/2Z)^2$-actions are generated by pairs $(A, -I)$ and $(-I, I)$, where $A^2 = I$. Since A and $-A$ give rise to the same subgroup, there are two free $(Z/2Z)^2$-actions. The orbit spaces are the total spaces of RP^2-bundles over RP^2.

If $(\tau(A, B))^4 = id$ then (BA, AB) is a fixed point free involution and so $BA = AB = -I$. Since $(A, I)\tau(A, -A^{-1})(A, I)^{-1} = \tau(I, -I)$ every free $Z/4Z$-action is conjugate to the one generated by $\tau(I, -I)$. The orbit space does not fibre over a surface. (See below.)

In the next section we shall see that these eight geometric manifolds may be distinguished by their fundamental group and Stiefel-Whitney classes. Note that if F is a finite group then $q(F) \geq 2/|F| > 0$, while $q^{SG}(F) \geq 2$. Thus S^4, RP^4 and the geometric manifolds with $|\pi| = 4$ have minimal Euler characteristic for their fundamental groups (i.e., $\chi(M) = q(\pi)$), while $S^2 \times S^2/(-I, -I)$ has minimal Euler characteristic among PD_4^+-complexes realizing $Z/2Z$.

12.3 Bundle spaces

There are two S^2-bundles over S^2, since $\pi_1(SO(3)) = Z/2Z$. The total space $S^2\tilde{\times}S^2$ of the nontrivial S^2-bundle over S^2 is determined up to homotopy equivalence by the conditions $\chi(S^2\tilde{\times}S^2) = 4$, $\pi_1(S^2\tilde{\times}S^2) = 1$, $w_2(S^2\tilde{\times}S^2) \neq 0$ and $\sigma(S^2\tilde{\times}S^2) = 0$, by Theorem 5.19. However there is one fake $S^2\tilde{\times}S^2$. The bundle space is homeomorphic to the connected sum $CP^2\sharp - CP^2$, while the fake version is homeomorphic to $CP^2\sharp - *CP^2$ and is not smoothable [FQ]. The manifolds $CP^2\sharp CP^2$ and $CP^2\sharp * CP^2$ also have $\pi_1 = 0$ and $\chi = 4$. However it is easily seen that any self homotopy equivalence of either of these manifolds has nonzero Lefshetz number, and so they do not properly cover any other 4-manifold.

Since the Kirby-Siebenmann obstruction of a closed 4-manifold is natural with respect to covering maps and dies on passage to 2-fold coverings, the nonsmoothable manifold $CP^2\sharp - *CP^2$ admits no nontrivial free involution. The following lemma implies that $S^2\tilde{\times}S^2$ admits no orientation preserving free involution, and hence no free action of $Z/4Z$ or $(Z/2Z)^2$.

Lemma 12.3 *Let M be a closed 4-manifold with fundamental group $\pi = Z/2Z$ and universal covering space $\widetilde{M}$. Then*

(1) $w_2(\widetilde{M}) = 0$ *if and only if* $w_2(M) = u^2$ *for some* $u \in H^1(M; \mathbb{F}_2)$*; and*

(2) *if* M *is orientable and* $\chi(M) = 2$ *then* $w_2(\widetilde{M}) = 0$ *and so* $\widetilde{M} \cong S^2 \times S^2$*.*

Proof The Cartan-Leray cohomology spectral sequence (with coefficients $\mathbb{F}_2$) for the projection $p : \widetilde{M} \to M$ gives an exact sequence

$$0 \to H^2(\pi; \mathbb{F}_2) \to H^2(M; \mathbb{F}_2) \to H^2(\widetilde{M}; \mathbb{F}_2),$$

in which the right hand map is induced by p and has image in the subgroup fixed under the action of π. Hence $w_2(\widetilde{M}) = p^* w_2(M)$ is 0 if and only if $w_2(M)$ is in the image of $H^2(\pi; \mathbb{F}_2)$. Since $\pi = Z/2Z$ this is so if and only if $w_2(M) = u^2$ for some $u \in H^1(M; \mathbb{F}_2)$.

Suppose that M is orientable and $\chi(M) = 2$. Then $H^2(\pi; \mathbb{Z}) = H^2(M; \mathbb{Z}) = Z/2Z$. Let x generate $H^2(M; \mathbb{Z})$ and let $\bar{x}$ be its image under reduction modulo (2) in $H^2(M; \mathbb{F}_2)$. Then $\bar{x} \cup \bar{x} = 0$ in $H^4(M; \mathbb{F}_2)$ since $x \cup x = 0$ in $H^4(M; \mathbb{Z})$. Moreover as M is orientable $w_2(M) = v_2(M)$ and so $w_2(M) \cup \bar{x} = \bar{x} \cup \bar{x} = 0$. Since the cup product pairing on $H^2(M; \mathbb{F}_2) \cong (Z/2Z)^2$ is nondegenerate it follows that $w_2(M) = \bar{x}$ or 0. Hence $w_2(\widetilde{M})$ is the reduction of $p^* x$ or is 0. The integral analogue of the above exact sequence implies that the natural map from $H^2(\pi; \mathbb{Z})$ to $H^2(M; \mathbb{Z})$ is an isomorphism and so $p^*(H^2(M; \mathbb{Z})) = 0$. Hence $w_2(\widetilde{M}) = 0$ and so $\widetilde{M} \cong S^2 \times S^2$. □

Since $\pi_1(BO(3)) = Z/2Z$ there are two S^2-bundles over the Möbius band Mb and each restricts to a trivial bundle over ∂Mb. Moreover a map from ∂Mb to $O(3)$ extends across Mb if and only if it homotopic to a constant map, since $\pi_1(O(3)) = Z/2Z$, and so there are four S^2-bundles over $RP^2 = Mb \cup D^2$. (See also Theorem 5.10.)

The orbit space $M = (S^2 \times S^2)/(A, -I)$ is orientable if and only if $det(A) = -1$. If A has a fixed point $P \in S^2$ then the image of $\{P\} \times S^2$ in M is an embedded projective plane which represents a nonzero class in $H_2(M; \mathbb{F}_2)$. If $A = I$ or is a reflection across a plane the fixed point set has dimension > 0 and so this projective plane has self intersection 0. As the fibre S^2 intersects this projective plane in one point and has self intersection 0 it follows that $v_2(M) = 0$ and so $w_2(M) = w_1(M)^2$ in these two cases. If A is a rotation about an axis then the projective plane has self intersection 1, by Lemma 10.14. Finally, if $A = -I$ then the image of the diagonal $\{(x, x) | x \in S^2\}$ is a projective plane in M with self intersection 1. Thus in these two cases $v_2(M) \neq 0$. Therefore, by part (1) of the lemma, $w_2(M)$ is the square of the nonzero element of $H^1(M; \mathbb{F}_2)$ if $A = -I$ and is 0 if A is a rotation. Thus these bundle spaces may be distinguished by their Stiefel-Whitney classes, and every S^2-bundle over RP^2 is geometric.

The group $E(RP^2)$ of self homotopy equivalences of RP^2 is connected and the natural map from $SO(3)$ to $E(RP^2)$ induces an isomorphism on π_1, by Lemma 5.15. Hence there are two RP^2-bundles over S^2, up to fibre homotopy equivalence. The total space of the nontrivial RP^2-bundle over S^2 is the quotient of $S^2\tilde{\times}S^2$ by the bundle involution which is the antipodal map on each fibre. If we observe that $S^2\tilde{\times}S^2 \cong CP^2\sharp - CP^2$ is the union of two copies of the D^2-bundle which is the mapping cone of the Hopf fibration and that this involution interchanges the hemispheres we see that this space is homeomorphic to $RP^4\sharp CP^2$.

There are two RP^2-bundles over RP^2. (The total spaces of each of the latter bundles have fundamental group $(Z/2Z)^2$, since $w_1 : \pi \to \pi_1(RP^2) = Z/2Z$ restricts nontrivially to the fibre, and so is a splitting homomorphism for the homomorphism induced by the inclusion of the fibre.) They may be distinguished by their orientation double covers, and each is geometric.

12.4 Cohomology and Stiefel-Whitney classes

We shall show that if M is a closed connected 4-manifold with finite fundamental group π such that $\chi(M)|\pi| = 4$ then $H^*(M;\mathbb{F}_2)$ is isomorphic to the cohomology ring of one of the above bundle spaces, as a module over the Steenrod algebra $\mathcal{A}_2$. (In other words, there is an isomorphism which preserves Stiefel-Whitney classes.) This is an elementary exercise in Poincaré duality and the Wu formulae.

The classifying map induces an isomorphism $H^1(\pi;\mathbb{F}_2) \cong H^1(M;\mathbb{F}_2)$ and a monomorphism $H^2(\pi;\mathbb{F}_2) \to H^2(M;\mathbb{F}_2)$. If $\pi = 1$ then M is homotopy equivalent to $S^2 \times S^2$, $S^2\tilde{\times}S^2$ or $CP^2\sharp CP^2$, and the result is clear.

$\pi = Z/2Z$ In this case $\beta_2(M;\mathbb{F}_2) = 2$. Let x generate $H^1(M;\mathbb{F}_2)$. Then $x^2 \neq 0$, so $H^2(M;\mathbb{F}_2)$ has a basis $\{x^2, u\}$. If $x^4 = 0$ then $x^2u \neq 0$, by Poincaré duality, and so $H^3(M;\mathbb{F}_2)$ is generated by xu. Hence $x^3 = 0$, for otherwise $x^3 = xu$ and $x^4 = x^2u \neq 0$. Therefore $v_2(M) = 0$ or x^2, and clearly $v_1(M) = 0$ or x. Since x restricts to 0 in $\widetilde{M}$ we must have $w_2(\widetilde{M}) = v_2(\tilde{M}) = 0$. (The four possibilities are realized by the four S^2-bundles over RP^2.)

If $x^4 \neq 0$ then we may assume that $x^2u = 0$ and that $H^3(M;\mathbb{F}_2)$ is generated by x^3. In this case $xu = 0$. Since $Sq^1(x^3) = x^4$ we have $v_1(M) = x$, and

$v_2(M) = u + x^2$. In this case $w_2(\widetilde{M}) \neq 0$, since $w_2(M)$ is not a square. (This possibility is realized by the nontrivial RP^2-bundle over S^2.)

$\pi \cong (Z/2Z)^2$ In this case $\beta_2(M;\mathbb{F}_2) = 3$ and $w_1(M) \neq 0$. Fix a basis $\{x, y\}$ for $H^1(M;\mathbb{F}_2)$. Then $\{x^2, xy, y^2\}$ is a basis for $H^2(M;\mathbb{F}_2)$, since $H^2(\pi;\mathbb{F}_2)$ and $H^2(M;\mathbb{F}_2)$ both have dimension 3.

If $x^3 = y^3$ then $x^4 = Sq^1(x^3) = Sq^1(y^3) = y^4$. Hence $x^4 = y^4 = 0$ and $x^2y^2 \neq 0$, by the nondegeneracy of cup product on $H^2(M;\mathbb{F}_2)$. Hence $x^3 = y^3 = 0$ and so $H^3(M;\mathbb{F}_2)$ is generated by $\{x^2y, xy^2\}$. Now $Sq^1(x^2y) = x^2y^2$ and $Sq^1(xy^2) = x^2y^2$, so $v_1(M) = x + y$. Also $Sq^2(x^2) = 0 = x^2xy$, $Sq^2(y^2) = 0 = y^2xy$ and $Sq^2(xy) = x^2y^2$, so $v_2(M) = xy$. Since the restrictions of x and y to the orientation cover M^+ agree we have $w_2(M^+) = x^2 \neq 0$. (This possibility is realized by $RP^2 \times RP^2$.)

If x^3, y^3 and $(x + y)^3$ are all distinct then we may assume that (say) y^3 and $(x+y)^3$ generate $H^3(M;\mathbb{F}_2)$. If $x^3 \neq 0$ then $x^3 = y^3+(x+y)^3 = x^3+x^2y+xy^2$ and so $x^2y = xy^2$. But then we must have $x^4 = y^4 = 0$, by the nondegeneracy of cup product on $H^2(M;\mathbb{F}_2)$. Hence $Sq^1(y^3) = y^4 = 0$ and $Sq^1((x + y)^3) = (x + y)^4 = x^4 + y^4 = 0$, and so $v_1(M) = 0$, which is impossible, as M is nonorientable. Therefore $x^3 = 0$ and so $x^2y^2 \neq 0$. After replacing y by $x + y$, if necessary, we may assume $xy^3 = 0$ (and hence $y^4 \neq 0$). Poincaré duality and the Wu relations then give $v_1(M) = x + y$, $v_2(M) = xy + x^2$ and hence $w_2(M^+) = 0$. (This possibility is realized by the nontrivial RP^2-bundle over RP^2.)

Note that if $\pi \cong (Z/2Z)^2$ then $H^*(M;\mathbb{F}_2)$ is generated by $H^1(M;\mathbb{F}_2)$ and so the image of $[M]$ in $H_4(\pi;\mathbb{F}_2)$ is uniquely determined.

In all cases, a class $x \in H^1(M;\mathbb{F}_2)$ such that $x^3 = 0$ may be realized by a map from M to $K(Z/2Z, 1) = RP^\infty$ which factors through $P_2(RP^2)$. However there are such 4-manifolds which do not fibre over RP^2.

12.5 The action of π on $\pi_2(M)$

Let M be a closed 4-manifold with finite fundamental group π and orientation character $w = w_1(M)$. The intersection form $S(\widetilde{M})$ on $\Pi = \pi_2(M) = H_2(\widetilde{M};\mathbb{Z})$ is unimodular and symmetric, and π acts w-isometrically (that is, $S(ga, gb) = w(g)S(a, b)$ for all $g \in \pi$ and a, $b \in \Pi$).

The two inclusions of S^2 as factors of $S^2 \times S^2$ determine the standard basis for $\pi_2(S^2 \times S^2)$. Let $J = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ be the matrix of the intersection form $\bullet$ on

$\pi_2(S^2\times S^2)$, with respect to this basis. The group $Aut(\pm\bullet)$ of automorphisms of $\pi_2(S^2\times S^2)$ which preserve this intersection form up to sign is the dihedral group of order eight, and is generated by the diagonal matrices and J or $K=\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$. The subgroup of strict isometries has order four, and is generated by $-I$ and J. (Note that the isometry J is induced by the involution τ.)

Let f be a self homeomorphism of $S^2\times S^2$ and let f_* be the induced automorphism of $\pi_2(S^2\times S^2)$. The Lefshetz number of f is $2+trace(f_*)$ if f is orientation preserving and $trace(f_*)$ if f is orientation reversing. As any self homotopy equivalence which induces the identity on π_2 has nonzero Lefshetz number the natural representation of a group π of fixed point free self homeomorphisms of $S^2\times S^2$ into $Aut(\pm\bullet)$ is faithful.

Suppose first that f is a free involution, so $f_*^2=I$. If f is orientation preserving then $trace(f_*)=-2$ so $f_*=-I$. If f is orientation reversing then $trace(f_*)=0$, so $f_*=\pm JK=\pm\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$. Note that if $f'=\tau f\tau$ then $f'_*=-f_*$, so after conjugation by τ, if necessary, we may assume that $f_*=JK$.

If f generates a free $Z/4Z$-action the induced automorphism must be $\pm K$. Note that if $f'=\tau f\tau$ then $f'_*=-f_*$, so after conjugation by τ, if necessary, we may assume that $f_*=K$.

Since the orbit space of a fixed point free action of $(Z/2Z)^2$ on $S^2\times S^2$ has Euler characteristic 1 it is nonorientable, and so the action is generated by two commuting involutions, one of which is orientation preserving and one of which is not. Since the orientation preserving involution must act via $-I$ and the orientation reversing involutions must act via $\pm JK$ the action of $(Z/2Z)^2$ is essentially unique.

The standard inclusions of $S^2=CP^1$ into the summands of $CP^2\sharp-CP^2\cong S^2\tilde{\times}S^2$ determine a basis for $\pi_2(S^2\tilde{\times}S^2)\cong Z^2$. Let $\tilde{J}=\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$ be the matrix of the intersection form $\tilde{\bullet}$ on $\pi_2(S^2\tilde{\times}S^2)$ with respect to this basis. The group $Aut(\pm\tilde{\bullet})$ of automorphisms of $\pi_2(S^2\tilde{\times}S^2)$ which preserve this intersection form up to sign is the dihedral group of order eight, and is also generated by the diagonal matrices and $J=\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. The subgroup of strict isometries has order four, and consists of the diagonal matrices. A nontrivial group of fixed point free self homeomorphisms of $S^2\tilde{\times}S^2$ must have order 2, since $S^2\tilde{\times}S^2$ admits no fixed point free orientation preserving involution. If f is an orientation reversing free involution of $S^2\tilde{\times}S^2$ then $f_*=\pm J$. Since the involution of CP^2 given by complex conjugation is orientation preserving it is isotopic to a selfhomeomorphism c which fixes a 4-disc. Let $g=c\sharp id_{CP^2}$. Then $g_*=\left(\begin{smallmatrix}-1&0\\0&1\end{smallmatrix}\right)$ and so $g_*Jg_*^{-1}=-J$. Thus after conjugating f by g, if necessary, we may assume that $f_*=J$.

All self homeomorphisms of $CP^2\sharp CP^2$ preserve the sign of the intersection form, and thus are orientation preserving. With part (2) of Lemma 12.3, this implies that no manifold in this homotopy type admits a free involution.

12.6 Homotopy type

The *quadratic 2-type* of M is the quadruple $[\pi,\pi_2(M),k_1(M),S(\widetilde{M})]$. Two such quadruples $[\pi,\Pi,\kappa,S]$ and $[\pi',\Pi',\kappa',S']$ with π a finite group, Π a finitely generated, $\mathbb{Z}$-torsion free $\mathbb{Z}[\pi]$-module, $\kappa\in H^3(\pi;\Pi)$ and $S:\Pi\times\Pi\to\mathbb{Z}$ a unimodular symmetric bilinear pairing on which π acts $\pm$-isometrically are *equivalent* if there is an isomorphism $\alpha:\pi\to\pi'$ and an (anti)isometry $\beta:(\Pi,S)\to(\Pi',(\pm)S')$ which is α-equivariant (i.e., such that $\beta(gm)=\alpha(g)\beta(m)$ for all $g\in\pi$ and $m\in\Pi$) and $\beta_*\kappa=\alpha^*\kappa'$ in $H^3(\pi,\alpha^*\Pi')$. Such a quadratic 2-type determines homomorphisms $w:\pi\to\mathbb{Z}^\times=Z/2Z$ and $v:\Pi\to Z/2Z$ by the equations $S(ga,gb)=w(g)S(a,b)$ and $v(a)\equiv S(a,a)\ mod\ (2)$, for all $g\in\pi$ and $a,\ b\in\Pi$. (These correspond to the orientation character $w_1(M)$ and the Wu class $v_2(\widetilde{M})=w_2(\widetilde{M})$, of course.)

Let $\gamma:A\to\Gamma(A)$ be the universal quadratic functor of Whitehead. Then the pairing S may be identified with an indivisible element of $\Gamma(Hom_{\mathbb{Z}}(\Pi,\mathbb{Z}))$. Via duality, this corresponds to an element $\widehat{S}$ of $\Gamma(\Pi)$, and the subgroup generated by the image of $\widehat{S}$ is a $\mathbb{Z}[\pi]$-submodule. Hence $\pi_3=\Gamma(\Pi)/\langle\widehat{S}\rangle$ is again a finitely generated, $\mathbb{Z}$-torsion free $\mathbb{Z}[\pi]$-module. Let B be the Postnikov 2-stage corresponding to the algebraic 2-type $[\pi,\Pi,\kappa]$. A *PD_4-polarization* of the quadratic 2-type $q=[\pi,\Pi,\kappa,S]$ is a 3-connected map $f:X\to B$, where X is a PD_4-complex, $w_1(X)=w\pi_1(f)$ and $\tilde{f}_*(\widehat{S}_{\widetilde{X}})=\widehat{S}$ in $\Gamma(\Pi)$. Let $S_4^{PD}(q)$ be the set of equivalence classes of PD_4-polarizations of q, where $f:X\to B\sim g:Y\to B$ if there is a map $h:X\to Y$ such that $f\simeq gh$.

Theorem 12.4 [Te] *There is an effective, transitive action of the torsion subgroup of* $\Gamma(\Pi)\otimes_{\mathbb{Z}[\pi]}Z^w$ *on* $S_4^{PD}(q)$.

Proof (We shall only sketch the proof.) Let $f:X\to B$ be a fixed PD_4-polarization of q. We may assume that $X=K\cup_g e^4$, where $K=X^{[3]}$ is the 3-skeleton and $g\in\pi_3(K)$ is the attaching map. Given an element α in $\Gamma(\Pi)$ whose image in $\Gamma(\Pi)\otimes_{\mathbb{Z}[\pi]}Z^w$ lies in the torsion subgroup, let $X_\alpha=K\cup_{g+\alpha}e^4$. Since $\pi_3(B)=0$ the map $f|_K$ extends to a map $f_\alpha:X_\alpha\to B$, which is again a PD_4-polarization of q. The equivalence class of f_α depends only on the image of α in $\Gamma(\Pi)\otimes_{\mathbb{Z}[\pi]}Z^w$. Conversely, if $g:Y\to B$ is another PD_4-polarization of q then $f_*[X]-g_*[Y]$ lies in the image of $Tors(\Gamma(\Pi)\otimes_{\mathbb{Z}[\pi]}Z^w)$ in $H_4(B;Z^w)$. See [Te] for the full details. □

Corollary 12.4.1 *If X and Y are PD_4-complexes with the same quadratic 2-type then each may be obtained by adding a single 4-cell to $X^{[3]} = Y^{[3]}$.* □

If $w = 0$ and the Sylow 2-subgroup of π has cohomological period dividing 4 then $Tors(\Gamma(\Pi) \otimes_{\mathbb{Z}[\pi]} Z^w) = 0$ [Ba88]. In particular, if M is orientable and π is finite cyclic then the equivalence class of the quadratic 2-type determines the homotopy type [HK88]. Thus in all cases considered here the quadratic 2-type determines the homotopy type of the orientation cover.

The group $Aut(B) = Aut([\pi, \Pi, \kappa])$ acts on $S_4^{PD}(q)$ and the orbits of this action correspond to the homotopy types of PD_4-complexes X admitting such polarizations f. When q is the quadratic 2-type of $RP^2 \times RP^2$ this action is nontrivial. (See below in this paragraph. Compare also Theorem 10.5.)

The next lemma shall enable us to determine the possible k-invariants.

Lemma 12.5 *Let M be a closed 4-manifold with fundamental group $\pi = Z/2Z$ and universal covering space $S^2 \times S^2$. Then the first k-invariant of M is a nonzero element of $H^3(\pi; \pi_2(M))$.*

Proof The first k-invariant is the primary obstruction to the existence of a cross-section to the classifying map $c_M : M \to K(Z/2Z, 1) = RP^\infty$ and is the only obstruction to the existence of such a cross-section for $c_{P_2(M)}$. The only nonzero differentials in the Cartan-Leray cohomology spectral sequence (with coefficients $Z/2Z$) for the projection $p : \widetilde{M} \to M$ are at the E_3^{**} level. By the results of Section 4, π acts trivially on $H^2(\widetilde{M}; \mathbb{F}_2)$, since $\widetilde{M} = S^2 \times S^2$. Therefore $E_3^{22} = E_2^{22} \cong (Z/2Z)^2$ and $E_3^{50} = E_2^{50} = Z/2Z$. Hence $E_\infty^{22} \neq 0$, so E_∞^{22} maps onto $H^4(M; \mathbb{F}_2) = Z/2Z$ and $d_3^{12} : H^1(\pi; H^2(\widetilde{M}; \mathbb{F}_2)) \to H^4(\pi; \mathbb{F}_2)$ must be onto. But in this region the spectral sequence is identical with the corresponding spectral sequence for $P_2(M)$. It follows that the image of $H^4(\pi; \mathbb{F}_2) = Z/2Z$ in $H^4(P_2(M); \mathbb{F}_2)$ is 0, and so $c_{P_2(M)}$ does not admit a cross-section. Thus $k_1(M) \neq 0$. □

If $\pi = Z/2Z$ and M is orientable then π acts via $-I$ on Z^2 and the k-invariant is a nonzero element of $H^3(Z/2Z; \pi_2(M)) = (Z/2Z)^2$. The isometry which transposes the standard generators of Z^2 is π-linear, and so there are just two equivalence classes of quadratic 2-types to consider. The k-invariant which is invariant under transposition is realised by $(S^2 \times S^2)/(-I, -I)$, while the other k-invariant is realized by the orientable bundle space with $w_2 = 0$. Thus M must be homotopy equivalent to one of these spaces.

If $\pi = Z/2Z$, M is nonorientable and $w_2(\widetilde{M}) = 0$ then $H^3(\pi;\pi_2(M)) = Z/2Z$ and there is only one quadratic 2-type to consider. There are four equivalence classes of PD_4-polarizations, as $Tors(\Gamma(\Pi) \otimes_{\mathbb{Z}[\pi]} Z^w) \cong (Z/2Z)^2$. The corresponding PD_4-complexes are all of the form $K \cup_f e^4$, where $K = (S^2 \times RP^2) - intD^4$ is the 3-skeleton of $S^2 \times RP^2$ and $f \in \pi_3(K)$. (In all cases $H^1(M;\mathbb{F}_2)$ is generated by an element x such that $x^3 = 0$.) Two choices for f give total spaces of S^2-bundles over RP^2, while a third choice gives $RP^4 \sharp_{S^1} RP^4$, which is the union of two disc bundles over RP^2, but is not a bundle space and is not geometric. There is a fourth homotopy type which has nontrivial Browder-Livesay invariant, and so is not realizable by a closed manifold [HM78]. The product space $S^2 \times RP^2$ is characterized by the additional conditions that $w_2(M) = w_1(M)^2 \neq 0$ (i.e., $v_2(M) = 0$) and that there is an element $u \in H^2(M;\mathbb{Z})$ which generates an infinite cyclic direct summand and is such that $u \cup u = 0$. (See Theorem 5.19.) The nontrivial nonorientable S^2-bundle over RP^2 has $w_2(M) = 0$. The manifold $RP^4 \sharp_{S^1} RP^4$ also has $w_2(M) = 0$, but it may be distinguished from the bundle space by the $Z/4Z$-valued quadratic function on $\pi_2(M) \otimes (Z/2Z)$ introduced in [KKR92].

If $\pi = Z/2Z$ and $w_2(\widetilde{M}) \neq 0$ then $H^3(\pi_1;\pi_2(M)) = 0$, and the quadratic 2-type is unique. (Note that the argument of Lemma 12.5 breaks down here because $E_\infty^{22} = 0$.) There are two equivalence classes of PD_4-polarizations, as $Tors(\Gamma(\Pi) \otimes_{\mathbb{Z}[\pi]} Z^w) = Z/2Z$. They are each of the form $K \cup_f e^4$, where $K = (RP^4 \sharp CP^2) - intD^4$ is the 3-skeleton of $RP^4 \sharp CP^2$ and $f \in \pi_3(K)$. The bundle space $RP^4 \sharp CP^2$ is characterized by the additional condition that there is an element $u \in H^2(M;\mathbb{Z})$ which generates an infinite cyclic direct summand and such that $u \cup u = 0$. (See Theorem 5.19.) In [HKT94] it is shown that any closed 4-manifold M with $\pi = Z/2Z$, $\chi(M) = 2$ and $w_2(\widetilde{M}) \neq 0$ is homotopy equivalent to $RP^4 \sharp CP^2$.

If $\pi \cong Z/4Z$ then $H^3(\pi;\pi_2(M)) \cong Z^2/(I-K)Z^2 = Z/2Z$, since $\Sigma_{k=1}^{k=4} f_*^k = \Sigma_{k=1}^{k=4} K^k = 0$. The k-invariant is nonzero, since it restricts to the k-invariant of the orientation double cover. In this case $Tors(\Gamma(\Pi) \otimes_{\mathbb{Z}[\pi]} Z^w) = 0$ and so M is homotopy equivalent to $(S^2 \times S^2)/\tau(I,-I)$.

Finally, let $\pi \cong (Z/2Z)^2$ be the diagonal subgroup of $Aut(\pm\bullet) < GL(2,\mathbb{Z})$, and let α be the automorphism induced by conjugation by J. The standard generators of $\pi_2(M) = Z^2$ generate complementary π-submodules, so that $\pi_2(M)$ is the direct sum $\tilde{Z} \oplus \alpha^*\tilde{Z}$ of two infinite cyclic modules. The isometry $\beta = J$ which transposes the factors is α-equivariant, and π and $V = \{\pm I\}$ act nontrivially on each summand. If ρ is the kernel of the action of π on $\tilde{Z}$ then $\alpha(\rho)$ is the kernel of the action on $\alpha^*\tilde{Z}$, and $\rho \cap \alpha(\rho) = 1$. Let $j_V : V \to \pi$ be the inclusion. As the projection of $\pi = \rho \oplus V$ onto V is compatible with the action, $H^*(j_V;\tilde{Z})$ is a split epimorphism and so $H^*(V;\tilde{Z})$ is a direct summand of $H^*(\pi;\tilde{Z})$. This implies in particular that the differentials in the LHSSS $H^p(V;H^q(\rho;\tilde{Z})) \Rightarrow H^{p+q}(\pi;\tilde{Z})$ which end on the row $q=0$ are all 0. Hence $H^3(\pi;\tilde{Z}) \cong H^1(V;\mathbb{F}_2) \oplus H^3(V;\tilde{Z}) \cong (Z/2Z)^2$. Similarly $H^3(\pi;\alpha^*\tilde{Z}) \cong (Z/2Z)^2$, and so $H^3(\pi;\pi_2(M)) \cong (Z/2Z)^4$. The k-invariant must restrict to the k-invariant of each double cover, which must be nonzero, by Lemma 12.5. Let K_V, K_ρ and $K_{\alpha(\rho)}$ be the kernels of the restriction homomorphisms from $H^3(\pi;\pi_2(M))$ to $H^3(V;\pi_2(M))$, $H^3(\rho;\pi_2(M))$ and $H^3(\alpha(\rho);\pi_2(M))$, respectively. Now $H^3(\rho;\tilde{Z}) = H^3(\alpha(\rho);\alpha^*\tilde{Z}) = 0$, $H^3(\rho;\alpha^*\tilde{Z}) = H^3(\alpha(\rho);\tilde{Z}) = Z/2Z$ and $H^3(V;\tilde{Z}) = H^3(V;\alpha^*\tilde{Z}) = Z/2Z$. Since the restrictions are epimorphisms $|K_V| = 4$ and $|K_\rho| = |K_{\alpha(\rho)}| = 8$. It is easily seen that $|K_\rho \cap K_{\alpha(\rho)}| = 4$. Moreover $\mathrm{Ker}(H^3(j_V;\tilde{Z})) \cong H^1(V;H^2(\rho;\tilde{Z})) \cong H^1(V;H^2(\rho;\mathbb{F}_2))$ restricts nontrivially to $H^3(\alpha(\rho);\tilde{Z}) \cong H^3(\alpha(\rho);\mathbb{F}_2)$, as can be seen by reduction modulo (2), and similarly $\mathrm{Ker}(H^3(j_V;\alpha^*\tilde{Z}))$ restricts nontrivially to $H^3(\rho;\alpha^*\tilde{Z})$. Hence $|K_V \cap K_\rho| = |K_V \cap K_\rho| = 2$ and $K_V \cap K_\rho \cap K_{\alpha(\rho)} = 0$. Thus $|K_V \cup K_\rho \cup K_{\alpha(\rho)}| = 8+8+4-4-2-2+1 = 13$ and so there are at most three possible k-invariants. Moreover the automorphism α and the isometry $\beta = J$ act on the k-invariants by transposing the factors. The k-invariant of $RP^2 \times RP^2$ is invariant under this transposition, while that of the nontrivial RP^2 bundle over RP^2 is not, for the k-invariant of its orientation cover is not invariant. Thus there are two equivalence classes of quadratic 2-types to be considered. Since $Tors(\Gamma(\Pi) \otimes_{\mathbb{Z}[\pi]} Z^w) \cong (Z/2Z)^2$ there are four equivalence classes of PD_4-polarizations of each of these quadratic 2-types. In each case the quadratic 2-type determines the cohomology ring, since it determines the orientation cover (see §4). The canonical involution of the direct product interchanges two of these polarizations in the $RP^2 \times RP^2$ case, and so there are seven homotopy types of PD_4-complexes X with $\pi \cong (Z/2Z)^2$ and $\chi(X) = 1$. Can the Browder-Livesay arguments of [HM78] be adapted to show that the two bundle spaces are the only such 4-manifolds?

12.7 Surgery

We may assume that M is a proper quotient of $S^2 \times S^2$ or of $S^2 \tilde{\times} S^2$, so $|\pi|\chi(M) = 4$ and $\pi \neq 1$. In the present context every homotopy equivalence is simple since $Wh(\pi) = 0$ for all groups π of order ≤ 4 [Hg40].

Suppose first that $\pi = Z/2Z$. Then $H^1(M;\mathbb{F}_2) = Z/2Z$ and $\chi(M) = 2$, so $H^2(M;\mathbb{F}_2) \cong (Z/2Z)^2$. The $\mathbb{F}_2$-Hurewicz homomorphism from $\pi_2(M)$ to $H_2(M;\mathbb{F}_2)$ has cokernel $H_2(\pi;\mathbb{F}_2) = Z/2Z$. Hence there is a map $\beta : S^2 \to M$

such that $\beta_*[S^2] \neq 0$ in $H_2(M;\mathbb{F}_2)$. If moreover $w_2(\widetilde{M}) = 0$ then $\beta^* w_2(M) = 0$, since β factors through $\widetilde{M}$. Then there is a self homotopy equivalence f_β of M with nontrivial normal invariant in $[M;G/TOP]$, by Lemma 6.5. Note also that M is homotopy equivalent to a PL 4-manifold (see §6 above).

If M is orientable $[M;G/TOP] \cong Z\oplus(Z/2Z)^2$. The surgery obstruction groups are $L_5(Z/2Z,+) = 0$ and $L_4(Z/2Z,+) \cong Z^2$, where the surgery obstructions are determined by the signature and the signature of the double cover, by Theorem 13.A.1 of [Wl]. Hence it follows from the surgery exact sequence that $S_{TOP}(M) \cong (Z/2Z)^2$. Since $w_2(\widetilde{M}) = 0$ (by Lemma 12.3) there is a self homotopy equivalence f_β of M with nontrivial normal invariant and so there are at most two homeomorphism classes within the homotopy type of M. Any $\alpha \in H^2(M;\mathbb{F}_2)$ is the codimension-2 Kervaire invariant of some homotopy equivalence $f : N \to M$. We then have $KS(N) = f^*(KS(M)+\alpha^2)$, by Lemma 15.5 of [Si71]. We may assume that M is PL. If $w_2(M) = 0$ then $KS(N) = f^*(KS(M)) = 0$, and so N is homeomorphic to M [Te97]. On the other hand if $w_2(M) \neq 0$ there is an $\alpha \in H^2(M;\mathbb{F}_2)$ such that $\alpha^2 \neq 0$ and then $KS(N) \neq 0$. Thus there are three homeomorphism classes of orientable closed 4-manifolds with $\pi = Z/2Z$ and $\chi = 2$. One of these is a fake $(S^2 \times S^2)/(-I,-I)$ and is not smoothable.

Nonorientable closed 4-manifolds with fundamental group $Z/2Z$ have been classified in [HKT94]. If M is nonorientable then $[M;G/TOP] \cong (Z/2Z)^3$, the surgery obstruction groups are $L_5(Z/2Z,-) = 0$ and $L_4(Z/2Z,-) = Z/2Z$, and $\sigma_4(\hat{g}) = c(\hat{g})$ for any normal map $\hat{g} : M \to G/TOP$, by Theorem 13.A.1 of [Wl]. Therefore $\sigma_4(\hat{g}) = (w_1(M)^2 \cup \hat{g}^*(k_2))[M]$, by Theorem 13.B.5 of [Wl]. (See also §2 of Chapter 6 above.) As $w_1(M)$ is not the reduction of a class in $H^1(M;\mathbb{Z}/4\mathbb{Z})$ its square is nonzero and so there is an element $\hat{g}^*(k_2)$ in $H^2(M;\mathbb{F}_2)$ such that this cup product is nonzero. Hence $S_{TOP}(M) \cong (Z/2Z)^2$. There are two homeomorphism types within each homotopy type if $w_2(\widetilde{M}) = 0$; if $w_2(\widetilde{M}) \neq 0$ (i.e., if $M \simeq RP^4\sharp CP^2$) there are four corresponding homeomorphism types [HKT94]. Thus there are eight homeomorphism classes of nonorientable closed 4-manifolds with $\pi = Z/2Z$ and $\chi = 2$.

The image of $[M;G/PL]$ in $[M;G/TOP]$ is a subgroup of index 2 (see Section 15 of [Si71]). It follows that if M is the total space of an S^2-bundle over RP^2 any homotopy equivalence $f : N \to M$ where N is also PL is homotopic to a homeomorphism. (For then $S_{TOP}(M)$ has order 4, and the nontrivial element of the image of $S_{PL}(M)$ is represented by an exotic self homotopy equivalence of M. The case $M = S^2 \times RP^2$ was treated in [Ma79]. See also [Te97] for the cases with $\pi = Z/2Z$ and $w_1(M) = 0$.) This is also true if $M = S^4$, RP^4, CP^2,

$S^2 \times S^2$ or $S^2 \tilde{\times} S^2$. The exotic homeomorphism types within the homotopy type of $RP^4 \sharp CP^2$ (the nontrivial RP^2-bundle over S^2) are $RP^4 \sharp * CP^2$, $*RP^4 \sharp CP^2$, which have nontrivial Kirby-Siebenmann invariant, and $(*RP^4) \sharp * CP^2$, which is smoothable [RS97]. Moreover $(*RP^4 \sharp * CP^2) \sharp (S^2 \times S^2) \cong (RP^4 \sharp CP^2) \sharp (S^2 \times S^2)$ [HKT94].

When $\pi \cong Z/4Z$ or $(Z/2Z)^2$ the manifold M is nonorientable, since $\chi(M) = 1$. As the $\mathbb{F}_2$-Hurewicz homomorphism is 0 in these cases Lemma 6.5 does not apply to give any exotic self homotopy equivalences.

If $\pi \cong Z/4Z$ then $[M; G/TOP] \cong (Z/2Z)^2$ and the surgery obstruction groups $L_4(Z/4Z, -)$ and $L_5(Z/4Z, -)$ are both 0, by Theorem 3.4.5 of [Wl76]. Hence $S_{TOP}(M) \cong (Z/2Z)^2$. Since $w_2(M) \neq 0$ there is a homotopy equivalence $f : N \to M$ where $KS(N) \neq KS(M)$. An argument of Fang using [Da95] shows that there is such a manifold N with $KS(N) = 0$ which is not homeomorphic to the geometric example. Thus there are either three or four homeomorphism classes of closed 4-manifolds with $\pi \cong Z/4Z$ and $\chi = 1$. In all cases the orientable double covering space has trivial Kirby-Siebenmann invariant and so is homeomorphic to $(S^2 \times S^2)/(-I, -I)$.

If $\pi \cong (Z/2Z)^2$ then $[M; G/TOP] \cong (Z/2Z)^4$ and the surgery obstruction groups are $L_5((Z/2Z)^2, -) = 0$ and $L_4((Z/2Z)^2, -) = Z/2Z$, by Theorem 3.5.1 of [Wl76]. Since $w_1(M)$ is a split epimorphism $L_4(w_1(M))$ is an isomorphism, so the surgery obstruction is detected by the Kervaire-Arf invariant. As $w_1(M)^2 \neq 0$ we find that $S_{TOP}(M) \cong (Z/2Z)^3$. Thus there are at most 56 homeomorphism classes of closed 4-manifolds with $\pi \cong (Z/2Z)^2$ and $\chi = 1$.

Chapter 13

Geometric decompositions of bundle spaces

We begin by considering which closed 4-manifolds with geometries of euclidean factor type are mapping tori of homeomorphisms of 3-manifolds. We also show that (as an easy consequence of the Kodaira classification of surfaces) a complex surface is diffeomorphic to a mapping torus if and only if its Euler characteristic is 0 and its fundamental group maps onto Z with finitely generated kernel, and we determine the relevant 3-manifolds and diffeomorphisms. In §2 we consider when an aspherical 4-manifold which is the total space of a surface bundle is geometric or admits a geometric decomposition. If the base and fibre are hyperbolic the only known examples are virtually products. In §3 we shall give some examples of torus bundles over closed surfaces which are not geometric, some of which admit geometric decompositions of type $\mathbb{F}^4$ and some of which do not. In §4 we apply some of our earlier results to the characterization of certain complex surfaces. In particular, we show that a complex surfaces fibres smoothly over an aspherical orientable 2-manifold if and only if it is homotopy equivalent to the total space of a surface bundle. In the final two sections we consider first S^1-bundles over geometric 3-manifolds and then the existence of symplectic structures on geometric 4-manifolds.

13.1 Mapping tori

In §3-5 of Chapter 8 and §3 of Chapter 9 we used 3-manifold theory to characterize mapping tori of homeomorphisms of geometric 3-manifolds which have product geometries. Here we shall consider instead which 4-manifolds with product geometries or complex structures are mapping tori.

Theorem 13.1 *Let M be a closed geometric 4-manifold with $\chi(M) = 0$ and such that $\pi = \pi_1(M)$ is an extension of Z by a finitely generated normal subgroup K. Then K is the fundamental group of a geometric 3-manifold.*

Proof Since $\chi(M) = 0$ the geometry must be either an infrasolvmanifold geometry or a product geometry $\mathbb{X}^3 \times \mathbb{E}^1$, where $\mathbb{X}^3$ is one of the 3-dimensional geometries $\mathbb{S}^3$, $\mathbb{S}^2\times\mathbb{E}^1$, $\mathbb{H}^3$, $\mathbb{H}^2\times\mathbb{E}^1$ or $\widetilde{\mathbb{SL}}$. If M is an infrasolvmanifold then π is torsion free and virtually poly-Z of Hirsch length 4, so K is torsion free and virtually poly-Z of Hirsch length 3, and the result is clear.

If $\mathbb{X}^3=\mathbb{S}^3$ then π is a discrete cocompact subgroup of $O(4)\times E(1)$. Since π maps onto Z it must in fact be a subgroup of $O(4)\times R$, and K is a finite subgroup of $O(4)$. Since π acts freely on $S^3\times R$ the subgroup K acts freely on S^3, and so K is the fundamental group of an $\mathbb{S}^3$-manifold. If $\mathbb{X}^3=\mathbb{S}^2\times\mathbb{E}^1$ it follows from Corollary 4.5.3 that $K\cong Z$, $Z\oplus(Z/2Z)$ or D, and so K is the fundamental group of an $\mathbb{S}^2\times\mathbb{E}^1$-manifold.

In the remaining cases $\mathbb{X}^3$ is of aspherical type. The key point here is that a discrete cocompact subgroup of the Lie group $Isom(\mathbb{X}^3\times\mathbb{E}^1)$ must meet the radical of this group in a lattice subgroup. Suppose first that $\mathbb{X}^3=\mathbb{H}^3$. After passing to a subgroup of finite index if necessary, we may assume that $\pi\cong H\times Z<PSL(2,\mathbb{C})\times R$, where H is a discrete cocompact subgroup of $PSL(2,\mathbb{C})$. If $K\cap(\{1\}\times R)=1$ then K is commensurate with H, and hence is the fundamental group of an X-manifold. Otherwise the subgroup generated by $K\cap H=K\cap PSL(2,\mathbb{C})$ and $K\cap(\{1\}\times R)$ has finite index in K and is isomorphic to $(K\cap H)\times Z$. Since K is finitely generated so is $K\cap H$, and hence it is finitely presentable, since H is a 3-manifold group. Therefore $K\cap H$ is a PD_2-group and so K is the fundamental group of a $\mathbb{H}^2\times\mathbb{E}^1$-manifold.

If $\mathbb{X}^3=\mathbb{H}^2\times\mathbb{E}^1$ then we may assume that $\pi\cong H\times Z^2<PSL(2,\mathbb{R})\times R^2$, where H is a discrete cocompact subgroup of $PSL(2,\mathbb{R})$. Since such groups do not admit nontrivial maps to Z with finitely generated kernel $K\cap H$ must be commensurate with H, and we again see that K is the fundamental group of an $\mathbb{H}^2\times\mathbb{E}^1$-manifold.

A similar argument applies if $\mathbb{X}^3=\widetilde{\mathbb{SL}}$. We may assume that $\pi\cong H\times Z$ where H is a discrete cocompact subgroup of $Isom(\widetilde{\mathbb{SL}})$. Since such groups H do not admit nontrivial maps to Z with finitely generated kernel K must be commensurate with H and so is the fundamental group of a $\widetilde{\mathbb{SL}}$-manifold. □

Corollary 13.1.1 *Suppose that M has a product geometry $X\times E^1$. If $\mathbb{X}^3=\mathbb{E}^3$, $\mathbb{S}^3$, $\mathbb{S}^2\times\mathbb{E}^1$, $\widetilde{\mathbb{SL}}$ or $\mathbb{H}^2\times\mathbb{E}^1$ then M is the mapping torus of an isometry of an $\mathbb{X}^3$-manifold with fundamental group K. (In the latter case we must assume that M is orientable.) If $\mathbb{X}^3=\mathbb{N}il^3$ or $\mathbb{S}ol^3$ then K is the fundamental group of an $\mathbb{X}^3$-manifold or of a $\mathbb{E}^3$-manifold. If $\mathbb{X}^3=\mathbb{H}^3$ then K is the fundamental group of a $\mathbb{H}^3$- or $\mathbb{H}^2\times\mathbb{E}^1$-manifold.*

Proof In all cases π is a semidirect product $K\times_\theta Z$ and may be realised by the mapping torus of a self homeomorphism of a closed 3-manifold with fundamental group K. If this manifold is an $\mathbb{X}^3$-manifold then the outer automorphism class of θ is finite (see Chapter 8) and θ may then be realized by an isometry of an $\mathbb{X}^3$-manifold. Infrasolvmanifolds are determined up to diffeomorphism by their fundamental groups. This is also true of $\mathbb{S}^2\times\mathbb{E}^2$- and $\mathbb{S}^3\times\mathbb{E}^1$-manifolds [Oh90], provided K is not finite cyclic, and $\widetilde{\mathbb{SL}}\times\mathbb{E}^1$- and orientable $\mathbb{H}^2\times\mathbb{E}^2$-manifolds [Ue90, 91]. (Note that $\widetilde{\mathbb{SL}}$-manifolds are orientable and self homeomorphisms of such manifolds are orientation preserving [NR78].) When K is finite cyclic it is still true that every such $\mathbb{S}^3\times\mathbb{E}^1$-manifold is a mapping torus of an isometry of a suitable lens space [Oh90]. Thus if M is an $\mathbb{X}^3\times\mathbb{E}^1$-manifold and K is the fundamental group of an $\mathbb{X}^3$-manifold M is the mapping torus of an isometry of an $\mathbb{X}^3$-manifold with fundamental group K. □

Does the Corollary remain true for nonorientable $\mathbb{H}^2\times\mathbb{E}^2$-manifolds?

There are (orientable) $\mathbb{N}il^3\times\mathbb{E}^1$- and $\mathbb{S}ol^3\times\mathbb{E}^1$-manifolds which are mapping tori of self homeomorphisms of flat 3-manifolds, but which are not mapping tori of self homeomorphisms of $\mathbb{N}il^3$- or $\mathbb{S}ol^3$-manifolds. (See Chapter 8.) There are analogous examples when $\mathbb{X}^3=\mathbb{H}^3$. (See §4 of Chapter 9.)

We may now improve upon the characterization of mapping tori up to homotopy equivalence from Chapter 4.

Theorem 13.2 *Let M be a closed 4-manifold with fundamental group π. Then M is homotopy equivalent to the mapping torus $M(\Theta)$ of a self homeomorphism of a closed 3-manifold with one of the geometries $\mathbb{E}^3$, $\mathbb{N}il^3$, $\mathbb{S}ol^3$, $\mathbb{H}^2\times\mathbb{E}^1$, $\widetilde{\mathbb{SL}}$ or $\mathbb{S}^2\times\mathbb{E}^1$ if and only if*

1. *$\chi(M)=0$;*
2. *π is an extension of Z by an FP_2 normal subgroup K; and*
3. *K has a nontrivial torsion free abelian normal subgroup A.*

If π is torsion free M is s-cobordant to $M(\Theta)$, while if moreover π is solvable M is homeomorphic to $M(\Theta)$.

Proof The conditions are clearly necessary. Since K has an infinite abelian normal subgroup it has one or two ends. If K has one end then M is aspherical and so K is a PD_3-group by Theorem 4.1. Condition (3) then implies that M' is homotopy equivalent to a closed 3-manifold with one of the first five of the geometries listed above, by Theorem 2.14. If K has two ends then M' is homotopy equivalent to $S^2\times S^1$, $S^2\tilde{\times}S^1$, $RP^2\times S^1$ or $RP^3\sharp RP^3$, by Corollary 4.5.3.

In all cases K is isomorphic to the fundamental group of a closed 3-manifold N which is either Seifert fibred or a $\mathbb{S}ol^3$-manifold, and the outer automorphism class $[\theta]$ determined by the extension may be realised by a self homeomorphism Θ of N. The manifold M is homotopy equivalent to the mapping torus $M(\Theta)$. Since $Wh(\pi)=0$, by Theorems 6.1 and 6.3, any such homotopy equivalence is simple.

If K is torsion free and solvable then π is virtually poly-Z, and so M is homeomorphic to $M(\Theta)$, by Theorem 6.11. Otherwise N is a closed $\mathbb{H}^2\times\mathbb{E}^1$- or $\widetilde{\mathbb{SL}}$-manifold. As $\mathbb{H}^2\times\mathbb{E}^1$ has a metric of nonpositive sectional curvature, the surgery obstruction homomorphisms σ_i^N are isomorphisms for i large in this case, by [FJ93']. This holds also for any irreducible, orientable 3-manifold N such that $\beta_1(N)>0$ [Ro00], and therefore also for all $\widetilde{\mathbb{SL}}$-manifolds, by the Dress induction argument of [NS85]. Comparison of the Mayer-Vietoris sequences for $\mathbb{L}_0$-homology and L-theory (as in Proposition 2.6 of [St84]) shows that σ_i^M and $\sigma_i^{M\times S^1}$ are also isomorphisms for i large, and so $S_{TOP}(M(\Theta)\times S^1)$ has just one element. Therefore M is s-cobordant to $M(\Theta)$. □

Mapping tori of self homeomorphisms of $\mathbb{H}^3$- and $\mathbb{S}^3$-manifolds satisfy conditions (1) and (2). In the hyperbolic case there is the additional condition

(3-H) *K has one end and no noncyclic abelian subgroup.*

If every PD_3-group is a 3-manifold group and the geometrization conjecture for atoroidal 3-manifolds is true then the fundamental groups of closed hyperbolic 3-manifolds may be characterized as PD_3-groups having no noncyclic abelian subgroup. Assuming this, and assuming also that group rings of such hyperbolic groups are regular coherent, Theorem 13.2 could be extended to show that a closed 4-manifold M with fundamental group π is s-cobordant to the mapping torus of a self homeomorphism of a hyperbolic 3-manifold if and only these three conditions hold.

In the spherical case the appropriate additional conditions are

(3-S) *K is a fixed point free finite subgroup of $SO(4)$ and (if K is not cyclic) the characteristic automorphism of K determining π is realized by an isometry of S^3/K*; and

(4-S) *the first nontrivial k-invariant of M is "linear".*

The list of fixed point free finite subgroups of $SO(4)$ is well known. (See Chapter 11.) If K is cyclic or $Q\times Z/p^jZ$ for some odd prime p or T_k^* then the second part of (3-S) and (4-S) are redundant, but the general picture is not yet clear [HM86].

The classification of complex surfaces leads easily to a complete characterization of the 3-manifolds and diffeomorphisms such that the corresponding mapping tori admit complex structures. (Since $\chi(M)=0$ for any mapping torus M we do not need to enter the imperfectly charted realm of surfaces of general type.)

Theorem 13.3 *Let N be a closed orientable 3-manifold with $\pi_1(N)=\nu$ and let $\theta : N \to N$ be an orientation preserving self diffeomorphism. Then the mapping torus $M(\theta)$ admits a complex structure if and only if one of the following holds:*

(1) *$N = S^3/G$ where G is a fixed point free finite subgroup of $U(2)$ and the monodromy is as described in* [Kt75]*;*

(2) *$N = S^2 \times S^1$ (with no restriction on θ);*

(3) *$N = S^1 \times S^1 \times S^1$ and the image of θ in $SL(3,\mathbb{Z})$ either has finite order or satisfies the equation $(\theta^2 - I)^2 = 0$;*

(4) *N is the flat 3-manifold with holonomy of order 2, θ induces the identity on ν/ν' and the absolute value of the trace of the induced automorphism of $\nu' \cong Z^2$ is at most 2;*

(5) *N is one of the flat 3-manifolds with holonomy cyclic of order 3, 4 or 6 and θ induces the identity on $H_1(N;\mathbb{Q})$;*

(6) *N is a $\mathbb{N}il^3$-manifold and either the image of θ in $Out(\nu)$ has finite order or $M(\theta)$ is a $\mathbb{S}ol_1^4$-manifold;*

(7) *N is a $\mathbb{H}^2 \times \mathbb{E}^1$- or $\widetilde{\mathbb{SL}}$-manifold and the image of θ in $Out(\nu)$ has finite order.*

Proof The mapping tori of these diffeomorphisms admit 4-dimensional geometries, and it is easy to read off which admit complex structures from [Wl86]. In cases (3), (4) and (5) note that a complex surface is Kähler if and only if its first Betti number is even, and so the parity of this Betti number is invariant under passage to finite covers. (See Proposition 4.4 of [Wl86].)

The necessity of these conditions follows from examining the list of complex surfaces X with $\chi(X)=0$ on page 188 of [BPV], in conjunction with Bogomolov's theorem on surfaces of class VII_0. (See [Tl94] for a clear account of the latter result.) □

In particular, N must be Seifert fibred and most orientable Seifert fibred 3-manifolds (excepting only $RP^3\sharp RP^3$ and the Hantzsche-Wendt flat 3-manifold) occur. Moreover, in most cases (with exceptions as in (3), (4) and (6)) the image of θ in $Out(\nu)$ must have finite order. Some of the resulting 4-manifolds arise as mapping tori in several distinct ways. The corresponding result for complex surfaces of the form $N \times S^1$ for which the obvious smooth S^1-action is holomorphic was given in [GG95]. In [EO94] it is shown that if N is a rational homology 3-sphere then $N \times S^1$ admits a complex structure if and only if N is Seifert fibred, and the possible complex structures on such products are determined.

Conversely, the following result is very satisfactory from the 4-dimensional point of view.

Theorem 13.4 *Let X be a complex surface. Then X is diffeomorphic to the mapping torus of a self diffeomorphism of a closed 3-manifold if and only if $\chi(X) = 0$ and $\pi = \pi_1(X)$ is an extension of Z by a finitely generated normal subgroup.*

Proof The conditions are clearly necessary. Sufficiency of these conditions again follows from the classification of complex surfaces, as in Theorem 13.3. □

13.2 Surface bundles and geometries

Let $p : E \to B$ be a bundle with base B and fibre F aspherical closed surfaces. Then p is determined up to bundle isomorphism by the group $\pi = \pi_1(E)$. If $\chi(B) = \chi(F) = 0$ then E has geometry $\mathbb{E}^4$, $\mathbb{N}il^3 \times \mathbb{E}^1$, $\mathbb{N}il^4$ or $\mathbb{S}ol^3 \times \mathbb{E}^1$, by Ue's Theorem. When the fibre is Kb the geometry must be $\mathbb{E}^4$ or $\mathbb{N}il^3 \times \mathbb{E}^1$, for then π has a normal chain $\zeta\pi_1(Kb) \cong Z < \sqrt{\pi_1(Kb)} \cong Z^2$, so $\zeta\sqrt{\pi}$ has rank at least 2. Hence a $\mathbb{S}ol^3 \times \mathbb{E}^1$- or $\mathbb{N}il^4$-manifold M is the total space of a T-bundle over T if and only if $\beta_1(\pi) = 2$. If $\chi(F) = 0$ but $\chi(B) < 0$ then E need not be geometric. (See Chapter 7 and §3 below.)

We shall assume henceforth that F is hyperbolic, i.e. that $\chi(F) < 0$. Then $\zeta\pi_1(F) = 1$ and so the characteristic homomorphism $\theta : \pi_1(B) \to Out(\pi_1(F))$ determines π up to isomorphism, by Theorem 5.2.

Theorem 13.5 *Let B and F be closed surfaces with $\chi(B) = 0$ and $\chi(F) < 0$. Let E be the total space of the F-bundle over B corresponding to a homomorphism $\theta : \pi_1(B) \to Out(\pi_1(F))$. Then E virtually has a geometric decomposition if and only if* $\mathrm{Ker}(\theta) \neq 1$. *Moreover*

(1) *E admits the geometry $\mathbb{H}^2 \times \mathbb{E}^2$ if and only if θ has finite image;*

(2) *E admits the geometry $\mathbb{H}^3 \times \mathbb{E}^1$ if and only if* $\mathrm{Ker}(\theta) \cong Z$ *and* $\mathrm{Im}(\theta)$ *contains the class of a pseudo-Anasov homeomorphism of F;*

(3) *otherwise E is not geometric.*

Proof Let $\pi = \pi_1(E)$. Since E is aspherical, $\chi(E) = 0$ and π is not solvable the only possible geometries are $\mathbb{H}^2 \times \mathbb{E}^2$, $\mathbb{H}^3 \times \mathbb{E}^1$ and $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$. If E has a proper geometric decomposition the pieces must all have $\chi = 0$, and the only other geometry that may arise is $\mathbb{F}^4$. In all cases the fundamental group of each piece has a nontrivial abelian normal subgroup.

If $\mathrm{Ker}(\theta) \neq 1$ then E is virtually a cartesian product $N \times S^1$, where N is the mapping torus of a self diffeomorphism ψ of F whose isotopy class in $\pi_0(Diff(F)) \cong Out(\pi_1(F))$ generates a subgroup of finite index in $\mathrm{Im}(\theta)$. Since N is a Haken 3-manifold it has a geometric decomposition and hence so does E. The mapping torus N is an $\mathbb{H}^3$-manifold if and only if ψ is pseudo-Anasov. In that case the action of $\pi_1(N) \cong \pi_1(F) \times_\psi Z$ on H^3 extends to an embedding $p : \pi/\sqrt{\pi} \to Isom(\mathbb{H}^3)$, by Mostow rigidity. Since $\sqrt{\pi} \neq 1$ we may also find a homomorphism $\lambda : \pi \to D < Isom(E^1)$ such that $\lambda(\sqrt{\pi}) \cong Z$. Then $\mathrm{Ker}(\lambda)$ is an extension of Z by F and is commensurate with $\pi_1(N)$, so is the fundamental group of a Haken $\mathbb{H}^3$-manifold, $\widehat{N}$ say. Together these homomorphisms determine a free cocompact action of π on $H^3 \times E^1$. If $\lambda(\pi) \cong Z$ then $M = \pi\backslash(H^3 \times E^1)$ is the mapping torus of a self homeomorphism of $\widehat{N}$; otherwise it is the union of two twisted I-bundles over $\widehat{N}$. In either case it follows from standard 3-manifold theory that since E has a similar structure E and M are diffeomorphic.

If θ has finite image then $\pi/C_\pi(\pi_1(F))$ is a finite extension of $\pi_1(F)$ and so acts properly and cocompactly on $\mathbb{H}^2$. We may therefore construct an $\mathbb{H}^2 \times \mathbb{E}^2$-manifold with group π and which fibres over B as in Theorems 7.3 and 9.8. Since such bundles are determined up to diffeomorphism by their fundamental groups E admits this geometry.

Conversely, if a finite cover of E has a geometric decomposition then we may assume that the cover is itself the total space of a surface bundle over the torus, and so we may assume that E has a geometric decomposition and that $B \cong S^1 \times S^1$. Let $\phi = \pi_1(F)$. Suppose first that E has a proper geometric decomposition. Then $\pi = \pi_1(E) \cong A *_C B$ or $A *_C$, where C is solvable and of Hirsch length 3, and where A is the fundamental group of one of the pieces of E. Note that $\sqrt{A} \neq 1$. Let $\bar{A} = A/A \cap \phi$, $\bar{B} = B/B \cap \phi$ and $\bar{C} = C/C \cap \phi$. Then $\bar{\pi} = \pi/\phi \cong Z^2$ has a similar decomposition as $\bar{A} *_{\bar{C}} \bar{B}$ or $\bar{A} *_{\bar{C}}$. Now

$C \cap \phi = 1$ or Z, since $\chi(F) < 0$. Hence $\bar{C} \cong Z^2$ and so $\bar{A} = \bar{C} = \bar{B}$. In particular, $\mathrm{Im}(\theta) = \theta(A)$. But as $\sqrt{A} \cap \phi \leq \sqrt{\phi} = 1$ and $\sqrt{A}$ and $A \cap \phi$ are normal subgroups of A it follows that $\sqrt{A}$ and $A \cap \phi$ commute. Hence $\theta(A)$ is a quotient of $A/\sqrt{A}.(A \cap \phi)$, which is abelian of rank at most 1, and so $\mathrm{Ker}(\theta) \neq 1$.

If E admits the geometry $\mathbb{H}^2 \times \mathbb{E}^2$ then $\sqrt{\pi} = \pi \cap Rad(Isom(\mathbb{H}^2 \times \mathbb{E}^2)) = \pi \cap (\{1\} \times R^2) \cong Z^2$, by Proposition 8.27 of [Rg]. Hence θ has finite image.

If E admits the geometry $\mathbb{H}^3 \times \mathbb{E}^1$ then $\sqrt{\pi} = \pi \cap (\{1\} \times R) \cong Z$, by Proposition 8.27 of [Rg]. Hence $\mathrm{Ker}(\theta) \cong Z$ and E is finitely covered a cartesian product $N \times S^1$, where N is a hyperbolic 3-manifold which is also an F-bundle over S^1. The geometric monodromy of the latter bundle is a pseudo-Anasov diffeomorphism of F whose isotopy class is in $\mathrm{Im}(\theta)$.

If ρ is the group of a $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$-manifold then $\sqrt{\rho} \cong Z^2$ and $\sqrt{\rho} \cap K' \neq 1$ for all subgroups K of finite index, and so E cannot admit this geometry. □

In particular, if $\chi(B) = 0$ and θ is injective E admits no geometric decomposition.

We shall assume henceforth that B is also hyperbolic. Then $\chi(E) > 0$ and $\pi_1(E)$ has no solvable subgroups of Hirsch length 3. Hence the only possible geometries on E are $\mathbb{H}^2 \times \mathbb{H}^2$, $\mathbb{H}^4$ and $\mathbb{H}^2(\mathbb{C})$. (These are the least well understood geometries, and little is known about the possible fundamental groups of the corresponding 4-manifolds.)

Theorem 13.6 *Let B and F be closed hyperbolic surfaces, and let E be the total space of the F-bundle over B corresponding to a homomorphism $\theta : \pi_1(B) \to Out(\pi_1(F))$. Then the following are equivalent:*

(1) *E admits the geometry $\mathbb{H}^2 \times \mathbb{H}^2$;*

(2) *E is finitely covered by a cartesian product of surfaces;*

(3) *θ has finite image.*

If $\mathrm{Ker}(\theta) \neq 1$ then E does not admit either of the geometries $\mathbb{H}^4$ or $\mathbb{H}^2(\mathbb{C})$.

Proof Let $\pi = \pi_1(E)$ and $\phi = \pi_1(F)$. If E admits the geometry $\mathbb{H}^2 \times \mathbb{H}^2$ it is virtually a cartesian product, by Corollary 9.8.1, and so (1) implies (2).

If π is virtually a direct product of PD_2-groups then $[\pi : C_\pi(\phi)] < \infty$, by Theorem 5.4. Therefore the image of θ is finite and so (2) implies (3).

If θ has finite image then $\mathrm{Ker}(\theta) \neq 1$ and $\pi/C_\pi(\phi)$ is a finite extension of ϕ. Hence there is a homomorphism $p : \pi \to Isom(\mathbb{H}^2)$ with kernel $C_\pi(\phi)$ and with image a discrete cocompact subgroup. Let $q : \pi \to \pi_1(B) < Isom(\mathbb{H}^2)$. Then (p, q) embeds π as a discrete cocompact subgroup of $Isom(\mathbb{H}^2 \times \mathbb{H}^2)$, and the closed 4-manifold $M = \pi\backslash(H^2 \times H^2)$ clearly fibres over B. Such bundles are determined up to diffeomorphism by the corresponding extensions of fundamental groups, by Theorem 5.2. Therefore E admits the geometry $\mathbb{H}^2 \times \mathbb{H}^2$ and so (3) implies (1).

If θ is not injective $Z^2 < \pi$ and so E cannot admit either of the geometries $\mathbb{H}^4$ or $\mathbb{H}^2(\mathbb{C})$, by Theorem 9 of [Pr43]. □

The mapping class group of a closed orientable surface has only finitely many conjugacy classes of finite groups [Ha71]. With the finiteness result for $\mathbb{H}^4$- and $\mathbb{H}^2(\mathbb{C})$-manifolds of [Wa72], this implies that only finitely many orientable bundle spaces with given Euler characteristic are geometric. In Corollary 13.7.2 we shall show that no such bundle space is homotopy equivalent to a $\mathbb{H}^2(\mathbb{C})$-manifold. Is there one which admits the geometry $\mathbb{H}^4$? If $\mathrm{Im}(\theta)$ contains the outer automorphism class determined by a Dehn twist on F then E admits no metric of nonpositive sectional curvature [KL96].

If E has a proper geometric decomposition the pieces are reducible $\mathbb{H}^2 \times \mathbb{H}^2$-manifolds and the inclusions of the cusps induce monomorphisms on π_1. Must E be a $\mathbb{H}^2 \times \mathbb{H}^2$-manifold?

Every closed orientable $\mathbb{H}^2 \times \mathbb{H}^2$-manifold has a 2-fold cover which is a complex surface, and has signature 0. Conversely, if E is a complex surface and p is a holomorphic submersion then $\sigma(E) = 0$ implies that the fibres are isomorphic, and so E is an $\mathbb{H}^2 \times \mathbb{H}^2$-manifold [Ko99]. This is also so if p is a holomorphic fibre bundle (see §V.6 of [BPV]). Any holomorphic submersion with base of genus at most 1 or fibre of genus at most 2 is a holomorphic fibre bundle [Ks68]. There are such holomorphic submersions in which $\sigma(E) \neq 0$ and so which are not virtually products. (See §V.14 of [BPV].) The image of θ must contain the outer automorphism class determined by a pseudo-Anasov homeomorphism and not be virtually abelian [Sh97].

Orientable $\mathbb{H}^4$-manifolds also have signature 0, but no closed $\mathbb{H}^4$-manifold admits a complex structure.

If B and E are orientable $\sigma(E) = -\theta^*\tau \cap [B]$, where $\tau \in H^2(Out(\pi_1(F)); \mathbb{Z})$ is induced from a universal class in $H^2(Sp_{2g}(\mathbb{Z}); \mathbb{Z})$ via the natural representation of $Out(\pi_1(F))$ as symplectic isometries of the intersection form on $H_1(F; \mathbb{Z}) \cong Z^{2g}$ [Me73]. In particular, if $g = 2$ then $\sigma(E) = 0$. Does the genus 2 mapping class group contain any subgroups which are hyperbolic PD_2-groups?

13.3 Geometric decompositions of torus bundles

In this section we shall give some examples of torus bundles over closed surfaces which are not geometric, some of which admit geometric decompositions of type $\mathbb{F}^4$ and some of which do not. If M is a compact manifold with boundary whose interior is an $\mathbb{F}^4$-manifold of finite volume then $\pi_1(M)$ is a semidirect product $Z^2 \times_\theta F$ where $\theta : F \to GL(2,\mathbb{Z})$ is a monomorphism with image of finite index. The double $DM = M \cup_\partial M$ is fibred over a hyperbolic base but is not geometric, since $\sqrt{\pi} \cong Z^2$ but $[\pi : C_\pi(\sqrt{\pi})]$ is infinite. The orientable surface of genus 2 can be represented as a double in two distinct ways; we shall give corresponding examples of nongeometric torus bundles which admit geometric decompositions of type $\mathbb{F}^4$. (Note that $\mathbb{F}^4$-manifolds are Seifert fibred with base a punctured hyperbolic orbifold.)

1. Let $F(2)$ be the free group of rank two and let $\gamma : F(2) \to SL(2,\mathbb{Z})$ have image the commutator subgroup $SL(2,\mathbb{Z})'$, which is freely generated by $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right)$. The natural surjection from $SL(2,\mathbb{Z})$ to $PSL(2,\mathbb{Z})$ induces an isomorphism of commutator subgroups. (See §2 of Chapter 1.) The parabolic subgroup $PSL(2,\mathbb{Z})' \cap Stab(0)$ is generated by the image of $\left(\begin{smallmatrix} -1 & 0 \\ -6 & -1 \end{smallmatrix}\right)$. Hence $[Stab(0) : PSL(2,\mathbb{Z})' \cap Stab(0)] = 6 = [PSL(2,\mathbb{Z}) : PSL(2,\mathbb{Z})']$, and so $PSL(2,\mathbb{Z})'$ has a single cusp at 0. The quotient space $PSL(2,\mathbb{Z})'\backslash H^2$ is the once-punctured torus. Let $N \subset PSL(2,\mathbb{Z})'\backslash H^2$ be the complement of an open horocyclic neighbourhood of the cusp. The double DN is the closed orientable surface of genus 2. The semidirect product $\Gamma = Z^2 \times_\gamma F(2)$ is a lattice in $Isom(\mathbb{F}^4)$, and the double of the bounded manifold with interior $\Gamma\backslash F^4$ is a torus bundle over DN.

2. Let $\delta : F(2) \to SL(2,\mathbb{Z})$ have image the subgroup which is freely generated by $U = \left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)$ and $V = \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$. Let $\bar{\delta} : F(2) \to PSL(2,\mathbb{Z})$ be the composed map. Then $\bar{\delta}$ is injective and $[PSL(2,\mathbb{Z}) : \bar{\delta}(F(2))] = 6$. (Note that $\delta(F(2))$ and $-I$ together generate the level 2 congruence subgroup.) Moreover $[Stab(0) : \bar{\delta}(F(2)) \cap Stab(0)] = 2$. Hence $\bar{\delta}(F(2))$ has three cusps, at 0, ∞ and 1, and $\bar{\delta}(F(2))\backslash H^2$ is the thrice-punctured sphere. The corresponding parabolic subgroups are generated by U, V and VU^{-1}, respectively. Doubling the complement N of disjoint horocyclic neighbourhoods of the cusps in $\bar{\delta}(F(2))\backslash H^2$ again gives a closed orientable surface of genus 2. The presentation for $\pi_1(DN)$ derived from this construction is

$$\langle U, V, U_1, V_1, s, t \mid s^{-1}Us = U_1, t^{-1}Vt = V_1, VU^{-1} = V_1U_1^{-1}\rangle,$$

which simplifies to the usual presentation $\langle U, V, s, t \mid s^{-1}V^{-1}sV = t^{-1}U^{-1}tU\rangle$. The semidirect product $\Delta = Z^2 \times_\delta F(2)$ is a lattice in $Isom(\mathbb{F}^4)$, and the double of the bounded manifold with interior $\Delta\backslash F^4$ is again a torus bundle over DN.

3. If G is an orientable PD_2-group which is not virtually Z^2 and $\lambda : G \to SL(2,\mathbb{Z})$ is a homomorphism whose image is infinite cyclic then $\pi = Z^2 \times_\lambda G$ is the fundamental group of a closed orientable 4-manifold which is fibred over an orientable hyperbolic surface but which has no geometric decomposition at all. (The only possible geometries are $\mathbb{F}^4$, $\mathbb{H}^2 \times \mathbb{E}^2$ and $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$. We may exclude pieces of type $\mathbb{F}^4$ as $Im(\lambda)$ has infinite index in $SL(2,\mathbb{Z})$, and we may exclude pieces of type $\mathbb{H}^2 \times \mathbb{E}^2$ or $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$ as $Im(\lambda) \cong Z$ is not generated by finite subgroups.)

13.4 Complex surfaces and fibrations

It is an easy consequence of the classification of surfaces that a minimal compact complex surface S is ruled over a curve C of genus ≥ 2 if and only if $\pi_1(S) \cong \pi_1(C)$ and $\chi(S) = 2\chi(C)$. (See Chapter VI of [BPV].) We shall give a similar characterization of the complex surfaces which admit holomorphic submersions to complex curves of genus ≥ 2, and more generally of quotients of such surfaces by free actions of finite groups. However we shall use the classification only to handle the cases of non-Kähler surfaces.

Theorem 13.7 *Let S be a complex surface. Then S has a finite covering space which admits a holomorphic submersion onto a complex curve, with base and fibre of genus ≥ 2, if and only if $\pi = \pi_1(S)$ has normal subgroups $K < \hat{\pi}$ such that K and $\hat{\pi}/K$ are PD_2^+-groups, $[\pi : \hat{\pi}] < \infty$ and $[\pi : \hat{\pi}]\chi(S) = \chi(K)\chi(\hat{\pi}/K) > 0$.*

Proof The conditions are clearly necessary. Suppose that they hold. Then S is aspherical, by Theorem 5.2. In particular, π is torsion free and $\pi_2(S) = 0$, so S is minimal. After enlarging K if necessary we may assume that π/K has no nontrivial finite normal subgroup. Let $\widehat{S}$ be the finite covering space corresponding to $\hat{\pi}$. Then $\beta_1(\widehat{S}) \geq 4$. If $\beta_1(\widehat{S})$ were odd then $\widehat{S}$ would be minimal properly elliptic, by the classification of surfaces. But then either $\chi(S) = 0$ or $\widehat{S}$ would have a singular fibre and the projection of $\widehat{S}$ to the base curve would induce an isomorphism on fundamental groups [CZ79]. Hence $\beta_1(\widehat{S})$ is even and so $\widehat{S}$ and S are Kähler (see Theorem 4.3 of [Wl86]). Since π/K is not virtually Z^2 it is isomorphic to a discrete group of isometries of the upper half plane $\mathbb{H}^2$ and $\beta_1^{(2)}(\pi/K) \neq 0$. Hence there is a properly discontinuous holomorphic action of π/K on $\mathbb{H}^2$ and a π/K-equivariant holomorphic map from the covering space S_K to $\mathbb{H}^2$, with connected fibres, by Theorems 4.1 and 4.2 of [ABR92]. Let B and $\widehat{B}$ be the complex curves $\mathbb{H}^2/(\pi/K)$ and $\mathbb{H}^2/(\hat{\pi}/K)$, respectively, and let $h : S \to B$ and $\hat{h} : \widehat{S} \to \widehat{B}$ be the induced maps. The quotient map from $\mathbb{H}^2$ to $\hat{B}$ is a covering projection, since $\hat{\pi}/K$ is torsion free, and so $\pi_1(\hat{h})$ is an epimorphism with kernel K.

The map h is a submersion away from the preimage of a finite subset $D \subset B$. Let F be the general fibre and F_d the fibre over $d \in D$. Fix small disjoint discs $\Delta_d \subset B$ about each point of D, and let $B^* = B - \cup_{d\in D}\Delta_d$, $S^* = h^{-1}(B^*)$ and $S_d = h^{-1}(\Delta_d)$. Since $h|_{S^*}$ is a submersion $\pi_1(S^*)$ is an extension of $\pi_1(B^*)$ by $\pi_1(F)$. The inclusion of ∂S_d into $S_d - F_d$ is a homotopy equivalence. Since F_d has real codimension 2 in S_d the inclusion of $S_d - F_d$ into S_d is 2-connected. Hence $\pi_1(\partial S_d)$ maps onto $\pi_1(S_d)$.

Let $m_d = [\pi_1(F_d)] : Im(\pi_1(F))]$. After blowing up S at singular points of F_d we may assume that it has only normal crossings. We may then pull $h|_{S_d}$ back over a suitable branched covering of Δ_d to obtain a singular fibre $\widetilde{F}_d$ with no multiple components and only normal crossing singularities. In that case $\widetilde{F}_d$ is obtained from F by shrinking vanishing cycles, and so $\pi_1(F)$ maps onto $\pi_1(\widetilde{F}_d)$. Since blowing up a point on a curve does not change the fundamental group it follows from §9 of Chapter III of [BPV] that in general m_d is finite.

We may regard B as an orbifold with cone singularities of order m_d at $d \in D$. By the Van Kampen theorem (applied to the space S and the orbifold B) the image of $\pi_1(F)$ in π is a normal subgroup and h induces an isomorphism from $\pi/\pi_1(F)$ to $\pi_1^{orb}(B)$. Therefore the kernel of the canonical map from $\pi_1^{orb}(B)$ to $\pi_1(B)$ is isomorphic to $K/\mathrm{Im}(\pi_1(F))$. But this is a finitely generated normal subgroup of infinite index in $\pi_1^{orb}(B)$, and so must be trivial. Hence $\pi_1(F)$ maps onto K, and so $\chi(F) \leq \chi(K)$.

Let $\widehat{D}$ be the preimage of D in $\widehat{B}$. The general fibre of $\hat{h}$ is again F. Let $\widehat{F}_d$ denote the fibre over $d \in \widehat{D}$. Then $\chi(\widehat{S}) = \chi(F)\chi(B) + \Sigma_{d\in\widehat{D}}(\chi(\widehat{F}_d) - \chi(F))$ and $\chi(\widehat{F}_d) \geq \chi(F)$, by Proposition III.11.4 of [BPV]. Moreover $\chi(\widehat{F}_d) > \chi(F)$ unless $\chi(\widehat{F}_d) = \chi(F) = 0$, by Remark III.11.5 of [BPV]. Since $\chi(\widehat{B}) = \chi(\hat{\pi}/K) < 0$, $\chi(\widehat{S}) = \chi(K)\chi(\hat{\pi}/K)$ and $\chi(F) \leq \chi(K)$ it follows that $\chi(F) = \chi(K) < 0$ and $\chi(\widehat{F}_d) = \chi(F)$ for all $d \in \widehat{D}$. Therefore $\widehat{F}_d \cong F$ for all $d \in \widehat{D}$ and so $\hat{h}$ is a holomorphic submersion. □

Similar results have been found independently by Kapovich and Kotschick [Ka98, Ko99]. Kapovich assumes instead that K is FP_2 and S is aspherical. As these hypotheses imply that K is a PD_2-group, by Theorem 1.19, the above theorem applies.

We may construct examples of such surfaces as follows. Let $n > 1$ and C_1 and C_2 be two curves such that Z/nZ acts freely on C_1 and with isolated fixed points on C_2. Then the quotient S of $C_1 \times C_2$ under the induced action is a complex surface and the projection from $C_1 \times C_2$ to C_2 induces a surjective holomorphic mappping from S to $C_2/(Z/nZ)$ with critical values corresponding to the fixed points.

Corollary 13.7.1 *The surface S admits such a holomorphic submersion onto a complex curve if and only if π/K is a PD_2^+-group.* □

Corollary 13.7.2 *No bundle space E is homotopy equivalent to a closed $\mathbb{H}^2(\mathbb{C})$-manifold.*

Proof Since $\mathbb{H}^2(\mathbb{C})$-manifolds have 2-fold coverings which are complex surfaces, we may assume that E is homotopy equivalent to a complex surface S. By the theorem, S admits a holomorphic submersion onto a complex curve. But then $\chi(S) > 3\sigma(S)$ [Li96], and so S cannot be a $\mathbb{H}^2(\mathbb{C})$-manifold. □

The relevance of Liu's work was observed by Kapovich, who has also found a cocompact $\mathbb{H}^2(\mathbb{C})$-lattice which is an extension of a PD_2^+-group by a finitely generated normal subgroup, but which is not almost coherent [Ka98].

Similar arguments may be used to show that a Kähler surface S is a minimal properly elliptic surface with no singular fibres if and only if $\chi(S) = 0$ and $\pi = \pi_1(S)$ has a normal subgroup $A \cong Z^2$ such that π/A is virtually torsion free and indicable, but is not virtually abelian. (This holds also in the non-Kähler case as a consequence of the classification of surfaces.) Moreover, if S is not a ruled surface then it is a complex torus, a hyperelliptic surface, an Inoue surface, a Kodaira surface or a minimal elliptic surface if and only if $\chi(S) = 0$ and $\pi_1(S)$ has a normal subgroup A which is poly-Z and not cyclic, and such that π/A is infinite and virtually torsion free indicable. (See Theorem X.5 of [H2].)

We may combine Theorem 13.7 with some observations deriving from the classification of surfaces for our second result.

Theorem 13.8 *Let S be a complex surface such that $\pi = \pi_1(S) \neq 1$. If S is homotopy equivalent to the total space E of a bundle over a closed orientable 2-manifold then S is diffeomorphic to E.*

Proof Let B and F be the base and fibre of the bundle, respectively. Suppose first that $\chi(F)=2$. Then $\chi(B)\leq 0$, for otherwise S would be simply-connected. Hence $\pi_2(S)$ is generated by an embedded S^2 with self-intersection 0, and so S is minimal. Therefore S is ruled over a curve diffeomorphic to B, by the classification of surfaces.

Suppose next that $\chi(B)=2$. If $\chi(F)=0$ and $\pi\not\cong Z^2$ then $\pi\cong Z\oplus(Z/nZ)$ for some $n>0$. Then S is a Hopf surface and so is determined up to diffeomorphism by its homotopy type, by Theorem 12 of [Kt75]. If $\chi(F)=0$ and $\pi\cong Z^2$ or if $\chi(F)<0$ then S is homotopy equivalent to $S^2\times F$, so $\chi(S)<0$, $w_1(S)=w_2(S)=0$ and S is ruled over a curve diffeomorphic to F. Hence E and S are diffeomorphic to $S^2\times F$.

In the remaining cases E and F are both aspherical. If $\chi(F)=0$ and $\chi(B)\leq 0$ then $\chi(S)=0$ and π has one end. Therefore S is a complex torus, a hyperelliptic surface, an Inoue surface, a Kodaira surface or a minimal properly elliptic surface. (This uses Bogomolov's theorem on class VII_0 surfaces [Te94].) The Inoue surfaces are mapping tori of self-diffeomorphisms of $S^1\times S^1\times S^1$, and their fundamental groups are not extensions of Z^2 by Z^2, so S cannot be an Inoue surface. As the other surfaces are Seifert fibred 4-manifolds E and S are diffeomorphic, by [Ue91].

If $\chi(F)<0$ and $\chi(B)=0$ then S is a minimal properly elliptic surface. Let A be the normal subgroup of the general fibre in an elliptic fibration. Then $A\cap\pi_1(F)=1$ (since $\pi_1(F)$ has no nontrivial abelian normal subgroup) and so $[\pi:A.\pi_1(F)]<\infty$. Therefore E is finitely covered by a cartesian product $T\times F$, and so is Seifert fibred. Hence E and S are diffeomorphic, by [Ue].

The remaining case ($\chi(B)<0$ and $\chi(F)<0$) is an immediate consequence of Theorem 13.7, since such bundles are determined by the corresponding extensions of fundamental groups (see Theorem 5.2). □

A simply-connected smooth 4-manifold which fibres over a 2-manifold must be homeomorphic to $CP^1\times CP^1$ or $CP^2\sharp\overline{CP^2}$. (See Chapter 12.) Is there such a surface of general type? (No surface of general type is diffeomorphic to $CP^1\times CP^1$ or $CP^2\sharp\overline{CP^2}$ [Qi93].)

Corollary 13.8.1 *If moreover the base has genus* 0 *or* 1 *or the fibre has genus* 2 *then* S *is finitely covered by a cartesian product.*

Proof A holomorphic submersion with fibre of genus 2 is the projection of a holomorphic fibre bundle and hence S is virtually a product, by [Ks68]. □

Up to deformation there are only finitely many algebraic surfaces with given Euler characteristic > 0 which admit holomorphic submersions onto curves [Pa68]. By the argument of the first part of Theorem 13.1 this remains true without the hypothesis of algebraicity, for any such complex surface must be Kähler, and Kähler surfaces are deformations of algebraic surfaces (see Theorem 4.3 of [Wl86]). Thus the class of bundles realized by complex surfaces is very restricted. Which extensions of PD_2^+-groups by PD_2^+-groups are realized by complex surfaces (i.e., not necessarily aspherical)?

The equivalence of the conditions "S is ruled over a complex curve of genus ≥ 2", "$\pi = \pi_1(S)$ is a PD_2^+-group and $\chi(S) = 2\chi(\pi) < 0$" and "$\pi_2(S) \cong Z$, π acts trivially on $\pi_2(S)$ and $\chi(S) < 0$" also follows by an argument similar to that used in Theorems 13.7 and 13.8. (See Theorem X.6 of [H2].)

If $\pi_2(S) \cong Z$ and $\chi(S) = 0$ then π is virtually Z^2. The finite covering space with fundamental group Z^2 is Kähler, and therefore so is S. Since $\beta_1(S) > 0$ and is even, we must have $\pi \cong Z^2$, and so S is either ruled over an elliptic curve or is a minimal properly elliptic surface, by the classification of complex surfaces. In the latter case the base of the elliptic fibration is CP^1, there are no singular fibres and there are at most 3 multiple fibres. (See [Ue91].) Thus S may be obtained from a cartesian product $CP^1 \times E$ by logarithmic transformations. (See §V.13 of [BPV].) Must S in fact be ruled?

If $\pi_2(S) \cong Z$ and $\chi(S) > 0$ then $\pi = 1$, by Theorem 10.1. Hence $S \simeq CP^2$ and so S is analytically isomorphic to CP^2, by a result of Yau (see Theorem I.1 of [BPV]).

13.5 S^1-Actions and foliations by circles

For each of the geometries $\mathbb{X}^4 = \mathbb{S}^3 \times \mathbb{E}^1$, $\mathbb{H}^3 \times \mathbb{E}^1$, $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$, $\mathbb{N}il^3 \times \mathbb{E}^1$, $\mathbb{S}ol^3 \times \mathbb{E}^1$, $\mathbb{N}il^4$ and $\mathbb{S}ol_1^4$ the real line R is a characteristic subgroup of the radical of $Isom(\mathbb{X}^4)$. (However the translation subgroup of the euclidean factor is *not* characteristic if $\mathbb{X}^4 = \widetilde{\mathbb{SL}} \times \mathbb{E}^1$ or $\mathbb{N}il^3 \times \mathbb{E}^1$.) The corresponding closed geometric 4-manifolds are foliated by circles, and the leaf space is a geometric 3-orbifold, with geometry $\mathbb{S}^3$, $\mathbb{H}^3$, $\mathbb{H}^2 \times \mathbb{E}^1$, $\mathbb{E}^3$, $\mathbb{S}ol^3$, $\mathbb{N}il^3$ and $\mathbb{S}ol^3$, respectively. In each case it may be verified that if π is a lattice in $Isom(\mathbb{X}^4)$ then $\pi \cap R \cong Z$. As this characteristic subgroup is central in the identity component of the isometry group such manifolds have double coverings which admit S^1-actions without fixed points. These actions lift to principal S^1-actions (without exceptional orbits) on suitable finite covering spaces. (This does not hold for all S^1-actions. For instance, S^3 admits non-principal S^1-actions without fixed points.)

Closed $\mathbb{E}^4$-, $\mathbb{S}^2\times\mathbb{E}^2$- or $\mathbb{H}^2\times\mathbb{E}^2$-manifolds all have finite covering spaces which are cartesian products with S^1, and thus admit principal S^1-actions. However these actions are not canonical. (There are also non-canonical S^1-actions on many $\widetilde{\mathbb{SL}}\times\mathbb{E}^1$- and $\mathbb{N}il^3\times\mathbb{E}^1$-manifolds.) No other closed geometric 4-manifold is finitely covered by the total space of an S^1-bundle. For if a closed manifold M is foliated by circles then $\chi(M)=0$. This excludes all other geometries except $\mathbb{S}ol^4_{m,n}$ and $\mathbb{S}ol^4_0$. If moreover M is the total space of an S^1-bundle and is aspherical then $\pi_1(M)$ has an infinite cyclic normal subgroup. As lattices in $Isom(\mathbb{S}ol^4_{m,n})$ or $Isom(\mathbb{S}ol^4_0)$ do not have such subgroups these geometries are excluded also. Does every geometric 4-manifold M with $\chi(M)=0$ nevertheless admit a foliation by circles?

In particular, a complex surface has a foliation by circles if and only if it admits one of the above geometries. Thus it must be Hopf, hyperelliptic, Inoue of type $S^{\pm}_{N...}$, Kodaira, minimal properly elliptic, ruled over an elliptic curve or a torus. With the exception of some algebraic minimal properly elliptic surfaces and the ruled surfaces over elliptic curves with $w_2\neq 0$ all such surfaces admit S^1-actions without fixed points.

Conversely, the total space E of an S^1-orbifold bundle ξ over a geometric 3-orbifold is geometric, except when the base B has geometry $\mathbb{H}^3$ or $\widetilde{\mathbb{SL}}$ and the characteristic class $c(\xi)$ has infinite order. More generally, E has a (proper) geometric decomposition if and only if B is a $\widetilde{\mathbb{SL}}$-orbifold and $c(\xi)$ has finite order or B has a (proper) geometric decomposition and the restrictions of $c(\xi)$ to the hyperbolic pieces of B each have finite order.

Total spaces of circle bundles over aspherical Seifert fibred 3-manifolds and $\mathbb{S}ol^3$-manifolds have a characterization parallel to that of Theorem 13.2.

Theorem 13.9 *Let M be a closed 4-manifold with fundamental group π. Then:*

(1) *M is simple homotopy equivalent to the total space E of an S^1-bundle over an aspherical closed Seifert fibred 3-manifold or a $\mathbb{S}ol^3$-manifold if and only if $\chi(M)=0$ and π has normal subgroups $A<B$ such that $A\cong Z$, π/A is torsion free and B/A is abelian.*
If $B/A\cong Z$ and is central in π/A then M is s-cobordant to E. If B/A has rank at least 2 then M is homeomorphic to E.

(2) *M is s-cobordant to the total space E of an S^1-bundle over the mapping torus of a self homeomorphism of an aspherical surface if and only if $\chi(M)=0$ and π has normal subgroups $A<B$ such that $A\cong Z$, π/A is torsion free, B is FP_2 and $\pi/B\cong Z$.*

Proof (1) The conditions are clearly necessary. If they hold then $h(\sqrt{\pi}) \geq h(B/A) + 1 \geq 2$, and so M is aspherical. If $h(\sqrt{\pi}) = 2$ then $\sqrt{\pi} \cong Z^2$, by Theorem 9.2. Hence $B/A \cong Z$ and $H^2(\pi/B; \mathbb{Z}[\pi/B]) \cong Z$, so π/B is virtually a PD_2-group, by Bowditch's Theorem. Since π/A is torsion free it is a PD_3-group, and so is the fundamental group of a closed Seifert fibred 3-manifold, N say, by Theorem 2.14. As $Wh(\pi) = 0$, by Theorem 6.4, M is simple homotopy equivalent to the total space E of an S^1-bundle over N. If moreover B/A is central in π/A then N admits an effective S^1-action, and $E \times S^1$ is an $S^1 \times S^1$-bundle over N. Hence $M \times S^1$ is homeomorphic to $E \times S^1$ (see Remark 3.4 of [NS85]), and so M is s-cobordant to E.

If B/A has rank at least 2 then $h(\sqrt{\pi}) > 2$ and so π is virtually poly-Z. Hence π/A is the fundamental group of a $\mathbb{E}^3$-, $\mathbb{N}il^3$- or $\mathbb{S}ol^3$-manifold and M is homeomorphic to such a bundle space E, by Theorem 6.11.

(2) The conditions are again necessary. If they hold then B/A is infinite, so B has one end and hence is a PD_3-group, by Theorem 4.1. Since B/A is torsion free it is a PD_2-group, by Bowditch's Theorem, and so π/A is the fundamental group of a mapping torus, N say. As $Wh(\pi) = 0$, by Theorem 6.4, M is simple homotopy equivalent to the total space E of an S^1-bundle over N. Since $\pi \times Z$ is square root closed accessible $M \times S^1$ is homeomorphic to $E \times S^1$ [Ca73], and so M is s-cobordant to E. □

If $B/A \cong Z$ and π/B acts nontrivially on B/A is M s-cobordant to E?

Simple homotopy equivalence implies s-cobordism for such bundles over other Haken bases (with square root closed accessible fundamental group or with $\beta_1 > 0$ and orientable) using [Ca73] or [Ro00]. However we do not yet have good intrinsic characterizations of the fundamental groups of such 3-manifolds.

If M fibres over a hyperbolic 3-manifold N then $\chi(M) = 0$, $\sqrt{\pi} \cong Z$ and $\pi/\sqrt{\pi}$ has one end, finite cohomological dimension and no noncyclic abelian subgroups. Conversely if π satisfies these conditions then $\rho = \pi/\sqrt{\pi}$ is a PD_3-group, by Theorem 4.12, and $\sqrt{\rho} = 1$. It may be conjectured that every such PD_3-group (with no nocyclic abelian subgroups and trivial Hirsch-Plotkin radical) is the fundamental group of a closed hyperbolic 3-manifold. If so, Theorem 13.9 may be extended to a characterization of such 4-manifolds up to s-cobordism, using Theorem 10.7 of [FJ89] instead of [NS85].

13.6 Symplectic structures

If M is a closed orientable 4-manifold which fibres over an orientable surface and the image of the fibre in $H_2(M; \mathbb{R})$ is nonzero then M has a symplectic structure [Th76]. The homological condition is automatic unless the fibre is a torus; some such condition is needed, as $S^3 \times S^1$ is the total space of a T-bundle over S^2 but $H^2(S^3 \times S^1;\mathbb{R}) = 0$, so it has no symplectic structure. If the base is also a torus then M admits a symplectic structure [Ge92]. Closed Kähler manifolds have natural symplectic structures. Using these facts, it is easy to show for most geometries that either every closed geometric manifold is finitely covered by one admitting a symplectic structure or no closed geometric manifold admits any symplectic structure.

If M is orientable and admits one of the geometries $\mathbb{CP}^2$, $\mathbb{S}^2 \times \mathbb{S}^2$, $\mathbb{S}^2 \times \mathbb{E}^2$, $\mathbb{S}^2 \times \mathbb{H}^2$, $\mathbb{H}^2 \times \mathbb{E}^2$, $\mathbb{H}^2 \times \mathbb{H}^2$ or $\mathbb{H}^2(\mathbb{C})$ then it has a 2-fold cover which is Kähler, and therefore symplectic. If it admits $\mathbb{E}^4$, $\mathbb{N}il^4$, $\mathbb{N}il^3 \times \mathbb{E}^1$ or $\mathbb{S}ol^3 \times \mathbb{E}^1$ then it has a finite cover which fibres over the torus, and therefore is symplectic. If all $\mathbb{H}^3$-manifolds are virtually mapping tori then $\mathbb{H}^3 \times \mathbb{E}^1$-manifolds would also be virtually symplectic. However, the question is not settled for this geometry.

As any closed orientable manifold with one of the geometries $\mathbb{S}^4$, $\mathbb{S}^3 \times \mathbb{E}^1$, $\mathbb{S}ol^4_{m,n}$ (with $m \neq n$), $\mathbb{S}ol^4_0$ or $\mathbb{S}ol^4_1$ has $\beta_2 = 0$ no such manifold can be symplectic. Nor are closed $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$-manifolds [Et01]. The question appears open for the geometry $\mathbb{H}^4$, as is the related question about bundles. (Note that symplectic 4-manifolds with index 0 have Euler characteristic divisible by 4, by Corollary 10.1.10 of [GS]. Hence covering spaces of odd degree of the Davis 120-cell space provide many examples of nonsymplectic $\mathbb{H}^4$-manifolds.)

If N is a 3-manifold which is a mapping torus then $S^1 \times N$ fibres over T, and so admits a symplectic structure. Taubes has asked whether the converse is true; if $S^1 \times N$ admits a symplectic structure must N fibre over S^1? More generally, one might ask which 4-dimensional mapping tori and S^1-bundles are symplectic?

Which manifolds with geometric decompositions are symplectic?

Part III

2-Knots

Chapter 14

Knots and links

In this chapter we introduce the basic notions and constructions of knot theory. Many of these apply equally well in all dimensions, and for the most part we have framed our definitions in such generality, although our main concern is with 2-knots (embeddings of S^2 in S^4). In particular, we show how the classification of higher dimensional knots may be reduced (essentially) to the classification of certain closed manifolds, and we give Kervaire's characterization of high dimensional knot groups.

In the final sections we comment briefly on links and link groups.

14.1 Knots

The standard orientation of R^n induces an orientation on the unit n-disc $D^n = \{(x_1, \dots x_n) \in R^n \mid \Sigma x_i^2 \leq 1\}$ and hence on its boundary $S^{n-1} = \partial D^n$, by the convention "outward normal first". We shall assume that standard discs and spheres have such orientations. Qualifications shall usually be omitted when there is no risk of amiguity. In particular, we shall often abbreviate $X(K)$, $M(K)$ and πK (defined below) as X, M and π, respectively.

An *n-knot* is a locally flat embedding $K : S^n \to S^{n+2}$. (We shall also use the terms "classical knot" when $n = 1$, "higher dimensional knot" when $n \geq 2$ and "high dimensional knot" when $n \geq 3$.) It is determined up to (ambient) isotopy by its image $K(S^n)$, considered as an oriented codimension 2 submanifold of S^{n+2}, and so we may let K also denote this submanifold. Let r_n be an orientation reversing self homeomorphism of S^n. Then K is *invertible, +amphicheiral* or *-amphicheiral* if it is isotopic to $rK = r_{n+2}K$, $K\rho = Kr_n$ or $-K = rK\rho$, respectively. An n-knot is *trivial* if it is isotopic to the composite of equatorial inclusions $S^n \subset S^{n+1} \subset S^{n+2}$.

Every knot has a product neighbourhood: there is an embedding $j : S^n \times D^2$ onto a closed neighbourhood N of K, such that $j(S^n \times \{0\}) = K$ and ∂N is bicollared in S^{n+2} [KS75,FQ]. We may assume that j is orientation preserving, and it is then unique up to isotopy *rel* $S^n \times \{0\}$. The *exterior* of K is the compact $(n+2)$-manifold $X(K) = S^{n+2} - intN$ with boundary $\partial X(K) \cong$

$S^n \times S^1$, and is well defined up to homeomorphism. It inherits an orientation from S^{n+2}. An n-knot K is trivial if and only if $X(K) \simeq S^1$; this follows from Dehn's Lemma if $n = 1$, is due to Freedman if $n = 2$ ([FQ] - see Corollary 17.1.1 below) and is an easy consequence of the s-cobordism theorem if $n \geq 3$.

The *knot group* is $\pi K = \pi_1(X(K))$. An oriented simple closed curve isotopic to the oriented boundary of a transverse disc $\{j\} \times S^1$ is called a *meridian* for K, and we shall also use this term to denote the corresponding elements of π. If μ is a meridian for K, represented by a simple closed curve on ∂X then $X \cup_\mu D^2$ is a deformation retract of $S^{n+2} - \{*\}$ and so is contractible. Hence π is generated by the conjugacy class of its meridians.

Assume for the remainder of this section that $n \geq 2$. The group of pseudoisotopy classes of self homeomorphisms of $S^n \times S^1$ is $(Z/2Z)^3$, generated by reflections in either factor and by the map τ given by $\tau(x, y) = (\rho(y)(x), y)$ for all x in S^n and y in S^1, where $\rho : S^1 \to SO(n + 1)$ is an essential map [Gl62, Br67, Kt69]. As any self homeomorphism of $S^n \times S^1$ extends across $D^{n+1} \times S^1$ the *knot manifold* $M(K) = X(K) \cup (D^{n+1} \times S^1)$ obtained from S^{n+2} by surgery on K is well defined, and it inherits an orientation from S^{n+2} via X. Moreover $\pi_1(M(K)) \cong \pi K$ and $\chi(M(K)) = 0$. Conversely, suppose that M is a closed orientable 4-manifold with $\chi(M) = 0$ and $\pi_1(M)$ is generated by the conjugacy class of a single element. (Note that each conjugacy class in π corresponds to an unique isotopy class of oriented simple closed curves in M.) Surgery on a loop in M representing such an element gives a 1-connected 4-manifold Σ with $\chi(\Sigma) = 2$ which is thus homeomorphic to S^4 and which contains an embedded 2-sphere as the cocore of the surgery. We shall in fact study 2-knots through such 4-manifolds, as it is simpler to consider closed manifolds rather than pairs.

There is however an ambiguity when we attempt to recover K from $M = M(K)$. The cocore $\gamma = \{0\} \times S^1 \subset D^{n+1} \times S^1 \subset M$ of the original surgery is well defined up to isotopy by the conjugacy class of a meridian in $\pi K = \pi_1(M)$. (In fact the orientation of γ is irrelevant for what follows.) Its normal bundle is trivial, so γ has a product neighbourhood, P say, and we may assume that $M - intP = X(K)$. But there are two essentially distinct ways of identifying ∂X with $S^n \times S^1 = \partial(S^n \times D^2)$, modulo self homeomorphisms of $S^n \times S^1$ that extend across $S^n \times D^2$. If we reverse the original construction of M we recover $(S^{n+2}, K) = (X \cup_j S^n \times D^2, S^n \times \{0\})$. If however we identify $S^n \times S^1$ with ∂X by means of $j\tau$ we obtain a new pair

$$(\Sigma, K^*) = (X \cup_{j\tau} S^n \times D^2, S^n \times \{0\}).$$

It is easily seen that $\Sigma \simeq S^{n+2}$, and hence $\Sigma \cong S^{n+2}$. We may assume that the homeomorphism is orientation preserving. Thus we obtain a new n-knot

K^*, which we shall call the *Gluck reconstruction* of K. The knot K is *reflexive* if it is determined as an unoriented submanifold by its exterior, i.e., if K^* is isotopic to K, rK, $K\rho$ or $-K$.

If there is an orientation preserving homeomorphism from $X(K_1)$ to $X(K)$ then K_1 is isotopic to K, K^*, $K\rho$ or $K^*\rho$. If the homeomorphism also preserves the homology class of the meridians then K_1 is isotopic to K or to K^*. Thus K is determined up to an ambiguity of order at most 2 by $M(K)$ together with the conjugacy class of a meridian.

A *Seifert hypersurface* for K is a locally flat, oriented codimension 1 submanifold V of S^{n+2} with (oriented) boundary K. By a standard argument these always exist. (Using obstruction theory it may be shown that the projection $pr_2 j^{-1} : \partial X \to S^n \times S^1 \to S^1$ extends to a map $p : X \to S^1$ [Ke65]. By topological transversality we may assume that $p^{-1}(1)$ is a bicollared, proper codimension 1 submanifold of X. The union $p^{-1}(1) \cup j(S^n \times [0,1])$ is then a Seifert hypersurface for K.) We shall say that V is *minimal* if the natural homomorphism from $\pi_1(V)$ to πK is a monomorphism.

In general there is no canonical choice of Seifert surface. However there is one important special case. An n-knot K is *fibred* if there is such a map $p : X \to S^1$ which is the projection of a fibre bundle. (Clearly K^* is then fibred also.) The exterior is then the mapping torus of a self homeomorphism θ of the fibre F of p. The isotopy class of θ is called the *(geometric) monodromy* of the bundle. Such a map p extends to a fibre bundle projection $q : M(K) \to S^1$, with fibre $\widehat{F} = F \cup D^{n+1}$, called the *closed fibre* of K. Conversely, if $M(K)$ fibres over S^1 then the cocore γ is homotopic (and thus isotopic) to a cross-section of the bundle projection, and so K is fibred. If the monodromy has finite order (and is nontrivial) then it has precisely two fixed points on ∂F, and we may assume that the closed monodromy also has finite order. However the converse is false; the closed monodromy may have finite order but not be isotopic to a map of finite order with nonempty fixed point set.

14.2 Covering spaces

Let K be an n-knot. Then $H_1(X(K);\mathbb{Z}) \cong Z$ and $H_i(X(K);\mathbb{Z}) = 0$ if $i > 1$, by Alexander duality. The meridians are all homologous and generate $\pi/\pi' = H_1(X;\mathbb{Z})$, and so determine a canonical isomorphism with Z. Moreover $H_2(\pi;\mathbb{Z}) = 0$, since it is a quotient of $H_2(X;\mathbb{Z}) = 0$.

We shall let $X'(K)$ and $M'(K)$ denote the covering spaces corresponding to the commutator subgroup. (The cover X'/X is also known as the *infinite* *cyclic cover* of the knot.) Since $\pi/\pi' = Z$ the (co)homology groups of X' are modules over the group ring $\mathbb{Z}[Z]$, which may be identified with the ring of integral Laurent polynomials $\Lambda = \mathbb{Z}[t, t^{-1}]$. If A is a Λ-module, let zA be the $\mathbb{Z}$-torsion submodule, and let $e^iA = Ext^i_\Lambda(A, \Lambda)$.

Since Λ is noetherian the (co)homology of a finitely generated free Λ-chain complex is finitely generated. The Wang sequence for the projection of X' onto X may be identified with the long exact sequence of homology corresponding to the exact sequence of coefficients

$$0 \to \Lambda \to \Lambda \to Z \to 0.$$

Since X has the homology of a circle it follows easily that multiplication by $t - 1$ induces automorphisms of the modules $H_i(X; \Lambda)$ for $i > 0$. Hence these homology modules are all finitely generated torsion Λ-modules. It follows that $Hom_\Lambda(H_i(X; \Lambda), \Lambda)$ is 0 for all i, and the UCSS collapses to a collection of short exact sequences

$$0 \to e^2H_{i-2} \to H^i(X; \Lambda) \to e^1H_{i-1} \to 0.$$

The infinite cyclic covering spaces X' and M' behave homologically much like $(n+1)$-manifolds, at least if we use field coefficients [Mi68, Ba80]. If $H_i(X; \Lambda) = 0$ for $1 \leq i \leq (n+1)/2$ then X' is acyclic; thus if also $\pi = Z$ then $X \simeq S^1$ and so K is trivial. All the classifications of high dimensional knots to date assume that $\pi = Z$ and that X' is highly connected.

When $n = 1$ or 2 knots with $\pi = Z$ are trivial, and it is more profitable to work with the universal cover $\widetilde{X}$ (or $\widetilde{M}$). In the classical case $\widetilde{X}$ is contractible [Pa57]. In higher dimensions X is aspherical only when the knot is trivial [DV73]. Nevertheless the closed 4-manifolds $M(K)$ obtained by surgery on 2-knots are often aspherical. (This asphericity is an additional reason for choosing to work with $M(K)$ rather than $X(K)$.)

14.3 Sums, factorization and satellites

The *sum* of two knots K_1 and K_2 may be defined (up to isotopy) as the n-knot $K_1\sharp K_2$ obtained as follows. Let $D^n(\pm)$ denote the upper and lower hemispheres of S^n. We may isotope K_1 and K_2 so that each $K_i(D^n(\pm))$ contained in $D^{n+2}(\pm)$, $K_1(D^n(+))$ is a trivial n-disc in $D^{n+2}(+)$, $K_2(D^n(-))$ is a trivial n-disc in $D^{n+2}(-)$ and $K_1|_{S^{n-1}} = K_2|_{S^{n-1}}$ (as the oriented boundaries of the images of $D^n(-)$). Then we let $K_1\sharp K_2 = K_1|_{D^n(-)} \cup K_2|_{D^n(+)}$. By van Kampen's theorem $\pi(K_1\sharp K_2) = \pi K_1 *_Z \pi K_2$ where the amalgamating subgroup is generated by a meridian in each knot group. It is not hard to see that $X'(K_1\sharp K_2) \simeq X'(K_1) \vee X'(K_2)$ and so in particular $\pi'(K_1\sharp K_2) \cong \pi'(K_1) * \pi'(K_2)$.

The knot K is *irreducible* if it is not the sum of two nontrivial knots. Every knot has a finite factorization into irreducible knots [DF87]. (For 1- and 2-knots whose groups have finitely generated commutator subgroups this follows easily from the Grushko-Neumann theorem on factorizations of groups as free products.) In the classical case the factorization is essentially unique, but for each $n \geq 3$ there are n-knots with several distinct such factorizations [BHK81]. Essentially nothing is known about uniqueness (or otherwise) of factorization when $n = 2$.

If K_1 and K_2 are fibred then so is their sum, and the closed fibre of $K_1\sharp K_2$ is the connected sum of the closed fibres of K_1 and K_2. However in the absence of an adequate criterion for a 2-knot to fibre, we do not know whether every summand of a fibred 2-knot is fibred. In view of the unique factorization theorem for oriented 3-manifolds we might hope that there would be a similar theorem for fibred 2-knots. However the closed fibre of an irreducible 2-knot need not be an irreducible 3-manifold. (For instance, the Artin spin of a trefoil knot is an irreducible fibred 2-knot, but its closed fibre is $(S^2 \times S^1)\sharp(S^2 \times S^1)$).

A more general method of combining two knots is the process of forming satellites. Although this process arose in the classical case, where it is intimately connected with the notion of torus decomposition, we shall describe only the higher-dimensional version of [Kn83]. Let K_1 and K_2 be n-knots (with $n \geq 2$) and let γ be a simple closed curve in $X(K_1)$, with a product neighbourhood U. Then there is a homeomomorphism h which carries $S^{n+2} - intU \cong S^n \times D^2$ onto a product neighbourhood of K_2. The knot $\Sigma(K_2; K_1, \gamma)$ is called the *satellite* of K_1 about K_2 relative to γ. We also call K_2 a *companion* of hK_1. If either $\gamma = 1$ or K_2 is trivial then $\Sigma(K_2; K_1, \gamma) = K_1$. If γ is a meridian for K_1 then $\Sigma(K_2; K_1, \gamma) = K_1\sharp K_2$. If γ has finite order in πK_1 let q be that order; otherwise let $q = 0$. Let w be a meridian in πK_2. Then $\pi = \pi K \cong (\pi K_2/\langle\langle w^q\rangle\rangle) *_{Z/qZ} \pi K_1$, where w is identified with γ in πK_1, by Van Kampen's theorem.

14.4 Spinning and twist spinning

The first nontrivial examples of higher dimensional knots were given by Artin [Ar25]. We may paraphrase his original idea as follows. As the half space $R^3_+ = \{(w, x, y, z) \in R^4 \mid w = 0, z \geq 0\}$ is spun about the axis $A = \{(0, x, y, 0)\}$

it sweeps out the whole of R^4, and any arc in R^3_+ with endpoints on A sweeps out a 2-sphere.

Fox incorporated a twist into Artin's construction [Fo66]. Let r be an integer and choose a small $(n+2)$-disc B^{n+2} which meets K in an n-disc B^n such that (B^{n+2}, B^n) is homeomorphic to the standard pair. Then $S^{n+2} - intB^{n+2} = D^n \times D^2$, and we may choose the homeomorphism so that $\partial(K - intB^n)$ lies in $\partial D^n \times \{0\}$. Let ρ_θ be the self homeomorphism of $D^n \times D^2$ that rotates the D^2 factor through θ radians. Then $\cup_{0\leq\theta<2\pi}(\rho_{r\theta}(K - intB^n) \times \{\theta\})$ is a submanifold of $(S^{n+2} - intB^{n+2}) \times S^1$ homeomorphic to $D^n \times S^1$ and which is standard on the boundary. The *r-twist spin* of K is the $(n+1)$-knot $\tau_r K$ with image

$$\tau_r K = \cup_{0\leq\theta<2\pi}(\rho_{r\theta}(K - intB^n) \times \{\theta\})) \cup (S^{n-1} \times D^2)$$

in $S^{n+3} = ((S^{n+2} - intB^{n+2}) \times S^1) \cup (S^{n+1} \times D^2)$.

The 0-twist spin is the *Artin spin* $\sigma K = \tau_0 K$, and $\pi\sigma K \cong \pi K$. The group of $\tau_r K$ is obtained from πK by adjoining the relation making the r^{th} power of (any) meridian central. Zeeman discovered the remarkable fact that if $r \neq 0$ then $\tau_r K$ is fibred, with geometric monodromy of order dividing r, and the closed fibre is the r-fold cyclic branched cover of S^{n+2}, branched over K [Ze65]. Hence $\tau_1 K$ is always trivial. Twist spins of -amphicheiral knots are -amphicheiral, while twist spinning interchanges invertibility and +amphicheirality [Li85].

If K is a classical knot the factors of the closed fibre of $\tau_r K$ are the cyclic branched covers of the prime factors of K, and are Haken, hyperbolic or Seifert fibred. With some exceptions for small values of r, the factors are aspherical, and $S^2 \times S^1$ is never a factor [Pl84]. If $r > 1$ and K is nontrivial then $\tau_r K$ is nontrivial, by the Smith Conjecture.

For other formulations and extensions of twist spinning see [GK78], [Li79], [Mo83,84] and [Pl84'].

14.5 Ribbon and slice knots

An n-knot K is a *slice* knot if it is concordant to the unknot; equivalently, if it bounds a properly embedded $(n+1)$-disc Δ in D^{n+3}. Such a disc is called a *slice disc* for K. Doubling the pair (D^{n+3}, Δ) gives an $(n+1)$-knot which meets the equatorial S^{n+2} of S^{n+3} transversally in K; if the $(n+1)$-knot can be chosen to be trivial then K is *doubly slice*. All even-dimensional knots are slice [Ke65], but not all slice knots are doubly slice, and no adequate criterion is yet known. The sum $K\sharp - K$ is a slice of $\tau_1 K$ and so is doubly slice [Su71].

An n-knot K is a *ribbon* knot if it is the boundary of an immersed $(n+1)$-disc Δ in S^{n+2} whose only singularities are transverse double points, the double point sets being a disjoint union of discs. Given such a "ribbon" $(n+1)$-disc Δ in S^{n+2} the cartesian product $\Delta \times D^p \subset S^{n+2} \times D^p \subset S^{n+2+p}$ determines a ribbon $(n+1+p)$-disc in S^{n+2+p}. All higher dimensional ribbon knots derive from ribbon 1-knots by this process [Yn77]. As the p-disc has an orientation reversing involution this easily imples that all ribbon n-knots with $n \geq 2$ are -amphicheiral. The Artin spin of a 1-knot is a ribbon 2-knot. Each ribbon 2-knot has a Seifert hypersurface which is a once-punctured connected sum of copies of $S^1 \times S^2$ [Yn69]. Hence such knots are reflexive. (See [Su76] for more on geometric properties of such knots.)

An n-knot K is a *homotopy ribbon* knot if it has a slice disc whose exterior W has a handlebody decomposition consisting of 0-, 1- and 2-handles. The dual decomposition of W relative to $\partial W = M(K)$ has only $(n+1)$- and $(n+2)$-handles, and so the inclusion of M into W is n-connected. (The definition of "homotopically ribbon" for 1-knots given in Problem 4.22 of [GK] requires only that this latter condition be satisfied.) Every ribbon knot is homotopy ribbon and hence slice [Hi79]. It is an open question whether every classical slice knot is ribbon. However in higher dimensions "slice" does not even imply "homotopy ribbon". (The simplest example is $\tau_2 3_1$ - see below.)

More generally, we shall say that K is π_1-*slice* if the inclusion of $M(K)$ into the exterior of some slice disc induces an isomorphism on fundamental groups. Nontrivial classical knots are never π_1-slice, since $H_2(\pi_1(M(K));\mathbb{Z}) \cong Z$ is nonzero while $H_2(\pi_1(D^4 - \Delta);\mathbb{Z}) = 0$. On the other hand higher-dimensional homotopy ribbon knots are π_1-slice.

Two 2-knots K_0 and K_1 are *s-concordant* if there is a concordance $\mathcal{K} : S^2 \times [0,1] \to S^4 \times [0,1]$ whose exterior is an s-cobordism (*rel* ∂) from $X(K_0)$ to $X(K_1)$. (In higher dimensions the analogous notion is equivalent to ambient isotopy, by the s-cobordism theorem.)

14.6 The Kervaire conditions

A group G has *weight* 1 if it has an element whose conjugates generate G. Such an element is called a *weight element* for G, and its conjugacy class is called a *weight class* for G. If G is solvable then it has weight 1 if and only if G/G' is cyclic, for a solvable group with trivial abelianization must be trivial.

If π is the group of an n-knot K then

(1) *π is finitely presentable;*

(2) *π is of weight 1;*

(3) *$H_1(\pi;\mathbb{Z}) = \pi/\pi' \cong Z$; and*

(4) *$H_2(\pi;\mathbb{Z}) = 0$.*

Kervaire showed that any group satisfying these conditions is an n-knot group, for every $n \geq 3$ [Ke65]. These conditions are also necessary when $n = 1$ or 2, but are then no longer sufficient, and there are as yet no corresponding characterizations for 1- and 2-knot groups. If (4) is replaced by the stronger condition that $\mathrm{def}(\pi) = 1$ then π is a 2-knot group, but this condition is not necessary [Ke65]. (See §9 of this chapter, §4 of Chapter 15 and §4 of Chapter 16 for examples with deficiency ≤ 0.) Gonzalez-Acuña has given a characterization of 2-knot groups as groups admitting certain presentations [GA94]. (Note also that if π is a high dimensional knot group then $q(\pi) \geq 0$, and $q(\pi) = 0$ if and only if π is a 2-knot group.)

If K is a nontrivial classical knot then πK has one end [Pa57], so $X(K)$ is aspherical, and $X(K)$ collapses to a finite 2-complex, so $g.d.\pi \leq 2$. Moreover π has a Wirtinger presentation of deficiency 1, i.e., a presentation of the form

$$\langle x_i, 0 \leq i \leq n \mid x_j = w_j x_0 w_j^{-1},\ 1 \leq j \leq n\rangle.$$

A group has such a presentation if and only if it has weight 1 and has a deficiency 1 presentation P such that the presentation of the trivial group obtained by adjoining the relation killing a weight element is AC-equivalent to the empty presentation [Yo82']. (See [Si80] for connections between Wirtinger presentations and the condition that $H_2(\pi;\mathbb{Z}) = 0$.) If G is an n-knot group then $g.d.G = 2$ if and only if $c.d.G = 2$ and $\mathrm{def}(G) = 1$, by Theorem 2.8.

Since the group of a homotopy ribbon n-knot (with $n \geq 2$) is the fundamental group of a $(n+3)$-manifold W with $\chi(W) = 0$ and which can be built with 0-, 1- and 2-handles only, such groups also have deficiency 1. Conversely, if a finitely presentable group G has weight 1 and and deficiency 1 then we use such a presentation to construct a 5-dimensional handlebody $W = D^5 \cup \{h_i^1\} \cup \{h_j^2\}$ with $\pi_1(\partial W) = \pi_1(W) \cong G$ and $\chi(W) = 0$. Adjoining another 2-handle h along a loop representing a weight class for $\pi_1(\partial W)$ gives a homotopy 5-ball B with 1-connected boundary. Thus $\partial B \cong S^4$, and the boundary of the cocore of the 2-handle h is clearly a homotopy ribbon 2-knot with group G. (In fact any group of weight 1 with a Wirtinger presentation of deficiency 1 is the group of a ribbon n-knot, for each $n \geq 2$ [Yj69] - see [H3].)

The deficiency may be estimated in terms of the minimum number of generators of the Λ-module $e^2(\pi'/\pi'')$. Using this observation, it may be shown that if K is the sum of $m+1$ copies of $\tau_2 3_1$ then $\mathrm{def}(\pi K) = -m$ [Le78]. Moreover there are irreducible 2-knots whose groups have deficiency $-m$, for each $m \geq 0$ [Kn83].

A knot group π has two ends if and only if π' is finite. We shall determine all such 2-knots in §4 of Chapter 15. Nontrivial torsion free knot groups have one end [Kl93]. There are also many 2-knot groups with infinitely many ends. The simplest is perhaps the group with presentation

$$\langle a, b, t \mid a^3 = b^7 = 1,\ ab = b^2 a,\ ta = a^2 t\rangle.$$

It is evidently an HNN extension of the metacyclic group generated by $\{a, b\}$, but is also the free product of such a metacyclic group with $\pi\tau_2 3_1$, amalgamated over a subgroup of order 3 [GM78].

14.7 Weight elements, classes and orbits

Two 2-knots K and K_1 have homeomorphic exteriors if and only if there is a homeomorphism from $M(K_1)$ to $M(K)$ which carries the conjugacy class of a meridian of K_1 to that of K (up to inversion). In fact if M is any closed orientable 4-manifold with $\chi(M) = 0$ and with $\pi = \pi_1(M)$ of weight 1 then surgery on a weight class gives a 2-knot with group π. Moreover, if t and u are two weight elements and f is a self homeomorphism of M such that u is conjugate to $f_*(t^{\pm 1})$ then surgeries on t and u lead to knots whose exteriors are homeomorphic (via the restriction of a self homeomorphism of M isotopic to f). Thus the natural invariant to distinguish between knots with isomorphic groups is not the weight class, but rather the orbit of the weight class under the action of self homeomorphisms of M. In particular, the orbit of a weight element under $Aut(\pi)$ is a well defined invariant, which we shall call the *weight orbit*. If every automorphism of π is realized by a self homeomorphism of M then the homeomorphism class of M and the weight orbit together form a complete invariant for the (unoriented) knot. (This is the case if M is an infrasolvmanifold.)

For oriented knots we need a refinement of this notion. If w is a weight element for π then we shall call the set $\{\alpha(w) \mid \alpha \in Aut(\pi),\ \alpha(w) \equiv w \ mod\ \pi'\}$ a *strict* weight orbit for π. A strict weight orbit determines a transverse orientation for the corresponding knot (and its Gluck reconstruction). An orientation for the ambient sphere is determined by an orientation for $M(K)$. If K is invertible or +amphicheiral then there is a self homeomorphism of M which is orientation preserving or reversing (respectively) and which reverses the transverse orientation of the knot, i.e., carries the strict weight orbit to its inverse. Similarly, if K is -amphicheiral there is an orientation reversing self homeomorphism of M which preserves the strict weight orbit.

Theorem 14.1 *Let G be a group of weight 1 and with $G/G' \cong Z$. Let t be an element of G whose image generates G/G' and let c_t be the automorphism of G' induced by conjugation by t. Then*

(1) *t is a weight element if and only if c_t is meridianal;*

(2) *two weight elements t, u are in the same weight class if and only if there is an inner automorphism c_g of G' such that $c_u = c_g c_t c_g^{-1}$;*

(3) *two weight elements t, u are in the same strict weight orbit if and only if there is an automorphism d of G' such that $c_u = dc_t d^{-1}$ and $dc_t d^{-1} c_t^{-1}$ is an inner automorphism;*

(4) *if t and u are weight elements then u is conjugate to $(g''t)^{\pm 1}$ for some g'' in G''.*

Proof The verification of (1-3) is routine. If t and u are weight elements then, up to inversion, u must equal $g't$ for some g' in G'. Since multiplication by $t-1$ is invertible on G'/G'' we have $g' = khth^{-1}t^{-1}$ for some h in G' and k in G''. Let $g'' = h^{-1}kh$. Then $u = g't = hg''th^{-1}$. □

An immediate consequence of this theorem is that if t and u are in the same strict weight orbit then c_t and c_u have the same order. Moreover if C is the centralizer of c_t in $Aut(G')$ then the strict weight orbit of t contains at most $[Aut(G') : C.Inn(G')] \leq |Out(G')|$ weight classes. In general there may be infinitely many weight orbits [Pl83']. However if π is metabelian the weight class (and hence the weight orbit) is unique up to inversion, by part (4) of the theorem.

14.8 The commutator subgroup

It shall be useful to reformulate the Kervaire conditions in terms of the automorphism of the commutator subgroup induced by conjugation by a meridian. An automorphism ϕ of a group G is *meridianal* if $\langle\langle g^{-1}\phi(g) \mid g \in G\rangle\rangle_G = G$. If H is a characteristic subgroup of G and ϕ is meridianal the induced automorphism of G/H is then also meridianal. In particular, $H_1(\phi) - 1$ maps

$H_1(G;\mathbb{Z}) = G/G'$ onto itself. If G is solvable an automorphism satisfying the latter condition is meridianal, for a solvable perfect group is trivial.

It is easy to see that no group G with $G/G' \cong Z$ can have $G' \cong Z$ or D. It follows that the commutator subgroup of a knot group never has two ends.

Theorem 14.2 [HK78, Le78] *A finitely presentable group π is a high dimensional knot group if and only if $\pi \cong \pi' \times_\theta Z$ for some meridianal automorphism θ of π' such that $H_2(\theta) - 1$ is an automorphism of $H_2(\pi';\mathbb{Z})$.* □

If π is a knot group then π'/π'' is a finitely generated Λ-module. Levine and Weber have made explicit the conditions under which a finitely generated Λ-module may be the commutator subgroup of a metabelian high dimensional knot group [LW78]. Leaving aside the Λ-module structure, Hausmann and Kervaire have characterized the finitely generated abelian groups A that may be commutator subgroups of high dimensional knot groups [HK78]. "Most" can occur; there are mild restrictions on 2- and 3-torsion, and if A is infinite it must have rank at least 3. We shall show that the abelian groups which are commutator subgroups of 2-knot groups are Z^3, $Z[\frac{1}{2}]$ (the additive group of dyadic rationals) and the cyclic groups of odd order. The commutator subgroup of a nontrivial classical knot group is never abelian.

Hausmann and Kervaire also showed that any finitely generated abelian group could be the centre of a high dimensional knot group [HK78']. We shall show that the centre of a 2-knot group is either Z^2, torsion free of rank 1, finitely generated of rank 1 or is a torsion group. (The only known examples are Z^2, $Z\oplus(Z/2Z)$, Z, $Z/2Z$ and 1.) The centre of a classical knot group is nontrivial if and only if the knot is a torus knot [BZ]; the centre is then Z.

Silver has given examples of high dimensional knot groups π with π' finitely generated but not finitely presentable [Si91]. He has also shown that there are embeddings $j : T \to S^4$ such that $\pi_1(S^4 - j(T))'$ is finitely generated but not finitely presentable [Si97]. However no such 2-knot groups are known. If the commutator subgroup is finitely generated then it is the unique HNN base [Si96]. Thus knots with such groups have no minimal Seifert hypersurfaces.

The first examples of high dimensional knot groups which are not 2-knot groups made use of Poincaré duality with coefficients Λ. Farber [Fa77] and Levine [Le77] independently found the following theorem.

Theorem 14.3 (Farber, Levine) *Let K be a 2-knot and $A = H_1(M(K);\Lambda)$. Then $H_2(M(K);\Lambda) \cong \overline{e^1A}$, and there is a nondegenerate $\mathbb{Z}$-bilinear pairing $[\ ,\ ] : zA \times zA \to \mathbb{Q}/\mathbb{Z}$ such that $[t\alpha, t\beta] = [\alpha,\beta]$ for all α and β in zA.* □

Most of this theorem follows easily from Poincaré duality with coefficients Λ, but some care is needed in order to establish the symmetry of the pairing. When K is a fibred 2-knot, with closed fibre $\widehat{F}$, the Farber-Levine pairing is just the standard linking pairing on the torsion subgroup of $H_1(\widehat{F};\mathbb{Z})$, together with the automorphism induced by the monodromy.

In particular, Farber observed that the group π with presentation

$$\langle a, t \mid tat^{-1} = a^2,\ a^5 = 1\rangle$$

is a high dimensional knot group but if ℓ is any nondegenerate $\mathbb{Z}$-bilinear pairing on $\pi' \cong Z/5Z$ with values in $\mathbb{Q}/\mathbb{Z}$ then $\ell(t\alpha, t\beta) = -\ell(\alpha,\beta)$ for all α, β in π', and so π is not a 2-knot group.

Corollary 14.3.1 [Le78] *$H_2(\pi';\mathbb{Z})$ is a quotient of $\overline{Hom_\Lambda(\pi'/\pi'', \mathbb{Q}(t)/\Lambda)}$.* □

In many cases every orientation preserving meridianal automorphism of a torsion free 3-manifold group is realizable by a fibred 2-knot.

Theorem 14.4 *Let N be a closed orientable 3-manifold whose prime factors are virtually Haken or $S^1 \times S^2$. If K is a 2-knot such that $(\pi K)' \cong \nu = \pi_1(N)$ then $M(K)$ is homotopy equivalent to the mapping torus of a self homeomorphism of N. If θ is a meridianal automorphism of ν then $\pi = \nu \times_\theta Z$ is a 2-knot group if and only if θ fixes the image of the fundamental class of N in $H_3(\nu;\mathbb{Z})$.*

Proof The first assertion follows from Corollary 4.6.1. The classifying maps for the fundamental groups induce a commuting diagram involving the Wang sequences of $M(K)$ and π from which the necessity of the orientation condition follows easily. (It is vacuous if ν is free group.)

If $\theta_*(c_{N*}[N]) = c_{N*}[N]$ then θ may be realized by an orientation preserving self homotopy equivalence g of N [Sw74]. Let $N = P\sharp R$ where P is a connected sum of copies of $S^1 \times S^2$ and R has no such factors. By the Splitting Theorem of [La74], g is homotopic to a connected sum of homotopy equivalences between the irreducible factors of R with a self homotopy equivalence of P. Every virtually Haken 3-manifold is either Haken, hyperbolic or Seifert-fibred, by [CS83] and [GMT96], and self homotopy equivalences of such manifolds are homotopic to homeomorphisms, by [Hm], Mostow rigidity and [Sc83], respectively. A similar result holds for $P = \sharp^r(S^1 \times S^2)$, by [La74]. Thus we may assume that g is a self homeomorphism of N. Surgery on a weight class in the mapping torus of g gives a fibred 2-knot with closed fibre N and group π. □

If Thurston's Geometrization Conjecture is true then it would suffice to assume that N is a closed orientable 3-manifold with $\pi_1(N)$ torsion free. The mapping torus is determined up to homeomorphism among fibred 4-manifolds with fibre N by its homotopy type if N is hyperbolic, Seifert fibred or if its prime factors are Haken or $S^1 \times S^2$, since homotopy implies isotopy in each case, by Mostow rigidity, [Sc85, BO91] and [HL74], respectively.

Yoshikawa has shown that a finitely generated abelian group is the base of some HNN extension which is a high dimensional knot group if and only if it satisfies the restrictions on torsion of [HK78], while if a knot group has a non-finitely generated abelian base then it is metabelian. Moreover a 2-knot group π which is an HNN extension with abelian base is either metabelian or has base $Z \oplus (Z/\beta Z)$ for some odd $\beta \geq 1$ [Yo86, Yo92]. In §6 of Chapter 15 we shall show that in the latter case β must be 1, and so π has a deficiency 1 presentation $\langle t, x \mid tx^n t^{-1} = x^{n+1} \rangle$. No nontrivial classical knot group is an HNN extension with abelian base. (This is implicit in Yoshikawa's work, and can also be deduced from the facts that classical knot groups have cohomological dimension ≤ 2 and symmetric Alexander polynomial.)

14.9 Deficiency and geometric dimension

J.H.C.Whitehead raised the question "is every subcomplex of an aspherical 2-complex also aspherical?" This is so if the fundamental group of the subcomplex is a 1-relator group [Go81] or is locally indicable [Ho82] or has no nontrivial superperfect normal subgroup [Dy87]. Whitehead's question has interesting connections with knot theory. (For instance, the exterior of a ribbon n-knot or of a ribbon concordance between classical knots is homotopy equivalent to such a 2-complex. The asphericity of such ribbon exteriors has been raised in [Co83] and [Go81].)

If the answer to Whitehead's question is YES, then a high dimensional knot group has geometric dimension at most 2 if and only if it has deficiency 1 (in which case it is a 2-knot group). For let G be a group of weight 1 and with $G/G' \cong Z$. If $C(P)$ is the 2-complex corresponding to a presentation of deficiency 1 then the 2-complex obtained by adjoining a 2-cell to $C(P)$ along a loop representing a weight element for G is 1-connected and has Euler characteristic 1, and so is contractible. The converse follows from Theorem 2.8. On the other hand a positive answer in general implies that there is a group G such that $c.d.G = 2$ and $g.d.G = 3$ [BB97].

If the answer is NO then either there is a finite nonaspherical 2-complex X such that $X \cup_f D^2$ is contractible for some $f : S^1 \to X$ or there is an infinite ascending chain of nonaspherical 2-complexes whose union is contractible [Ho83]. In the finite case $\chi(X) = 0$ and so $\pi = \pi_1(X)$ has deficiency 1; moreover, π has weight 1 since it is normally generated by the conjugacy class represented by f. Such groups are 2-knot groups. Since X is not aspherical $\beta_1^{(2)}(\pi) \neq 0$, by Theorem 2.4, and so π' cannot be finitely generated, by Lemma 2.1.

A group is called *knot-like* if it has abelianization Z and deficiency 1. If the commutator subgroup of a classical knot group is finitely generated then it is free; Rapaport asked whether this is true of all knot-like groups G, and established this in the 2-generator, 1-relator case [Rp60]. This is true also if G' is FP_2, by Corollary 2.5.1. If every knot-like group has a finitely presentable HNN base then this Corollary would settle Rapaport's question completely, for if G' is finitely generated then it is the unique HNN base for G [Si96].

In particular, if the group of a fibred 2-knot has a presentation of deficiency 1 then its commutator subgroup must be free. Any 2-knot with such a group is s-concordant to a fibred homotopy ribbon knot (see §6 of Chapter 17). Must it in fact be a ribbon knot?

It follows also that if $\tau_r K$ is a nontrivial twist spin then $\mathrm{def}(\pi\tau_r K) \leq 0$ and $\tau_r K$ is not a homotopy ribbon 2-knot. For $S^2 \times S^1$ is never a factor of the closed fibre of $\tau_r K$ [Pl84], and so $(\pi\tau_r K)'$ is never a nontrivial free group.

The next result is a consequence of Theorem 2.5, but the argument below is self contained.

Lemma 14.5 *If G is a group with $\mathrm{def}(G) = 1$ and $e(G) = 2$ then $G \cong Z$.*

Proof The group G has an infinite cyclic subgroup A of finite index, since $e(G) = 2$. Let C be the finite 2-complex corresponding to a presentation of deficiency 1 for G, and let D be the covering space corresponding to A. Then D is a finite 2-complex with $\pi_1(D) = A \cong Z$ and $\chi(D) = [\pi : A]\chi(C) = 0$. Since $H_2(D; \mathbb{Z}[A]) = H_2(\widetilde{D}; \mathbb{Z})$ is a submodule of a free $\mathbb{Z}[A]$-module and is of rank $\chi(D) = 0$ it is 0. Hence $\widetilde{D}$ is contractible, and so G must be torsion free and hence abelian. □

It follows immediately that $\mathrm{def}(\pi\tau_2 3_1) = 0$, since $\pi\tau_2 3_1 \cong (Z/3Z) \times_{-1} Z$. Moreover, if K is a nontrivial classical knot then π' is infinite. Hence if π' is finitely generated then $H^1(\pi; \mathbb{Z}[\pi]) = 0$, and so $X(K)$ is aspherical, by Poincaré duality.

Theorem 14.6 *Let K be a 2-knot with group π. Then $\pi \cong Z$ if and only if* $\mathrm{def}(\pi) = 1$ *and* $\pi_2(M(K)) = 0$.

Proof The conditions are necessary, by Theorem 11.1. If they hold then $\beta_j^{(2)}(M) = \beta_j^{(2)}(\pi)$ for $j \leq 2$, by Theorem 6.54 of [Lü], and so $0 = \chi(M) = \beta_2^{(2)}(\pi) - 2\beta_1^{(2)}(\pi)$. Now $\beta_1^{(2)}(\pi) - \beta_2^{(2)}(\pi) \geq \mathrm{def}(\pi) - 1 = 0$, by Corollary 2.4.1. Therefore $\beta_1^{(2)}(\pi) = \beta_2^{(2)}(\pi) = 0$ and so $g.d.\pi \leq 2$, by the same Corollary. Since $\mathrm{def}(\pi) = 1$ the manifold M is not aspherical, by Theorem 3.6. Hence $H^1(\pi; \mathbb{Z}[\pi]) \cong H_3(M; \mathbb{Z}[\pi]) \neq 0$. Since π is torsion free it is indecomposable as a free product [Kl93]. Therefore $e(\pi) = 2$ and so $\pi \cong Z$, by Lemma 14.5. □

In fact K must be trivial ([FQ] - see Corollary 17.1.1). A simpler argument is used in [H1] to show that if $\mathrm{def}(\pi) = 1$ then $\pi_2(M)$ maps onto $H_2(M; \Lambda)$, which is nonzero if $\pi' \neq \pi''$.

14.10 Asphericity

The outstanding property of the exterior of a classical knot is that it is aspherical. Swarup extended the classical Dehn's lemma criterion for unknotting to show that if K is an n-knot such that the natural inclusion of S^n (as a factor of $\partial X(K)$) into $X(K)$ is null homotopic then $X(K) \simeq S^1$, provided πK is accessible [Sw75]. Since it is now known that finitely presentable groups are accessible [DD], it follows that the exterior of a higher dimensional knot is aspherical if and only if the knot is trivial. Nevertheless, we shall see that the closed 4-manifolds $M(K)$ obtained by surgery on 2-knots are often aspherical.

Theorem 14.7 *Let K be a 2-knot. Then $M(K)$ is aspherical if and only if πK is a PD_4-group (which must then be orientable).*

Proof The condition is clearly necessary. Suppose that it holds. Let M^+ be the covering space associated to $\pi^+ = \mathrm{Ker}(w_1(\pi))$. Then $[\pi : \pi^+] \leq 2$, so $\pi' < \pi^+$. Since $\pi/\pi' \cong Z$ and $t - 1$ acts invertibly on $H_1(\pi'; \mathbb{Z})$ it follows that $\beta_1(\pi^+) = 1$. Hence $\beta_2(M^+) = 0$, since M^+ is orientable and $\chi(M^+) = 0$. Hence $\beta_2(\pi^+)$ is also 0, so $\chi(\pi^+) = 0$, by Poincaré duality for π^+. Therefore $\chi(\pi) = 0$ and so M must be aspherical, by Corollary 3.5.1. □

We may use this theorem to give more examples of high dimensional knot groups which are not 2-knot groups. Let $A \in GL(3, \mathbb{Z})$ be such that $det(A) = -1$,

$det(A-I) = \pm 1$ and $det(A+I) = \pm 1$. The characteristic polynomial of A must be either $f_1(X) = X^3 - X^2 - 2X + 1$, $f_2(X) = X^3 - X^2 + 1$, $f_3(X) = X^3 f_1(X^{-1})$ or $f_4(X) = X^3 f_2(X^{-1})$. It may be shown that the rings $\mathbb{Z}[X]/(f_i(X))$ are principal ideal domains. Hence there are only two conjugacy classes of such matrices, up to inversion. The Kervaire conditions hold for $Z^3 \times_A Z$, and so it is a 3-knot group. However it cannot be a 2-knot group, since it is a PD_4-group of nonorientable type. (Such matrices have been used to construct fake RP^4s [CS76'].)

Is every (torsion free) 2-knot group with $H^s(\pi; \mathbb{Z}[\pi]) = 0$ for $s \leq 2$ a PD_4-group? Is every 3-knot group which is also a PD_4-group a 2-knot group? (Note that by Theorem 3.6 such a group cannot have deficiency 1.)

We show next that knots with such groups cannot be a nontrivial satellite.

Theorem 14.8 *Let $K = \Sigma(K_2; K_1, \gamma)$ be a satellite 2-knot. If πK is a PD_4-group then $K = K_1$ or K_2.*

Proof Let q be the order of γ in πK_1. Then $\pi = \pi K \cong \pi K_1 *_C B$, where $B = \pi K_2 / \langle\langle w^q \rangle\rangle$, and C is cyclic. Since π is torsion free $q = 0$ or 1. Suppose that $K \neq K_1$. Then $q = 0$, so $C \cong Z$, while $B \neq C$. If $\pi K_1 \neq C$ then πK_1 and B have infinite index in π, and so $c.d.\pi K_1 \leq 3$ and $c.d.B \leq 3$, by Strebel's Theorem. A Mayer-Vietoris argument then gives $4 = c.d.\pi \leq 3$, which is impossible. Therefore K_1 is trivial and so $K = K_2$. □

In particular if πK is a PD_4-group then K is irreducible.

14.11 Links

A *μ-component n-link* is a locally flat embedding $L : \mu S^n \to S^{n+2}$. The *exterior* of L is $X(L) = S^{n+2} \setminus intN(L)$, where $N(L) \cong \mu S^n \times D^2$ is a regular neighbourhood of the image of L, and the *group* of L is $\pi L = \pi_1(X(L))$. Let $M(L) = X(L) \cup \mu D^{n+1} \times S^1$ be the closed manifold obtained by surgery on L in S^{n+2}.

An n-link L is *trivial* if it bounds a collection of μ disjoint locally flat 2-discs in S^n. It is *split* if it is isotopic to one which is the union of nonempty sublinks L_1 and L_2 whose images lie in disjoint discs in S^{n+2}, in which case we write $L = L_1 \amalg L_2$, and it is a *boundary* link if it bounds a collection of μ disjoint hypersurfaces in S^{n+2}. Clearly a trivial link is split, and a split link is a boundary link; neither implication can be reversed if $\mu > 1$. Knots are boundary links, and many arguments about knots that depend on Seifert hypersurfaces extend readily to boundary links. The definitions of slice and ribbon knots and s-concordance extend naturally to links.

A 1-link is trivial if and only if its group is free, and is split if and only if its group is a nontrivial free product, by the Loop Theorem and Sphere Theorem, respectively. (See Chapter 1 of [H3].) Gutiérrez has shown that if $n \geq 4$ an n-link L is trivial if and only if πL is freely generated by meridians and the homotopy groups $\pi_j(X(L))$ are all 0, for $2 \leq j \leq (n+1)/2$ [Gu72]. His argument applies also when $n = 3$. While the fundamental group condition is necessary when $n = 2$, we cannot yet use surgery to show that it is a complete criterion for triviality of 2-links with more than one component. We shall settle for a weaker result.

Theorem 14.9 *Let M be a closed 4-manifold with $\pi_1(M)$ free of rank r and $\chi(M) = 2(1-r)$. If M is orientable it is s-cobordant to $\sharp^r(S^1 \times S^3)$, while if it is nonorientable it is s-cobordant to $(S^1 \tilde{\times} S^3)\sharp(\sharp^{r-1}(S^1 \times S^3))$.*

Proof We may assume without loss of generality that $\pi_1(M)$ has a free basis $\{x_1, ...x_r\}$ such that x_i is an orientation preserving loop for all $i > 1$, and we shall use c_{M*} to identify $\pi_1(M)$ with $F(r)$. Let $N = \sharp^r(S^1 \times S^3)$ if M is orientable and let $N = (S^1 \tilde{\times} S^3)\sharp(\sharp^{r-1}(S^1 \times S^3))$ otherwise. (Note that $w_1(N) = w_1(M)$ as homomorphisms from $F(r)$ to $\{\pm 1\}$.) Since $c.d.\pi_1(M) \leq 2$ and $\chi(M) = 2\chi(\pi_1(M))$ we have $\pi_2(M) \cong \overline{H^2(F(r); \mathbb{Z}[F(r)])}$, by Theorem 3.12. Hence $\pi_2(M) = 0$ and so $\pi_3(M) \cong H_3(\widetilde{M}; \mathbb{Z}) \cong D = \overline{H^1(F(r); \mathbb{Z}[F(r)])}$, by the Hurewicz theorem and Poincaré duality. Similarly, we have $\pi_2(N) = 0$ and $\pi_3(N) \cong D$.

Let $c_M = g_M h_M$ be the factorization of c_M through $P_3(M)$, the third stage of the Postnikov tower for M. Thus $\pi_i(h_M)$ is an isomorphism if $i \leq 3$ and $\pi_j(P_3(M)) = 0$ if $j > 3$. As $K(F(r), 1) = \vee^r S^1$ each of the fibrations g_M and g_N clearly have cross-sections and so there is a homotopy equivalence $k : P_3(M) \to P_3(N)$ such that $g_M = g_N k$. (See Section 5.2 of [Ba].) We may assume that k is cellular. Since $P_3(M) = M \cup \{cells\ of\ dimension \geq 5\}$ it follows that $kh_M = h_N f$ for some map $f : M \to N$. Clearly $\pi_i(f)$ is an isomorphism for $i \leq 3$. Since the universal covers $\widetilde{M}$ and $\widetilde{N}$ are 2-connected open 4-manifolds the induced map $\tilde{f} : \widetilde{M} \to \widetilde{N}$ is an homology isomorphism, and so is a homotopy equivalence. Hence f is itself a homotopy equivalence. As $Wh(F(r)) = 0$ any such homotopy equivalence is simple.

If M is orientable $[M, G/TOP] \cong Z$, since $H^2(M; \mathbb{Z}/2\mathbb{Z}) = 0$. As the surgery obstruction in $L_4(F(r)) \cong Z$ is given by a signature difference, it is a bijection, and so the normal invariant of f is trivial. Hence there is a normal cobordism $F : P \to N \times I$ with $F|\partial_- P = f$ and $F|\partial_+ P = id_N$. There is another normal cobordism $F' : P' \to N \times I$ from id_N to itself with surgery obstruction $\sigma_5(P', F') = -\sigma_5(P, F)$ in $L_5(F(r))$, by Theorem 6.7 and Lemma 6.9. The union of these two normal cobordisms along $\partial_+ P = \partial_- P'$ is a normal cobordism from f to id_N with surgery obstruction 0, and so we may obtain an s-cobordism W by 5-dimensional surgery (rel ∂).

A similar argument applies in the nonorientable case. The surgery obstruction is then a bijection from $[N; G/TOP]$ to $L_4(F(r), -) = Z/2Z$, so f is normally cobordant to id_N, while $L_5(Z, -) = 0$, so $L_5(F(r), -) \cong L_5(F(r-1))$ and the argument of [FQ] still applies. $\square$

Corollary 14.9.1 *Let L be a μ-component 2-link such that πL is freely generated by μ meridians. Then L is s-concordant to the trivial μ-component link.*

Proof Since $M(L)$ is orientable, $\chi(M(L)) = 2(1-\mu)$ and $\pi_1(M(L)) \cong \pi L = F(\mu)$, there is an s-cobordism W with $\partial W = M(L) \cup M(\mu)$, by Theorem 14.9. Moreover it is clear from the proof of that theorem that we may assume that the elements of the meridianal basis for πL are freely homotopic to loops representing the standard basis for $\pi_1(M(\mu))$. We may realise such homotopies by μ disjoint embeddings of annuli running from meridians for L to such standard loops in $M(\mu)$. Surgery on these annuli (i.e., replacing $D^3 \times S^1 \times [0,1]$ by $S^2 \times D^2 \times [0,1]$) then gives an s-concordance from L to the trivial μ-component link. $\square$

A similar strategy may be used to give an alternative proof of the higher dimensional unlinking theorem of [Gu72] which applies uniformly for $n \geq 3$. The hypothesis that πL be freely generated by meridians cannot be dropped entirely [Po71]. On the other hand, if L is a 2-link whose longitudes are all null homotopic then the pair $(X(L), \partial X(L))$ is homotopy equivalent to the pair $(\sharp^\mu S^1 \times D^3, \partial(\sharp^\mu S^1 \times D^3))$ [Sw77], and hence the Corollary applies.

There is as yet no satisfactory splitting criterion for higher-dimensional links. However we can give a stable version for 2-links.

Theorem 14.10 *Let M be a closed 4-manifold such that $\pi = \pi_1(M)$ is isomorphic to a nontrivial free product $G * H$. Then M is stably homeomorphic to a connected sum $M_G \sharp M_H$ with $\pi_1(M_G) \cong G$ and $\pi_1(M_H) \cong H$.*

Proof Let $K = K_G \cup [-1,1] \cup K_H/(*_G \sim -1, +1 \sim *_H)$, where K_G and K_H are $K(G,1)$- and $K(H,1)$-spaces with basepoints $*_G$ and $*_H$ (respectively). Then K is a $K(\pi,1)$-space and so there is a map $f : M \to K$ which induces an isomorphism of fundamental groups. We may assume that f is transverse to $0 \in [-1,1]$, so $V = f^{-1}(0)$ is a submanifold of M with a product neighbourhood $V \times [-\epsilon, \epsilon]$. We may also assume that V is connected, by the arc-chasing argument of Stallings' proof of Kneser's conjecture. (See page 67 of [Hm].) Let $j : V \to M$ be the inclusion. Since fj is a constant map and $\pi_1(f)$ is an isomorphism $\pi_1(j)$ is the trivial homomorphism, and so $j^* w_1(M) = 0$. Hence V is orientable and so there is a framed link $L \subset V$ such that surgery on L in V gives S^3 [Li62]. The framings of the components of L in V extend to framings in M. Let $W = M \times [0,1] \cup_{L \times D^2 \times [-\epsilon,\epsilon] \times \{1\}} (\mu D^2 \times D^2 \times [-\epsilon, \epsilon])$, where μ is the number of components of L. Note that if $w_2(M) = 0$ then we may choose the framed link L so that $w_2(W) = 0$ also [Kp79]. Then $\partial W = M \cup \widehat{M}$, where $\widehat{M}$ is the result of surgery on L in M. The map f extends to a map $F : W \to K$ such that $\pi_1(F|_{\widehat{M}})$ is an isomorphism and $(F|_{\widehat{M}})^{-1}(0) \cong S^3$. Hence $\widehat{M}$ is a connected sum as in the statement. Since the components of L are null-homotopic in M they may be isotoped into disjoint discs, and so $\widehat{M} \cong M \sharp (\sharp^\mu S^2 \times S^2)$. This proves the theorem. ∎

Note that if V is a homotopy 3-sphere then M is a connected sum, for $V \times R$ is then homeomorphic to $S^3 \times R$, by 1-connected surgery.

Theorem 14.11 *Let L be a μ-component 2-link with sublinks L_1 and $L_2 = L \backslash L_1$ such that there is an isomorphism from πL to $\pi L_1 * \pi L_2$ which is compatible with the homomorphisms determined by the inclusions of $X(L)$ into $X(L_1)$ and $X(L_2)$. Then $X(L)$ is stably homeomorphic to $X(L_1 \amalg L_2)$.*

Proof By Theorem 14.10, $M(L)\sharp(\sharp^a S^2 \times S^2) \cong N \sharp P$, where $\pi_1(N) \cong \pi L_1$ and $\pi_1(P) \cong \pi L_2$. On undoing the surgeries on the components of L_1 and L_2, respectively, we see that $M(L_2)\sharp(\sharp^a S^2 \times S^2) \cong N \sharp \bar{P}$, and $M(L_1)\sharp(\sharp^a S^2 \times S^2) \cong \bar{N} \sharp P$, where $\bar{N}$ and $\bar{P}$ are simply connected. Since undoing the surgeries on all the components of L gives $\sharp^a S^2 \times S^2 \cong \bar{N} \sharp \bar{P}$, $\bar{N}$ and $\bar{P}$ are each connected sums of copies of $S^2 \times S^2$, so N and P are stably homeomorphic to $M(L_1)$ and $M(L_2)$, respectively. The result now follows easily. ∎

Similar arguments may be used to show that, firstly, if L is a 2-link such that $c.d.\pi L \leq 2$ and there is an isomorphism $\theta : \pi L \to \pi L_1 * \pi L_2$ which is compatible with the natural maps to the factors then there is a map f_o :

$M(L)_o = M(L)\backslash intD^4 \to M(L_1)\sharp M(L_2)$ such that $\pi_1(f_o) = \theta$ and $\pi_2(f_o)$ is an isomorphism; and secondly, if moreover f_o extends to a homotopy equivalence $f : M(L) \to M(L_1)\sharp M(L_2)$ and the factors of πL are either classical link groups or are square root closed accessible then L is s-concordant to the split link $L_1 \amalg L_2$. (The surgery arguments rely on [AFR97] and [Ca73], respectively.) However we do not know how to bridge the gap between the algebraic hypothesis and obtaining a homotopy equivalence.

14.12 Link groups

If π is the group of a μ-component n-link L then

(1) *π is finitely presentable;*

(2) *π is of weight μ;*

(3) *$H_1(\pi;\mathbb{Z}) = \pi/\pi' \cong Z^\mu$; and*

(4) *(if $n > 1$) $H_2(\pi;\mathbb{Z}) = 0$.*

Conversely, any group satisfying these conditions is the group of an n-link, for every $n \geq 3$ [Ke 65']. (Note that $q(\pi) \geq 2(1-\mu)$, with equality if and only if π is the group of a 2-link.) If (4) is replaced by the stronger condition that $\mathrm{def}(\pi) = \mu$ (and π has a deficiency μ Wirtinger presentation) then π is the group of a (ribbon) 2-link which is a sublink of a (ribbon) link whose group is a free group. (See Chapter 1 of [H3].) The group of a classical link satisfies (4) if and only if the link splits completely as a union of knots in disjoint balls. If subcomplexes of aspherical 2-complexes are aspherical then a higher-dimensional link group group has geometric dimension at most 2 if and only if it has deficiency μ (in which case it is a 2-link group).

A link L is a boundary link if and only if there is an epimorphism from $\pi(L)$ to the free group $F(\mu)$ which carries a set of meridians to a free basis. If the latter condition is dropped L is said to be an homology boundary link. Although sublinks of boundary links are clearly boundary links, the corresponding result is not true for homology boundary links. It is an attractive conjecture that every even-dimensional link is a slice link. This has been verified under additional hypotheses on the link group. For a 2-link L it suffices that there be a homomorphism $\phi : \pi L \to G$ where G is a high-dimensional link group such that $H_3(G;\mathbb{F}_2) = H_4(G;\mathbb{Z}) = 0$ and where the normal closure of the image of ϕ is G [Co84]. In particular, sublinks of homology boundary 2-links are slice links.

A choice of (based) meridians for the components of a link L determines a homomorphism $f : F(\mu) \to \pi L$ which induces an isomorphism on abelianization. If L is a higher dimensional link $H_2(\pi L;\mathbb{Z}) = H_2(F(\mu);\mathbb{Z}) = 0$ and hence f induces isomorphisms on all the nilpotent quotients $F(\mu)/F(\mu)_{[n]} \cong \pi L/(\pi L)_{[n]}$, and a monomorphism $F(\mu) \to \pi L/(\pi L)_{[\omega]} = \pi L/\cap_{n\geq 1}(\pi L)_{[n]}$ [St65]. (In particular, if $\mu \geq 2$ then πL contains a nonabelian free subgroup.) The latter map is an isomorphism if and only if L is a homology boundary link. In that case the homology groups of the covering space $X(L)^\omega$ corresponding to $\pi L/(\pi L)_{[\omega]}$ are modules over $\mathbb{Z}[\pi L/(\pi L)_{[\omega]}] \cong \mathbb{Z}[F(\mu)]$, which is a coherent ring of global dimension 2. Poincaré duality and the UCSS then give rise to an isomorphism $e^2e^2(\pi L/(\pi L)_{[\omega]}) \cong \overline{e^2(\pi L/(\pi L)_{[\omega]})}$, where $e^i(M) = Ext^i_{\mathbb{Z}[F(\mu)]}(M,\mathbb{Z}[F(\mu)])$, which is the analogue of the Farber-Levine pairing for 2-knots.

The argument of [HK78'] may be adapted to show that every finitely generated abelian group is the centre of the group of some μ-component boundary n-link, for any $\mu \geq 1$ and $n \geq 3$. However the centre of the group of a 2-link with more than one component must be finite. (In all known examples the centre is trivial.)

Theorem 14.12 *Let L be a μ-component 2-link with group π. If $\mu > 1$ then*

(1) *π has no infinite amenable normal subgroup;*

(2) *π is not an ascending HNN extension over a finitely generated base.*

Proof If (1) or (2) is false then $\beta_1^{(2)}(\pi) = 0$ (see §2 of Chapter 2), and clearly $\mu > 0$. Since $\beta_2^{(2)}(M(L)) = \chi(M(L)) + 2\beta_1^{(2)}(\pi) = 2(1-\mu)$, we must have $\mu = 1$. ∎

In particular, the exterior of a 2-link with more than one component never fibres over S^1. (This is true of all higher dimensional links: see Theorem 5.12 of [H3].) Moreover a 2-link group has finite centre and is never amenable. In contrast, we shall see that there are many 2-knot groups which have infinite centre or are solvable.

The exterior of a classical link is aspherical if and only the link is unsplittable, while the exterior of a higher dimensional link with more than one component is never aspherical [Ec76]. Is $M(L)$ ever aspherical?

14.13 Homology spheres

A closed connected n-manifold M is an homology n-sphere if $H_q(M;\mathbb{Z}) = 0$ for $0 < q < n$. In particular, it is orientable and so $H_n(M;\mathbb{Z}) \cong Z$. If π is the group of an homology n-sphere then

(1) *π is finitely presentable;*

(2) *π is perfect, i.e., $\pi = \pi'$; and*

(3) $H_2(\pi;\mathbb{Z}) = 0$.

A group satisfying the latter two conditions is said to be superperfect. Every finitely presentable superperfect group is the group of an homology n-sphere, for every $n \geq 5$ [Ke69], but in low dimensions more stringent conditions hold. As any closed 3-manifold has a handlebody structure with one 0-handle and equal numbers of 1- and 2-handles, homology 3-sphere groups have deficiency 0. Every perfect group with a presentation of deficiency 0 is an homology 4-sphere group (and therefore is superperfect) [Ke69]. However none of the implications "G is an homology 3-sphere group" $\Rightarrow$ "G is finitely presentable, perfect and $\mathrm{def}(G) = 0$" $\Rightarrow$ "G is an homology 4-sphere group" $\Rightarrow$ "G is finitely presentable and superperfect" can be reversed, as we shall now show.

Although the finite groups $SL(2,\mathbb{F}_p)$ are perfect and have deficiency 0 for each prime $p \geq 5$ [CR80] the binary icosahedral group $I^* = SL(2,\mathbb{F}_5)$ is the only nontrivial finite perfect group with cohomological period 4, and thus is the only finite homology 3-sphere group.

Let $G = \langle x, s \mid x^3 = 1, sxs^{-1} = x^{-1}\rangle$ be the group of $\tau_2 3_1$ and let $H = \langle a, b, c, d \mid bab^{-1} = a^2, cbc^{-1} = b^2, dcd^{-1} = c^2, ada^{-1} = d^2\rangle$ be the Higman group [Hg51]. Then H is perfect and $\mathrm{def}(H) = 0$, so there is an homology 4-sphere Σ with group H. Surgery on a loop representing sa^{-1} in $\Sigma\sharp M(\tau_2 3_1)$ gives an homology 4-sphere with group $\pi = (G * H)/\langle\langle sa^{-1}\rangle\rangle$. Then π is the semidirect product $\rho \rtimes H$, where $\rho = \langle\langle G'\rangle\rangle_\pi$ is the normal closure of the image of G' in π. The obvious presentation for this group has deficiency -1. We shall show that this is best possible.

Let $\Gamma = \mathbb{Z}[H]$. Since H has cohomological dimension 2 [DV73'] the augmentation ideal $I = \mathrm{Ker}(\varepsilon : \Gamma \to \mathbb{Z})$ has a short free resolution

$$C_* : 0 \to \Gamma^4 \to \Gamma^4 \to I \to 0.$$

Let $B = H_1(\pi;\Gamma) \cong \rho/\rho'$. Then $B \cong \Gamma/\Gamma(3, a + 1)$ as a left Γ-module and there is an exact sequence

$$0 \to B \to A \to I \to 0,$$

in which $A = H_1(\pi, 1; \Gamma)$ is a relative homology group [Cr61]. Since $B \cong \Gamma \otimes_\Lambda (\Lambda/\Lambda(3, a+1))$, where $\Lambda = \mathbb{Z}[a, a^{-1}]$, there is a free resolution

$$0 \to \Gamma \xrightarrow{(3,a+1)} \Gamma^2 \xrightarrow{\left(\begin{smallmatrix} a+1 \\ -3 \end{smallmatrix}\right)} \Gamma \to B \to 0.$$

Suppose that π has deficiency 0. Evaluating the Jacobian matrix associated to an optimal presentation for π via the natural epimorphism from $\mathbb{Z}[\pi]$ to Γ gives a presentation matrix for A as a module (see [Cr61] or [Fo62]). Thus there is an exact sequence

$$D_* : \cdots \to \Gamma^n \to \Gamma^n \to A \to 0.$$

A mapping cone construction leads to an exact sequence of the form

$$D_1 \to C_1 \oplus D_0 \to B \oplus C_0 \to 0$$

and hence to a presentation of deficiency 0 for B of the form

$$D_1 \oplus C_0 \to C_1 \oplus D_0 \to B.$$

Hence there is a free resolution

$$0 \to L \to \Gamma^p \to \Gamma^p \to B \to 0.$$

Schanuel's Lemma gives an isomorphism $\Gamma^{1+p+1} \cong L \oplus \Gamma^{p+2}$, on comparing these two resolutions of B. Since Γ is weakly finite the endomorphism of Γ^{p+2} given by projection onto the second summand is an automorphism. Hence $L = 0$ and so B has a short free resolution. In particular, $Tor_2^\Gamma(R, B) = 0$ for any right Γ-module R. But it is easily verified that if $\overline{B} \cong \Gamma/(3, a+1)\Gamma$ is the conjugate right Γ-module then $Tor_2^\Gamma(\overline{B}, B) \neq 0$. Thus our assumption was wrong, and $\mathrm{def}(\pi) = -1 < 0$.

If $k \geq 0$ let $G_k = (\mathbb{F}_5^2)^k \rtimes I^*$, where I^* acts diagonally on $(\mathbb{F}_5^2)^k$, with respect to the standard action on $\mathbb{F}_5^2$, and let H_k be the subgroup generated by $\mathbb{F}_5^{2k}$ and $\left(\begin{smallmatrix} -1 & 2 \\ 0 & -1 \end{smallmatrix}\right)$. Then G_k is a finite superperfect group, $[G_k : H_k] = 12$, $\beta_1(H_k; \mathbb{F}_5) = 1$ and $\beta_2(H_k; \mathbb{F}_5) = k^2$. Applying part (1) of Lemma 3.11 we find that $\mathrm{def} G_k < 0$ if $k > 3$ and $q^{SG}(G_k) > 2$ if $k > 4$. In the latter case G_k is not realized by any homology 4-sphere. (This argument derives from [HW85].)

Does every *finite* homology 4-sphere group have deficiency 0? Our example above is "very infinite" in the sense that the Higman group H has no finite quotients, and therefore no finite-dimensional representations over any field [Hg51]. The smallest finite superperfect group which is not known to have deficiency 0 nor to be an homology 4-sphere group is G_1, which has order 3000 and has the deficiency -2 presentation

$$\langle x, y, e \mid x^2 = y^3 = (xy)^5, xex^{-1} = yey^{-1}, eyey^{-1} = yey^{-1}e, ey^2e = yey \rangle.$$

Kervaire's criteria may be extended further to the groups of links in homology spheres. Unfortunately, the condition $\chi(M) = 0$ is central to most of our arguments, and is satisfied only by the manifolds arising from knots in homology 4-spheres.

Chapter 15

Restrained normal subgroups

It is plausible that if K is a 2-knot whose group $\pi = \pi K$ has an infinite restrained normal subgroup N then either π' is finite or $\pi \cong \Phi$ (the group of Fox's Example 10) or $M(K)$ is aspherical and $\sqrt{\pi} \neq 1$ or N is virtually Z and π/N has infinitely many ends. In this chapter we shall give some evidence in this direction. In order to clarify the statements and arguments in later sections, we begin with several characterizations of Φ, which plays a somewhat exceptional role. In §2 we assume that N is almost coherent and locally virtually indicable, but not locally finite. In §3 we assume that N is abelian of positive rank and almost establish the tetrachotomy in this case. In §4 we determine all such π with π' finite, and in §5 we give a version of the Tits alternative for 2-knot groups. In §6 we shall complete Yoshikawa's determination of the 2-knot groups which are HNN extensions over abelian bases. We conclude with some observations on 2-knot groups with infinite locally finite normal subgroups.

15.1 The group Φ

Let $\Phi \cong Z*_2$ be the group with presentation $\langle a, t \mid tat^{-1} = a^2\rangle$. This group is an ascending HNN extension with base Z, is metabelian, and has commutator subgroup isomorphic to $\mathbb{Z}[\frac{1}{2}]$. The 2-complex corresponding to this presentation is aspherical and so $g.d.\Phi = 2$.

The group Φ is the group of Example 10 of Fox, which is the boundary of the ribbon D^3 in S^4 obtained by "thickening" a suitable immersed ribbon D^2 in S^3 for the stevedore's knot 6_2 [Fo62]. Such a ribbon disc may be constructed by applying the method of §7 of Chapter 1 of [H3] to the equivalent presentation $\langle t, u, v \mid vuv^{-1} = t,\ tut^{-1} = v\rangle$ for Φ (where $u = ta$ and $v = t^2at^{-1}$).

Theorem 15.1 *Let π be a 2-knot group such that $c.d.\pi = 2$ and π has a nontrivial normal subgroup E which is either elementary amenable or almost coherent, locally virtually indicable and restrained. Then either $\pi \cong \Phi$ or π is an iterated free product of (one or more) torus knot groups, amalgamated over central subgroups. In either case* $\mathrm{def}(\pi) = 1$.

Proof If π is solvable then $\pi \cong Z*_m$, for some $m \neq 0$, by Corollary 2.6.1. Since $\pi/\pi' \cong Z$ we must have $m = 2$ and so $\pi \cong \Phi$.

Otherwise $E \cong Z$, by Theorem 2.7. Then $[\pi : C_\pi(E)] \leq 2$ and $C_\pi(E)'$ is free, by Bieri's Theorem. This free subgroup must be nonabelian for otherwise π would be solvable. Hence $E \cap C_\pi(E)' = 1$ and so E maps injectively to $H = \pi/C_\pi(E)'$. As H has an abelian normal subgroup of index at most 2 and $H/H' \cong Z$ we must in fact have $H \cong Z$. It follows easily that $C_\pi(E) = \pi$, and so π' is free. The further structure of π is then due to Strebel [St76]. The final observation follows readily. □

The following alternative characterizations of Φ shall be useful.

Theorem 15.2 *Let π be a 2-knot group with maximal locally finite normal subgroup T. Then $\pi/T \cong \Phi$ if and only if π is elementary amenable and $h(\pi) = 2$. Moreover the following are equivalent:*

1. *π has an abelian normal subgroup A of rank 1 such that π/A has two ends;*
2. *π is elementary amenable, $h(\pi) = 2$ and π has an abelian normal subgroup A of rank 1;*
3. *π is almost coherent, elementary amenable and $h(\pi) = 2$;*
4. *$\pi \cong \Phi$.*

Proof Since π is finitely presentable and has infinite cyclic abelianization it is an HNN extension $\pi \cong H*_\phi$ with base H a finitely generated subgroup of π', by Theorem 1.13. Since π is elementary amenable the extension must be ascending. Since $h(\pi'/T) = 1$ and π'/T has no nontrivial locally-finite normal subgroup $[\pi'/T : \sqrt{\pi'/T}] \leq 2$. The meridianal automorphism of π' induces a meridianal automorphism on $(\pi'/T)/\sqrt{\pi'/T}$ and so $\pi'/T = \sqrt{\pi'/T}$. Hence π'/T is a torsion free rank 1 abelian group. Let $J = H/H \cap T$. Then $h(J) = 1$ and $J \leq \pi'/T$ so $J \cong Z$. Now ϕ induces a monomorphism $\psi : J \to J$ and $\pi/T \cong J*_\psi$. Since $\pi/\pi' \cong Z$ we must have $J*_\psi \cong \Phi$.

If (1) holds then π is elementary amenable and $h(\pi) = 2$. Suppose (2) holds. We may assume without loss of generality that A is the normal closure of an element of infinite order, and so π/A is finitely presentable. Since π/A is elementary amenable and $h(\pi/A) = 1$ it is virtually Z. Therefore π is virtually an HNN extension with base a finitely generated subgroup of A, and so is coherent. If (3) holds then $\pi \cong \Phi$, by Corollary 3.17.1. Since Φ clearly satisfies conditions (1-3) this proves the theorem. □

Corollary 15.2.1 *If T is finite and $\pi/T \cong \Phi$ then $T = 1$ and $\pi \cong \Phi$.* □

15.2 Almost coherent, restrained and locally virtually indicable

We shall show that the basic tetrachotomy of the introduction is essentially correct, under mild coherence hypotheses on πK or N. Recall that a restrained group has no noncyclic free subgroups. Thus if N is a countable restrained group either it is elementary amenable and $h(N) \leq 1$ or it is an increasing union of finitely generated one-ended groups.

Theorem 15.3 *Let K be a 2-knot whose group $\pi = \pi K$ is an ascending HNN extension over an FP_2 base H with finitely many ends. Then either π' is finite or $\pi \cong \Phi$ or $M(K)$ is aspherical.*

Proof This follows from Theorem 3.17, since a group with abelianization Z cannot be virtually Z^2. □

Is $M(K)$ still aspherical if we assume only that H is finitely generated and one-ended?

Corollary 15.3.1 *If H is FP_3 and has one end then $\pi' = H$ and is a PD_3^+-group.*

Proof This follows from Lemma 3.4 of [BG85], as in Theorem 2.13. □

Does this remain true if we assume only that H is FP_2 and has one end?

Corollary 15.3.2 *If π is an ascending HNN extension over an FP_2 base H and has an infinite restrained normal subgroup A then either π' is finite or $\pi \cong \Phi$ or $M(K)$ is aspherical or $\pi' \cap A = 1$ and π/A has infinitely many ends.*

Proof If H is finite or $A \cap H$ is infinite then H has finitely many ends (cf. Corollary 1.16.1) and Theorem 15.3 applies. Therefore we may assume that H has infinitely many ends and $A \cap H$ is finite. But then $A \not\leq \pi'$, so π is virtually $\pi' \times Z$. Hence $\pi' = H$ and $M(K)'$ is a PD_3-complex. In particular $\pi' \cap A = 1$ and π/A has infinitely many ends. □

In §4 we shall determine all 2-knot groups with π' finite. If K is the r-twist spin of an irreducible 1-knot then the r^{th} power of a meridian is central in π and either π' is finite or $M(K)$ is aspherical. (See §3 of Chapter 16.) The final possibility is realized by Artin spins of nontrivial torus knots.

Theorem 15.4 *Let K be a 2-knot whose group $\pi = \pi K$ is an HNN extension with FP_2 base B and associated subgroups I and $\phi(I) = J$. If π has a restrained normal subgroup N which is not locally finite and $\beta_1^{(2)}(\pi) = 0$ then either π' is finite or $\pi \cong \Phi$ or $M(K)$ is aspherical or N is locally virtually Z and π/N has infinitely many ends.*

Proof If $\pi' \cap N$ is locally finite then it follows from Britton's lemma (on normal forms in HNN extensions) that either $B \cap N = I \cap N$ or $B \cap N = J \cap N$. Moreover $N \not\leq \pi'$ (since N is not locally finite), and so $\pi'/\pi' \cap N$ is finitely generated. Hence $B/B \cap N \cong I/I \cap N \cong J/J \cap N$. Thus either $B = I$ or $B = J$ and so the HNN extension is ascending. If B has finitely many ends we may apply Theorem 15.3. Otherwise $B \cap N$ is finite, so $\pi' \cap N = B \cap N$ and N is virtually Z. Hence π/N is commensurable with $B/B \cap N$, and $e(\pi/N) = \infty$.

If $\pi' \cap N$ is locally virtually Z and $\pi/\pi' \cap N$ has two ends then π is elementary amenable and $h(\pi) = 2$, so $\pi \cong \Phi$. Otherwise we may assume that either $\pi/\pi' \cap N$ has one end or $\pi' \cap N$ has a finitely generated, one-ended subgroup. In either case $H^s(\pi; \mathbb{Z}[\pi]) = 0$ for $s \leq 2$, by Theorem 1.18, and so $M(K)$ is aspherical, by Theorem 3.5. □

Note that $\beta_1^{(2)}(\pi) = 0$ if N is amenable. Every knot group is an HNN extension with finitely generated base and associated subgroups, by Theorem 1.13, and in all known cases these subgroups are FP_2.

Theorem 15.5 *Let K be a 2-knot such that $\pi = \pi K$ has an almost coherent, locally virtually indicable, restrained normal subgroup E which is not locally finite. Then either π' is finite or $\pi \cong \Phi$ or $M(K)$ is aspherical or E is abelian of rank 1 and π/E has infinitely many ends or E is elementary amenable, $h(E) = 1$ and π/E has one or infinitely many ends.*

Proof Let F be a finitely generated subgroup of E. Since F is FP_2 and virtually indicable it has a subgroup of finite index which is an HNN extension over a finitely generated base, by Theorem 1.13. Since F is restrained the HNN extension is ascending, and so $\beta_1^{(2)}(F) = 0$, by Lemma 2.1. Hence $\beta_1^{(2)}(E) = 0$ and so $\beta_1^{(2)}(\pi) = 0$, by Theorem 7.2 of [Lü].

If every finitely generated infinite subgroup of E has two ends, then E is elementary amenable and $h(E) = 1$. If π/E is finite then π' is finite. If π/E has two ends then π is almost coherent, elementary amenable and $h(\pi) = 2$, and so $\pi \cong \Phi$, by Theorem 15.2. If E is abelian and π/E has one end, or if E

has a finitely generated, one-ended subgroup and π is not elementary amenable of Hirsch length 2 then $H^s(\pi;\mathbb{Z}[\pi]) = 0$ for $s \leq 2$, by Theorem 1.17. Hence $M(K)$ is aspherical, by Theorem 3.5.

The remaining possibilities are that either π/E has infinitely many ends or that E is locally virtually Z but nonabelian and π/E has one end. □

Does this theorem hold without any coherence hypothesis? Note that the other hypotheses hold if E is elementary amenable and $h(E) \geq 2$. If E is elementary amenable, $h(E) = 1$ and π/E has one end is $H^2(\pi;\mathbb{Z}[\pi]) = 0$?

Corollary 15.5.1 *Let K be a 2-knot with group $\pi = \pi K$. Then either π' is finite or $\pi \cong \Phi$ or $M(K)$ is aspherical and $\sqrt{\pi} \cong Z^2$ or $M(K)$ is homeomorphic to an infrasolvmanifold or $h(\sqrt{\pi}) = 1$ and $\pi/\sqrt{\pi}$ has one or infinitely many ends or $\sqrt{\pi}$ is locally finite.*

Proof Finitely generated nilpotent groups are polycyclic. If $\pi/\sqrt{\pi}$ has two ends we may apply Theorem 15.3. If $h(\sqrt{\pi}) = 2$ then $\sqrt{\pi} \cong Z^2$, by Theorem 9.2, while if $h > 2$ then π is virtually poly-Z, by Theorem 8.1. □

Under somewhat stronger hypotheses we may assume that π has a nontrivial torsion free abelian normal subgroup.

Theorem 15.6 *Let N be a group which is either elementary amenable or is locally FP_3, virtually indicable and restrained. If c.d.$N \leq 3$ then N is virtually solvable.*

Proof Suppose first that N is locally FP_3 and virtually indicable, and let E be a finitely generated subgroup of N which maps onto Z. Then E is an ascending HNN extension $H*_\phi$ with FP_3 base H and associated subgroups. If *c.d.*$H = 3$ then $H^3(H;\mathbb{Z}[E]) \cong H^3(H;\mathbb{Z}[H]) \otimes_H \mathbb{Z}[E] \neq 0$ and the homomorphism $H^3(H;\mathbb{Z}[E]) \to H^3(H;\mathbb{Z}[E])$ in the Mayer-Vietoris sequence for the HNN extension is not onto, by Lemma 3.4 and the subsequent Remark 3.5 of [BG85]. But then $H^4(E;\mathbb{Z}[E]) \neq 0$, contrary to *c.d.*$N \leq 3$. Therefore *c.d.*$H \leq 2$, and so H is elementary amenable, by Theorem 2.7. Hence N is elementary amenable, and so is virtually solvable by Theorem 1.11. □

In particular, $\zeta\sqrt{N}$ is a nontrivial, torsion free abelian characteristic subgroup of N. A similar argument shows that if N is locally FP_n, virtually indicable, restrained and *c.d.*$N \leq n$ then N is virtually solvable.

15.3 Abelian normal subgroups

In this section we shall consider 2-knot groups with infinite abelian normal subgroups. The class with rank 1 abelian normal subgroups includes the groups of torus knots and twist spins, the group Φ, and all 2-knot groups with finite commutator subgroup. If there is such a subgroup of rank > 1 the knot manifold is aspherical; this case is considered further in Chapter 16.

Theorem 15.7 *Let K be a 2-knot whose group $\pi = \pi K$ has an infinite abelian normal subgroup A, of rank r. Then $r \leq 4$ and*

(1) *if A is a torsion group then π' is not FP_2;*

(2) *if $r = 1$ either π' is finite or $\pi \cong \Phi$ or $M(K)$ is aspherical or $e(\pi/A) = \infty$;*

(3) *if $r = 1$, $e(\pi/A) = \infty$ and $\pi' \leq C_\pi(A)$ then A and $\sqrt{\pi}$ are virtually Z;*

(4) *if $r = 1$ and $A \not\leq \pi'$ then $M(K)$ is a PD_3^+-complex, and is aspherical if and only if π' is a PD_3^+-group if and only if $e(\pi') = 1$;*

(5) *if $r = 2$ then $A \cong Z^2$ and $M(K)$ is aspherical;*

(6) *if $r = 3$ then $A \cong Z^3$, $A \leq \pi'$ and $M(K)$ is homeomorphic to an infrasolvmanifold;*

(7) *if $r = 4$ then $A \cong Z^4$ and $M(K)$ is homeomorphic to a flat 4-manifold.*

Proof If π' is FP_2 then $M(K)'$ is a PD_3-complex, by Corollary 4.5.2, and so locally finite normal subgroups of π are finite.

The four possibilities in case (2) correspond to whether π/A is finite or has one, two or infinitely many ends, by Theorem 15.5. These possibilities are mutually exclusive; if $e(\pi/A) = \infty$ then a Mayer-Vietoris argument as in Lemma 14.8 implies that π cannot be a PD_4-group.

Suppose that $r = 1$, and $A \leq \zeta\pi'$. Then A is a module over $\mathbb{Z}[\pi/\pi'] \cong \Lambda$. On replacing A by a subgroup, if necessary, we may assume that A is cyclic as a Λ-module and is $\mathbb{Z}$-torsion free. If moreover $e(\pi/A) = \infty$ then $\sqrt{\pi}/A$ must be finite and $K = \pi'/A$ is not finitely generated. We may write K as an increasing union of finitely generated subgroups $K = \cup_{n\geq 1} K_n$. Let S be an infinite cyclic subgroup of A and let $G = \pi'/S$. Then G is an extension of K by A/S, and so is an increasing union $G = \cup G_n$, where G_n is an extension of K_n by A/S. If A is not finitely generated then A/S is an infinite abelian normal subgroup. Therefore if some G_n is finitely generated then it has one end, and so $H^1(G_n; F) = 0$ for any free $\mathbb{Z}[G_n]$-module F. Otherwise we may write G_n as an increasing union of finitely generated subgroups $G_n = \cup_{m\geq 1} G_{nm}$, where

G_{nm} is an extension of K_n by a finite cyclic group Z/d_mZ, d_m divides d_{m+1} for all $m \geq 1$, and $A/S = \cup Z/d_mZ$. Let u be a generator of the subgroup Z/d_1Z, and let $\bar{G}_n = G_n/\langle u\rangle$ and $\bar{G}_{nm} = G_{nm}/\langle u\rangle$ for all $m \geq 1$. Then $\bar{G}_{n1} \cong K_n$, and so $\bar{G}_n \cong K_n \times (A/d_1^{-1}S)$. Since K_n is finitely generated and $A/d_1^{-1}S$ is infinite we again find that $H^1(\bar{G}_n; F) = 0$ for any free $\mathbb{Z}[\bar{G}]$-module F. It now follows from Theorem 1.16 that $H^1(\bar{G}; F) = 0$ for any free $\mathbb{Z}[\bar{G}]$-module F. An application of the LHSSS for π' as an extension of $\bar{G}$ by the normal subgroup $d_1^{-1}S \cong Z$ then gives $H^s(\pi'; \mathbb{Z}[\pi]) = 0$ for $s \leq 2$. Another LHSSS argument then gives $H^s(\pi; \mathbb{Z}[\pi]) = 0$ for $s \leq 2$ and so $M(K)$ is aspherical. As observed above, this contradicts the hypothesis $e(\pi/A) = \infty$.

Suppose next that $r = 1$ and A is not contained in π'. Let $x_1, \dots x_n$ be a set of generators for π and let s be an element of A which is not in π'. As each commutator $[s, x_i]$ is in $\pi' \cap A$ it has finite order, e_i say. Let $e = \Pi e_i$. Then $[s^e, x] = s^e(xs^{-1}x^{-1})^e = (sxs^{-1}x^{-1})^e$, so s^e commutes with all the generators. The subgroup generated by $\{s^e\} \cup \pi'$ has finite index in π and is isomorphic to $Z \times \pi'$, so π' is finitely presentable. Hence $M(K)'$ is an orientable PD_3-complex, by Corollary 4.5.2, and $M(K)$ is aspherical if and only if π' has one end, by Theorem 4.1. (In particular, A is finitely generated.)

If $r = 2$ then $A \cong Z^2$ and $M(K)$ is aspherical by Theorem 9.2. If $r > 2$ then $r \leq 4$, $A \cong Z^r$ and $M(K)$ is homeomorphic to an infrasolvmanifold by Theorem 8.1. In particular, π is virtually poly-Z and $h(\pi) = 4$. If $r = 3$ then $A \leq \pi'$, for otherwise $h(\pi/\pi' \cap A) = 2$, which is impossible for a group with abelianization Z. If $r = 4$ then $[\pi : A] < \infty$ and so $M(K)$ is homeomorphic to a flat 4-manifold. □

It remains an open question whether abelian normal subgroups of PD_n groups must be finitely generated. If this is so, Φ is the only 2-knot group with an abelian normal subgroup of positive rank which is not finitely generated.

The argument goes through with A a nilpotent normal subgroup. Can it be extended to the Hirsch-Plotkin radical? The difficulties are when $h(\sqrt{\pi}) = 1$ and $e(\pi/\sqrt{\pi}) = 1$ or ∞.

Corollary 15.7.1 *If A has rank 1 its torsion subgroup T is finite, and if moreover π' is infinite and π'/A is finitely generated $T = 1$.* □

The evidence suggests that if π' is finitely generated and infinite then A is free abelian. Little is known about the rank 0 case. All the other possibilities allowed by this theorem occur. (We shall consider the cases with rank ≥ 2

further in Chapter 16.) In particular, if π is torsion free and $\pi'\cap A = 1$ then π' is a free product of PD_3^+-groups and free groups, and the various possibilities (π' finite, $e(\pi') = 1$ or $e(\pi') = \infty$) are realized by twists spins of classical knots. Is every 2-knot K such that $\zeta\pi \not\leq \pi'$ and π is torsion free s-concordant to a fibred knot?

Corollary 15.7.2 *If π' finitely generated then either π' is finite or $\pi'\cap A = 1$ or $M(K)$ is aspherical. If moreover $\pi'\cap A$ has rank 1 then $\zeta\pi' \neq 1$.*

Proof As $\pi'\cap A$ is torsion free $Aut(\pi'\cap A)$ is abelian. Hence $\pi'\cap A \leq \zeta\pi'$. □

If π' is FP_2 and $\pi'\cap A$ is infinite then π' is the fundamental group of an aspherical Seifert fibred 3-manifold. There are no known examples of 2-knot groups π with π' finitely generated but not finitely presentable.

We may construct examples of 2-knots with such groups as follows. Let N be a closed 3-manifold such that $\nu = \pi_1(N)$ has weight 1 and $\nu/\nu' \cong Z$, and let $w = w_1(N)$. Then $H^2(N; Z^w) \cong Z$. Let M_e be the total space of the S^1-bundle over N with Euler class $e \in H^2(N; Z^w)$. Then M_e is orientable, and $\pi_1(M_e)$ has weight 1 if $e = \pm 1$ or if $w \neq 0$ and e is odd. In such cases surgery on a weight class in M_e gives S^4, so $M_e \cong M(K)$ for some 2-knot K.

In particular, we may take N to be the result of 0-framed surgery on a classical knot. If the classical knot is 3_1 or 4_1 (i.e., is fibred of genus 1) then the resulting 2-knot group has commutator subgroup Γ_1. For examples with $w \neq 0$ we may take one of the nonorientable surface bundles with group $\langle t, a_i, b_i\ (1 \leq i \leq n) \mid \Pi[a_i, b_i] = 1,\ ta_it^{-1} = b_i,\ tb_it^{-1} = a_ib_i\ (1 \leq i \leq n)\rangle$, where n is odd. (When $n = 1$ we get the third of the three 2-knot groups with commutator subgroup Γ_1. See Theorem 16.13.)

Theorem 15.8 *Let K be a 2-knot with a minimal Seifert hypersurface, and such that $\pi = \pi K$ has an abelian normal subgroup A. Then $A\cap\pi'$ is finite cyclic or is torsion free, and $\zeta\pi$ is finitely generated.*

Proof By assumption, $\pi = HNN(H; \phi : I \cong J)$ for some finitely presentable group H and isomorphism of ϕ of subgroups I and J, where $I \cong J \cong \pi_1(V)$ for some Seifert hypersurface V. Let $t \in \pi$ be the stable letter. Either $H\cap A = I\cap A$ or $H\cap A = J\cap A$ (by Britton's Lemma). Hence $\pi'\cap A = \cup_{n\in Z} t^n(I\cap A)t^{-n}$ is a monotone union. Since $I\cap A$ is an abelian normal subgroup of a 3-manifold group it is finitely generated [Ga92], and since V is orientable $I\cap A$ is torsion free or finite. If $A\cap I$ is finite cyclic or is central in π then $A\cap I = t^n(A\cap I)t^{-n}$, for all n, and so $A\cap\pi' = A\cap I$. (In particular, $\zeta\pi$ is finitely generated.) Otherwise $A\cap\pi'$ is torsion free. $\square$

This argument derives from [Yo92,97], where it was shown that if A is a finitely generated abelian normal subgroup then $\pi'\cap A \leq I\cap J$.

Corollary 15.8.1 *Let K be a 2-knot with a minimal Seifert hypersurface. If $\pi = \pi K$ has a nontrivial abelian normal subgroup A then $\pi'\cap A$ is finite cyclic or is torsion free. Moreover $\zeta\pi \cong 1$, $Z/2Z$, Z, $Z\oplus(Z/2Z)$ or Z^2.* $\square$

The knots $\tau_0 4_1$, the trivial knot, $\tau_3 3_1$ and $\tau_6 3_1$ are fibred and their groups have centres 1, Z, $Z\oplus(Z/2Z)$ and Z^2, respectively. A 2-knot with a minimal Seifert hypersurface and such that $\zeta\pi = Z/2Z$ is constructed in [Yo82]. This paper also gives an example with $\zeta\pi \cong Z$, $\zeta\pi < \pi'$ and such that $\pi/\zeta\pi$ has infinitely many ends. In all known cases the centre of a 2-knot group is cyclic, $Z\oplus(Z/2Z)$ or Z^2.

15.4 Finite commutator subgroup

It is a well known consequence of the asphericity of the exteriors of classical knots that classical knot groups are torsion free. The first examples of higher dimensional knots whose groups have nontrivial torsion were given by Mazur [Mz62] and Fox [Fo62]. These examples are 2-knots whose groups have finite commutator subgroup. We shall show that if π is such a group π' must be a CK group, and that the images of meridianal automorphisms in $Out(\pi')$ are conjugate, up to inversion. In each case there is just one 2-knot group with given finite commutator subgroup. Many of these groups can be realized by twist spinning classical knots. Zeeman introduced twist spinning in order to study Mazur's example; Fox used hyperplane cross sections, but his examples (with $\pi' \cong Z/3Z$) were later shown to be twist spins [Kn83'].

Lemma 15.9 *An automorphism of $Q(8)$ is meridianal if and only if it is conjugate to σ.*

Proof Since $Q(8)$ is solvable an automorphism is meridianal if and only if the induced automorphism of $Q(8)/Q(8)'$ is meridianal. It is easily verified that all such elements of $Aut(Q(8)) \cong (Z/2Z)^2 \rtimes SL(2,\mathbb{F}_2)$ are conjugate to σ. $\square$

Lemma 15.10 *All nontrivial automorphisms of I^* are meridianal. Moreover each automorphism is conjugate to its inverse. The nontrivial outer automorphism class of I^* cannot be realised by a 2-knot group.*

Proof Since the only nontrivial proper normal subgroup of I^* is its centre ($\zeta I^* = Z/2Z$) the first assertion is immediate. Since $Aut(I^*) \cong S_5$ and the conjugacy class of a permutation is determined by its cycle structure each automorphism is conjugate to its inverse. Consideration of the Wang sequence for the projection of $M(K)'$ onto $M(K)$ shows that the meridianal automorphism induces the identity on $H_3(\pi';\mathbb{Z})$, and so the nontrivial outer automorphism class cannot occur, by Lemma 11.4. □

The elements of order 2 in $A_5 \cong Inn(I^*)$ are all conjugate, as are the elements of order 3. There are two conjugacy classes of elements of order 5.

Lemma 15.11 *An automorphism of T_k^* is meridianal if and only if it is conjugate to $\rho^{3^{k-1}}$ or $\rho^{3^{k-1}}\eta$. All such automorphisms have the same image in $Out(T_k^*)$.*

Proof Since T_k^* is solvable an automorphism is meridianal if and only if the induced automorphism of $T_k^*/(T_k^*)'$ is meridianal. Any such automorphism is conjugate to either ρ^{2j+1} or to $\rho^{2j+1}\eta$ for some $0 \leq j < 3^{k-1}$. (Note that 3 divides $2^{2j}-1$ but does not divide $2^{2j+1}-1$.) However among them only those with $2j+1 = 3^{k-1}$ satisfy the isometry condition of Theorem 14.3. □

Theorem 15.12 *Let K be a 2-knot with group $\pi = \pi K$. If π' is finite then $\pi' \cong P \times (Z/nZ)$ where $P = 1$, $Q(8)$, I^* or T_k^*, and $(n, 2|P|) = 1$, and the meridianal automorphism sends x and y in $Q(8)$ to y and xy, is conjugation by a noncentral element on I^*, sends x, y and z in T_k^* to y^{-1}, x^{-1} and z^{-1}, and is -1 on the cyclic factor.*

Proof Since $\chi(M(K)) = 0$ and π has two ends π' has cohomological period dividing 4, by Theorem 11.1, and so is among the groups listed in §2 of Chapter 11. As the meridianal automorphism of π' induces a meridianal automorphism on the quotient by any characteristic subgroup, we may eliminate immediately the groups $O^*(k)$ and $A(m,e)$ and direct products with $Z/2nZ$ since these all have abelianization cyclic of even order. If $k > 1$ the subgroup generated by x in $Q(8k)$ is a characteristic subgroup of index 2. Since $Q(2^n a)$ is a quotient of $Q(2^n a, b, c)$ by a characteristic subgroup (of order bc) this eliminates this class also. Thus there remain only the above groups.

It is clear that automorphisms of a group $G = H \times J$ such that $(|H|, |J|) = 1$ correspond to pairs of automorphisms ϕ_H and ϕ_J of H and J, respectively, and ϕ is meridianal if and only if ϕ_H and ϕ_J are. Multiplication by s induces a meridianal automorphism of Z/mZ if and only if $(s-1, m) = (s, m) = 1$. If Z/mZ is a direct factor of π' then it is a direct summand of $\pi'/\pi'' = H_1(M(K); \Lambda)$ and so $s^2 \equiv 1$ modulo (m), by Theorem 14.3. Hence we must have $s \equiv -1$ modulo (m). The theorem now follows from Lemmas 15.9-15.11. □

Finite cyclic groups are realized by the 2-twist spins of 2-bridge knots, while the commutator subgroups of $\tau_3 3_1$, $\tau_4 3_1$ and $\tau_5 3_1$ are $Q(8)$, T_1^* and I^*, respectively. As the groups of 2-bridge knots have 2 generator 1 relator presentations the groups of these twist spins have 2 generator presentations of deficiency 0. The groups with $\pi' \cong Q(8) \times (Z/nZ)$ also have such presentations, namely $\langle t, u \mid tu^2t^{-1} = u^{-2}, t^2u^nt^{-2} = u^ntu^nt^{-1}\rangle$. They are realized by fibred 2-knots [Yo82], but if $n > 1$ no such group can be realized by a twist spin (see §3 of Chapter 16). An extension of the twist spin construction may be used to realize such groups by smooth fibred knots in the standard S^4, if $n = 3$, 5, 11, 13, 19, 21 or 27 [Kn88,Tr90]. Is this so in general? The direct products of T_k^* and I^* with cyclic groups are realized by the 2-twist spins of certain pretzel knots [Yo82]. The corresponding knot groups have presentations $\langle t, x, y, z \mid z^\alpha = 1, x = ztzt^{-1}, y = z^2tzt^{-1}z^{-1}, zyz^{-1} = xy, tx = xt\rangle$ and $\langle t, w \mid tw^nt^{-1} = w^nt^2w^nt^{-2}, t^5w^n = w^nt^5, tw^{10}t^{-1} = w^{-10}\rangle$, respectively. We may easily eliminate the generators x and y from the former presentations to obtain 2 generator presentations of deficiency -1. It is not known whether any of these groups (other than those with $\pi' \cong T_1^*$ or I^*) have deficiency 0. Note that when $P = I^*$ there is an isomorphism $\pi \cong I^* \times (\pi/I^*)$.

If $P = 1$ or $Q(8)$ the weight class is unique up to inversion, while T_k^* and I^* have 2 and 4 weight orbits, respectively, by Theorem 14.1. If $\pi' = T_1^*$ or I^* each weight orbit is realized by a branched twist spun torus knot [PS87].

The group $\pi\tau_5 3_1 \cong Z \times I^* = Z \times SL(2, \mathbb{F}_5)$ is the common member of two families of high dimensional knot groups which are not otherwise 2-knot groups. If p is a prime greater than 3 then $SL(2, \mathbb{F}_p)$ is a finite superperfect group. Let $e_p = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Then $(1, e_p)$ is a weight element for $Z \times SL(2, \mathbb{F}_p)$. Similarly, $(I^*)^m$ is superperfect and $(1, e_5, \dots, e_5)$ is a weight element for $G = Z \times (I^*)^m$, for any $m \geq 0$. However $SL(2, \mathbb{F}_p)$ has cohomological period $p-1$ (see Corollary 1.27 of [DM85]), while $\zeta(I^*)^m \cong (Z/2Z)^m$ and so $(I^*)^m$ does not have periodic cohomology if $m > 1$.

Kanenobu has shown that for every $n > 0$ there is a 2-knot group with an element of order exactly n [Kn80].

15.5 The Tits alternative

An HNN extension (such as a knot group) is restrained if and only if it is ascending and the base is restrained. The class of groups considered in the next result probably includes all restrained 2-knot groups.

Theorem 15.13 *Let π be a 2-knot group. Then the following are equivalent:*

(1) *π is restrained, locally FP_3 and locally virtually indicable;*

(2) *π is an ascending HNN extension $H*_\phi$ where H is FP_3, restrained and virtually indicable;*

(3) *π is elementary amenable and has an abelian normal subgroup of rank > 0;*

(4) *π is elementary amenable and is an ascending HNN extension $H*_\phi$ where H is FP_2;*

(5) *π' is finite or $\pi \cong \Phi$ or π is torsion free virtually poly-Z and $h(\pi) = 4$.*

Proof Condition (1) implies (2) by Corollary 3.17.1. If (2) holds and H has one end then $\pi' = H$ and is a PD_3-group, by Corollary 15.3.1. Since H is virtually indicable and admits a meridianal automorphism, it must have a subgroup of finite index which maps onto Z^2. Hence H is virtually poly-Z, by Corollary 2.13.1 (together with the remark following it). Hence (2) implies (5). Conditions (3) and (4) imply (5) by Theorems 15.2 and 15.3, respectively. On the other hand (5) implies (1-4). □

In particular, if K is a 2-knot with a minimal Seifert hypersurface, πK is restrained and the Alexander polynomial of K is nontrivial then either $\pi \cong \Phi$ or π is torsion free virtually poly-Z and $h(\pi) = 4$.

15.6 Abelian HNN bases

We shall complete Yoshikawa's study of 2-knot groups which are HNN extensions with abelian base. The first four paragraphs of the following proof outline the arguments of [Yo86,92]. (Our contribution is the argument in the final paragraph, eliminating possible torsion when the base has rank 1.)

Theorem 15.14 *Let π be a 2-knot group which is an HNN extension with abelian base. Then either π is metabelian or it has a deficiency 1 presentation $\langle t, x \mid tx^n t^{-1} = x^{n+1}\rangle$ for some $n > 1$.*

Proof Suppose that $\pi = HNN(A; \phi : B \to C)$ where A is abelian. Let j and j_C be the inclusions of B and C into A, and let $\tilde{\phi} = j_C\phi$. Then $\tilde{\phi} - j : B \to A$ is an isomorphism, by the Mayer-Vietoris sequence for homology with coefficients $\mathbb{Z}$ for the HNN extension. Hence $rank(A) = rank(B) = r$, say, and the torsion subgroups TA, TB and TC of A, B and C coincide.

Suppose first that A is not finitely generated. Since π is finitely presentable and $\pi/\pi' \cong Z$ it is also an HNN extension with finitely generated base and associated subgroups, by the Bieri-Strebel Theorem (1.13). Moreover we may assume the base is a subgroup of A. Considerations of normal forms with respect to the latter HNN structure imply that it must be ascending, and so π is metabelian [Yo92].

Assume now that A is finitely generated. Then the image of TA in π is a finite normal subgroup N, and π/N is a torsion free HNN extension with base $A/TA \cong Z^r$. Let j_F and ϕ_F be the induced inclusions of B/TB into A/TA, and let $M_j = |det(j_F)|$ and $M_\phi = |det(\phi_F)|$. Applying the Mayer-Vietoris sequence for homology with coefficients Λ, we find that $t\tilde{\phi} - j$ is injective and $\pi'/\pi'' \cong H_1(\pi; \Lambda)$ has rank r as an abelian group. Now $H_2(A; \mathbb{Z}) \cong A \wedge A$ (see page 334 of [Ro]) and so $H_2(\pi; \Lambda) \cong \mathrm{Cok}(t \wedge_2 \tilde{\phi} - \wedge_2 j)$ has rank $\binom{r}{2}$. Let $\delta_i(t) = \Delta_0(H_i(\pi; \Lambda))$, for $i = 1$ and 2. Then $\delta_1(t) = det(t\phi_F - j_F)$ and $\delta_2(t) = det(t\phi_F \wedge \phi_F - j_F \wedge j_F)$. Moreover $\delta_2(t^{-1})$ divides $\delta_1(t)$, by Theorem 14.3. In particular, $\binom{r}{2} \leq r$, and so $r \leq 3$.

If $r = 0$ then clearly $B = A$ and so π is metabelian. If $r = 2$ then $\binom{r}{2} = 1$ and $\delta_2(t) = \pm(tM_\phi - M_j)$. Comparing coefficients of the terms of highest and lowest degree in $\delta_1(t)$ and $\delta_2(t^{-1})$, we see that $M_j = M_\phi$, so $\delta_2(1) \equiv 0 \mod (2)$, which is impossible since $|\delta_1(1)| = 1$. If $r = 3$ a similar comparison of coefficients in $\delta_1(t)$ and $\delta_2(t^{-1})$ shows that M_j^3 divides M_ϕ and M_ϕ^3 divides M_j, so $M_j = M_\phi = 1$. Hence ϕ is an isomorphism, and so π is metabelian.

There remains the case $r = 1$. Yoshikawa used similar arguments involving coefficients $\mathbb{F}_p\Lambda$ instead to show that in this case $N \cong Z/\beta Z$ for some odd $\beta \geq 1$. The group π/N then has a presentation $\langle t, x \mid tx^n t^{-1} = x^{n+1}\rangle$ (with $n \geq 1$). Let p be a prime. There is an isomorphism of the subfields $\mathbb{F}_p(X^n)$ and $\mathbb{F}_p(X^{n+1})$ of the rational function field $\mathbb{F}_p(X)$ which carries X^n to X^{n+1}. Therefore $\mathbb{F}_p(X)$ embeds in a skew field L containing an element t such that $tX^n t^{-1} = X^{n+1}$, by Theorem 5.5.1 of [Cn]. It is clear from the argument of this theorem that the group ring $\mathbb{F}_p[\pi/N]$ embeds as a subring of L, and so this group ring is weakly finite. Therefore so is the subring $\mathbb{F}_p[C_\pi(N)/N]$. It now follows from Lemma 3.15 that N must be trivial. Since π is metabelian if $n = 1$ this completes the proof. □

15.7 Locally finite normal subgroups

Let K be a 2-knot such that $\pi = \pi K$ has an infinite locally finite normal subgroup T, which we may assume maximal. As π has one end and $\beta_1^{(2)}(\pi) = 0$, by Gromov's Theorem (2.3), $H^2(\pi;\mathbb{Z}[\pi]) \neq 0$. For otherwise $M(K)$ would be aspherical and so π would be torsion free, by Theorem 3.5. Moreover $T < \pi'$ and π/T is not virtually Z, so $e(\pi/T) = 1$ or ∞. (No examples of such 2-knot groups are known, and we expect that there are none with $e(\pi/T) = 1$).

If $H_1(T;R) = 0$ for some subring R of $\mathbb{Q}$ and $\mathbb{Z}[\pi/T]$ embeds in a weakly finite ring S with an involution extending that of the group ring, which is flat as a right $\mathbb{Z}[\pi/T]$-module and such that $S \otimes_{\mathbb{Z}[\pi/T]} Z = 0$ then either π/T is a PD_4^+-group over R or $H^2(\pi/T;R[\pi/T]) \neq 0$, or $e(\pi/T) = \infty$, by the Addendum to Theorem 2.7 of [H2]. This applies in particular if π/T has a nontrivial locally nilpotent normal subgroup U/T, for then U/T is torsion free. (See Proposition 5.2.7 of [Ro].) Moreover $e(\pi/T) = 1$. An iterated LHSSS argument shows that if $h(U/T) > 1$ or if $U/T \cong Z$ and $e(\pi/U) = 1$ then $H^2(\pi/T;\mathbb{Q}[\pi/T]) = 0$. (This is also the case if $h(U/T) = 1$, $e(\pi/U) = 1$ and π/T is finitely presentable, by Theorem 1 of [Mi87] with [GM86].) Thus if $H^2(\pi/T;\mathbb{Q}[\pi/T]) \neq 0$ then U/T is abelian of rank 1 and either $e(\pi/U) = 2$ (in which case $\pi/T \cong \Phi$, by Theorem 15.2), $e(\pi/U) = 1$ (and U/T not finitely generated and π/U not finitely presentable) or $e(\pi/U) = \infty$. As $Aut(U/T)$ is then abelian U/T is central in π'/T. Moreover π/U can have no nontrivial locally finite normal subgroups, for otherwise T would not be maximal in π, by an easy extension of Schur's Theorem (Proposition 10.1.4 of [Ro]).

Hence if π has an ascending series whose factors are either locally finite or locally nilpotent then either $\pi/T \cong \Phi$ or $h(\sqrt{\pi/T}) \geq 2$ and so π/T is a PD_4^+-group over Q. Since $J = \pi/T$ is elementary amenable and has no nontrivial locally finite normal subgroup it is virtually solvable and $h(J) = 4$, by Theorem 1.11. It can be shown that J is virtually poly-Z and $J' \cap \sqrt{J} \cong Z^3$ or Γ_q for some $q \geq 1$. (See Theorem VI.2 of [H1].) The possibilities for J' are examined in Theorems VI.3-5 and VI.9 of [H1]. We shall not repeat this discussion here as we expect that if G is finitely presentable and T is an infinite locally finite normal subgroup such that $e(G/T) = 1$ then $H^2(G;\mathbb{Z}[G]) = 0$.

The following lemma suggests that there may be a homological route to showing that solvable 2-knot groups are virtually torsion free.

Lemma 15.15 *Let G be an FP_2 group with a torsion normal subgroup T such that either $G/T \cong Z*_m$ for some $m \neq 0$ or G/T is virtually poly-Z. Then T/T' has finite exponent as an abelian group. In particular, if π is solvable then $T = 1$ if and only if $H_1(T;\mathbb{F}_p) = 0$ for all primes p.*

Proof Let C_* be a free $\mathbb{Z}[G]$-resolution of the augmentation module Z which is finitely generated in degrees ≤ 2. Since $\mathbb{Z}[G/T]$ is coherent [BS79], $T/T' = H_1(\mathbb{Z}[G/T] \otimes_G C_*)$ is finitely presentable as a $\mathbb{Z}[G/T]$-module. If T/T' is generated by elements t_i of order e_i then Πe_i is a finite exponent for T/T'.

If π is solvable then so is T, and $T = 1$ if and only if $T/T' = 1$. Since T/T' has finite exponent $T/T' = 1$ if and only if $H_1(T;\mathbb{F}_p) = 0$ for all primes p. □

Note also that $\mathbb{F}_p[Z*_m]$ is a coherent Ore domain of global dimension 2, while if J is a torsion free virtually poly-Z group then $\mathbb{F}_p[J]$ is a noetherian Ore domain of global dimension $h(J)$. (See §4.4 and §13.3 of [Pa].)

Chapter 16

Abelian normal subgroups of rank ≥ 2

If K is a 2-knot such that $h(\sqrt{\pi K}) = 2$ then $\sqrt{\pi K} \cong Z^2$, by Corollary 15.5.1. The main examples are the branched twist spins of torus knots, whose groups usually have centre of rank 2. (There are however examples in which $\sqrt{\pi}$ is not central.) Although we have not been able to show that all 2-knot groups with centre of rank 2 are realized by such knots, we have a number of partial results that suggest strongly that this may be so. Moreover we can characterize the groups which arise in this way (obvious exceptions aside) as being the 3-knot groups which are PD_4^+-groups and have centre of rank 2, with some power of a weight element being central. The strategy applies to other twist spins of prime 1-knots; however in general we do not have satisfactory algebraic characterizations of the 3-manifold groups involved. If $h(\sqrt{\pi K}) > 2$ then $M(K)$ is homeomorphic to an infrasolvmanifold. We shall determine the groups of such knots and give optimal presentations for them in §4 of this chapter. Two of these groups are virtually Z^4; in all other cases $h(\sqrt{\pi K}) = 3$.

16.1 The Brieskorn manifolds $M(p,q,r)$

Let $M(p,q,r) = \{(u,v,w) \in \mathbb{C}^3 \mid u^p + v^q + w^r = 0\} \cap S^5$. Thus $M(p,q,r)$ is a *Brieskorn* 3*-manifold* (the link of an isolated singularity of the intersection of n algebraic hypersurfaces in $\mathbb{C}^{n+2}$, for some $n \geq 1$). It is clear that $M(p,q,r)$ is unchanged by a permutation of $\{p,\ q,\ r\}$.

Let $s = hcf\{pq, pr, qr\}$. Then $M(p,q,r)$ admits an effective S^1-action given by $z(u,v,w) = (z^{qr/s}u, z^{pr/s}v, z^{pq/s}w)$ for $z \in S^1$ and $(u,v,w) \in M(p,q,r)$, and so is Seifert fibred. More precisely, let $\ell = lcm\{p,q,r\}$, $p' = lcm\{q,r\}$, $q' = lcm\{p,r\}$ and $r' = lcm\{p,q\}$, $s_1 = qr/p'$, $s_2 = pr/q'$ and $s_3 = pq/r'$ and $t_1 = \ell/p'$, $t_2 = \ell/q'$ and $t_3 = \ell/r'$. Let $g = (2 + (pqr/\ell) - s_1 - s_2 - s_3)/2$. Then $M(p,q,r) = M(g; s_1(t_1,\beta_1), s_2(t_2,\beta_2), s_3(t_3,\beta_3))$, in the notation of [NR78], where the coefficients β_i are determined modulo t_i by the equation

$$e = -(qr\beta_1 + pr\beta_2 + pq\beta_3)/\ell) = -pqr/\ell^2$$

for the generalized Euler number. (See [NR78].) If $p^{-1}+q^{-1}+r^{-1}\leq 1$ the Seifert fibration is essentially unique. (See Theorem 3.8 of [Sc83'].) In most cases the triple $\{p,q,r\}$ is determined by the Seifert structure of $M(p,q,r)$. (Note however that, for example, $M(2,9,18)\cong M(3,5,15)$ [Mi75].)

The map $f: M(p,q,r)\to\mathbb{CP}^1$ given by $f(u,v,w)=[u^p:v^q]$ is constant on the orbits of the S^1-action, and the exceptional fibres are those above 0, -1 and ∞ in $\mathbb{CP}^1$. In particular, if p, q and r are pairwise relatively prime f is the orbit map and $M(p,q,r)$ is Seifert fibred over the orbifold $S^2(p,q,r)$. The involution c of $M(p,q,r)$ induced by complex conjugation in $\mathbb{C}^3$ is orientation preserving and is compatible with f and complex conjugation in $\mathbb{CP}^1$.

The 3-manifold $M(p,q,r)$ is a $\mathbb{S}^3$-manifold if and only if $p^{-1}+q^{-1}+r^{-1}>1$. The triples $(2,2,r)$ give lens spaces. The other triples with $p^{-1}+q^{-1}+r^{-1}>1$ are permutations of $(2,3,3)$, $(2,3,4)$ or $(2,3,5)$, and give the three CK 3-manifolds with fundamental groups $Q(8)$, T_1^* and I^*. The manifolds $M(2,3,6)$, $M(3,3,3)$ and $M(2,4,4)$ are $\mathbb{N}il^3$-manifolds; in all other cases $M(p,q,r)$ is a $\widetilde{\mathbb{SL}}$-manifold (in fact, a coset space of $\widetilde{SL}$ [Mi75]), and $\sqrt{\pi_1(M(p,q,r))}\cong Z$.

Let $A(u,v,w)=(u,v,e^{2\pi i/r}w)$ and $g(u,v,w)=(u,v)/(|u|^2+|v|^2)$, for $(u,v,w)\in M(p,q,r)$. Then A generates a Z/rZ-action which commutes with the above S^1-action, and these actions agree on their subgroups of order r/s. The projection to the orbit space $M(p,q,r)/\langle A\rangle$ may be identified with the map $g: M(p,q,r)\to S^3$, which is an r-fold cyclic branched covering, branched over the (p,q)-torus link. (See Lemma 1.1 of [Mi75].)

16.2 Rank 2 subgroups

In this section we shall show that an abelian normal subgroup of rank 2 in a 2-knot group is free abelian and not contained in the commutator subgroup.

Lemma 16.1 *Let ν be the fundamental group of a closed $\mathbb{H}^2\times\mathbb{E}^1$-, $\mathbb{S}ol^3$- or $\mathbb{S}^2\times\mathbb{E}^1$-manifold. Then ν admits no meridianal automorphism.*

Proof The fundamental group of a closed $\mathbb{S}ol^3$- or $\mathbb{S}^2\times\mathbb{E}^1$-manifold has a characteristic subgroup with quotient having two ends. If ν is a lattice in $Isom^+(\mathbb{H}^2\times\mathbb{E}^1)$ then $\sqrt{\nu}\cong Z$ and either $\sqrt{\nu}=\zeta\nu$ and is not contained in ν' or $C_\nu(\sqrt{\nu})$ is a characteristic subgroup of index 2 in ν. In none of these cases can ν admit a meridianal automorphism. ∎

Theorem 16.2 *Let K be a 2-knot whose group $\pi = \pi K$ has an abelian normal subgroup A of rank 2. Then π is a PD_4^+-group, $A \cong Z^2$, $\pi' \cap A \cong Z$, $\pi' \cap A \leq \zeta\pi' \cap I(\pi')$, $[\pi : C_\pi(A)] \leq 2$ and $\pi' = \pi_1(N)$, where N is a $\mathbb{N}il^3$- or $\widetilde{\mathbb{SL}}$-manifold. If π is virtually solvable then $M(K)$ is homeomorphic to a $\mathbb{N}il^3 \times \mathbb{E}^1$-manifold. If π is not virtually solvable then $M(K)$ is s-cobordant to the mapping torus $M(\Theta)$ of a self homeomorphism Θ of a $\widetilde{\mathbb{SL}}$-manifold; $M(\Theta)$ is a $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$-manifold if $\zeta\pi \cong Z^2$.*

Proof The first two assertions follow from Theorem 9.2, where it is also shown that π/A is virtually a PD_2-group. If $A < \pi'$ then π/A has infinite abelianization and so maps onto some planar discontinuous group, with finite kernel [EM82]. As the planar discontinuous group is virtually a surface group it has a compact fundamental region. But no such group has abelianization Z. (This follows for instance from consideration of the presentations given in Theorem 4.5.6 of [ZVC].) Therefore $\pi' \cap A \cong Z$. If τ is the meridianal automorphism of $\pi'/I(\pi')$ then $\tau - 1$ is invertible, and so cannot have ± 1 as an eigenvalue. Hence $\pi' \cap A \leq I(\pi')$. In particular, π' is not abelian.

The image of $\pi/C_\pi(A)$ in $Aut(A) \cong GL(2, \mathbb{Z})$ is triangular, since $\pi' \cap A \cong Z$ is normal in π. Therefore as $\pi/C_\pi(A)$ has finite cyclic abelianization it must have order at most 2. Thus $[\pi : C_\pi(A)] \leq 2$, so $\pi' < C_\pi(A)$ and $\pi' \cap A < \zeta\pi'$. The subgroup H generated by $\pi' \cup A$ has finite index in π and so is also a PD_4^+-group. Since A is central in H and maps onto H/π' we have $H \cong \pi' \times Z$. Hence π' is a PD_3^+-group with nontrivial centre. As the nonabelian flat 3-manifold groups either admit no meridianal automorphism or have trivial centre, $\pi' = \pi_1(N)$ for some $\mathbb{N}il^3$- or $\widetilde{\mathbb{SL}}$-manifold N, by Theorem 2.14 and Lemma 16.1.

The manifold $M(K)$ is s-cobordant to the mapping torus $M(\Theta)$ of a self homeomorphism of N, by Theorem 13.2. If N is a $\mathbb{N}il^3$-manifold $M(K)$ is homeomorphic to $M(\Theta)$, by Theorem 8.1, and $M(K)$ must be a $\mathbb{N}il^3 \times \mathbb{E}^1$-manifold, since the groups of $\mathbb{S}ol_1^4$-manifolds do not have rank 2 abelian normal subgroups, while the groups of $\mathbb{N}il^4$-manifolds cannot have abelianization Z, as they have characteristic rank 2 subgroups contained in their commutator subgroups.

We may assume also that $M(\Theta)$ is Seifert fibred over a 2-orbifold B. If moreover $\zeta\pi \cong Z^2$ then B must be orientable, and the monodromy representation of $\pi_1^{orb}(B)$ in $Aut(\zeta\pi) \cong GL(2, \mathbb{Z})$ is trivial. Therefore if N is an $\widetilde{\mathbb{SL}}$-manifold and $\zeta\pi \cong Z^2$ then $M(\Theta)$ is a $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$-manifold, by Theorem B of [Ue91] and Lemma 16.1. $\square$

If p, q and r are pairwise relatively prime $M(p,q,r)$ is a $\mathbb{Z}$-homology 3-sphere and $\pi_1(M(p,q,r))$ has a presentation

$$\langle a_1, a_2, a_3, h \mid a_1^p = a_2^q = a_3^r = a_1a_2a_3 = h\rangle$$

(see [Mi75]). The automorphism c_* of $\nu = \pi_1(M(p,q,r))$ induced by the involution c is determined by $c_*(a_1) = a_1^{-1}$, $c_*(a_2) = a_2^{-1}$ and $c_*(h) = h^{-1}$, and hence $c_*(a_3) = a_2a_3^{-1}a_2^{-1}$. If one of p, q and r is even c_* is meridianal. Surgery on the mapping torus of c gives rise to a 2-knot whose group $\nu \times_{c_*} Z$ has an abelian normal subgroup $A = \langle t^2, h\rangle$. If moreover $p^{-1} + q^{-1} + r^{-1} < 1$ then $A \cong Z^2$, but is not central.

The only virtually poly-Z groups with noncentral rank 2 abelian normal subgroups are the groups $\pi(b,\epsilon)$ discussed in §4 below.

Theorem 16.3 *Let π be a 2-knot group such that $\zeta\pi$ has rank greater than 1. Then $\zeta\pi \cong Z^2$, $\zeta\pi' = \pi' \cap \zeta\pi \cong Z$, and $\zeta\pi' \leq \pi''$.*

Proof If $\zeta\pi$ had rank greater than 2 then $\pi' \cap \zeta\pi$ would contain an abelian normal subgroup of rank 2, contrary to Theorem 16.2. Therefore $\zeta\pi \cong Z^2$ and $\pi' \cap \zeta\pi \cong Z$. Moreover $\pi' \cap \zeta\pi \leq \pi''$, since $\pi/\pi' \cong Z$. In particular π' is nonabelian and π'' has nontrivial centre. Hence π' is the fundamental group of a $\mathbb{N}il^3$- or $\widetilde{\mathbb{SL}}$-manifold, by Theorem 16.2, and so $\zeta\pi' \cong Z$. It follows easily that $\pi' \cap \zeta\pi = \zeta\pi'$. ∎

The proof of this result in [H1] relied on the theorems of Bieri and Strebel, rather than Bowditch's Theorem.

16.3 Twist spins of torus knots

The commutator subgroup of the group of the r-twist spin of a classical knot K is the fundamental group of the r-fold cyclic branched cover of S^3, branched over K [Ze65]. The r-fold cyclic branched cover of a sum of knots is the connected sum of the r-fold cyclic branched covers of the factors, and is irreducible if and only if the knot is prime. Moreover the cyclic branched covers of a prime knot are either aspherical or finitely covered by S^3; in particular no summand has free fundamental group [Pl84]. The cyclic branched covers of prime knots with nontrivial companions are Haken 3-manifolds [GL84]. The r-fold cyclic branched cover of a simple nontorus knot is a hyperbolic 3-manifold if $r \geq 3$, excepting only the 3-fold cyclic branched cover of the figure-eight knot, which is the Hanztsche-Wendt flat 3-manifold [Du83]. The r-fold cyclic branched cover of the (p,q)-torus knot $k_{p,q}$ is the Brieskorn manifold $M(p,q,r)$ [Mi75]. (In particular, there are only four r-fold cyclic branched covers of nontrivial knots for any $r > 2$ which have finite fundamental group.)

Theorem 16.4 *Let M be the r-fold cyclic branched cover of S^3, branched over a knot K, and suppose that $r > 2$ and that $\sqrt{\pi_1(M)} \neq 1$. Then K is uniquely determined by M and r, and either K is a torus knot or $K \cong 4_1$ and $r = 3$.*

Proof As the connected summands of M are the cyclic branched covers of the factors of K, any homotopy sphere summand must be standard, by the proof of the Smith conjecture. Therefore M is aspherical, and is either Seifert fibred or is a $\mathbb{S}ol^3$-manifold, by Theorem 2.14. It must in fact be a $\mathbb{E}^3$-, $\mathbb{N}il^3$- or $\widetilde{\mathbb{SL}}$-manifold, by Lemma 16.1. If there is a Seifert fibration which is preserved by the automorphisms of the branched cover the fixed circle (the branch set of M) must be a fibre of the fibration (since $r > 2$) which therefore passes to a Seifert fibration of $X(K)$. Thus K must be a (p,q)-torus knot, for some relatively prime integers p and q [BZ]. These integers may be determined arithmetically from r and the formulae for the Seifert invariants of $M(p,q,r)$ given in §1. Otherwise M is flat [MS86] and so $K \cong 4_1$ and $r = 3$, by [Du83]. $\square$

All the knots whose 2-fold branched covers are Seifert fibred are torus knots or Montesinos knots. (This class includes the 2-bridge knots and pretzel knots, and was first described in [Mo73].) The number of distinct knots whose 2-fold branched cover is a given Seifert fibred 3-manifold can be arbitrarily large [Be84]. Moreover for each $r \geq 2$ there are distinct simple 1-knots whose r-fold cyclic branched covers are homeomorphic [Sa81, Ko86].

If K is a fibred 2-knot with monodromy of finite order r and if $(r,s) = 1$ then the s-fold cyclic branched cover of S^4, branched over K is again a 4-sphere and so the branch set gives a new 2-knot, which we shall call the s-fold cyclic branched cover of K. This new knot is again fibred, with the same fibre and monodromy the s^{th} power of that of K [Pa78, Pl86]. If K is a classical knot we shall let $\tau_{r,s}K$ denote the s-fold cyclic branched cover of the r-twist spin of K. We shall call such knots *branched twist spins*, for brevity.

Using properties of S^1-actions on smooth homotopy 4-spheres, Plotnick obtains the following result [Pl86].

Theorem (Plotnick) *A 2-knot is fibred with periodic monodromy if and only if it is a branched twist spin of a knot in a homotopy 3-sphere.* $\square$

Here "periodic monodromy" means that the fibration of the exterior of the knot has a characteristic map of finite order. It is not in general sufficient that the *closed* monodromy be represented by a map of finite order. (For instance, if K is a fibred 2-knot with $\pi' \cong Q(8) \times (Z/nZ)$ for some $n > 1$ then the meridianal automorphism of π' has order 6, and so it follows from the observations above that K is not a twist spin.)

In our application in the next theorem we are able to show directly that the homotopy 3-sphere arising there may be assumed to be standard.

Theorem 16.5 *A group G which is not virtually solvable is the group of a branched twist spin of a torus knot if and only if it is a 3-knot group and a PD_4^+-group with centre of rank 2, some nonzero power of a weight element being central.*

Proof If K is a cyclic branched cover of the r-twist spin of the (p,q)-torus knot then $M(K)$ fibres over S^1 with fibre $M(p,q,r)$ and monodromy of order r, and so the r^{th} power of a meridian is central. Moreover the monodromy commutes with the natural S^1-action on $M(p,q,r)$ (see Lemma 1.1 of [Mi75]) and hence preserves a Seifert fibration. Hence the meridian commutes with $\zeta\pi_1(M(p,q,r))$, which is therefore also central in G. Since (with the above exceptions) $\pi_1(M(p,q,r))$ is a PD_3^+-group with infinite centre and which is virtually representable onto Z, the necessity of the conditions is evident.

Conversely, if G is such a group then G' is the fundamental group of a Seifert fibred 3-manifold, N say, by Theorem 2.14. Moreover N is "sufficiently complicated" in the sense of [Zi79], since G' is not virtually solvable. Let t be an element of G whose normal closure is the whole group, and such that t^n is central for some $n > 0$. Let θ be the automorphism of G' determined by t, and let m be the order of the outer automorphism class $[\theta] \in Out(G')$. By Corollary 3.3 of [Zi79] there is a fibre preserving self homeomorphism τ of N inducing $[\theta]$ such that the group of homeomorphisms of $\widetilde{N} \cong R^3$ generated by the covering group G' together with the lifts of τ is an extension of Z/mZ by G', and which is a quotient of the semidirect product $\hat{G} = G/\langle\langle t^n \rangle\rangle \cong G' \times_\theta (Z/nZ)$. Since the self homeomorphism of $\widetilde{N}$ corresponding to the image of t has finite order it has a connected 1-dimensional fixed point set, by Smith theory. The image P of a fixed point in N determines a cross-section $\gamma = \{P\} \times S^1$ of the mapping torus $M(\tau)$. Surgery on γ in $M(\tau)$ gives a 2-knot with group G which is fibred with monodromy (of the fibration of the exterior X) of finite order. We may then apply Plotnick's Theorem to conclude that the 2-knot is a branched twist spin of a knot in a homotopy 3-sphere. Since the monodromy respects the Seifert fibration and leaves the centre of G' invariant, the branch set must be a fibre, and the orbit manifold a Seifert fibred homotopy 3-sphere. Therefore the orbit knot is a torus knot in S^3, and the 2-knot is a branched twist spin of a torus knot. □

Can we avoid the appeal to Plotnick's Theorem in the above argument?

If p, q and r are pairwise relatively prime then $M(p,q,r)$ is an homology sphere and the group π of the r-twist spin of the (p,q)-torus knot has a central element which maps to a generator of π/π'. Hence $\pi \cong \pi' \times Z$ and π' has weight 1. Moreover if t is a generator for the Z summand then an element h of π' is a weight element for π' if and only if ht is a weight element for π. This correspondance also gives a bijection between conjugacy classes of such weight elements. If we exclude the case $(2,3,5)$ then π' has infinitely many distinct weight orbits, and moreover there are weight elements such that no power is central [Pl83]. Therefore we may obtain many 2-knots whose groups are as in Theorem 16.5 but which are not themselves branched twist spins by surgery on weight elements in $M(p,q,r) \times S^1$.

If K is a 2-knot with group as in Theorem 16.5 then $M(K)$ is aspherical, and so is homotopy equivalent to $M(K_1)$ for some K_1 which is a branched twist spin of a torus knot. If we assume that K is fibred, with irreducible fibre, we get a stronger result. The next theorem is a version of Proposition 6.1 of [Pl86], starting from more algebraic hypotheses.

Theorem 16.6 *Let K be a fibred 2-knot whose group π has centre of rank 2, some power of a weight element being central. Suppose that the fibre is irreducible. Then $M(K)$ is homeomorphic to $M(K_1)$, where K_1 is some branched twist spin of a torus knot.*

Proof Let F be the closed fibre and $\phi : F \to F$ the characteristic map. Then F is a Seifert fibred manifold, as above. Now the automorphism of F constructed as in Theorem 16.5 induces the same outer automorphism of $\pi_1(F)$ as ϕ, and so these maps must be homotopic. Therefore they are in fact isotopic [Sc85, BO91]. The theorem now follows. □

We may apply Plotnick's theorem in attempting to understand twist spins of other knots. As the arguments are similar to those of Theorems 16.5 and 16.6, except in that the existence of homeomorphisms of finite order and "homotopy implies isotopy" require different justifications, while the conclusions are less satisfactory, we shall not give prooofs for the following assertions.

Let G be a 3-knot group such that G' is the fundamental group of a hyperbolic 3-manifold and in which some nonzero power of a weight element is central. If the 3-dimensional Poincaré conjecture is true then we may use Mostow rigidity to show that G is the group of some branched twist spin K of a simple non-torus knot. Moreover if K_1 is any other fibred 2-knot with group G and hyperbolic fibre then $M(K_1)$ is homeomorphic to $M(K)$. In particular the simple knot and the order of the twist are uniquely determined by G.

Similarly if G' is the fundamental group of a Haken 3-manifold which is not Seifert fibred and the 3-dimensional Poincaré conjecture is true then we may use [Zi82] to show that G is the group of some branched twist spin of a prime non-torus knot. If moreover all finite group actions on the fibre are geometric the prime knot and the order of the twist are uniquely determined by G' [Zi86].

16.4 Solvable PD_4-groups

If π is a 2-knot group such that $h(\sqrt{\pi}) > 2$ then π is virtually poly-Z and $h(\pi) = 4$, by Theorem 8.1. In this section we shall determine all such 2-knot groups.

Lemma 16.7 *Let G be torsion free and virtually poly-Z with $h(G) = 4$ and $G/G' \cong Z$. Then $G' \cong Z^3$ or G_6 or $\sqrt{G'} \cong \Gamma_q$ (for some $q > 0$) and $G'/\sqrt{G'} \cong Z/3Z$ or 1.*

Proof Let $H = G/\sqrt{G'}$. Then $H/H' \cong Z$ and $h(H') \leq 1$, since $\sqrt{G'} = G' \cap \sqrt{G}$ and $h(G' \cap \sqrt{G}) \geq h(G) - 1 \geq 2$. Hence $H' = G'/\sqrt{G'}$ is finite.

If $\sqrt{G'} \cong Z^3$ then $G' \cong Z^3$ or G_6, since these are the only flat 3-manifold groups which admit meridianal automorphisms.

If $\sqrt{G'} \cong \Gamma_q$ for some $q > 0$ then $\zeta\sqrt{G'} \cong Z$ is normal in G and so is central in G'. Using the known structure of automorphisms of Γ_q, it follows that the finite group $G'/\sqrt{G'}$ must act on $\sqrt{G'}/\zeta\sqrt{G'} \cong Z^2$ via $SL(2,\mathbb{Z})$ and so must be cyclic. Moreover it must be of odd order, and hence 1 or $Z/3Z$, since $G/\sqrt{G'}$ has infinite cyclic abelianization. □

Such a group G is the group of a fibred 2-knot if and only if it is orientable, by Theorems 14.4 and 14.7.

Theorem 16.8 *Let π be a 2-knot group with $\pi' \cong Z^3$, and let C be the image of the meridianal automorphism in $SL(3,\mathbb{Z})$. Then $\Delta_C(t) = det(tI - C)$ is irreducible, $|\Delta_C(1)| = 1$ and π' is isomorphic to an ideal in the domain $R = \Lambda/(\Delta_C(t))$. Two such groups are isomorphic if and only if the polynomials are equal (after inverting t, if necessary) and the ideal classes then agree. There are finitely many ideal classes for each such polynomial and each class (equivalently, each such matrix) is realized by some 2-knot group. Moreover $\sqrt{\pi} = \pi'$ and $\zeta\pi = 1$. Each such group π has two strict weight orbits.*

Proof Let t be a weight element for π and let C be the matrix of the action of t by conjugation on π', with respect to some basis. Then $det(C - I) = \pm 1$, since $t-1$ acts invertibly. Moreover if K is a 2-knot with group π then $M(K)$ is orientable and aspherical, so $det(C) = +1$. Conversely, surgery on the mapping torus of the self homeomorphism of $S^1 \times S^1 \times S^1$ determined by such a matrix C gives a 2-knot with group $Z^3 \times_C Z$.

The Alexander polynomial of K is the characteristic polynomial $\Delta_K(t) = det(tI - C)$ which has the form $t^3 - at^2 + bt - 1$, for some a and $b = a \pm 1$. It is irreducible, since it does not vanish at ± 1. Since π' is annihilated by $\Delta_K(t)$ it is an R-module; moreover as it is torsion free it embeds in $\mathbb{Q} \otimes \pi'$, which is a vector space over the field of fractions $\mathbb{Q} \otimes R$. Since π' is finitely generated and π' and R each have rank 3 as abelian groups it follows that π' is isomorphic to an ideal in R. Moreover the characteristic polynomial of C cannot be cyclotomic and so no power of t can commute with any nontrivial element of π'. Hence $\sqrt{\pi} = \pi'$ and $\zeta\pi = 1$.

By Lemma 1.1 two such semidirect products are isomorphic if and only if the matrices are conjugate up to inversion. The conjugacy classes of matrices in $SL(3,\mathbb{Z})$ with given irreducible characteristic polynomial $\Delta(t)$ correspond to the ideal classes of $\Lambda/(\Delta(t))$, by Theorem 1.4. Therefore π is determined by the ideal class of π', and there are finitely many such 2-knot groups with given Alexander polynomial.

Since $\pi'' = 1$ the final observation follows from Theorem 14.1. □

We shall call 2-knots with such groups "Cappell-Shaneson" 2-knots.

Lemma 16.9 *Let $\Delta_a(t) = t^3 - at^2 + (a-1)t - 1$ for some $a \in \mathbb{Z}$. Then every ideal in the domain $R = \Lambda/(\Delta_a(t))$ can be generated by 2 elements as an R-module.*

Proof In this lemma "cyclic" shall mean "cyclic as an R-module" or equivalently "cyclic as a Λ-module". Let M be an ideal in R. We shall show that we can choose a nonzero element $x \in M$ such that $M/(Rx + pM)$ is cyclic, for all primes p. The result will then follow via Nakayama's Lemma and the Chinese Remainder Theorem.

Let D be the discriminant of $\Delta_a(t)$. Then $D = a(a-2)(a-3)(a-5) - 23$. If p does not divide D then $\Delta_a(t)$ has no repeated roots modulo p. If p divides D choose integers α_p, β_p such that $\Delta_a(t) \equiv (t-\alpha_p)^2(t-\beta_p)$ modulo (p), and let $K_p = \{m \in M \mid (t-\beta_p)m \in pM\}$. If $\beta_p \not\equiv \alpha_p$ modulo (p) then $K_p = (p, t-\alpha_p)M$ and has index p^2 in M.

If $\beta_p \equiv \alpha_p$ modulo (p) then $\alpha_p^3 \equiv 1$ and $(1-\alpha_p)^3 \equiv -1$ modulo (p). Together these congruences imply that $3\alpha_p \equiv -1$ modulo (p), and hence that $p = 7$ and $\alpha_p \equiv 2$ modulo (7). If $M/7M \cong (\Lambda/(7, t-2))^3$ then the automorphism τ of $M/49M$ induced by t is congruent to multiplication by 2 modulo (7). But $M/49M \cong (Z/49Z)^3$ as an abelian group, and so $det(\tau) = 8$ in $\mathbb{Z}/49\mathbb{Z}$, contrary to t being an automorphism of M. Therefore

$$M/7M \cong (\Lambda/(7, t-2)) \oplus (\Lambda/(7, (t-2)^2))$$

and K_7 has index 7 in M, in this case.

The set $M - \cup_{p|D} K_p$ is nonempty, since

$$\frac{1}{7} + \Sigma_{p|D, p\neq 7}\frac{1}{p^2} < \frac{1}{7} + \int_2^\infty \frac{1}{t^2}dt < 1.$$

Let x be an element of $M - \cup_{p|D} K_p$ which is not $\mathbb{Z}$-divisible in M. Then $N = M/Rx$ is finite, and is generated by at most two elements as an abelian group, since $M \cong Z^3$ as an abelian group. For each prime p the $\Lambda/p\Lambda$-module M/pM is an extension of N/pN by the submodule X_p generated by the image of x and its order ideal is generated by the image of $\Delta_a(t)$ in the P.I.D. $\Lambda/p\Lambda \cong \mathbb{F}_p[t, t^{-1}]$.

If p does not divide D the image of $\Delta_a(t)$ in $\Lambda/p\Lambda$ is square free. If $p|D$ and $\beta_p \neq \alpha_p$ the order ideal of X_p is divisible by $t - \alpha_p$. If $\beta_7 = \alpha_7 = 2$ the order ideal of X_7 is $(t-2)^2$. In all cases the order ideal of N/pN is square free and so N/pN is cyclic. By the Chinese Remainder Theorem there is an element $y \in M$ whose image is a generator of N/pN, for each prime p dividing the order of N. The image of y in N generates N, by Nakayama's Lemma. □

In [AR84] matrix calculations are used to show that any matrix C as in Theorem 16.8 is conjugate to one with first row $(0, 0, 1)$. (The prime 7 also needs special consideration in their argument.) This is equivalent to showing that M has an element x such that the image of tx in $M/\mathbb{Z}x$ is indivisible, from which it follows that M is generated as an abelian group by x, tx and some third element y. Given this, it is easy to see that the corresponding Cappell-Shaneson 2-knot group has a presentation

$$\langle t, x, y, z \mid xy = yx,\ xz = zx,\ txt^{-1} = z,\ tyt^{-1} = x^m y^n z^p,\ tzt^{-1} = x^q y^r z^s \rangle.$$

Since p and s must be relatively prime these relations imply $yz = zy$. We may reduce the number of generators and relations on setting $z = txt^{-1}$.

Lemma 16.10 *Let π be a finitely presentable group such that $\pi/\pi' \cong Z$, and let $R = \Lambda$ or $\Lambda/p\Lambda$ for some prime $p \geq 2$. Then*

(1) *if π can be generated by 2 elements $H_1(\pi; R)$ is cyclic as an R-module;*

(2) *if $\mathrm{def}(\pi) = 0$ then $H_2(\pi; R)$ is cyclic as an R-module.*

Proof If π is generated by two elements t and x, say, we may assume that the image of t generates π/π' and that $x \in \pi'$. Then π' is generated by the conjugates of x under powers of t, and so $H_1(\pi; R) = R \otimes_\Lambda (\pi'/\pi'')$ is generated by the image of x.

If X is the finite 2-complex determined by a deficiency 0 presentation for π then $H_0(X; R) = R/(t-1)$ and $H_1(X; R)$ are R-torsion modules, and $H_2(X; R)$ is a submodule of a finitely generated free R-module. Hence $H_2(X; R) \cong R$, as it has rank 1 and R is an UFD. Therefore $H_2(\pi; R)$ is cyclic as an R-module, since it is a quotient of $H_2(X; R)$, by Hopf's Theorem. ∎

Theorem 16.11 *Let $\pi = Z^3 \times_C Z$ be the group of a Cappell-Shaneson 2-knot, and let $\Delta(t) = det(tI - C)$. Then π has a 3 generator presentation of deficiency -2. Moreover the following are equivalent.*

(1) *π has a 2 generator presentation of deficiency 0;*

(2) *π is generated by 2 elements;*

(3) *$\mathrm{def}(\pi) = 0$;*

(4) *π' is cyclic as a Λ-module.*

Proof The first assertion follows immediately from Lemma 16.9. Condition (1) implies (2) and (3), since $\mathrm{def}(\pi) \leq 0$, by Theorem 2.5, while (2) implies (4), by Lemma 16.10. If $\mathrm{def}(\pi) = 0$ then $H_2(\pi; \Lambda)$ is cyclic as a Λ-module, by Lemma 16.10. Since $\pi' = H_1(\pi; \Lambda) \cong \overline{H^3(\pi; \Lambda)} \cong \overline{Ext^1_\Lambda(H_2(\pi; \Lambda), \Lambda)}$, by

Poincaré duality and the UCSS, it is also cyclic and so (3) also implies (4). If π' is generated as a Λ-module by x then it is easy to see that π has a presentation of the form

$$\langle t, x \mid xtxt^{-1} = txt^{-1}x,\ t^3xt^{-3} = t^2x^at^{-2}tx^bt^{-1}x\rangle,$$

for some integers a, b, and so (1) holds. $\square$

In fact Theorem A.3 of [AR84] implies that any such group has a 3 generator presentation of deficiency -1, as remarked before Lemma 16.10.

The isomorphism class of the Λ-module π' is that of its Steinitz-Fox-Smythe row invariant, which is the ideal $(r, t-n)$ in the domain $\Lambda/(\Delta(t))$ (see Chapter 3 of [H3]). Thus π' is cyclic if and only if this ideal is principal. In particular, this is not so for the concluding example of [AR84], which gives rise to the group with presentation

$$\langle t, x, y, z \mid xz = zx,\ yz = zy,\ txt^{-1} = y^{-5}z^{-8},\ tyt^{-1} = y^2z^3,\ tzt^{-1} = xz^{-7}\rangle.$$

Let $G(+)$ and $G(-)$ be the extensions of Z by G_6 with presentations

$$\langle t, x, y \mid xy^2x^{-1}y^2 = 1, txt^{-1} = (xy)^{\mp 1}, tyt^{-1} = x^{\pm 1}\rangle.$$

(These presentations have optimal deficiency, by Theorem 2.5.) The group $G(+)$ is the group of the 3-twist spin of the figure eight knot ($G(+) \cong \pi\tau_3 4_1$.)

Theorem 16.12 *Let π be a 2-knot group with $\pi' \cong G_6$. Then $\pi \cong G(+)$ or $G(-)$. In each case π is virtually Z^4, $\pi' \cap \zeta\pi = 1$ and $\zeta\pi \cong Z$.*

Proof Since $Out(G_6)$ is finite π is virtually $G_6 \times Z$ and hence is virtually Z^4. The groups $G(+)$ and $G(-)$ are the only orientable flat 4-manifold groups with $\pi/\pi' \cong Z$. The next assertion ($\pi' \cap \zeta\pi = 1$) follows as $\zeta G_6 = 1$. It is easily seen that $\zeta G(+)$ and $\zeta G(-)$ are generated by the images of t^3 and $t^6x^{-2}y^2(xy)^{-2}$, respectively, and so in each case $\zeta\pi \cong Z$. $\square$

Although $G(-)$ is the group of a fibred 2-knot, by Theorem 14.4, it can be shown that no power of any weight element is central and so it is not the group of any twist spin. (This also follows from Theorem 16.4 above.)

Theorem 16.13 *Let π be a 2-knot group with $\pi' \cong \Gamma_q$ for some $q > 0$, and let θ be the image of the meridianal automorphism in $Out(\Gamma_q)$. Then either $q = 1$ and θ is conjugate to $[\left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right), 0]$ or $[\left(\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right), 0]$, or q is odd and θ is conjugate to $[\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right), 0]$ or its inverse. Each such group π has two strict weight orbits.*

Proof If (A, μ) is a meridianal automorphism of Γ_q then the induced automorphisms of $\Gamma_q/\zeta\Gamma_q \cong Z^2$ and $tors(\Gamma_q/\Gamma'_q) \cong Z/qZ$ are also meridianal. Therefore $det(A - I) = \pm 1$ and $det(A) - 1$ is a unit modulo (q), so q must be odd and $det(A) = -1$ if $q > 1$. The characteristic polynomial $\Delta_A(X)$ of such a 2×2 matrix must be $X^2 - X + 1$, $X^2 - 3X + 1$, $X^2 - X - 1$ or $X^2 + X - 1$. The corresponding rings $\mathbb{Z}[X]/(\Delta_A(X))$ are principal ideal domains (namely $\mathbb{Z}[(1 + \sqrt{-3})/2]$ and $\mathbb{Z}[(1 + \sqrt{5})/2]$) and so A is conjugate to one of $\left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, or $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & -1 \end{smallmatrix}\right)$, by Theorem 1.4. Now

$$[A, \mu][A, 0][A, \mu]^{-1} = [A, \mu(I - det(A)A)^{-1}]$$

in $Out(\Gamma_q)$. (See §7 of Chapter 8.) As in each case $I - det(A)A$ is invertible, it follows that θ is conjugate to $[A, 0]$ or to $[A^{-1}, 0] = [A, 0]^{-1}$. Since $\pi'' \leq \zeta\pi'$ the final observation follows from Theorem 14.1. ∎

The groups Γ_q are discrete cocompact subgroups of the Lie group Nil^3 and the coset spaces are S^1-bundles over the torus. Every automorphism of Γ_q is orientation preserving and each of the groups allowed by Theorem 16.13 is the group of some fibred 2-knot, by Theorem 14.4. The group of the 6-twist spin of the trefoil has commutator subgroup Γ_1 and monodromy $[\left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right), 0]$. In all the other cases the meridianal automorphism has infinite order and the group is not the group of any twist spin.

The groups with commutator subgroup Γ_1 have presentations

$$\langle t, x, y \mid xyxy^{-1} = yxy^{-1}x,\ txt^{-1} = xy,\ tyt^{-1} = w \rangle,$$

where $w = x^{-1}$, xy^2 or x (respectively), while those with commutator subgroup Γ_q with $q > 1$ have presentations

$$\langle t, u, v, z \mid uvu^{-1}v^{-1} = z^q,\ tut^{-1} = v,\ tvt^{-1} = zuv,\ tzt^{-1} = z^{-1} \rangle.$$

(Note that as $[v, u] = t[u, v]t^{-1} = [v, zuv] = [v, z]z[v, u]z^{-1} = [v, z][v, u]$, we have $vz = zv$ and hence $uz = zu$ also.) These are easily seen to have 2 generator presentations of deficiency 0 also.

The other $\mathbb{N}il^3$-manifolds which arise as the closed fibres of fibred 2-knots are Seifert fibred over S^2 with 3 exceptional fibres of type $(3, \beta_i)$, with $\beta_i = \pm 1$. Hence they are 2-fold branched covers of S^3, branched over a Montesinos link $K(0|e; (3, \beta_1), (3, \beta_2), (3, \beta_3))$ [Mo73]. If e is even this link is a knot, and is invertible, but not amphicheiral (see §12E of [BZ]). (This class includes the knots 9_{35}, 9_{37}, 9_{46}, 9_{48}, 10_{74} and 10_{75}.)

Let $\pi(e, \eta)$ be the group of the 2-twist spin of $K(0|e; (3, 1), (3, 1), (3, \eta))$.

Theorem 16.14 *Let π be a 2-knot group such that $\sqrt{\pi'} \cong \Gamma_q$ (for some $q \geq 1$) and $\pi'/\sqrt{\pi'} \cong Z/3Z$. Then q is odd and $\pi \cong \pi(e,\eta)$, for some $e \in 2Z$ and $\eta = 1$ or -1.*

Proof It follows easily from Lemma 16.7 that $\zeta\sqrt{\pi'} = \zeta\pi'$ and $G = \pi'/\zeta\pi'$ is isomorphic to $Z^2 \times_{-B} (Z/3Z)$, where $B = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 1 \end{smallmatrix}\right)$. Thus G may be identified with the orbifold fundamental group of the flat 2-orbifold $S(3,3,3)$, and so is a discrete subgroup of $Isom(\mathbb{E}^2)$. As remarked above, π' is the fundamental group of the 2-fold branched cover of $K(0|e;(3,1),(3,1),(3,\eta))$, for some e and $\eta = \pm 1$. Hence it has a presentation

$$\langle h, x, y, z, \mid x^{3\eta} = y^3 = z^3 = h,\ xyz = h^e\rangle.$$

(This can also be seen algebraically as π' is a torsion free central extension of G by Z.) The image of h in π' generates $\zeta\pi'$, and the images of $x^{-1}y$ and yx^{-1} in $G = \pi'/\langle h\rangle$ form a basis for the translation subgroup $T(G) \cong Z^2$ of G. Since $\pi'/(\pi')^2 \cong Z/(2, e-1)$ and π' admits a meridianal automorphism e must be even.

The isometry group $E(2) = Isom(\mathbb{E}^2) = R^2 \tilde{\times} O(2)$ embeds in the affine group $Aff(2) = R^2 \tilde{\times} GL(2,\mathbb{R})$. The normalizer of G in $Aff(2)$ is the semidirect product of the dihedral subgroup of $GL(2,\mathbb{Z})$ generated by B and $R = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ with the normal subgroup $(I+B)^{-1}Z^2$, and its centralizer there is trivial. It follows from the Bieberbach theorems (and is easily verified directly) that $Aut(G) \cong N_{Aff(2)}(G)$. Let b, r, k represent the classes of $(0,B)$, $(0,R)$ and $((-\frac{1}{3}, \frac{1}{3}), I)$ in $Out(G)$. Then $Out(G) \cong S_3 \times (Z/2Z)$, and has a presentation

$$\langle b, r, k \mid b^2 = r^2 = k^3 = 1,\ br = rb,\ bkb = rkr = k^{-1}\rangle$$

Since π'/π'' is finite $Hom(\pi', \zeta\pi') = 1$ and so the natural homomorphism from $Out(\pi')$ to $Out(G)$ is injective. As each of the automorphisms b, r and k lifts to an automorphism of π' this homomorphism is an isomorphism. On considering the effect of an automorphism of π' on its characteristic quotients $\pi'/\sqrt{\pi'} = G/T(G) \cong Z/3Z$ and $G/G' = (Z/3Z)^2$, we see that the only outer automorphism classes which contain meridianal automorphisms are rb, rbk and rbk^2. Since these are conjugate in $Out(G)$ and $\pi' \cong \pi(e,\eta)'$ the theorem now follows from Lemma 1.1. □

The subgroup $A = \langle t^2, x^3\rangle < \pi(e,\eta)$ is abelian of rank 2 and normal but is not central. As $H_1(\pi;\Lambda/3\Lambda) \cong H_2(\pi;\Lambda/3\Lambda) \cong (\Lambda/(3,t+1))^2$ in all cases the presentations

$$\langle t, x, y \mid x^3 = y^3 = (x^{1-3e}y)^{-3\eta},\ txt^{-1} = x^{-1},\ tyt^{-1} = xy^{-1}x^{-1}\rangle$$

are optimal, by Lemma 16.10.

We may refine the conclusions of Theorem 15.7 as follows. If K is a 2-knot whose group π has an abelian normal subgroup of rank ≥ 3 then either K is a Cappell-Shaneson 2-knot or $\pi K \cong G(+)$ or $G(-)$.

Chapter 17

Knot manifolds and geometries

In this chapter we shall attempt to characterize certain 2-knots in terms of algebraic invariants. As every 2-knot K may be recovered (up to orientations and Gluck reconstruction) from $M(K)$ together with the orbit of a weight class in $\pi = \pi K$ under the action of self homeomorphisms of M, we need to characterize $M(K)$ up to homeomorphism. After some general remarks on the algebraic 2-type in §1, and on surgery in §2, we shall concentrate on three special cases: when $M(K)$ is aspherical, when π' is finite and when $g.d.\pi = 2$.

When π is torsion free and virtually poly-Z the surgery obstructions vanish, and when it is poly-Z the weight class is unique. When π has torsion the surgery obstruction groups are notoriously difficult to compute. However we can show that there are infinitely many distinct 2-knots K such that $M(K)$ is simple homotopy equivalent to $M(\tau_2 3_1)$; if the 3-dimensional Poincaré conjecture is true then among these knots only $\tau_2 3_1$ has a minimal Seifert hypersurface. In the case of Φ the homotopy type of $M(K)$ determines the exterior of the knot. The difficulty here is in finding a homotopy equivalence from $M(K)$ to a standard model.

In the final sections we shall consider which knot manifolds are homeomorphic to geometric 4-manifolds or complex surfaces. If $M(K)$ is geometric then either K is a Cappell-Shaneson knot or the geometry must be one of $\mathbb{E}^4$, $\mathbb{N}il^3 \times \mathbb{E}^1$, $\mathbb{S}ol_1^4$, $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$, $\mathbb{H}^3 \times \mathbb{E}^1$ or $\mathbb{S}^3 \times \mathbb{E}^1$. If $M(K)$ is homeomorphic to a complex surface then either K is a branched twist spin of a torus knot or $M(K)$ admits one of the geometries $\mathbb{N}il^3 \times \mathbb{E}^1$, $\mathbb{S}ol_0^4$ or $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$.

17.1 Homotopy classification of $M(K)$

Let K and K_1 be 2-knots and suppose that $\alpha : \pi = \pi K \to \pi K_1$ and $\beta : \pi_2(M) \to \pi_2(M_1)$ determine an isomorphism of the algebraic 2-types of $M = M(K)$ and $M_1 = M(K_1)$. Since the infinite cyclic covers M' and M_1' are homotopy equivalent to 3-complexes there is a map $h : M' \to M_1'$ such that $\pi_1(h) = \alpha|_\pi$ and $\pi_2(h) = \beta$. If $\pi = \pi K$ has one end then $\pi_3(M) \cong \Gamma(\pi_2(M))$ and so h is a homotopy equivalence. Let t and $t_1 = \alpha(t)$ be corresponding generators of $Aut(M'/M)$ and $Aut(M_1'/M_1)$, respectively. Then $h^{-1}t_1^{-1}ht$ is a self homotopy equivalence of M' which fixes the algebraic 2-type. If this is homotopic to $id_{M'}$ then M and M_1 are homotopy equivalent, since up to homotopy they are the mapping tori of t and t_1, respectively. Thus the homotopy classification of such knot manifolds may be largely reduced to determining the obstructions to homotoping a self-map of a 3-complex to the identity.

We may use a similar idea to approach this problem in another way. Under the same hypotheses on K and K_1 there is a map $f_o : M - intD^4 \to M_1$ inducing isomorphisms of the algebraic 2-types. If π has one end $\pi_3(f_o)$ is an epimorphism, and so f_o is 3-connected. If there is an extension $f : M \to M_1$ then it is a homotopy equivalence, as it induces isomorphisms on the homology of the universal covering spaces.

If $g.d.\pi \leq 2$ the algebraic 2-type is determined by π, for then $\pi_2(M) = H^2(\pi;\mathbb{Z}[\pi])$, by Theorem 3.12, and the k-invariant is 0. In particular, if π' is free of rank r then $M(K)$ is homotopy equivalent to the mapping torus of a self-homeomorphism of $\sharp^r S^1 \times S^2$, by Corollary 4.5.1. On the other hand, the group Φ has resisted attack thus far.

The related problem of determining the homotopy type of the exterior of a 2-knot has been considered in [Lo81], [Pl83] and [PS85]. In each of the examples considered in [Pl83] either π' is finite or $M(K)$ is aspherical, so they do not test the adequacy of the algebraic 2-type for the present problem. The examples of [PS85] probably show that in general $M(K)$ is not determined by π and $\pi_2(M(K))$ alone.

17.2 Surgery

Recall from Chapter 6 that we may define natural transformations $I_G : G \to L_5^s(G)$ for any group G, which clearly factor through G/G'. If $\alpha : G \to Z$ induces an isomorphism on abelianization the homomorphism $\hat{I}_G = I_G\alpha^{-1}I_Z^{-1}$ is a canonical splitting for $L_5(\alpha)$.

Theorem 17.1 *Let K be a 2-knot with group π. If $L_5^s(\pi) \cong Z$ and N is simple homotopy equivalent to $M(K)$ then N is s-cobordant to $M(K)$.*

Proof Since $M = M(K)$ is orientable and $[M, G/TOP] \cong H^4(M;\mathbb{Z}) \cong Z$ the surgery obstruction map $\sigma_4 : [M(K), G/TOP] \to L_4^s(\pi)$ is injective, by Theorem 6.6. The image of $L_5(Z)$ under $\hat{I}_\pi$ acts trivially on $S_{TOP}(M(K))$, by Theorem 6.7. Hence there is a normal cobordism with obstruction 0 from any simple homotopy equivalence $f : N \to M$ to id_M. ∎

This theorem applies if π is square root closed accessible [Ca73], or if π is a classical knot group [AFR97].

Corollary 17.1.1 (Freedman) *A 2-knot K is trivial if and only if $\pi K \cong Z$.*

Proof The condition is clearly necessary. Conversely, if $\pi K \cong Z$ then $M(K)$ is homeomorphic to $S^3 \times S^1$, by Theorem 6.11. Since the meridian is unique up to inversion and the unknot is clearly reflexive the result follows. □

Surgery on an s-concordance $\mathcal{K}$ from K_0 to K_1 gives an s-cobordism from $M(K_0)$ to $M(K_1)$ in which the meridians are conjugate. Conversely, if $M(K)$ and $M(K_1)$ are s-cobordant via such an s-cobordism then K_1 is s-concordant to K or K^*. In particular, if K is reflexive then K and K_1 are s-concordant.

Lemma 17.2 *Let K be a 2-knot. Then K has a Seifert hypersurface which contains no fake 3-cells.*

Proof Every 2-knot has a Seifert hypersurface, by the standard obstruction theoretical argument and TOP transversality. Thus K bounds a locally flat 3-submanifold V which has trivial normal bundle in S^4. If Δ is a homotopy 3-cell in V then $\Delta \times R \cong D^3 \times R$, by simply connected surgery, and the submanifold $\partial\Delta$ of $\partial(\Delta \times R) = \partial(D^3 \times R)$ is isotopic there to the boundary of a standard 3-cell in $D^3 \times R$ which we may use instead of Δ. □

The modification in this lemma clearly preserves minimality. (Every 2-knot has a closed Seifert hypersurface which is a hyperbolic 3-manifold [Ru90], and so contains no fake 3-cells, but these are rarely minimal).

17.3 The aspherical cases

Whenever the group of a 2-knot K contains a sufficiently large abelian normal subgroup then $M(K)$ is aspherical. This is notably the case for most twist spins of prime knots.

Theorem 17.3 *Let K be a 2-knot with group π and suppose that either $\sqrt{\pi}$ is torsion free abelian of rank 1 and $\pi/\sqrt{\pi}$ has one end or $h(\sqrt{\pi}) \geq 2$. Then the universal cover $\widetilde{M}(K)$ is homeomorphic to R^4.*

Proof If $\sqrt{\pi}$ is torsion free abelian of rank 1 and $\pi/\sqrt{\pi}$ has one end M is aspherical, by Theorem 15.5, and π is 1-connected at ∞, by Theorem 1 of [Mi87]. If $h(\sqrt{\pi}) = 2$ then $\sqrt{\pi} \cong Z^2$ and M is s-cobordant to the mapping torus of a self homeomorphism of a $\widetilde{\mathbb{SL}}$-manifold, by Theorem 16.2. If $h(\sqrt{\pi}) \geq 3$ then M is homeomorphic to an infrasolvmanifold, by Theorem 8.1. In all cases, $\widetilde{M}$ is contractible and 1-connected at ∞, and so is homeomorphic to R^4 by [Fr82]. □

Is there a 2-knot K with $\widetilde{M}(K)$ contractible but not 1-connected at ∞?

Theorem 17.4 *Let K be a 2-knot such that $\pi = \pi K$ is torsion free and virtually poly-Z. Then K is determined up to Gluck reconstruction by π together with a generator of $H_4(\pi;\mathbb{Z})$ and the strict weight orbit of a meridian.*

Proof If $\pi \cong Z$ then K must be trivial, and so we may assume that π is torsion free and virtually poly-Z of Hirsch length 4. Hence $M(K)$ is aspherical and is determined up to homeomorphism by π, and every automorphism of π may be realized by a self homeomorphism of $M(K)$, by Theorem 6.11. Since $M(K)$ is aspherical orientations of $M(K)$ correspond to generators of $H_4(\pi;\mathbb{Z})$. □

This theorem applies in particular to the Cappell-Shaneson 2-knots, which have an unique strict weight orbit, up to inversion. (A similar argument applies to Cappell-Shaneson n-knots with $n > 2$, provided we assume also that $\pi_i(X(K)) = 0$ for $2 \leq i \leq (n+1)/2$).

Theorem 17.5 *Let K be a 2-knot with group π. Then K is s-concordant to a fibred knot with closed fibre a $\widetilde{\mathbb{SL}}$-manifold if and only if π is not virtually solvable, π' is FP_2 and $\zeta\pi' \cong Z$. The fibred knot is determined up to Gluck reconstruction by π together with a generator of $H_4(\pi;\mathbb{Z})$ and the strict weight orbit of a meridian.*

Proof The conditions are clearly necessary. Suppose that they hold. The manifold $M(K)$ is aspherical, by Theorem 15.7, so every automorphism of π is induced by a self homotopy equivalence of $M(K)$. Moreover as π is not virtually solvable π' is the fundamental group of a $\widetilde{\mathbb{SL}}$-manifold. Therefore $M(K)$ is determined up to s-cobordism by π, by Theorem 13.2. The rest is standard. □

Branched twist spins of torus knots are perhaps the most important examples of such knots, but there are others. (See Chapter 16).

Is every 2-knot K such that $\pi = \pi K$ is a PD_4^+-group determined up to s-concordance and Gluck reconstruction by π together with a generator of $H_4(\pi;\mathbb{Z})$ and a strict weight orbit? Is K s-concordant to a fibred knot with aspherical closed fibre if and only if π' is FP_2 and has one end? (This is surely true if $\pi' \cong \pi_1(N)$ for some virtually Haken 3-manifold N).

17.4 Quasifibres and minimal Seifert hypersurfaces

Let M be a closed 4-manifold with fundamental group π. If $f : M \to S^1$ is a map which is transverse to $p \in S^1$ then $\widehat{V} = f^{-1}(p)$ is a codimension 1 submanifold with a product neighbourhood $N \cong \widehat{V} \times [-1,1]$. If moreover the induced homomorphism $f_* : \pi \to Z$ is an epimorphism and each of the inclusions $j_\pm : \widehat{V} \cong \widehat{V} \times \{\pm 1\} \subset W = M \backslash V \times (-1,1)$ induces monomorphisms on fundamental groups then we shall say that $\widehat{V}$ is a *quasifibre* for f. The group π is then an HNN extension with base $\pi_1(W)$ and associated subgroups $j_{\pm *}(\pi_1(\widehat{V})$, by Van Kampen's Theorem. Every fibre of a bundle projection is a quasifibre. We may use the notion of quasifibre to interpolate between the homotopy fibration theorem of Chapter 4 and a TOP fibration theorem. (See also Theorem 6.12 and Theorem 17.7).

Theorem 17.6 *Let M be a closed 4-manifold with $\chi(M) = 0$ and such that $\pi = \pi_1(M)$ is an extension of Z by a finitely generated normal subgroup ν. If there is a map $f : M \to S^1$ inducing an epimorphism with kernel ν and which has a quasifibre $\widehat{V}$ then the infinite cyclic covering space M_ν associated with ν is homotopy equivalent to $\widehat{V}$.*

Proof As ν is finitely generated the monomorphisms $j_{\pm *}$ must be isomorphisms. Therefore ν is finitely presentable, and so M_ν is a PD_3-complex, by Theorem 4.5. Now $M_\nu \cong W \times Z/\sim$, where $(j_+(v), n) \sim (j_-(v), n+1)$ for all $v \in \widehat{V}$ and $n \in Z$. Let $\tilde{j}(v)$ be the image of $(j_+(v), 0)$ in M_ν. Then $\pi_1(\tilde{j})$ is an isomorphism. A Mayer-Vietoris argument shows that $\tilde{j}$ has degree 1, and so $\tilde{j}$ is a homotopy equivalence. □

One could use duality instead to show that $H_s = H_s(W, \partial_\pm W; \mathbb{Z}[\pi]) = 0$ for $s \neq 2$, while H_2 is a stably free $\mathbb{Z}[\pi]$-module, of rank $\chi(W, \partial_\pm W) = 0$. Since $\mathbb{Z}[\pi]$ is weakly finite this module is 0, and so W is an h-cobordism.

Corollary 17.6.1 *let K be a 2-knot such that π' is finitely generated, and which has a minimal Seifert hypersurface V. If every self homotopy equivalence of $\widehat{V}$ is homotopic to a homeomorphism then $M(K)$ is homotopy equivalent to $M(K_1)$, where $M(K_1)$ is a fibred 2-knot with fibre V.*

Proof Let $j_+^{-1}: M(K)' \to \widehat{V}$ be a homotopy inverse to the homotopy equivalence j_+, and let θ be a self homeomorphism of $\widehat{V}$ homotopic to $j_+^{-1}j_-$. Then $j_+\theta j_+^{-1}$ is homotopic to a generator of $Aut(M(K)'/M(K))$, and so the mapping torus of θ is homotopy equivalent to $M(K)$. Surgery on this mapping torus gives such a knot K_1. □

If a Seifert hypersurface V for a 2-knot has fundamental group Z then the Mayer-Vietoris sequence for $H_*(M(K);\Lambda)$ gives $H_1(X') \cong \Lambda/(ta_+ - a_-)$, where $a_\pm : H_1(V) \to H_1(S^4 - V)$. Since $H_1(X) = Z$ we must have $a_+ - a_- = \pm 1$. If $a_+a_- \neq 0$ then V is minimal. However one of a_+ or a_- could be 0, in which case V may not be minimal. The group Φ is realized by ribbon knots with such minimal Seifert hypersurfaces (homeomorphic to $S^2 \times S^1 - intD^3$) [Fo62]. Thus minimality does not imply that π' is finitely generated.

It remains an open question whether every 2-knot has a minimal Seifert hypersurface, or indeed whether every 2-knot group is an HNN extension with finitely presentable base and associated subgroups. (There are high dimensional knot groups which are not of this type [Si91, 96]). Yoshikawa has shown that there are ribbon 2-knots whose groups are HNN extensions with base a torus knot group and associated subgroups Z but which cannot be expressed as HNN extensions with base a free group [Yo88].

17.5 The spherical cases

Let π be a 2-knot group with commutator subgroup $\pi' \cong P \times (Z/(2r+1)Z)$, where $P = 1$, $Q(8)$, T_k^* or I^*. The meridianal automorphism induces the identity on the set of irreducible real representations of π', except when $P = Q(8)$. (It permutes the three nontrivial 1-dimensional representations when $\pi' \cong Q(8)$, and similarly when $\pi' \cong Q(8) \times (Z/nZ)$). It then follows as in Chapter 11 that $L_5^s(\pi)$ has rank $r+1$, $3(r+1)$, $3^{k-1}(5+7r)$ or $9(r+1)$, respectively. Hence if $\pi' \neq 1$ then there are infinitely many distinct 2-knots with group π, since the group of self homotopy equivalences of $M(K)$ is finite.

The simplest nontrivial such group is $\pi = (Z/3Z) \times_{-1} Z$. If K is any 2-knot with this group then $M(K)$ is homotopy equivalent to $M(\tau_2 3_1)$. Since $Wh(Z/3Z) = 0$ [Hi40] and $L_5(Z/3Z) = 0$ [Ba75] we have $L_5^s(\pi) \cong L_4(\pi') \cong Z^2$, but we do not know whether $Wh(\pi) = 0$.

Theorem 17.7 *Let K be a 2-knot with group $\pi = \pi K$ such that $\pi' \cong Z/3Z$, and which has a minimal Seifert hypersurface. Then K is fibred.*

Proof Let V be a minimal Seifert hypersurface for K. Then we may assume V is irreducible. Let $\widehat{V} = V \cup D^3$ and $W = M(K)\backslash V \times (-1,1)$. Then W is an h-cobordism from $\widehat{V}$ to itself (see the remark following Theorem 6). Therefore $W \cong \widehat{V} \times I$, by surgery over $Z/3Z$. (Note that $Wh(Z/3Z) = L_5(Z/3Z) = 0$). Hence M fibres over S^1 and so K is fibred also. □

Free actions of $Z/3Z$ on S^3 are conjugate to the standard orthogonal action, by a result of Rubinstein (see [Th]). If the 3-dimensional Poincaré conjecture is true then the closed fibre must be the lens space $L(3,1)$, and so K must be $\tau_2 3_1$. None of the other 2-knots with this group could have a minimal Seifert surface, and so we would have (infinitely many) further counter-examples to the most natural 4-dimensional analogue of Farrell's fibration theorem. We do not know whether any of these knots (other than $\tau_2 3_1$) is PL in some PL structure on S^4.

Let F be an $\mathbb{S}^3$-group, and let $W = (W; j_\pm)$ be an h-cobordism with homeomorphisms $j_\pm : N \to \partial_\pm W$, where $N = S^3/F$. Then W is an s-cobordism [KS92]. The set of such s-cobordisms from N to itself is a finite abelian group with respect to stacking of cobordisms. All such s-cobordisms are products if F is cyclic, but there are nontrivial examples if $F \cong Q(8) \times (Z/pZ)$, for any odd prime p [KS95]. If ϕ is a self-homeomorphism of N the closed 4-manifold Z_ϕ obtained by identifying the ends of W via $j_+ \phi j_-^{-1}$ is homotopy equivalent to $M(\phi)$. However if Z_ϕ is a mapping torus of a self-homeomorphism of N then W is trivial. In particular, if ϕ induces a meridianal automorphism of F then $Z_\phi \cong M(K)$ for an exotic 2-knot K with $\pi' \cong F$ and which has a minimal Seifert hypersurface, but which is not fibred with geometric fibre.

17.6 Finite geometric dimension 2

Knot groups with finite 2-dimensional Eilenberg-Mac Lane complexes have deficiency 1, by Theorem 2.8, and so are 2-knot groups. This class includes all classical knot groups, all knot groups with free commutator subgroup and all knot groups in the class $\mathcal{X}$. (The latter class includes all those as in Theorem 15.1).

Theorem 17.8 *Let K be a 2-knot with group π. If π is a 1-knot group or a $\mathcal{X}$-group then $M(K)$ is determined up to s-cobordism by its homotopy type.*

Proof This is an immediate consequence of Lemma 6.9, if π is a $\mathcal{X}$-group. If π is a nontrivial classical knot group it follows from Theorem 17.1, since $Wh(\pi) = 0$ [Wd78] and $L_5^s(\pi) \cong Z$ [AFR97]. □

Does the conclusion of this theorem hold for every knot whose group has geometric dimension 2?

Corollary 17.8.1 *A ribbon* 2*-knot* K *with group* Φ *is determined by the oriented homotopy type of* $M(K)$*.*

Proof Since Φ is metabelian s-cobordism implies homeomorphism and there is an unique weight class up to inversion, so the knot exterior is determined by the homotopy type of $M(K)$, and since K is a ribbon knot it is -amphicheiral and is determined by its exterior. □

Examples 10 and 11 of [Fo62] are ribbon knots with group Φ, and are mirror images of each other. Although they are -amphicheiral they are not invertible, since their Alexander polynomials are asymmetric. Thus they are not isotopic. Are there any other 2-knots with this group? In particular, is there one which is not a ribbon knot?

Theorem 17.9 *A* 2*-knot* K *with group* π *is* s*-concordant to a fibred knot with closed fibre* $\sharp^r(S^1 \times S^2)$ *if and only if* $\mathrm{def}(\pi) = 1$ *and* π' *is* FP_2*. Moreover any such fibred* 2*-knot is reflexive and homotopy ribbon.*

Proof The conditions are clearly necessary. If they hold then $\pi' \cong F(r)$, for some $r \geq 0$, by Corollary 2.5.1. Then $M(K)$ is homotopy equivalent to a PL 4-manifold N which fibres over S^1 with fibre $\sharp^r(S^1 \times S^2)$, by Corollary 4.5.1. Moreover $Wh(\pi) = 0$, by Lemma 6.3, and π is square root closed accessible, so I_π is an isomorphism, by Lemma 6.9, so there is an s-cobordism W from M to N, by Theorem 17.1. We may embed an annulus $A = S^1 \times [0,1]$ in W so that $M \cap A = S^1 \times \{0\}$ is a meridian for K and $N \cap A = S^1 \times \{1\}$. Surgery on A in W then gives an s-concordance from K to such a fibred knot K_1, which is reflexive [Gl62] and homotopy ribbon [Co83]. □

The group of isotopy classes of self homeomorphisms of $\sharp^r(S^1 \times S^2)$ which induce the identity in $Out(F(r))$ is generated by twists about nonseparating 2-spheres, and is isomorphic to $(Z/2Z)^r$. Thus given a 2-knot group $\pi \cong F(r) \times_\alpha Z$ there are 2^r corresponding homotopy types of knot manifolds $M(K)$.

Is every automorphism of π induced by a self-homeomorphism of each such fibred manifold? If so, the knot is determined (among such fibred knots) up to finite ambiguity by its group together with the weight orbit of a meridian. (However, the group $\pi 3_1$ has infinitely many weight orbits [Su85]).

The theorem implies there is a slice disc Δ for K such that the inclusion of $M(K)$ into $D^5 - \Delta$ is 2-connected. Is K itself homotopy ribbon? (This would follow from "homotopy connectivity implies geometric connectivity", but our situation is just beyond the range of known results). Is every such group the group of a ribbon knot? Which are the groups of classical fibred knots? If $K = \sigma k$ is the Artin spin of a fibred 1-knot then $M(K)$ fibres over S^1 with fibre $\sharp^r(S^2 \times S^1)$. However not all such fibred 2-knots arise in this way. (For instance, the Alexander polynomial need not be symmetric [AY81]). There are just three groups G with $G/G' \cong Z$ and G' free of rank 2, namely $\pi 3_1$ (the trefoil knot group), $\pi 4_1$ (the figure eight knot group) and the group with presentation

$$\langle x, y, t \mid txt^{-1} = y, tyt^{-1} = xy\rangle.$$

(Two of the four presentations given in [Rp60] present isomorphic groups). The group with presentation

$$\langle x, y \mid x^2y^2x^2 = y\rangle$$

is the group of a fibred knot in the homology 3-sphere $M(2,3,11)$, but is not a classical knot group [Rt83].

Part of Theorem 17.9 also follows from an argument of Trace [Tr86]. The embedding of a Seifert hypersurface V for an n-knot K in X extends to an embeding of $\widehat{V} = V \cup D^{n+1}$ in M, which lifts to an embedding in M'. Since the image of $[\widehat{V}]$ in $H_{n+1}(M;\mathbb{Z})$ is Poincaré dual to a generator of $H^1(M;\mathbb{Z}) = Hom(\pi,\mathbb{Z}) = [M, S^1]$ its image in $H_{n+1}(M';\mathbb{Z}) \cong Z$ is a generator. Thus if K is fibred, so M' is homotopy equivalent to the closed fibre $\widehat{F}$, there is a degree 1 map from $\widehat{V}$ to $\widehat{F}$, and hence to any factor of $\widehat{F}$. In particular, if $\widehat{F}$ has a summand which is aspherical or whose fundamental group is a nontrivial finite group then $\pi_1(V)$ cannot be free. (In particular, K cannot be a ribbon knot). Similarly, as the Gromov norm of a 3-manifold does not increase under degree 1 maps, if $\widehat{F}$ is a $\mathbb{H}^3$-manifold then $\widehat{V}$ cannot be a graph manifold [Ru90]. Rubermann observes also that the "Seifert volume" of [BG84] may be used instead to show that if $\widehat{F}$ is a $\widetilde{\mathbb{SL}}$-manifold then $\widehat{V}$ must have nonzero Seifert volume. (Connected sums of $\mathbb{E}^3$-, $\mathbb{S}^3$-, $\mathbb{N}il^3$-, $\mathbb{S}ol^3$-, $\mathbb{S}^2 \times \mathbb{E}^1$- or $\mathbb{H}^2 \times \mathbb{E}^1$-manifolds all have Seifert volume 0 [BG84]).

We conclude this section by showing that π_1-slice fibred 2-knots have groups with free commutator subgroup.

Theorem 17.10 *Let K be a 2-knot with group $\pi = \pi K$. If K is π_1-slice then the homomorphism from $H_3(M';\mathbb{Z}) = H_3(M(K);\Lambda)$ to $H_3(\pi';\mathbb{Z}) = H_3(\pi;\Lambda)$ induced by c_M is trivial. If moreover M' is a PD_3-complex and π is torsion free then π' is a free group.*

Proof Let Δ and R be chosen as above. Since c_M factors through $D^5 - R$ the first assertion follows from the exact sequence of homology (with coefficients Λ) for the pair $(D^5 - R, M)$. If M' is a PD_3^+-complex with torsion free fundamental group then it is a connected sum of aspherical PD_3-complexes with handles $S^2 \times S^1$, by Turaev's theorem. It is easily seen that if $H_3(c_M;\Lambda) = 0$ there is no aspherical summand, and so π' is free. □

We may broaden the question raised earlier to ask whether every π_1-slice 2-knot is a homotopy ribbon knot. (Every homotopy ribbon n-knot with $n > 1$ is clearly π_1-slice).

17.7 Geometric 2-knot manifolds

The 2-knots K for which $M(K)$ is homeomorphic to an infrasolvmanifold are essentially known. There are three other geometries which may be realized by such knot manifolds. All known examples are fibred, and most are derived from twist spins of classical knots. However there are examples (for instance, those with $\pi' \cong Q(8) \times (Z/nZ)$ for some $n > 1$) which cannot be constructed from twist spins. The remaining geometries may be eliminated very easily; only $\mathbb{H}^2 \times \mathbb{E}^2$ and $\mathbb{S}^2 \times \mathbb{E}^2$ require a little argument.

Theorem 17.11 *Let K be a 2-knot with group $\pi = \pi K$. If $M(K)$ admits a geometry then the geometry is one of $\mathbb{E}^4$, $\mathbb{N}il^3 \times \mathbb{E}^1$, $\mathbb{S}ol_0^4$, $\mathbb{S}ol_1^4$, $\mathbb{S}ol_{m,n}^4$ (for certain $m \neq n$ only), $\mathbb{S}^3 \times \mathbb{E}^1$, $\mathbb{H}^3 \times \mathbb{E}^1$ or $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$. All these geometries occur.*

Proof The knot manifold $M(K)$ is homeomorphic to an infrasolvmanifold if and only if $h(\sqrt{\pi}) \geq 3$, by Theorem 8.1. It is then determined up to homeomorphism by π. We may then use the observations of §10 of Chapter 8 to show that $M(K)$ admits a geometry of solvable Lie type. By Lemma 16.7 and Theorems 16.12 and 16.14 π must be either $G(+)$ or $G(-)$, $\pi(e,\eta)$ for some even b and $\epsilon = \pm 1$ or $\pi' \cong Z^3$ or Γ_q for some odd q. We may identify the geometry on looking more closely at the meridianal automorphism.

If $\pi \cong G(+)$ or $G(-)$ then $M(K)$ admits the geometry $\mathbb{E}^4$. If $\pi \cong \pi(e,\eta)$ then $M(K)$ is the mapping torus of an involution of a $\mathbb{N}il^3$-manifold, and so admits the geometry $\mathbb{N}il^3 \times \mathbb{E}^1$. If $\pi' \cong Z^3$ then $M(K)$ is homeomorphic to a $\mathbb{S}ol^4_{m,n}$- or $\mathbb{S}ol^4_0$-manifold. More precisely, we may assume (up to change of orientations) that the Alexander polynomial of K is $X^3-(m-1)X^2+mX-1$ for some integer m. If $m \geq 6$ all the roots of this cubic are positive and the geometry is $\mathbb{S}ol^4_{m-1,m}$. If $0 \leq m \leq 5$ two of the roots are complex conjugates and the geometry is $\mathbb{S}ol^4_0$. If $m < 0$ two of the roots are negative and π has a subgroup of index 2 which is a discrete cocompact subgroup of $Sol^4_{m',n'}$, where $m' = m^2-2m+2$ and $n' = m^2-4m+1$, so the geometry is $\mathbb{S}ol^4_{m',n'}$.

If $\pi' \cong \Gamma_q$ and the image of the meridianal automorphism in $Out(\Gamma_q)$ has finite order then $q = 1$ and $K = \tau_6 3_1$ or $(\tau_6 3_1)^* = \tau_{6,5} 3_1$. In this case $M(K)$ admits the geometry $\mathbb{N}il^3 \times \mathbb{E}^1$. Otherwise (if $\pi' \cong \Gamma_q$ and the order of the image of the meridianal automorphism in $Out(\Gamma_q)$ is infinite) $M(K)$ admits the geometry $\mathbb{S}ol^4_1$.

If K is a branched r-twist spin of the (p,q)-torus knot then $M(K)$ is a $\mathbb{S}^3 \times \mathbb{E}^1$-manifold if $p^{-1}+q^{-1}+r^{-1} > 1$, and is a $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$-manifold if $p^{-1}+q^{-1}+r^{-1} < 1$. (The case $p^{-1}+q^{-1}+r^{-1} = 1$ gives the $\mathbb{N}il^3 \times \mathbb{E}^1$-manifold $M(\tau_6 3_1)$). The manifolds obtained from 2-twist spins of 2-bridge knots and certain other "small" simple knots also have geometry $\mathbb{S}^3 \times \mathbb{E}^1$. Branched r-twist spins of simple (nontorus) knots with $r > 2$ give $\mathbb{H}^3 \times \mathbb{E}^1$-manifolds, excepting $M(\tau_3 4_1) \cong M(\tau_{3,2} 4_1)$, which is the $\mathbb{E}^4$-manifold with group $G(+)$.

Every orientable $\mathbb{H}^2 \times \mathbb{E}^2$-manifold is double covered by a Kähler surface [Wl86]. Since the unique double cover of a 2-knot manifold $M(K)$ has first Betti number 1 no such manifold can be an $\mathbb{H}^2 \times \mathbb{E}^2$-manifold. (If K is fibred we could use Lemma 16.1 instead to exclude this geometry). Since π is infinite and $\chi(M(K)) = 0$ we may exclude the geometries $\mathbb{S}^4$, $\mathbb{CP}^2$ and $\mathbb{S}^2 \times \mathbb{S}^2$, and $\mathbb{H}^4$, $\mathbb{H}^2(\mathbb{C})$, $\mathbb{H}^2 \times \mathbb{H}^2$ and $\mathbb{S}^2 \times \mathbb{H}^2$, respectively. The geometry $\mathbb{S}^2 \times \mathbb{E}^2$ may be excluded by Theorem 10.10 or Lemma 16.1 (no group with two ends admits a meridianal automorphism), while $\mathbb{F}^4$ is not realized by any closed 4-manifold. □

In particular, no knot manifold is a $\mathbb{N}il^4$-manifold or a $\mathbb{S}ol^3 \times \mathbb{E}^1$-manifold, and many of the other $\mathbb{S}ol^4_{m,n}$-geometries do not arise in this way. The knot manifolds which are infrasolvmanifolds or have geometry $\mathbb{S}^3 \times \mathbb{E}^1$ are essentially known, by Theorems 8.1, 11.1, 15.12 and §4 of Chapter 16. The knot is uniquely determined up to Gluck reconstruction and change of orientations if $\pi' \cong Z^3$ (see Theorem 17.4 and the subsequent remarks above), Γ_q (see §3 of Chapter 18) or $Q(8) \times (Z/nZ)$ (since the weight class is then unique up to inversion). If it is fibred with closed fibre a lens space it is a 2-twist spin of a 2-bridge knot [Te89]. The other knot groups corresponding to infrasolvmanifolds have infinitely many weight orbits.

Corollary 17.11.1 *If $M(K)$ admits a geometry then it fibres over S^1.*

Proof This is clear if $M(K)$ is an infrasolvmanifold or if the geometry is $\mathbb{S}^3 \times \mathbb{E}^1$. If the geometry is $\mathbb{H}^3 \times \mathbb{E}^1$ then $\sqrt{\pi} = \pi \cap (\{1\} \times R)$, by Proposition 8.27 of [Rg]. Let $\sigma = \pi \cap (Isom(\mathbb{H}^3) \times R)$. Then $[\pi : \sigma] \leq 2$. Since $\pi/\pi' \cong Z$ it follows that $\beta_1(\sigma) = 1$ and hence that $\sqrt{\pi}$ maps injectively to $\sigma/I(\sigma) \leq \pi/\pi'$. Hence π has a subgroup of finite index which is isomorphic to $\pi' \times Z$, and so π' is the fundamental group of a closed $\mathbb{H}^3$-manifold. If the geometry is $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$ then π' is the fundamental group of a closed $\widetilde{\mathbb{SL}}$-manifold, by Theorem 16.2. In each case $M(K)$ fibres over S^1, by Corollary 13.1.1. □

If the geometry is $\mathbb{H}^3 \times \mathbb{E}^1$ is $M(K) \cong M(K_1)$ for some branched twist spin of a simple non-torus knot? (See §3 of Chapter 16).

If $M(K)$ is Seifert fibred must it be geometric? If so it is a $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$-, $\mathbb{N}il^3 \times \mathbb{E}^1$- or $\mathbb{S}^3 \times \mathbb{E}^1$-manifold. (See §4 of Chapter 7).

17.8 Complex surfaces and 2-knot manifolds

If a complex surface S is homeomorphic to a 2-knot manifold $M(K)$ then S is minimal, since $\beta_2(S) = 0$, and has Kodaira dimension $\kappa(S) = 1$, 0 or $-\infty$, since $\beta_1(S) = 1$ is odd. If $\kappa(S) = 1$ or 0 then S is elliptic and admits a compatible geometric structure, of type $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$ or $\mathbb{N}il^3 \times \mathbb{E}^1$, respectively [Ue90,91, Wl86]. The only complex surfaces with $\kappa(S) = -\infty$, $\beta_1(S) = 1$ and $\beta_2(S) = 0$ are Inoue surfaces, which are not elliptic, but admit compatible geometries of type $\mathbb{S}ol_0^4$ or $\mathbb{S}ol_1^4$, and Hopf surfaces [Tl94]. An elliptic surface with Euler characteristic 0 has no exceptional fibres other than multiple tori.

If $M(K)$ has a complex structure compatible with a geometry then the geometry is one of $\mathbb{S}ol_0^4$, $\mathbb{S}ol_1^4$, $\mathbb{N}il^3 \times \mathbb{E}^1$, $\mathbb{S}^3 \times \mathbb{E}^1$ or $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$, by Theorem 4.5 of [Wl86]. Conversely, if $M(K)$ admits one of the first three of these geometries then it is homeomorphic to an Inoue surface of type S_M, an Inoue surface of type $S^{(+)}_{N,p,q,r;t}$ or $S^{(-)}_{N,p,q,r}$ or an elliptic surface of Kodaira dimension 0, respectively. (See [In74], [EO94] and Chapter V of [BPV]).

Lemma 17.12 *Let K be a branched r-twist spin of the (p,q)-torus knot. Then $M(K)$ is homeomorphic to an elliptic surface.*

Proof We shall adapt the argument of Lemma 1.1 of [Mi75]. (See also [Ne83]). Let $V_0 = \{(z_1, z_2, z_3) \in C^3\backslash\{0\} | z_1^p + z_2^q + z_3^r = 0\}$, and define an action of $\mathbb{C}^\times$ on V_0 by $u.v = (u^{qr}z_1, u^{pr}z_2, u^{pq}z_3)$ for all u in $\mathbb{C}^\times$ and $v = (z_1, z_2, z_3)$ in V_0. Define functions $m : V_0 \to R^+$ and $n : V_0 \to m^{-1}(1)$ by $m(v) = (|z_1|^p + |z_2|^q + |z_3|^r)^{1/pqr}$ and $n(v) = m(v)^{-1}.v$ for all v in V_0. Then the map $(m, n) : V_0 \to m^{-1}(1) \times R^+$ is an R^+-equivariant homeomorphism, and so $m^{-1}(1)$ is homeomorphic to V_0/R^+. Therefore there is a homeomorphism from $m^{-1}(1)$ to the Brieskorn manifold $M(p, q, r)$, under which the action of the group of r^{th} roots of unity on $m^{-1}(1) = V_0/R^+$ corresponds to the group of covering homeomorphisms of $M(p, q, r)$ as the branched cyclic cover of S^3, branched over the (p, q)-torus knot [Mi75]. The manifold $M(K)$ is the mapping torus of some generator of this group of self homeomorphisms of $M(p, q, r)$. Let ω be the corresponding primitive r^{th} root of unity. If $t > 1$ then $t\omega$ generates a subgroup Ω of $\mathbb{C}^\times$ which acts freely and holomorphically on V_0, and the quotient V_0/Ω is an elliptic surface over the curve V_0/Ω. Moreover V_0/Ω is homeomorphic to the mapping torus of the self homeomorphism of $m^{-1}(1)$ which maps v to $m(t\omega.v)^{-1}.t\omega.v = \omega m(t.v)^{-1}t.v$. Since this map is isotopic to the map sending v to $\omega.v$ this mapping torus is homeomorphic to $M(K)$. This proves the Lemma. □

The Kodaira dimension of the elliptic surface in the above lemma is 1, 0 or $-\infty$ according as $p^{-1} + q^{-1} + r^{-1}$ is < 1, 1 or > 1. In the next theorem we shall settle the case of elliptic surfaces with $\kappa = -\infty$.

Theorem 17.13 *Let K be a 2-knot. Then $M(K)$ is homeomorphic to a Hopf surface if and only if K or its Gluck reconstruction is a branched r-twist spin of the (p, q)-torus knot for some p, q and r such that $p^{-1} + q^{-1} + r^{-1} > 1$.*

Proof If $K = \tau_{r,s}k_{p,q}$ then $M(K)$ is homeomorphic to an elliptic surface, by Lemma 17.13, and the surface must be a Hopf surface if $p^{-1} + q^{-1} + r^{-1} > 1$.

If $M(K)$ is homeomorphic to a Hopf surface then π has two ends, and there is a monomorphism $h : \pi = \pi K \to GL(2, \mathbb{C})$ onto a subgroup which contains a contraction c (Kodaira - see [Kt75]). Hence π' is finite and $h(\pi') = h(\pi) \cap SL(2, \mathbb{C})$, since $det(c) \neq 1$ and $\pi/\pi' \cong Z$. Finite subgroups of $SL(2, \mathbb{C})$ are conjugate to subgroups of $SU(2) = S^3$, and so are cyclic, binary dihedral or isomorphic to T_1^*, O_1^* or I^*. Therefore $\pi \cong \pi\tau_2 k_{2,n}$, $\pi\tau_3 3_1$, $\pi\tau_4 3_1$ or $\pi\tau_5 3_1$, by Theorem 15.12 and the subsequent remarks. Hopf surfaces with $\pi \cong Z$ or π nonabelian are determined up to diffeomorphism by their fundamental groups, by Theorem 12 of [Kt75]. Therefore $M(K)$ is homeomorphic to the manifold of the corresponding torus knot. If π' is cyclic there is an unique weight orbit. The weight orbits of $\tau_4 3_1$ are realized by $\tau_2 k_{3,4}$ and $\tau_4 3_1$, while the weight orbits of T_1^* are realized by $\tau_2 k_{3,5}$, $\tau_3 k_{2,5}$, $\tau_5 3_1$ and $\tau_{5,2} 3_1$ [PS87]. Therefore K agrees up to Gluck reconstruction with a branched twist spin of a torus knot. □

The Gluck reconstruction of a branched twist spin of a classical knot is another branched twist spin of that knot, by §6 of [Pl84'].

Elliptic surfaces with $\beta_1 = 1$ and $\kappa = 0$ are $\mathbb{N}il^3 \times \mathbb{E}^1$-manifolds, and so a knot manifold $M(K)$ is homeomorphic to such an elliptic surface if and only if πK is virtually poly-Z and $\zeta\pi K \cong Z^2$. For minimal properly elliptic surfaces (those with $\kappa = 1$) we must settle for a characterization up to s-cobordism.

Theorem 17.14 *Let K be a 2-knot with group $\pi = \pi K$. Then $M(K)$ is s-cobordant to a minimal properly elliptic surface if and only if $\zeta\pi \cong Z^2$ and π' is not virtually poly-Z.*

Proof If $M(K)$ is a minimal properly elliptic surface then it admits a compatible geometry of type $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$ and π is isomorphic to a discrete cocompact subgroup of $Isom_o(\widetilde{\mathbb{SL}}) \times R$, the maximal connected subgroup of $Isom_o(\widetilde{\mathbb{SL}} \times \mathbb{E}^1)$, for the other components consist of orientation reversing or antiholomorphic isometries (see Theorem 3.3 of [Wl86]). Since π meets $\zeta(Isom_o(\widetilde{\mathbb{SL}}) \times R)) \cong R^2$ in a lattice subgroup $\zeta\pi \cong Z^2$ and projects nontrivially onto the second factor $\pi' = \pi \cap Isom_o(\widetilde{\mathbb{SL}})$ and is the fundamental group of a $\widetilde{\mathbb{SL}}$-manifold. Thus the conditions are necessary.

Suppose that they hold. Then $M(K)$ is s-cobordant to a $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$-manifold which is the mapping torus $M(\Theta)$ of a self homeomorphism of a $\widetilde{\mathbb{SL}}$-manifold, by Theorem 16.2. As Θ must be orientation preserving and induce the identity on $\zeta\pi' \cong Z$ the group π is contained in $Isom_o(\widetilde{\mathbb{SL}}) \times R$. Hence $M(\Theta)$ has a compatible structure as an elliptic surface, by Theorem 3.3 of [Wl86]. □

An elliptic surface with Euler characteristic 0 is a Seifert fibred 4-manifold, and so is determined up to diffeomorphism by its fundamental group if the base orbifold is euclidean or hyperbolic [Ue90,91]. Using this result (instead of [Kt75]) together with Theorem 16.6 and Lemma 17.12 it may be shown that if $M(K)$ is homeomorphic to a minimal properly elliptic surface and some power of a weight element is central in πK then $M(K)$ is homeomorphic to $M(K_1)$, where K_1 is some branched twist spin of a torus knot. However in general there may be infinitely many algebraically distinct weight classes in πK and we cannot conclude that K is itself such a branched twist spin.

Chapter 18

Reflexivity

The most familiar invariants of knots are derived from the knot complements, and so it is natural to ask whether every knot is determined by its complement. This has been confirmed for classical knots [GL89]. Given a higher dimensional knot there is at most one other knot (up to change of orientations) with homeomorphic exterior. The first examples of non-reflexive 2-knots were given by Cappell and Shaneson [CS76]; these are fibred with closed fibre R^3/Z^3. Gordon gave a different family of examples [Go76], and Plotnick extended his work to show that no fibred 2-knot with monodromy of odd order is reflexive. It is plausible that this may be so whenever the order is greater than 2, but this is at present unknown.

We shall consider 2-knots which are fibred with closed fibre a geometric 3-manifold. A nontrivial cyclic branched cover of S^3, branched over a knot, admits a geometry if and only if the knot is a prime simple knot. The geometry is then $\widetilde{\mathbb{SL}}$, $\mathbb{S}^3$, $\mathbb{H}^3$, $\mathbb{E}^3$ or $\mathbb{N}il^3$. We shall show that no branched r-twist spin of such a knot is ever reflexive, if $r > 2$. (Our argument also explains why fibred knots with monodromy of order 2 are reflexive). If the 3-dimensional Poincaré conjecture is true then all fibred 2-knots with monodromy of finite order are branched twist spins, by Plotnick's theorem (see Chapter 16). The remaining three geometries may be excluded without reference to this conjecture, by Lemma 15.7.

This chapter is based on joint work with Plotnick and Wilson (in [HP88] and [HW89], respectively).

18.1 Reflexivity for fibred 2-knots

Let N be a closed oriented 3-manifold and θ an orientation preserving self diffeomorphism of N which fixes a basepoint P and induces a meridianal automorphism of $\nu = \pi_1(N)$. Let

$$M = M(\theta) = N \times_\theta S^1 = N \times [0,1]/((n,0) \sim (\theta(n),1)),$$

and let t be the weight element of $\pi = \pi_1(M) = \nu \times_{\theta_*} Z$ represented by the loop sending $[u] = e^{2\pi i u}$ to $[*, u]$ in the mapping torus, for all $0 \leq u \leq 1$. The image

$C = \{P\} \times S^1$ of this loop is the *canonical* cross-section of the mapping torus. Let $\widetilde{N}$ be the universal covering space of N, and let $\tilde{\theta}$ be the lift of θ which fixes some chosen basepoint. Let $\widehat{M} = \widetilde{N} \times_{\tilde{\theta}} S^1$ be the (irregular) covering space corresponding to the subgroup of π generated by t. This covering space shall serve as a natural model for a regular neighbourhood of C in our geometric arguments below.

Choose an embedding $J : D^3 \times S^1 \to M$ onto a regular neighbourhood R of C. Let $M_o = M - intR$ and let $j = J|_{\partial D^3 \times S^1}$. Then $\Sigma = M_o \cup_j S^2 \times D^2$ and $\Sigma_\tau = M_o \cup_{j\tau} S^2 \times D^2$ are homotopy 4-spheres and the images of $S^2 \times \{0\}$ represent 2-knots K and K^* with group π.

If K is reflexive there is a homeomorphism f of $X = X(K)$ which (up to changes of orientations) restricts to the nontrivial twist τ on $\partial X \cong S^2 \times S^1$. (See §1 of Chapter 14). This extends to a homeomorphism of (M, C) via the "radial" extension of τ to $D^3 \times S^1$. If f preserves the homology class of the meridians (i.e., if it induces the identity on π/π') then we may assume this extension fixes C pointwise. Now $\partial X \cong S^2 \times_A S^1$, where A is the restriction of the monodromy to $\partial(N - intD^3) \cong S^2$. Roughly speaking, the local situation - the behaviour of f and A on $D^3 \times S^1$ - determines the global situation. Assume that f is a fibre preserving self homeomorphism of $D^3 \times_A S^1$ which induces a linear map B on each fibre D^3. If A has infinite order, the question as to when f "changes the framing", i.e., induces τ on $\partial D^3 \times_A S^1$ is delicate. (See §2 and §3 below). But if A has finite order we have the following easy result.

Lemma 18.1 *Let A in $SO(3)$ be a rotation of order $r \geq 2$ and let B in $O(3)$ be such that $BAB^{-1} = A^{\pm 1}$, so that B induces a diffeomorphism f_B of $D^3 \times_A S^1$. If f_B changes the framing then $r = 2$.*

Proof We may choose coordinates for R^3 so that $A = \rho_{s/r}$, where ρ_u is the matrix of rotation through $2\pi u$ radians about the z-axis in R^3, and $0 < s < r$. Let $\rho : D^3 \times_A S^1 \to D^3 \times S^1$ be the diffeomorphism given by $\rho([x, u]) = (\rho_{-su/r}, \theta)$, for all $x \in D^3$ and $0 \leq u \leq 1$.

If $BA = AB$ then $f_B([x, u]) = [Bx, u]$ and $\rho f_B \rho^{-1}(x, u) = (\rho_{-su/r} B \rho_{su/r} x, u)$. If $r \geq 3$ then $B = \rho_v$ for some v, and so $\rho f_B \rho^{-1}(x, u) = (Bx, u)$ does not change the framing. But if $r = 2$ then $A = diag[-1, -1, 1]$ and there is more choice for B. In particular, $B = diag[1, -1, 1]$ acts dihedrally: $\rho_{-u} B \rho_u = \rho_{-2u} B$, and so $\rho_{-u} f_B \rho_u(x, u) = (\rho_{-u} x, u)$, i.e. $\rho_{-u} f_B \rho_u$ is the twist τ.

If $BAB^{-1} = B^{-1}$ then $f_B([x, u]) = [Bx, 1 - u]$. In this case $\rho f_B \rho^{-1}(x, u) = (\rho_{-s(1-u)/r} B \rho_{su/r} x, 1 - u)$. If $r \geq 3$ then B must act as a reflection in the first two coordinates, so $\rho f_B\rho^{-1}(x,u) = (\rho_{-s/r}Bx, 1-u)$ does not change the framing. But if $r=2$ we may take $B=I$, and then $\rho f_B\rho^{-1}(x,u) = (\rho_{(u-1)/2}\rho_{u/2}x, 1-u) = (\rho_{(u-\frac{1}{2})}x, 1-u)$, which after reversing the S^1 factor is just τ. □

Note this explains why $r=2$ is special. If $\alpha^2 = id$ the diffeomorphism of $N\times_\alpha S^1$ sending $[x,\theta]$ to $[x,1-\theta]$ which "turns the bundle upside down" also changes the framing. This explains why 2-twist spins (in any dimension) are reflexive.

Lemma 18.2 *Let τ be the nontrivial twist map of $S^3\times S^1$. Then τ is not homotopic to the identity.*

Proof Let p be the projection of $S^3\times S^1$ onto S^3. The suspension of $p\tau$, restricted to the top cell of $\Sigma(S^3\times S^1) = S^2\vee S^4\vee S^5$ is the nontrivial element of $\pi_5(S^4)$, whereas the corresponding restriction of the suspension of p is trivial. (See [CS76], [Go76]). □

The hypotheses in the next lemma seem very stringent, but are satisfied by most aspherical geometric 3-manifolds.

Lemma 18.3 *Suppose that $\widetilde{N}\cong R^3$ and that every automorphism of ν which commutes with θ_* is induced by a diffeomorphism of N which commutes with θ. Suppose also that for any homeomorphism ω of N which commutes with θ there is an isotopy γ from $id_{\widetilde{N}}$ to $\tilde\theta$ which commutes with the lift $\tilde\omega$. Then no orientation preserving self homeomorphism of M which fixes C pointwise changes the framing.*

Proof Let h be an orientation preserving self homeomorphism of M which fixes C pointwise. Suppose that h changes the framing. We may assume that $h|_R$ is a bundle automorphism and hence that it agrees with the radial extension of τ from $\partial R = S^2\times S^1$ to R. Since $h_*(t)=t$ we have $h_*\theta_* = \theta_* h_*$. Let ω be a basepoint preserving self diffeomorphism of N which induces h_* and commutes with θ. Then we may define a self diffeomorphism h_ω of M by $h_\omega([n,s]) = [\omega(n), s]$ for all $[n,s]$ in $M = N\times_\theta S^1$.

Since $h_{\omega *} = h_*$ and M is aspherical, h and h_ω are homotopic. Therefore the lifts $\hat h$ and $\hat h_\omega$ to basepoint preserving maps of $\widehat{M}$ are properly homotopic. Let $\tilde\omega$ be the lift of ω to a basepoint preserving map of $\widetilde{N}$. Note that $\tilde\omega$ is orientation preserving, and so is isotopic to $id_{\widetilde{N}}$.

Given an isotopy γ from $\gamma(0) = id_{\widetilde{N}}$ to $\gamma(1) = \tilde{\theta}$ we may define a diffeomorphism $\rho_\gamma : \widetilde{N} \times S^1 \to \widetilde{M}$ by $\rho_\gamma(x, e^{2\pi it}) = [\gamma(t)(x), t]$. Now $\rho_\gamma^{-1}\hat{h}_\omega\rho_\gamma(l, [u]) = (\gamma(u)^{-1}\tilde{\omega}\gamma(u)(l), [u])$. Thus if $\gamma(t)\tilde{\omega} = \tilde{\omega}\gamma(t)$ for all t then $\rho_\gamma^{-1}\hat{h}_\omega\rho_\gamma = \tilde{\omega} \times id_{S^1}$, and so $\hat{h}$ is properly homotopic to $id_{\widehat{M}}$.

Since the radial extension of τ and $\rho_\gamma^{-1}\hat{h}\rho_\gamma$ agree on $D^3 \times S^1$ they are properly homotopic on $R^3 \times S^1$ and so τ is properly homotopic to the identity. Now τ extends uniquely to a self diffeomorphism τ of $S^3 \times S^1$, and any such proper homotopy extends to a homotopy from τ to the identity. But this is impossible, by Lemma 18.2. Therefore h cannot change the framing. □

Note that in general there is no isotopy from id_N to θ.

We may use a similar argument to give a sufficient condition for knots constructed from mapping tori to be -amphicheiral. As we shall not use this result below we shall only sketch a proof.

Lemma 18.4 *Let N be a closed orientable 3-manifold with universal cover $\widetilde{N} \cong R^3$. Suppose now that there is an orientation reversing self diffeomorphism $\psi : N \to N$ which commutes with θ and which fixes P. If there is a path γ from I to $\Theta = D\theta(P)$ which commutes with $\Psi = D\psi(P)$ then each of K and K^* is -amphicheiral.*

Proof The map ψ induces an orientation reversing self diffeomorphism of M which fixes C pointwise. We may use such a path γ to define a diffeomorphism $\rho_\gamma : \widetilde{N} \times S^1 \to \widetilde{M}$. We may then verify that $\rho_\gamma^{-1}\hat{h}\rho_\gamma$ is isotopic to $\Psi \times id_{S^1}$, and so $\rho_\gamma^{-1}\hat{h}\rho_\gamma|_{\partial D^3 \times S^1}$ extends across $S^2 \times D^2$. □

18.2 Cappell-Shaneson knots

Let $A \in SL(3, \mathbb{Z})$ be such that $det(A - I) = \pm 1$. Then A determines an orientation preserving self homeomorphism of R^3/Z^3, and the mapping torus $M = (R^3/Z^3) \times_A S^1$ is a 2-knot manifold. All such knots are -amphicheiral, since inversion in each fibre gives an involution of $M(K)$ fixing a circle, which readily passes to orientation reversing fixed point free involutions of (Σ, K) and (Σ^*, K^*). However such knots are not invertible, for the Alexander polynomial is $det(XI - A)$, which has odd degree and does not vanish at ± 1, and so cannot be symmetric.

Cappell and Shaneson showed that if none of the eigenvalues of the monodromy of such a knot are negative then it is not reflexive. In a footnote they observed that the two knots obtained from a matrix A in $SL(3,\mathbb{Z})$ such that $det(A-I)=\pm 1$ and with negative eigenvalues are equivalent if and only if there is a matrix B in $GL(3,\mathbb{Z})$ such that $AB=BA$ and the restriction of B to the negative eigenspace of A has negative determinant. We shall translate this matrix criterion into one involving algebraic numbers and settle the issue by showing that up to change of orientations there is just one reflexive Cappell-Shaneson 2-knot.

We note first that on replacing A by A^{-1} if necessary (which corresponds to changing the orientation of the knot) we may assume that $det(A-I)=+1$.

Theorem 18.5 *Let $A\in SL(3,\mathbb{Z})$ satisfy $det(A-I)=1$. If A has trace -1 then the corresponding Cappell-Shaneson knot is reflexive, and is determined up to change of orientations among all 2-knots with metabelian group by its Alexander polynomial X^3+X^2-2X-1. If the trace of A is not -1 then the corresponding Cappell-Shaneson knots are not reflexive.*

Proof Let a be the trace of A. Then the characteristic polynomial of A is $f_a(X)=X^3-aX^2+(a-1)X-1=X(X-1)(X-a+1)-1$. It is easy to see that f_a is irreducible; indeed, it is irreducible modulo (2). Since the leading coefficient of f_a is positive and $f_a(1)<0$ there is at least one positive eigenvalue. If $a>5$ all three eigenvalues are positive (since $f_a(0)=-1$, $f_a(\frac{1}{2})=(2a-11)/8>0$ and $f_a(1)=-1$). If $0\leq a\leq 5$ there is a pair of complex eigenvalues.

Thus if $a\geq 0$ there are no negative eigenvalues, and so $\gamma(t)=tA+(1-t)I$ (for $0\leq t\leq 1$) defines an isotopy from I to A in $GL(3,\mathbb{R})$. Let h be a self homeomorphism of (M,C) such that $h(*)=*$. We may assume that h is orientation preserving and that $h_*(t)=t$. Since M is aspherical h is homotopic to a map h_B, where $B\in SL(3,\mathbb{Z})$ commutes with A. Hence K is not reflexive, by Lemma 18.3.

We may assume henceforth that $a<0$. There are then three real roots λ_i, for $1\leq i\leq 3$, such that $a-1<\lambda_3<a<\lambda_2<0<1<\lambda_1<2$. Note that the products $\lambda_i(\lambda_i-1)$ are all positive, for $1\leq i\leq 3$.

Since the eigenvalues of A are real and distinct there is a matrix P in $GL(3,\mathbb{R})$ such that $\tilde{A}=PAP^{-1}$ is the diagonal matrix $diag[\lambda_1,\lambda_2,\lambda_3]$. If B in $GL(3,\mathbb{Z})$ commutes with A then $\tilde{B}=PBP^{-1}$ commutes with $\tilde{A}$ and hence is also diagonal (as the λ_i are distinct). Suppose that $\tilde{B}=diag[\beta_1,\beta_2,\beta_3]$. We may isotope PAP^{-1} linearly to $diag[1,-1,-1]$. If $\beta_2\beta_3>0$ for all such B then PBP^{-1} is isotopic to I through block diagonal matrices and we may again conclude that the knot is not reflexive. On the other hand if there is such a B with $\beta_2\beta_3 < 0$ then the knot is reflexive. On replacing B by $-B$ if necessary we may assume that $det(B) = +1$ and the criterion for reflexivity then becomes $\beta_1 < 0$.

If $a = -1$ the ring $\mathbb{Z}[X]/(f_{-1}(X))$ is integrally closed. (For the discriminant D of the integral closure $\tilde{R}$ of $R = \mathbb{Z}[X]/(f_{-1}(X))$ divides 49, the discriminant of $f_{-1}(X)$, and $49/D = [\tilde{R} : R]^2$. As the discriminant must be greater than 1, by a classical result of Minkowski, this index must be 1). As this ring has class number 1 (see the tables of [AR84]) it is a PID. Hence any two matrices in $SL(3,\mathbb{Z})$ with this characteristic polynomial are conjugate, by Theorem 1.4. Therefore the knot group is unique and determines K up to Gluck reconstruction and change of orientations, by Theorem 17.5. Since $B = -A - I$ has determinant 1 and $\beta_1 = -\lambda_1 - 1 < 0$, the corresponding knot is reflexive.

Suppose now that $a < -1$. Let F be the field $\mathbb{Q}[X]/(f_a(X))$ and let λ be the image of X in F. We may view Q^3 as a $\mathbb{Q}[X]$-module and hence as a 1-dimensional F-vector space via the action of A. If B commutes with A then it induces an automorphism of this vector space which preserves a lattice and so determines a unit $u(B)$ in O_F, the ring of integers in F. Moreover $det(B) = N_{F/\mathbb{Q}}u(B)$. If σ is the embedding of F in R which sends λ to λ_1 and P and B are as above we must have $\sigma(u(B)) = \beta_1$.

Let $U = O_F^\times$ be the group of all units in O_F, and let U^ν, U^σ, U^+ and U^2 be the subgroups of units of norm 1, units whose image under σ is positive, totally positive units and squares, respectively. Then $U \cong Z^2 \times \{\pm1\}$, since F is a totally real cubic number field, and so $[U : U^2] = 8$. The unit -1 has norm -1, and λ is a unit of norm 1 in U^σ which is not totally positive. Hence $[U : U^\nu] = [U^\nu \cap U^\sigma : U^+] = 2$. It is now easy to see that there is a unit of norm 1 that is not in U^σ (i.e., $U^\nu \neq U^\nu \cap U^\sigma$) if and only if every totally positive unit is a square (i.e., $U^+ = U^2$).

The image of $X(X-1)$ in F is $\lambda(\lambda-1)$, which is totally positive and is a unit (since $X(X-1)(X-a+1) = 1 + f_a(X)$). Suppose that it is a square in F. Then $\phi = \lambda - (a-1)$ is a square (since $\lambda(\lambda-1)(\lambda-(a-1)) = 1$). The minimal polynomial of ϕ is $g(Y) = Y^3 + (2a-3)Y^2 + (a^2-3a+2)Y - 1$. If $\phi = \psi^2$ for some ψ in F then ψ is a root of $h(Z) = g(Z^2)$ and so the minimal polynomial of ψ divides h. This polynomial has degree 3 also, since $Q(\psi) = F$, and so $h(Z) = p(Z)q(Z)$ for some polynomials $p(Z) = Z^3 + rZ^2 + sZ + 1$ and $q(Z) = Z^3 + r'Z^2 + s'Z - 1$ with integer coefficients. Since the coefficients of Z and Z^5 in h are 0 we must have $r' = -r$ and $s' = -s$. Comparing the coefficients of Z^2 and Z^4 then gives the equations $2s - r^2 = 2a - 3$ and

$s^2 - 2r = a^2 - 3a + 2$. Eliminating s we find that $r(r^3 + (4a-6)r - 8) = -1$ and so $1/r$ is an integer. Hence $r = \pm 1$ and so $a = -1$ or 3, contrary to hypothesis. Thus there is no such matrix B and so the Cappell-Shaneson knots corresponding to A are not reflexive. □

The other fibred 2-knots with closed fibre a flat 3-manifold have group $G(+)$ or $G(-)$. We shall show below that one of these ($\tau_3 4_1$) is not reflexive. The question remains open for the other knots with these groups.

18.3 $\mathbb{N}il^3$-fibred knots

The group $Nil = Nil^3$ is a subgroup of $SL(3,\mathbb{R})$ and is diffeomorphic to R^3, with multiplication given by $[r,s,t][r',s',t'] = [r+r', s+s', rs'+t+t']$. (See Chapter 7). The kernel of the natural homomorphism from $Aut_{Lie}(Nil)$ to $Aut_{Lie}(R^2) = GL(2,\mathbb{R})$ induced by abelianization ($Nil/Nil' \cong R^2$) is isomorphic to $Hom_{Lie}(Nil, \zeta Nil) \cong R^2$. The set underlying the group $Aut_{Lie}(Nil)$ is the cartesian product $GL(2,\mathbb{R}) \times R^2$, with $(A,\mu) = ((\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}), (m_1, m_2))$ acting via $(A,\mu)([r,s,t]) =$

$$[ar+cs, br+ds, m_1 r + m_2 s + (ad-bc)t + bcrs + ab\binom{r}{2} + cd\binom{s}{2}].$$

The Jacobian of such an automorphism is $(ad-bc)^2$, and so it is orientation preserving. Let $(B,\nu) = ((\begin{smallmatrix} g & j \\ h & k \end{smallmatrix}), (n_1, n_2))$ be another automorphism, and let

$$\eta(A,B) = (abg(1-g) + cdh(1-h) - 2bcgh, abj(1-j) + cdk(1-k) - 2bcjk).$$

Then $(A,\mu)\circ(B,\nu) = (AB, \mu B + det(A)\nu + \frac{1}{2}\eta(A,B))$. In particular, $Aut_{Lie}(Nil)$ is *not* a semidirect product of $GL(2,\mathbb{R})$ with R^2. For each $q > 0$ in Z the stabilizer of Γ_q in $Aut_{Lie}(Nil)$ is the subgroup $GL(2,\mathbb{Z}) \times (q^{-1}Z^2)$, and this is easily verified to be $Aut(\Gamma_q)$. (See §7 of Chapter 8). Thus every automorphism of Γ_q extends to an automorphism of Nil. (This is a special case of a theorem of Malcev on embeddings of torsion free nilpotent groups in 1-connected nilpotent Lie groups - see [Rg]).

Let the identity element $[0,0,0]$ and its images in $N_q = Nil/\Gamma_q$ be the basepoints for Nil and for these coset spaces. The extension of each automorphism of Γ_q to Nil induces a basepoint and orientation preserving self homeomorphism of N_q.

If K is a 2-knot with group $\pi = \pi K$ and $\pi' \cong \Gamma_q$ then $M = M(K)$ is homeomorphic to the mapping torus of such a self homeomorphism of N_q. (In fact, such mapping tori are determined up to diffeomorphism by their fundamental groups). Up to conjugacy and involution there are just three classes of meridianal automorphisms of Γ_1 and one of Γ_q, for each odd $q > 1$. (See Theorem 16.13). Since $\pi'' \leq \zeta\pi'$ it is easily seen that π has just two strict weight orbits. Hence K is determined up to Gluck reconstruction and changes of orientation by π alone, by Theorem 17.5. (Instead of appealing to 4-dimensional surgery to realize automorphisms of π by basepoint and orientation preserving self homeomorphisms of M we may use the S^1-action on N_q to construct such a self homeomorphism which in addition preserves the fibration over S^1).

We shall show that the knots with $\pi' \cong \Gamma_1$ and whose characteristic polynomials are $X^2 - X + 1$ and $X^2 - 3X + 1$ are not reflexive, while for all other groups the corresponding knots are reflexive.

The polynomial $X^2 - X + 1$ is realized by $\tau_6 3_1$ and its Gluck reconstruction. Since the trefoil knot 3_1 is strongly invertible $\tau_6 3_1$ is strongly +amphicheiral [Li85]. The involution of $X(\tau_6 3_1)$ extends to an involution of $M(\tau_6 3_1)$ which fixes the canonical section C pointwise and does not change the framing of the normal bundle, and hence $(\tau_6 3_1)^*$ is also +amphicheiral. (We shall see below that these knots are distinct).

Lemma 18.6 *Let K be a fibred 2-knot with closed fibre N_1 and Alexander polynomial $X^2 - 3X + 1$. Then K is +amphicheiral.*

Proof Let $\Theta = (A, (0,0))$ be the automorphism of Γ_1 with $A = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right)$. Then Θ induces a basepoint and orientation preserving self diffeomorphism θ of N_1. Let $M = N_1 \times_\theta S^1$ and let C be the canonical section. A basepoint and orientation preserving self diffeomorphism ψ of N_1 such that $\psi\theta\psi^{-1} = \theta^{-1}$ induces a self diffeomorphism of M which reverses the orientations of M and C. If moreover it does not twist the normal bundle of C then each of the 2-knots K and K^* obtained by surgery on C is +amphicheiral. We may check the normal bundle condition by using an isotopy from Θ to id_{Nil} to identify $\widehat{M}$ with $Nil \times S^1$.

Thus we seek an automorphism $\Psi = (B, \mu)$ of Γ_1 such that $\Psi\Theta_t\Psi^{-1} = \Theta_t^{-1}$, or equivalently $\Theta_t\Psi\Theta_t = \Psi$, for some isotopy Θ_t from $\Theta_0 = id_{Nil}$ to $\Theta_1 = \Theta$.

Let $P = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Then $PAP^{-1} = A^{-1}$, or $APA = P$. It may be checked that the equation $\Theta(P, \mu)\Theta = (P, \mu)$ reduces to a linear equation for μ with unique solution $\mu = -(2,3)$. Let $\Psi = (P, -(2,3))$ and let h be the induced diffeomorphism of M.

As the eigenvalues of A are both positive it lies on a 1-parameter subgroup, determined by $L=\ln(A)=m\left(\begin{smallmatrix}1 & -2\\ -2 & -1\end{smallmatrix}\right)$, where $m=(\ln((3+\sqrt{5})/2))/\sqrt{5}$. Now $PLP^{-1}=-L$ and so $P\exp(tL)P^{-1}=\exp(-tL)=(\exp(tL)^{-1}$, for all t. We seek an isotopy $\Theta_t=(\exp(tL),v_t)$ from id_{Nil} to Θ such that $\Theta_t\Psi\Theta_t=\Psi$ for all t. It is easily seen that this imposes a linear condition on v_t which has an unique solution, and moreover $v_0=v_1=(0,0)$.

Now $\rho^{-1}h\rho(x,u)=(\Theta_{1-u}\Psi\Theta_u(x),1-u)=(\Psi\Theta_{1-u}\Theta_u,1-u)$. Since $\exp((1-u)L)\exp(uL)=\exp(L)$ the loop $u\mapsto\Theta_{1-u}\Theta_u$ is freely contractible in the group $Aut_{Lie}(Nil)$. It follows easily that h does not change the framing of C. □

Instead of using the one-parameter subgroup determined by $L=\ln(A)$ we may use the polynomial isotopy given by $A_t=\left(\begin{smallmatrix}1 & t\\ t & 1+t^2\end{smallmatrix}\right)$, for $0\leq t\leq 1$. A similar argument could be used for the polynomial X^2-X+1.

On the other hand, the polynomial X^2+X-1 is not symmetric and so the corresponding knots are not +amphicheiral. Since every automorphism of Γ_q is orientation preserving no such knot is -amphicheiral or invertible.

Theorem 18.7 *Let K be a fibred 2-knot with closed fibre N_q.*

(1) *If the fibre is N_1 and the monodromy has characteristic polynomial X^2-X+1 or X^2-3X+1 then K is not reflexive;*

(2) *If the fibre is N_q (q odd) and the monodromy has characteristic polynomial $X^2\pm X-1$ then K is reflexive.*

Proof As $\tau_6 3_1$ is shown to be not reflexive in §4 below, we shall concentrate on the knots with polynomial X^2-3X+1, and then comment on how our argument may be modified to handle the other cases.

Let Θ, θ and $M=N_1\times_\theta S^1$ be as in Lemma 18.6, and let $\widehat{M}=Nil\times_\Theta S^1$ be as in §1. We shall take $[0,0,0,0]$ as the basepoint of $\widehat{M}$ and its image in M as the basepoint there.

Suppose that $\Omega=(B,\nu)$ is an automorphism of Γ_1 which commutes with Θ. Since the eigenvalues of A are both positive the matrix $A(u)=uA+(1-u)I$ is invertible and $A(u)B=BA(u)$, for all $0\leq u\leq 1$. We seek a path of the form $\gamma(u)=(A(u),\mu(u))$ with commutes with Ω. On equating the second elements of the ordered pairs $\gamma(u)\Omega$ and $\Omega\gamma(u)$ we find that $\mu(u)(B-det(B)I)$ is uniquely determined. If $det(B)$ is an eigenvalue of B then there is a corresponding eigenvector ξ in Z^2. Then $BA\xi=AB\xi=det(B)A\xi$, so $A\xi$ is also an eigenvector of B. Since the eigenvalues of A are irrational we must have $B = det(B)I$ and so $B = I$. But then $\Omega\Theta = (A, \nu A)$ and $\Theta\Omega = (A, \nu)$, so $\nu(A - I) = 0$ and hence $\nu = 0$. Therefore $\Omega = id_{Nil}$ and there is no difficulty in finding such a path. Thus we may assume that $B - det(B)I$ is invertible, and then $\mu(u)$ is uniquely determined. Moreover, by the uniqueness, when $A(u) = A$ or I we must have $\mu(u) = (0, 0)$. Thus γ is an isotopy from $\gamma(0) = id_{Nil}$ to $\gamma(1) = \Theta$ (through diffeomorphisms of Nil) and so determines a diffeomorphism ρ_γ from $R^3 \times S^1$ to $\widehat{M}$ via $\rho_\gamma(r, s, t, [u]) = [\gamma(u)([r, s, t]), u]$.

A homeomorphism f from Σ to Σ_τ carrying K to K_τ (as unoriented submanifolds) extends to a self homeomorphism h of M which leaves C invariant, but changes the framing. We may assume that h preserves the orientations of M and C, by Lemma 18.6. But then h must preserve the framing, by Lemma 18.3. Hence there is no such homeomorphism and such knots are not reflexive.

If $\pi \cong \pi\tau_6 3_1$ then we may assume that the meridianal automorphism is $\Theta = (\left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right), (0, 0))$. As an automorphism of Nil, Θ fixes the centre pointwise, and it has order 6. Moreover $(\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), (0, 0)$ is an involution of Nil which conjugates Θ to its inverse, and so M admits an orientation reversing involution. It can easily be seen that any automorphism of Γ_1 which commutes with Θ is a power of Θ, and the rest of the argument is similar.

If the monodromy has characteristic polynomial $X^2 \pm X - 1$ we may assume that the meridianal automorphism is $\Theta = (D, (0, 0))$, where $D = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ or its inverse. As $\Omega = (-I, (-1, 1))$ commutes with Θ (in either case) it determines a self homeomorphism h_ω of $M = N_q \times_\theta S^1$ which leaves the meridianal circle $\{0\} \times S^1$ pointwise fixed. The action of h_ω on the normal bundle may be detected by the induced action on $\widehat{M}$. In each case there is an isotopy from Θ to $\Upsilon = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ which commutes with Ω, and so we may replace $\widehat{M}$ by the mapping torus $Nil \times_\Upsilon S^1$. (Note also that Υ and Ω act linearly under the standard identification of Nil with R^3).

Let $R(u) \in SO(2)$ be rotation through πu radians, and let $v(u) = (0, u)$, for $0 \leq u \leq 1$. Then $\gamma(u) = \left(\begin{smallmatrix} 1 & v(u) \\ 0 & R(u) \end{smallmatrix}\right)$ defines a path γ in $SL(3, \mathbb{R})$ from $\gamma(0) = id_{Nil}$ to $\gamma(1) = \Upsilon$ which we may use to identify the mapping torus of Υ with $R^3 \times S^1$. In the "new coordinates" h_ω acts by sending $(r, s, t, e^{2\pi iu})$ to $(\gamma(u)^{-1}\Omega\gamma(u)(r, s, t), e^{2\pi iu})$. The loop sending $e^{2\pi iu}$ in S^1 to $\gamma(u)^{-1}\Omega\gamma(u)$ in $SL(3, \mathbb{R})$ is freely homotopic to the loop $\gamma_1(u)^{-1}\Omega_1\gamma_1(u)$, where $\gamma_1(u) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & R(u) \end{smallmatrix}\right)$ and $\Omega_1 = diag[-1, -1, 1]$. These loops are essential in $SL(3, \mathbb{R})$, since on multiplying the latter matrix product on the left by $diag[-1, 1, -1]$ we obtain $\left(\begin{smallmatrix} 1 & 0 \\ 0 & R(2u) \end{smallmatrix}\right)$. Thus h_ω induces the twist τ on the normal bundle of the meridian, and so the knot is equivalent to its Gluck reconstruction. ∎

The other fibred 2-knots with closed fibre a $\mathbb{N}il^3$-manifold have group $\pi(b,\epsilon)$, for some even b and $\epsilon=\pm1$. The 2-twist spins of Montesinos knots are reflexive (by Lemma 18.1). Are the other knots with these groups also reflexive?

It has been shown that for many of the Cappell-Shaneson knots at least one of the (possibly two) corresponding smooth homotopy 4-spheres is the standard S^4 [AR84]. Can a similar study be made in the Nil cases?

18.4 Other geometrically fibred knots

We shall assume henceforth throughout this section that k is a prime simple 1-knot, i.e., that k is either a torus knot or a hyperbolic knot.

Lemma 18.8 *Let A and B be automorphisms of a group π such that $AB=BA$, $A(h)=h$ for all h in $\zeta\pi$ and the images of A^i and B in $Aut(\pi/\zeta\pi)$ are equal. Let $[A]$ denote the induced automorphism of π/π'. If $I-[A]$ is invertible in $End(\pi/\pi')$ then $B=A^i$ in $Aut(\pi)$.*

Proof There is a homomorphism $\epsilon:\pi\to\zeta\pi$ such that $BA^{-i}(x)=x\epsilon(x)$ for all x in π. Moreover $\epsilon A=\epsilon$, since $BA=AB$. Equivalently, $[\epsilon](I-[A])=0$, where $[\epsilon]:\pi/\pi'\to\zeta\pi$ is induced by ϵ. If $I-[A]$ is invertible in $End(\pi/\pi')$ then $[\epsilon]=0$ and so $B=A^i$. □

Let $p=ap'$, $q=bq'$ and $r=p'q'c$, where $(a,qc)=(b,pc)=1$. Let A denote both the canonical generator of the Z/rZ action on $M(p,q,r)=\{(u,v,w)\in\mathbb{C}^3\mid u^p+v^q+w^r=0\}\cap S^5$ given by $A(u,v,w)=(u,v,e^{2\pi i/r}w)$ and its effect on $\pi_1(M(p,q,r))$. Then the image of the Seifert fibration of $M(p,q,r)$ under the projection to the orbit space $M(p,q,r)/\langle A\rangle\cong S^3$ is the Seifert fibration of S^3 with one fibre of multiplicity p and one of multiplicity q. The quotient of $M(p,q,r)$ by the subgroup generated by $A^{p'q'}$ may be identified with $M(p,q,p'q')$. (Note that $S^2(p,q,r)\cong S^2(p,q,p'q')$). Sitting above the fibre in S^3 of multiplicity p in both M's we find q' fibres of multiplicity a, and above the fibre of multiplicity q we find p' fibres of multiplicity b. But above the branch set, a principal fibre in S^3, we have one fibre of multiplicity c in $M(p,q,r)$, but a principal fibre in $M(p,q,p'q')$.

We may display the factorization of these actions as follows:

$$\begin{array}{ccc} M(p,q,r) & \xrightarrow{/S^1} & S^2(p,q,r) \\ \downarrow & & \downarrow \\ M(p,q,p'q') & \xrightarrow{/S^1} & S^2(p,q,p'q') \\ \downarrow & & \downarrow \\ (S^3,(p,q)) & \xrightarrow{/S^1} & S^2 \end{array}$$

We have the following characterization of the centralizer of A in $Aut(\pi)$.

Theorem 18.9 *Assume that $p^{-1}+q^{-1}+r^{-1} \leq 1$, and let A be the automorphism of $\pi = \pi_1(M(p,q,r))$ of order r induced by the canonical generator of the branched covering transformations. If B in $Aut(\pi)$ commutes with A then $B = A^i$ for some $0 \leq i < r$.*

Proof The 3-manifold $M = M(p,q,r)$ is aspherical, with universal cover R^3, and π is a central extension of $Q(p,q,r)$ by an infinite cyclic normal subgroup. Here $Q = Q(p,q,r)$ is a discrete planar group with signature $((1-p')(1-q')/2; a\dots a, b\dots b, c)$ (where there are q' entries a and p' entries b). Note that Q is Fuchsian except for $Q(2,3,6) \cong Z^2$. (In general, $Q(p,q,pq)$ is a PD_2^+-group of genus $(1-p)(1-q)/2$).

There is a natural homomorphism from $Aut(\pi)$ to $Aut(Q) = Aut(\pi/\zeta\pi)$. The strategy shall be to show first that $B = A^i$ in $Aut(Q)$ and then lift to $Aut(\pi)$. The proof in $Aut(Q)$ falls naturally into three cases.

Case 1. $r = c$. In this case M is a homology 3-sphere, fibred over S^2 with three exceptional fibres of multiplicity p, q and r. Thus $Q \cong \Delta(p,q,r) = \langle q_1, q_2, q_3 \mid q_1^p = q_2^q = q_3^r = q_1q_2q_3 = 1\rangle$, the group of orientation preserving symmetries of a tesselation of H^2 by triangles with angles π/p, π/q and π/r. Since Z_r is contained in S^1, A is inner. (In fact it is not hard to see that the image of A in $Aut(Q)$ is conjugation by q_3^{-1}. See §3 of [Pl83]).

It is well known that the automorphisms of a triangle group correspond to symmetries of the tessellation (see Chapters V and VI of [ZVC]). Since p, q and r are pairwise relatively prime there are no self symmetries of the (p,q,r) triangle. So, fixing a triangle T, all symmetries take T to another triangle. Those that preserve orientation correspond to elements of Q acting by inner automorphisms, and there is one nontrivial outerautomorphism, R say, given by reflection in one of the sides of T. We can assume $R(q_3) = q_3^{-1}$.

Let B in $Aut(Q)$ commute with A. If B is conjugation by b in Q then $BA = AB$ is equivalent to $bq_3 = q_3b$, since Q is centreless. If B is R followed by conjugation by b then $bq_3 = q_3^{-1}b$. But since $\langle q_3\rangle = Z_r$ in Q is generated by an elliptic element the normalizer of $\langle q_3\rangle$ in $PSL(2,\mathbb{R})$ consists of elliptic elements with the same fixed point as q_3. Hence the normalizer of $\langle q_3\rangle$ in Q is just $\langle q_3\rangle$. Since $r > 2$ $q_3 \neq q_3^{-1}$ and so we must have $bq_3 = q_3b$, $b = q_3^i$ and $B = A^i$. (Note that if $r = 2$ then R commutes with A in $Aut(Q)$).

Case 2. $r = p'q'$ so that $Z_r \cap S^1 = 1$. The map from $S^2(p,q,p'q')$ to S^2 is branched over three points in S^2. Over the point corresponding to the fibre of multiplicity p in S^3 the map is p'-fold branched; it is q'-fold branched over the point corresponding to the fibre of multiplicity q in S^3, and it is $p'q'$-fold branched over the point $*$ corresponding to the branching locus of M over S^3.

Represent S^2 as a hyperbolic orbifold $H^2/\Delta(p,q,p'q')$. (If $(p,q,r) = (2,3,6)$ we use instead the flat orbifold $E^2/\Delta(2,3,6)$). Lift this to an orbifold structure on $S^2(p,q,p'q')$, thereby representing $Q = Q(p,q,p'q')$ into $PSL(2,\mathbb{R})$. Lifting the $Z_{p'q'}$-action to H^2 gives an action of the semidirect product $Q\tilde{\times}Z_{p'q'}$ on H^2, with $Z_{p'q'}$ acting as rotations about a point $\tilde{*}$ of H^2 lying above $*$. Since the map from H^2 to $S^2(p,q,p'q')$ is unbranched at $\tilde{*}$ (equivalently, $Z_r \cap S^1 = 1$), $Q \cap Z_{p'q'} = 1$. Thus $Q\tilde{\times}Z_{p'q'}$ acts effectively on H^2, with quotient S^2 and three branch points, of orders p, q and $p'q'$.

In other words, $Q\tilde{\times}Z_{p'q'}$ is isomorphic to $\Delta(p,q,p'q')$. The automorphism A extends naturally to an automorphism of Δ, namely conjugation by an element of order $p'q'$, and B also extends to $Aut(\Delta)$, since $BA = AB$.

We claim $B = A^i$ in $Aut(\Delta)$. We cannot directly apply the argument in Case 1, since $p'q'$ is not prime to pq. We argue as follows. In the notation of Case 1, A is conjugation by q_3^{-1}. Since $BA = AB$, $B(q_3) = q_3^{-1}B(q_3)q_3$, which forces $B(q_3) = q_3^j$. Now $q_3^{-1}B(q_2)q_3 = AB(q_2) = B(q_3^{-1})B(q_2)B(q_3) = q_3^{-j}B(q_2)q_3^j$, or $B(q_2) = q_3^{1-j}B(q_3)q_3^{j-1}$. But $B(q_2)$ is not a power of q_3, so $q_3^{1-j} = 1$, or $j \equiv 1$ modulo (r). Thus $B(q_3) = q_3$. This means that the symmetry of the tessellation that realizes B shares the same fixed point as A, so B is in the dihedral group fixing that point, and now the proof is as before.

Case 3. $r = p'q'c$ (the general case). We have $Z_{p'q'c}$ contained in $Aut(\pi)$, but $Z_{p'q'c} \cap S^1 = Z_c$, so that Z_c is the kernel of the composition

$$Z_r \to Out(\pi) \to Out(Q).$$

Let $\bar{Q}$ be the extension corresponding to the abstract kernel $Z_{p'q'} \to Out(Q)$. (The extension is unique since $\zeta Q = 1$). Then $\bar{Q}$ is a quotient of the semidirect product $Q(p,q,r)\tilde{\times}(Z/rZ)$ by a cyclic normal subgroup of order c.

Geometrically, this corresponds to the following. The map from $S^2(p,q,r)$ to S^2 is branched as in Case 2, over three points with branching indices p, q and $p'q'$. This time, represent S^2 as $H^2/\Delta(p,q,p'q')$. Lift to an orbifold structure on $S^2(p,q,r)$ with one cone point of order c. Lifting an elliptic element of order r in $\Delta(p,q,r)$ to the universal orbifold cover of $S^2(p,q,r)$ gives Z_r contained in $Aut(Q(p,q,r))$ defining the semidirect product. But $Q(p,q,r) \cap Z_r = Z_c$, so the action is ineffective. Projecting to $Z_{p'q'}$ and taking the extension $\bar{Q}$ kills the ineffective part of the action. Note that $Q(p,q,r)$ and Z_r inject into $\bar{Q}$.

As in Case 2, $\bar{Q} \cong \Delta(p,q,r)$, A extends to conjugation by an element of order r in $\bar{Q}$, and B extends to an automorphism of $Q(p,q,r)\tilde{\times}Z_r$, since $BA = AB$. Now $(q_3, p'q')$ in $Q(p,q,r)\tilde{\times}Z_r$ normally generates the kernel of $Q(p,q,r)\tilde{\times}Z_r \to \bar{Q}$, where q_3 is a rotation of order c with the same fixed point as the generator of Z_r. In other words, A in $Aut(Q(p,q,r))$ is such that $A^{p'q'}$ is conjugation by q_3. Since $BA^{p'q'} = A^{p'q'}B$ the argument in Case 2 shows that $B(q_3) = q_3$. So B also gives an automorphism of $\bar{Q}$, and now the argument of Case 2 finishes the proof.

We have shown that $B = A^i$ in $Aut(Q)$. Since A in $Aut(\pi)$ is the monodromy of a fibred knot in S^4 (or, more directly, since A is induced by a branched cover of a knot in a homology sphere), $I - [A]$ is invertible. Thus the Theorem now follows from Lemma 18.8. □

Theorem 18.10 *Let k be a prime simple knot in S^3. Let $0 < s < r$, $(r,s) = 1$ and $r > 2$. Then $\tau_{r,s}k$ is not reflexive.*

Proof We shall consider separately the three cases (a) k a torus knot and the branched cover aspherical; (b) k a torus knot and the branched cover spherical; and (c) k a hyperbolic knot.

Aspherical branched covers of torus knots Let $K = \tau_{r,s}(k_{p,q})$ where $r > 2$ and $M(p,q,r)$ is aspherical. Then $X(K) = (M(p,q,r) - intD^3) \times_{A^s} S^1$, $M = M(K) = M(p,q,r) \times_{A^s} S^1$ and $\pi = \pi K \cong \pi_1(M(p,q,r)) \times_{A^s} Z$.

If K is reflexive there is a homeomorphism f of X which changes the framing on ∂X. Now $k_{p,q}$ is strongly invertible - there is an involution of $(S^3, k_{p,q})$ fixing two points of the knot and reversing the meridian. This lifts to an involution of $M(p,q,r)$ fixing two points of the branch set and conjugating A^s to A^{-s}, thus inducing a diffeomorphism of $X(K)$ which reverses the meridian. By Lemma 18.1 this preserves the framing, so we can assume that f preserves the meridian of K. Since $M(p,q,r)$ is an aspherical Seifert fibred 3-manifold $\widetilde{M(p,q,r)} \cong R^3$ and all automorphisms of $\pi_1(M(p,q,r))$ are induced by self-diffeomorphisms [Hm]. Hence f must be orientation preserving also, as all self homeomorphisms of $\widetilde{\mathbb{SL}}$-manifolds are orientation preserving [NR78]. The remaining hypothesis of Lemma 18.3 is satisfied, by Theorem 18.9. Therefore there is no such self homeomorphism f, and K is not reflexive.

Spherical branched covers of torus knots We now adapt the previous argument to the spherical cases. The analogue of Theorem 18.9 is valid, except for $(2,5,3)$. We sketch the proofs.

$(2,3,3)$: $M(2,3,3 = S^3/Q(8)$. The image in $Aut(Q(8)/\zeta Q(8)) \cong S_3$ of the automorphism A induced by the 3-fold cover of the trefoil knot has order 3 and so generates its own centralizer.

$(2,3,4)$: $M(2,3,4) = S^3/T_1^*$. In this case the image of A in $Aut(T_1^*) \cong S_4$ must be a 4-cycle, and generates its own centralizer.

$(2,3,5)$: $M(2,3,5) = S^3/I^*$. In this case the image of A in $Aut(I^*) \cong S_5$ must be a 5-cycle, and generates its own centralizer.

$(2,5,3)$: We again have I^*, but in this case $A^3 = I$, say $A = (123)(4)(5)$. Suppose $BA = AB$. If B fixes 4 and 5 then it is a power of A. But B may transpose 4 and 5, and then $B = A^iC$, where $C = (1)(2)(3)(45)$ represents the nontrivial outer automorphism class of I^*.

Now let $K = \tau_{r,s}(k_{p,q})$ as usual, with (p,q,r) one of the above four triples, and let $M = M(p,q,r) \times_{A^s} S^1$. As earlier, if K is reflexive we have a homeomorphism f which preserves the meridian t and changes the framing on $D^3 \times_{A^s} S^1$.

Let $\widehat{M}$ be the cover of M corresponding to the meridian subgroup, so $\widehat{M} = S^3 \times_{\hat{A}^s} S^1$, where $\hat{A}$ is a rotation about an axis. Let f be a basepoint preserving self homotopy equivalence of M such that $f_*(t) = t$ in π. Let B in $Aut(\pi_1(M(p,q,r))$ be induced by f_*, so $BA^s = A^sB$. The discussion above shows that $B = A^{si}$ except possibly for $(2,5,3)$. But if B represented the outer automorphism of I^* then after lifting to infinite cyclic covers we would have a homotopy equivalence of S^3/I^* inducing C, contradicting Lemma 11.5. So we have an obvious fibre preserving diffeomorphism f_B of M.

The proof that $\hat{f}_B$ is homotopic to $id_{\widehat{M}}$ is exactly as in the aspherical case. To see that $\hat{f}_B$ is homotopic to $\hat{f}$ (the lift of f to a basepoint preserving proper self homotopy equivalence of $\widehat{M}$) we investigate whether f_B is homotopic to

f. Since $\pi_2(M) = 0$ we can homotope f_B to f on the 2-skeleton of M. On the 3-skeleton we meet an obstruction in $H^3(M;\pi_3) \cong H^3(M;\mathbb{Z}) = Z$, since M has the homology of $S^3 \times S^1$. But this obstruction is detected on the top cell of $M(p,q,r)$ and just measures the difference of the degrees of f and f_B on the infinite cyclic covers [Ol53]. Since both f and f_B are orientation preserving homotopy equivalences this obstruction vanishes. On the 4-skeleton we have an obstruction in $H^4(M;\pi_4) = Z/2Z$, which may not vanish. But this obstruction is killed when we lift to $\widehat{M}$, since the map from $\widehat{M}$ to M has even degree, proving that $\hat{f}_B \simeq \hat{f}$.

We now use radial homotopies on $S^3 \times S^1$ to finish, as before.

Branched covers of hyperbolic knots Let k be hyperbolic. Excluding $N_3(4_1)$ (the 3-fold cyclic branched cover of the figure eight knot), $N = N_r(k)$ is a closed hyperbolic 3-manifold, with $\langle\alpha\rangle \cong Z/rZ$ acting by isometries. As usual, we assume there is a homeomorphism f of $M = M(\tau_{r,s}(k))$ which changes the framing on $D^3 \times_{A^s} S^1$. As in the aspherical torus knot case, it shall suffice to show that the lift $\hat{f}$ on $\widehat{M}$ is properly homotopic to a map of $(R^3 \times S^1, D^3 \times S^1)$ that does not change the framing on $D^3 \times S^1$.

Letting $B = f_*$ on $\nu = \pi_1(N)$, we have $BA^sB^{-1} = A^{\pm s}$, depending on whether $f_*(t) = t^{\pm 1}$ in $\pi = \nu \times_{A^s} Z$. There is an unique isometry β of N realizing the class of B in $Out(\nu)$, by Mostow rigidity, and $\beta\alpha^s\beta^{-1} = \alpha^{\pm s}$. Hence there is an induced self diffeomorphism f_β of $M = N \times_{\alpha^s} S^1$. Note that $f_* = (f_\beta)_*$ in $Out(\pi)$, so f is homotopic to f_β. We cannot claim that β fixes the basepoint of N, but β preserves the closed geodesic fixed by α^s.

Now $\widehat{M} = H^3 \times_{\hat{\alpha}^s} S^1$ where $\hat{\alpha}^s$ is an elliptic rotation about an axis L, and $\hat{f}_\beta$ is fibrewise an isometry $\hat{\beta}$ preserving L. We can write $H^3 = R^2 \times L$ (non-metrically!) by considering the family of hyperplanes perpendicular to L, and then $\hat{\beta}$ is just an element of $O(2) \times E(1)$ and $\hat{\alpha}^s$ is an element of $SO(2) \times \{1\}$. The proof of Lemma 18.1, with trivial modifications, shows that, after picking coordinates and ignoring orientations, $\hat{f}_\beta$ is the identity. This completes the proof of the theorem. □

The manifolds $M(p,q,r)$ with $p^{-1} + q^{-1} + r^{-1} < 1$ are coset spaces of $\widetilde{SL}$ [Mi75]. Conversely, let K be a 2-knot obtained by surgery on the canonical cross-section of $N \times_\theta S^1$, where N is such a coset space. If θ is induced by an automorphism of $\widetilde{SL}$ which normalizes $\nu = \pi_1(N)$ then it has finite order, since $N_{\widetilde{SL}}(\nu)/\nu \cong N_{PSL(2,\mathbb{R})}(\nu/\zeta\nu)/(\nu/\zeta\nu)$. Thus if θ has infinite order we cannot expect to use such geometric arguments to analyze the question of reflexivity.

Bibliography

Books

[BPV] Barth, W., Peters, C. and Van de Ven, A. *Compact Complex Surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete 3 Folge, Bd 4, Springer-Verlag, Berlin - Heidelberg - New York - Tokyo (1984).

[Ba] Baues, H.J. *Obstruction Theory*, Lecture Notes in Mathematics 628, Springer-Verlag, Berlin - Heidelberg - New York (1977).

[Ba'] Baues, H.J. *Combinatorial Homotopy and 4-Dimensional Complexes*, De Gruyter Expositions in Mathematics 2, Walter De Gruyter, Berlin - New York (1991).

[Bi] Bieri, R. *Homological Dimension of Discrete Groups*, Queen Mary College Mathematics Notes, London (1976).

[B-Z] Brown, H., Bülow, R., Neubüser, J., Wondratschek, H. and Zassenhaus, H. *Crystallographic Groups of Four-Dimensional Space*, John Wiley and Sons Inc., New York - London - Sydney - Toronto (1978).

[BZ] Burde, G. and Zieschang, H. *Knots*, de Gruyter Studies in Mathematics 5, W. de Gruyter, Berlin - New York (1985).

[Cb] Cobb, R. *Infrasolvmanifolds of Dimension Four*, PhD thesis, University of Sydney (1999).

[Co] Cohen, M.M. *A Course in Simple Homotopy Theory*, Graduate Texts in Mathematics 10, Springer-Verlag, Berlin - Heidelberg - New York (1973).

[Cn] Cohn, P.M. *Skew Fields. Theory of General Division Rings*, Cambridge University Press (1995).

[De] Dekimpe. K. *Almost-Bieberbach Groups: Affine and Polynomial Structures*, Lecture Notes in Mathematics 1639, Springer-Verlag, Berlin - Heidelberg - New York (1996).

[DD] Dicks, W. and Dunwoody, M.J. *Groups acting on Graphs*, Cambridge studies in advanced mathematics 17, Cambridge University Press, Cambridge - New York - Melbourne (1989).

[FJ] Farrell, F.T. and Jones, L.E. *Classical Aspherical Manifolds*, CBMS Regional Conference Series 75, American Mathematical Society, Providence (1990).

[Fi] Filipkiewicz, R.O. *Four-Dimensional Geometries*, Ph.D thesis, University of Warwick (1984).

[FQ] Freedman, M.H. and Quinn, F. *Topology of 4-Manifolds*, Princeton University Press, Princeton (1990).

[Go] Goldman, W.M. *Complex Hyperbolic Geometry*, Oxford Mathematical Monographs, Oxford University Press, Oxford - New York (1999).

[GS] Gompf, R. and Stipsicz, A. *4-Manifolds and Kirby Calculus*, Graduate Studies in Mathematics 20, American Mathematical Society, Providence (1999).

[GK] Gordon, C. McA. and Kirby, R.C. (editors) *Four-Manifold Theory*, CONM 35, American Mathematical Society, Providence (1984).

[Gr] Gromov, M. *Asymptotic Invariants of Infinite Groups*, London Mathematical Society Lecture Note Series 182, Cambridge University Press, Cambridge - New York - Melbourne (1993).

[Hm] Hempel, J. *3-Manifolds*, Annals of Mathematics Study 86, Princeton University Press, Princeton (1976).

[Hn] Hendriks, H. *Applications de la théorie d'obstruction en dimension 3*, Bull. Soc. Math. France Memoire 53 (1977), 1-86.

[H1] Hillman, J.A. *2-Knots and their Groups*, Australian Mathematical Society Lecture Series 5, Cambridge University Press, Cambridge - New York - Melbourne (1989).

[H2] Hillman, J.A. *The Algebraic Characterization of Geometric 4-Manifolds*, London Mathematical Society Lecture Note Series 198, Cambridge University Press, Cambridge - New York - Melbourne (1994).

[H3] Hillman, J.A. *Algebraic Invariants of Links*, Series on Knots and Everything, Vol. 32, World Scientific Publishing Co, Singapore (2002).

[JS] Jaco, W. and Shalen, P.B. *Seifert fibred spaces in 3-manifolds*, MEMO 220, American Mathematical Society, Providence (1979).

[KS] Kirby, R.C. and Siebenmann, L. *Foundational Essays on Topological Manifolds, Smoothings, and Triangulations*, Annals of Mathematics Study 88, Princeton University Press, Princeton (1977).

[La] Laudenbach, F. *Topologie de la Dimension Trois: Homotopie et Isotopie*, Astérisque 12 (1974).

[Lu] Lück, W. *L^2-Invariants: Theory and Applications to Geometry and K-Theory*, Volume 44, Ergebnisse der Mathematik und ihrer Grenzgebiete 3 Folge, Springer-Verlag, Berlin - Heidelberg - New York (2002).

[Ma] Margulis, G. *Discrete Subgroups of Semisimple Lie Groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete 3 Folge Bd. 17, Springer-Verlag, Berlin - Heidelberg - New York (1991).

[Mc] McCleary, J. *User's Guide to Spectral Sequences*, Mathematics Lecture Series 12, Publish or Perish, Inc., Wilmington (1985).

[Mi] Milnor, J.W. *Introduction to Algebraic K-Theory*, Annals of Mathematics Study 72, Princeton University Press, Princeton (1971).

[Mo] Montesinos, J.M. *Classical Tessellations and Three-Manifolds*, Springer-Verlag, Berlin - Heidelberg - New York (1987).

[Pa] Passman, D.S. *The Algebraic Structure of Group Rings*, John Wiley and Sons Inc., New York - London - Sydney - Toronto (1977).

[Pi] Pier, J. *Amenable Locally Compact Groups*, John Wiley and Sons Inc., New York - London - Sydney - Toronto (1984).

[Rg] Raghunathan, M.S. *Discrete Subgroups of Lie Groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete 2 Folge, Bd 68, Springer-Verlag, Berlin - Heidelberg - New York (1972).

[Rn] Ranicki, A. *Algebraic L-Theory and Topological Manifolds*, Cambridge University Press, Cambridge - New York - Melbourne (1992).

[Rt] Ratcliffe, J.G. *Foundations of Hyperbolic Manifolds*, Graduate Texts in Mathematics 149, Springer-Verlag, Berlin - Heidelberg - New York (1994).

[Ro] Robinson, D.J.S. *A Course in the Theory of Groups*, Graduate Texts in Mathematics 80, Springer-Verlag, Berlin - Heidelberg - New York (1982).

[Se] Serre, J.P. *Linear Representations of Finite Groups*, Graduate Texts in Mathematics 42, Springer-Verlag, Berlin - Heidelberg - New York (1977).

[Sp] Spanier, E.H. *Algebraic Topology*, McGraw-Hill, New York (1966).

[St] Stallings, J.R. *Group Theory and Three-Dimensional Manifolds*, Yale Mathematical Monographs 4, Yale University Press, New Haven - London (1971).

[Su] Sunder, V.S. *An Invitation to Von Neumann Algebras*, Springer-Verlag, Berlin - Heidelberg - New York (1987).

[Te] Teichner, P. *Topological Four-Manifolds with Finite Fundamental Group*, Verlag Shaker, Aachen (1992).

[Th] Thomas, C.B. *Elliptic Structures on 3-Manifolds*, London Mathematical Society Lecture Note Series 104, Cambridge University Press, Cambridge - New York - Melbourne (1986).

[Vi] Vignéras, M.-F. *Arithmétique des Algèbres de Quaternions*, Lecture Notes in Mathematics 800, Springer-Verlag, Berlin - Heidelberg - New York (1980).

[Wl] Wall, C.T.C. *Surgery on Compact Manifolds*, Academic Press, New York - London (1970).

[Wo] Wolf, J.A. *Spaces of Constant Curvature*, fifth edition, Publish or Perish Inc., Wilmington (1984).

[Zi] Zieschang, H. *Finite Groups of Mapping Classes of Surfaces*, Lecture Notes in Mathematics 875, Springer-Verlag, Berlin - Heidelberg - New York (1981).

[ZVC] Zieschang, H., Vogt, E. and Coldewey, H.D. *Surfaces and Planar Discontinuous Groups*, Lecture Notes in Mathematics 835, Springer-Verlag, Berlin - Heidelberg - New York (1980).

Journal articles

[AR84] Aitchison, I.R. and Rubinstein, J.H. Fibered knots and involutions on homotopy spheres, in [GK], 1-74.

[ABR92] Arapura, D., Bressler, P. and Ramachandran, M. On the fundamental group of a compact Kähler manifold, Duke Math. J. 68 (1993), 477-488.

[AFR97] Aravinda, C.S., Farrell, F.T. and Roushon, S.K. Surgery groups of knot and link complements, Bull. London Math. Soc. 29 (1997), 400-406.

[Ar25] Artin, E. Zur Isotopie Zwei-dimensionaler Flächen im R_4, Abh. Math. Sem. Univ. Hamburg 4 (1925), 174-177.

[AY81] Asano, K. and Yoshikawa, K. On polynomial invariants of fibred 2-knots, Pacific J. Math. 97 (1981), 267-269.

[At76] Atiyah, M.F. Elliptic operators, discrete groups and von Neumann algebras, Astérisque 32/33 (1976), 43-72.

[Au73] Auslander, L. An exposition of the structure of solvmanifolds, I, Bull. Amer. Math. Soc. 79 (1973), 227-261; II, *ibid.* 262-285.

[AJ76] Auslander, L. and Johnson, F.E.A. On a conjecture of C.T.C.Wall, J. London Math. Soc. 14 (1976), 331-332.

[BH72] Baer, R. and Heineken, H. Radical groups of finite abelian section rank, Illinois J. Math. 16 (1972), 533-589.

[Ba75] Bak, A. Odd dimension surgery groups of odd torsion groups vanish, Topology 14 (1975), 367-374.

[Ba80] Barge, J. An algebraic proof of a theorem of J.Milnor, in *Proceedings of the Topology Symposium, Siegen, 1979*, (edited by U. Koschorke and W.D.Neumann), Lecture Notes in Mathematics 788, Springer-Verlag, Berlin - Heidelberg - New York (1980), 396-398.

[Ba80'] Barge, J. Dualité dans les revêtements galoisiens, Inventiones Math. 58 (1980), 101-106.

[Ba88] Bauer, S. The homotopy type of a four-manifold with finite fundamental group, in *Algebraic Topology and Transformation Groups* (edited by T. tom Dieck), Lecture Notes in Mathematics 1361, Springer-Verlag, Berlin - Heidelberg - New York (1988), 1-6.

[BHK81] Bayer, E., Hillman, J.A. and Kearton, C. The factorization of simple knots, Math. Proc. Camb. Phil. Soc. 90 (1981), 495-506.

[Be84] Bedient, R.E. Double branched covers and pretzel knots, Pacific J. Math. 112 (1984), 265-272.

[BB97] Bestvina, M. and Brady, N. Morse theory and finiteness properties of groups, Inventiones Math. 129 (1997), 445-470.

[BH91] Bieri, R. and Hillman, J.A. Subnormal subgroups of 3-dimensional Poincaré duality groups, Math. Z. 206 (1991), 67-69.

[BNS87] Bieri, R., Neumann, W.D. and Strebel, R. A geometric invariant of discrete groups, Inventiones Math. 90 (1987), 451-477.

[BS78] Bieri, R. and Strebel, R. Almost finitely presentable soluble groups, Comment. Math. Helvetici 53 (1978), 258-278.

[BS79] Bieri, R. and Strebel, R. Soluble groups with coherent group rings, in *Homological Group Theory* (edited by C.T.C.Wall), London Mathematical Society Lecture Note Series 36, Cambridge University Press, Cambridge (1979), 235-240.

[BO91] Boileau, M. and Otal, J.-P. Scindements de Heegaard et groupe des homéotopies des petites variétés de Seifert, Inventiones Math. 106 (1991), 85-107.

[Bo99] Bowditch, B.H. Planar groups and the Seifert conjecture, preprint, University of Southampton (1999).

[BM94] Bowditch, B. and Mess, G. A 4-dimensional Kleinian group, Trans. Amer. Math. Soc. 344 (1994), 391-405.

[BG84] Brooks, R. and Goldman, W. Volumes in Seifert space. Duke Math. J. 51 (1984), 529-545.

[Br67] Browder, W. Diffeomorphisms of 1-connected manifolds, Trans. Amer. Math. Soc. 128 (1967), 155-163.

[Br72] Browder, W. Poincaré spaces, their normal fibrations and surgery, Inventiones Math. 17 (1972), 191-202.

[Br75] Brown, K.S. A homological criterion for finiteness, Comment. Math. Helvetici 50 (1975), 129-135.

[BG85] Brown, K.S. and Geoghegan, R. Cohomology with free coefficients of the fundamental group of a graph of groups, Comment. Math. Helvetici 60 (1985), 31-45.

[BM70] Burde, G. and Murasugi, K. Links and Seifert fibre spaces, Duke Math. J. 37 (1970), 89-93.

[Ca90] Cairns, G. Compact 4-manifolds that admit totally umbilic metric foliations, in *Differential Geometry and its Applications (Brno 1989)* (edited by Josef Januška and Demeter Krupka), World Scientific Publishing Inc., Teaneck N.J. (1990), 9-16.

[Ca73] Cappell, S.E. Mayer-Vietoris sequences in Hermitean K-theory, in *Hermitean K-Theory and Geometric Applications* (edited by H.Bass), Lecture Notes in Mathematics 343, Springer-Verlag, Berlin - Heidelberg - New York (1973), 478-512.

[Ca76] Cappell, S.E. A splitting theorem for manifolds, Inventiones Math. 33 (1976), 69-170.

[CS71] Cappell, S.E. and Shaneson, J.L. On four-dimensional surgery and applications, Comment. Math. Helvetici 46 (1971), 500-528.

[CS76] Cappell, S.E. and Shaneson, J.L. There exist inequivalent knots with the same complement, Ann. Math. 103 (1976), 349-353.

[CS76'] Cappell, S.E. and Shaneson, J.L. Some new four-manifolds, Ann. Math. 104 (1976), 61-72.

[CS85] Cappell, S.E. and Shaneson, J.L. On four-dimensional s-cobordisms, J.Diff. Geom 22 (1985), 97-115.

[CG86] Cheeger, J. and Gromov, M. L_2-Cohomology and group cohomology, Topology 25 (1986), 189-215.

[CH96] Cobb, R. and Hillman, J.A. S^2-bundles over aspherical closed surfaces are geometric, Proc. Amer. Math. Soc. 125 (1997), 3415-3422.

[Co83] Cochran, T.D. Ribbon knots in S^4, J. London Math. Soc. 28 (1983), 563-576.

[Co84] Cochran, T.D. Slice links in S^4, Trans. Amer. Math. Soc. 285 (1984), 389-401.

[CH90] Cochran, T.D. and Habegger, N. On the homotopy theory of simply-connected four-manifolds, Topology 29 (1990), 419-440.

[CR77] Conner, P.E. and Raymond, F. Deforming homotopy equivalences to homeomorphisms in aspherical manifolds, Bull. Amer. Math. Soc. 83 (1977), 36-85.

[CZ79] Cox, D.A. and Zucker, S. Intersection numbers of sections of elliptic surfaces, Inventiones Math. 53 (1979), 1-44.

[Cr00] Crisp, J.S. The decomposition of 3-dimensional Poincaré complexes, Comment. Math. Helvetici 75 (2000), 232-246.

[Cr61] Crowell, R.H. Corresponding group and module sequences, Nagoya Math. J. 19 (1961), 27-40.

[CS83] Culler, M. and Shalen, P.B. Varieties of group representations and splittings of three-manifolds, Ann. Math. 117 (1983), 109-146.

[Da95] Davis, J. The Borel/Novikov conjectures and stable diffeomorphisms of 4-manifolds, preprint, Johannes Gutenberg-Universität Mainz (1995).

[DM85] Davis, J.F. and Milgram, R.J. A survey of the spherical space form problem, Math. Reports 2(1985), 223-283.

[Da83] Davis, M.W. Groups generated by reflections and aspherical manifolds not covered by Euclidean space, Ann. Math. 117 (1983), 293-325.

[Da85] Davis, M.W. A hyperbolic 4-manifold, Proc. Amer. Math. Soc. 93 (1985), 325-328.

[Da98] Davis, M.W. The cohomology of a Coxeter group with group ring coefficients, Duke Math. J. 91 (1998), 397-314.

[DJ91] Davis, M.W. and Januszkiewicz, T. Hyperbolization of polyhedra, J. Diff. Geom. 34 (1991), 347-388.

[De97] Dekimpe, K. Determining the translational part of the fundamental group of an infrasolvmanifold of type (R), Math. Proc. Camb. Phil. Soc. 122 (1997), 515-524.

[Du83] Dunbar, W. Geometric orbifolds, Revista Matematica de la Universidad Complutense de Madrid 1 (1988), 67-99.

[DF87] Dunwoody, M.J. and Fenn, R.A. On the finiteness of higher knot sums, Topology 26 (1987), 337-343.

[DST96] Dwyer, W.G., Stolz, S and Taylor, L.R. On the dimension of infinite covers, Proc. Amer. Math. Soc. 124 (1996), 2235-2239.

[DV73] Dyer, E. and Vasquez, A.T. The sphericity of higher dimensional knots, Canadian J. Math. 25 (1973), 1132-1136.

[DV73'] Dyer, E. and Vasquez, A.T. Some small aspherical spaces, J. Aust. Math. Soc. 16 (1973), 332-352.

[Dy87] Dyer, M. Localization of group rings and applications to 2-complexes, Comment. Math. Helvetici 62 (1987), 1-17.

[Dy87'] Dyer, M. Euler characteristics of groups, Quarterly J. Math. Oxford 38 (1987), 35-44.

[EE69] Earle, C. and Eells, J. A fibre bundle description of Teichmüller theory, J. Diff. Geom. 3 (1969), 19-43.

[EO94] Ebeling, W. and Okonek, C. Homology Hopf surfaces, Compositio Math. 91 (1994), 271-304.

[Eb80] Eberlein, P. Lattices in spaces of nonpositive curvature, Ann. Math. 111 (1980), 435-476.

[Ec76] Eckmann, B. Aspherical manifolds and higher-dimensional knots, Comment. Math. Helvetici 51 (1976), 93-98.

[Ec86] Eckmann, B. Cyclic homology of groups and the Bass conjecture, Comment. Math. Helvetici 61 (1986), 193-202.

[Ec92] Eckmann, B. Amenable groups and Euler characteristic, Comment. Math. Helvetici 67 (1992), 383-393.

[Ec94] Eckmann, B. Manifolds of even dimension with amenable fundamental group, Comment. Math. Helvetici 69 (1994), 501-511.

[Ec96] Eckmann, B. Projective and Hilbert modules over group algebras, and finitely dominated spaces, Comment. Math. Helvetici 71 (1996), 453-462. Addendum, *ibid.* 72 (1996), 329.

[EM82] Eckmann, B. and Müller, H. Plane motion groups and virtual Poincaré duality groups of dimension two, Inventiones Math. 69 (1982), 293-310.

[En98] Endo, H. A construction of surface bundles over surfaces with nonzero signature, Osaka J. Math. 35 (1998), 915-930.

[Et01] Etgü, T. Lefschetz fibrations, complex structures and Seifert fibrations on $S^1 \times M^3$, Alg. Geom. Top. 1 (2001), 469-489.

[EJ73] Evans, B. and Jaco, W. Varieties of groups and 3-manifolds, Topology 12 (1973), 83-97.

[Fa77] Farber, M.S. Duality in an infinite cyclic covering and even-dimensional knots, Math. USSR Izvestija 11 (1977), 749-781.

[Fa71] Farrell, F.T. The obstruction to fibering a manifold over a circle, Indiana Univ. J. Math. 21 (1971), 315-346.

[Fa74] Farrell, F.T. The second cohomology group of G with coefficients $Z/2Z[G]$, Topology 13 (1974), 313-326.

[Fa75] Farrell, F.T. Poincaré duality and groups of type FP, Comment. Math. Helvetici 50 (1975), 187-195.

[FJ86] Farrell, F.T. and Jones, L.E. K-Theory and dynamics.I, Ann. Math. 124 (1986), 531-569.

[FJ89] Farrell, F.T. and Jones, L.E. A topological analogue of Mostow's rigidity theorem, J. Amer. Math. Soc. 2 (1989), 257-370.

[FJ93] Farrell, F.T. and Jones, L.E. Isomorphism conjectures in algebraic K-theory, J. Amer. Math. Soc. 6 (1993), 249-297.

[FJ93'] Farrell, F.T. and Jones, L.E. Topological rigidity for compact nonpositively curved manifolds, in *Proceedings of Symposia in Pure Mathematics 54*, Part 3, American Mathematical Society (1993), 229-274.

[FJ97] Farrell, F.T. and Jones, L.E. Compact infrasolvmanifolds are smoothly rigid, in *Geometry from the Pacific Rim* (edited by A.J.Berrick and H.-Y. Wang), W.de Gruyter (1997), 85-97.

[FS93] Farrell, F.T. and Stark, C.W. Compact spherical-Euclidean spaceform groups of infinite VCD, Bull. London Math. Soc. 25 (1993), 189-192.

[Fi78] Fintushel, R. Classification of circle actions on 4-manifolds, Trans. Amer. Math. Soc. 242 (1978), 377-390.

[Fo62] Fox, R.H. A quick trip through knot theory, in *Topology of 3-Manifolds and Related Topics* (edited by M.K.Fort, Jr), Prentice-Hall, Englewood Cliffs, N.J.(1962), 120-167.

[Fo66] Fox, R.H. Rolling, Bull. Amer. Math. Soc. 72 (1966), 162-164.

[FQ80] Freedman, M.H. and Quinn, F. A quick proof of the 4-dimensional stable surgery theorem, Comment. Math. Helvetici 55 (1980), 668-671.

[FT95] Freedman, M.H. and Teichner, P. 4-Manifold topology I: subexponential groups, Inventiones Math. 122 (1995), 509-529.

[Ga97] Gabai, D. On the geometric and topological rigidity of hyperbolic 3-manifolds, J. Amer. Math. Soc. 10 (1997), 37-74.

[GMT96] Gabai, D., Meyerhoff, G.R. and Thurston, N. Homotopy hyperbolic 3-manifolds are hyperbolic, Ann. Math., to appear (preprint, MSRI 1996-058).

[Ga00] Gaboriau, D. Sur la (co-)homologie L^2 des actions préservant une mesure, C.R. Acad. Sci. Paris 330 (2000), 365-370.

[Ge92] Geiges, H. Symplectic structures on T^2-bundles over T^2, Duke Math. J. 67 (1992), 539-555.

[GG95] Geiges, H. and Gonzalo, J. Contact geometry and complex surfaces I, Inventiones Math. 121 (1995), 147-209.

[GM86] Geoghegan, R. and Mihalik, M.L. A note on the vanishing of $H^n(G;Z[G])$, J. Pure Appl. Alg. 39 (1986), 301-304.

[Gi79] Gildenhuys, D. Classification of soluble groups of cohomological dimension two, Math. Z. 166 (1979), 21-25.

[GS81] Gildenhuys, D. and Strebel, R. On the cohomological dimension of soluble groups, Canad. Math. Bull. 24 (1981), 385-392.

[Gl62] Gluck, H. The embedding of two-spheres in the four-sphere, Trans. Amer. Math. Soc. 104 (1962), 308-333.

[GK78] Goldsmith, D. and Kauffman, L.H. Twist spinning revisited, Trans. Amer. Math. Soc. 239 (1978), 229-251.

[GA94] González-Acuña, F. A characterization of 2-knot groups, Revista Mat. Iberoamericana 10 (1994), 221-228.

[GM78] González-Acuña, F. and Montesinos, J.M. Ends of knot groups, Ann. Math. 108 (1978), 91-96.

[Go71] Gorbatsevich, V.V. Discrete subgroups of solvable Lie groups of type R, Math. USSR Sbornik 14 (1971), 233-251.

[Go76] Gordon, C. McA. Knots in the 4-sphere, Comment. Math. Helvetici 51 (1976), 585-596.

[Go81] Gordon, C. McA. Ribbon concordance of knots in the 3-sphere, Math. Ann. 257 (1981), 157-170.

[GL84] Gordon, C. McA. and Litherland, R.A. Incompressible surfaces in branched coverings, in *The Smith Conjecture* (edited by J. Morgan and H. Bass), Academic Press, New York - London (1984), 139-152.

[GL89] Gordon, C. McA. and Luecke, J, Knots are determined by their complements, J. Amer. Math. Soc. 2 (1989), 371-415.

[Go65] Gottlieb, D.H. A certain subgroup of the fundamental group, Amer. J. Math. 87 (1965), 840-856.

[Go68] Gottlieb, D.H. On fibre spaces and the evaluation map, Ann. Math. 87 (1968), 42-55.

[Go79] Gottlieb, D.H. Poincaré duality and fibrations, Proc. Amer. Math. Soc. 76 (1979), 148-150.

[Gr98] Grigorchuk, R.I. An example of a finitely presentable amenable group which is not in the class EG, Mat. Sbornik 198 (1998), 79-100.

[Gu72] Gutiérrez, M.A. Boundary links and an unlinking theorem, Trans. Amer. Math. Soc. 171 (1972), 491-499.

[Ha82] Habegger, N. Une variété de dimension 4 avec forme d'intersection paire et signature -8, Comment. Math. Helvetici 57 (1982), 22-24.

[HK88] Hambleton, I. and Kreck, M. On the classification of topological 4-manifolds with finite fundamental group, Math. Ann. 280 (1988), 85-104.

[HKT94] Hambleton, I., Kreck, M. and Teichner, P. Nonorientable four-manifolds with fundamental group of order 2, Trans. Amer. Math. Soc. 344 (1994), 649-665.

[HM86] Hambleton, I. and Madsen, I. Actions of finite groups on R^{n+k} with fixed point set R^k, Canad. J. Math. 38 (1986), 781-860.

[HM86'] Hambleton, I. and Madsen, I. Local surgery obstructions and space forms, Math. Z. 193 (1986), 191-214.

[HM78] Hambleton, I. and Milgram, R.J., Poincaré transversality for double covers, Canad. J. Math. 30 (1978), 1319-1330.

[Ha71] Harvey, W.J. Branch loci in Teichmüller space, Trans. Amer. Math. Soc. 153 (1971), 387-399.

[HK78] Hausmann, J.-C. and Kervaire, M. Sous-groupes derivés des groupes de noeuds, L'Enseignement Math, 24 (1978), 111-123.

[HK78'] Hausmann, J.-C. and Kervaire, M. Sur le centre des groupes de noeuds multidimensionelles, C.R. Acad. Sci. Paris 287 (1978), 699-702.

[HW85] Hausmann, J.-C. and Weinberger, S. Caractérisques d'Euler et groupes fondamentaux des variétés de dimension 4, Comment. Math. Helvetici 60 (1985), 139-144.

[He87] Hempel, J. Residual finiteness for 3-manifolds, in *Combinatorial Group Theory and Topology* (edited by S.M.Gersten and J.R.Stallings), Annals of Mathematics Study 111, Princeton University Press, Princeton (1987), 379-396.

[Hn77] Hendriks, H. Obstruction theory in 3-dimensional topology: an extension theorem, J. London Math. Soc. 16 (1977), 160-164. Corrigendum, *ibid.* 18 (1978), 192.

[HL74] Hendriks, H. and Laudenbach, F. Scindement d'une équivalence d'homotopie en dimension 3, Ann. Sci. Ecole Norm. Sup. 7 (1974), 203-217.

[HP96] Hersonsky, S. and Paulin, F. On the volumes of complex hyperbolic manifolds, Duke J. Math. 84 (1996), 719-737.

[Hg40] Higman, G. The units of group rings, Proc. London Math. Soc. 46 (1940), 231-248.

[Hg51] Higman, G. A finitely generated infinite simple group, J.London Math. Soc. 26 (1951), 61-64.

[Hi87] Hillman, J.A. 3-dimensional Poincaré duality groups which are extensions, Math. Z. 195 (1987), 89-92.

[Hi89] Hillman, J.A. A homotopy fibration theorem in dimension four, Topology Appl. 33 (1989), 151-161.

[Hi91] Hillman, J.A. Elementary amenable groups and 4-manifolds with Euler characteristic 0, J. Austral. Math. Soc. 50 (1991), 160-170.

[Hi93] Hillman, J.A. On 3-dimensional Poincaré duality complexes and 2-knot groups, Math. Proc. Camb. Phil. Soc. 114 (1993), 215-218.

[Hi94] Hillman, J.A. Minimal 4-manifolds for groups of cohomological dimension 2, Proc. Edinburgh Math. Soc. 37 (1994), 455-462.

[Hi94'] Hillman, J.A. Finitely dominated subnormal covers of 4-manifolds, Research Report 94-23, The University of Sydney, 1994.

[Hi95] Hillman, J.A. Free products and 4-dimensional connected sums, Bull. London Math. Soc. 27 (1995), 387-391.

[Hi95'] Hillman, J.A. Flat 4-manifold groups, New Zealand Math. J. 24 (1995), 29-40.

[Hi97] Hillman, J.A. L^2-homology and asphericity, Israel J. Math. 99 (1997), 271-283.

[Hi97'] Hillman, J.A. On the homotopy types of closed 4-manifolds covered by $S^2 \times R^2$, Topology Appl. 75 (1997), 287-295.

[Hi98] Hillman, J.A. On 4-dimensional mapping tori and product geometries, J. London Math. Soc. 58 (1998), 229-238.

[Hi02] Hillman, J.A. An homology 4-sphere group with negative deficiency, L'Enseignement Math., to appear.

[Hi03] Hillman, J.A. Tits alternatives for groups of finite cohomological dimension, J. Math. Soc. Japan, to appear

[HL92] Hillman, J.A. and Linnell, P.A. Elementary amenable groups of finite Hirsch length are locally-finite by virtually-solvable, J. Austral. Math. Soc. 52 (1992), 237-241.

[HP88] Hillman, J.A. and Plotnick, S.P. Geometrically fibred 2-knots, Math. Ann. 287 (1990), 259-273.

[HW89] Hillman, J.A. and Wilson, S.M.J. On the reflexivity of Cappell-Shaneson 2-knots, Bull. London Math. Soc. 21 (1989), 591-593.

[Hi79] Hitt, L.R. Examples of higher-dimensional slice knots which are not ribbon knots, Proc. Amer. Math. Soc. 77 (1979), 291-297.

[HR92] Holt, D.F. and Rees, S. Testing for isomorphism between finitely presented groups, in *Groups, Combinatorics and Geometry* (edited by M.W. Liebeck and J. Saxl), Cambridge University Press, Cambridge - New York - Melbourne (1992), 459-475.

[Ho77] Houghton, C.H. Cohomology and the behaviour at infinity of finitely presentable groups, J. London Math. Soc. 15 (1977), 465-471.

[Ho82] Howie, J. On locally indicable groups, Math. Z. 180 (1982), 445-461.

[Ho83] Howie, J. Some remarks on a problem of J.H.C.Whitehead, Topology 22 (1983), 475-485.

[In74] Inoue, I. On surfaces of class VII_o, Inventiones Math. 24 (1974), 269-310.

[Jo79] Johnson, F.E.A. On the realisability of poly-surface groups, J. Pure Appl. Math. 15 (1979), 235-241.

[Jo93] Johnson, F.E.A. Surface fibrations and automorphisms of non-abelian extensions, Quarterly J. Math. Oxford (2) 44 (1993), 199-214.

[Jo94] Johnson, F.E.A. A group-theoretic analogue of the Parshin-Arakelov theorem, Archiv Math. (Basel) 63 (1994), 354-361.

[Jo96] Johnson, F.E.A. Finite coverings and surface fibrations, J. Pure Appl. Algebra 112 (1996), 41-52.

[Jo99] Johnson, F.E.A. A rigidity theorem for group extensions, Archiv Math. (Basel) 73 (1999), 81-89.

[JK93] Johnson, F.E.A. and Kotschick, D. On the signature and Euler characteristic of certain four-manifolds, Math. Proc. Camb. Phil. Soc. 114 (1993), 431-438.

[KLR83] Kamashima, Y., Lee, K.B. and Raymond, F. The Seifert construction and its applications to infranilmanifolds, Quarterly J. Math. Oxford 34 (1983), 433-452.

[Kn80] Kanenobu, T. 2-knot groups with elements of finite order, Math. Sem. Notes Kobe University 8 (1980), 557-560.

[Kn83] Kanenobu, T. Groups of higher dimensional satellite knots, J. Pure Appl. Alg. 28 (1983), 179-188.

[Kn83'] Kanenobu, T. Fox's 2-spheres are twist spun knots, Mem. Fac. Sci. Kyushu University (Ser. A) 37 (1983), 81-86.

[Kn88] Kanenobu, T. Deforming twist spun 2-bridge knots of genus one, Proc. Japan Acad. (Ser. A) 64 (1988), 98-101.

[Kp79] Kaplan, S.J. Constructing framed 4-manifolds with given almost framed boundaries, Trans. Amer. Math. Soc. 254 (1979), 237-263.

[Ka98] Kapovich, M. On normal subgroups in the fundamental groups of complex surfaces, preprint, University of Utah (August, 1998).

[KK99] Kapovich, M. and Kleiner, B. Coarse Alexander duality and duality groups, preprint, University of Utah (1999).

[HL96] Kapovich, M. and Leeb, B. Actions of discrete groups on nonpositively curved spaces, Math. Ann. 306 (1996), 341-352.

[Ks68] Kas, A. On deformations of a certain type of irregular algebraic surface, Amer. J. Math. 90 (1968), 789-804.

[Kt69] Kato, M. A concordance classification of PL homeomorphisms of $S^p \times S^q$, Topology 8 (1969), 371-383.

[Kt75] Kato, M. Topology of Hopf surfaces, J. Math. Soc. Japan 27 (1975), 222-238. Erratum, *ibid.* 41 (1989), 173.

[Ke83] Kerckhoff, S.P. The Nielsen realization problem, Ann. Math. 117 (1983), 235-265.

[Ke65] Kervaire, M.A. Les noeuds de dimensions supérieures, Bull. Soc. Math. France 93 (1965), 225-271.

[Ke65'] Kervaire, M.A. On higher dimensional knots, in *Differential and Combinatorial Topology (A Symposium in Honor of Marston Morse)* (edited by S.S.Cairns), Princeton University Press, Princeton (1965), 105-109.

[Ke69] Kervaire, M.A. Smooth homology spheres and their fundamental groups, Trans. Amer. Math. Soc. 144 (1969), 67-72.

[KKR92] Kim, M.H., Kojima, S. and Raymond, F. Homotopy invariants of nonorientable 4-manifolds, Trans. Amer. Math. Soc. 333 (1992), 71-83.

[KR90] Kim, M.H. and Raymond, F. The diffeotopy group of the twisted 2-sphere bundle over the circle, Trans. Amer. Math. Soc. 322 (1990), 159-168.

[KS75] Kirby, R.C. and Siebenmann, L. Normal bundles for codimension 2 locally flat embeddings, in *Geometric Topology* (edited by L.C.Glaser and T.B.Rushing), Lecture Notes in Mathematics 438, Springer-Verlag, Berlin - Heidelberg - New York (1975), 310-324.

[Kl93] Klyachko, A. Funny property of sphere and equations over groups, Comm. Alg. 21 (1993), 2555-2575.

[Ko86] Kojima, S. Determining knots by branched covers, in *Low-dimensional Topology and Kleinian Groups* (edited by D.B.A.Epstein), London Mathematical Society Lecture Note Series 112, Cambridge University Press, Cambridge - New York - Melbourne (1986), 193-207.

[Ko92] Kotschick, D. Remarks on geometric structures on compact complex surfaces, Topology 31 (1992), 317-321.

[Ko94] Kotschick, D. Four-manifold invariants of finitely presentable groups, in *Topology, Geometry and Field Theory* (edited by K.Fukuya, M.Fukuya, T.Kohno and D.Kotschick), World Scientific Publishing Co., Teaneck, N.J. (1994), 89-99.

[Ko99] Kotschick, D. On regular fibred complex surfaces, in *Proceedings of the Kirbyfest* (edited by Joel Hass and Martin Scharlemann), Geometry and Topology Monographs, Volume 2 (1999), 291-298.

[Kr99] Kreck, M. *Surgery and Duality*, Ann. Math. 149 (1999), 707-754.

[Kr86] Kropholler, P.H. Cohomological dimensions of soluble groups, J. Pure Appl. Alg. 43 (1986), 281-287.

[Kr90] Kropholler, P.H. An analogue of the torus decomposition theorem for certain Poincaré duality groups, Proc. London Math. Soc. 60 (1990), 503-529.

[Kr90'] Kropholler, P.H. Baumslag-Solitar groups and some groups of cohomological dimension two, Comment. Math. Helvetici 65 (1990), 547-558.

[Kr90a] Kropholler, P.H. A note on centrality in 3-manifold groups, Math. Proc. Camb. Phil. Soc. 107(1990), 261-266.

[Kr93] Kropholler, P.H. A group theoretic proof of the torus theorem, in *Geometric Group Theory* vol. 1 (edited by G.A.Niblo and M.A.Roller), London Mathematical Society Lecture Note Series 181, Cambridge University Press, Cambridge - New York - Melbourne (1993), 138-158.

[KR88] Kropholler, P.H. and Roller, M.A. Splittings of Poincaré duality groups I, Math. Z. 197 (1988), 421-438.

[KR89] Kropholler, P.H. and Roller, M.A. Splittings of Poincaré duality groups III, J. London Math. Soc. 39 (1989), 271-284.

[Kw86] Kwasik, S. On low dimensional s-cobordisms, Comment. Math. Helvetici 61 (1986), 415-428.

[KS88] Kwasik, S. and Schultz, R. Desuspensions of group actions and the ribbon theorem, Topology 27 (1988), 443-457.

[KS92] Kwasik, S. and Schultz, R. Vanishing of Whitehead torsion in dimension four, Topology 31 (1992), 735-756.

[KS95] Kwasik, S. and Schultz, R. Visible surgery, four-dimensional s-cobordisms and related questions in geometric topology, K-Theory 9 (1995), 323-352.

[LS71] Lashof, R. and Shaneson, J.L. Smoothing 4-manifolds, Inventiones Math. 14 (1971), 197-210

[La79] Lawson, T.J. Trivializing 5-dimensional h-cobordisms by stabilization, Manuscripta Math. 29 (1979), 305-321.

[Le95] Lee, K.B. Infrasolvmanifolds of type (R), Quarterly J. Math. Oxford 46 (1995), 185-195.

[LS00] Leeb, B. and Scott, P. A geometric characteristic splitting in all dimensions, Comment. Math. Helvetici 75 (2000), 201-215.

[Le77] Levine, J. Knot modules. I, Trans. Amer. Math. Soc. 229 (1977), 1-50.

[Le78] Levine, J. Some results on higher dimensional knot groups, in *Knot Theory, Proceedings, Plans-sur-Bex, Switzerland 1977* (edited by J.C.Hausmann), Lecture Notes in Mathematics 685, Springer-Verlag, Berlin - Heidelberg - New York (1978), 243-269.

[LW78] Weber, C. Appendix to [Le78], *ibid.* 270-273.

[Li62] Lickorish, W.B.R. A representation of orientable combinatorial 3-manifolds, Ann. Math. 76 (1962), 531-538.

[Li79] Litherland, R.A. Deforming twist-spun knots, Trans. Amer. Math. Soc. 250 (1979), 311-331.

[Li85] Litherland, R.A. Symmetries of twist-spun knots, in *Knot Theory and Manifolds* (edited by D.Rolfsen), Lecture Notes in Mathematics 1144, Springer-Verlag, Berlin - Heidelberg - New York (1985), 97-107.

[Li96] Liu Kefeng, Geometric height inequalities, Math. Research Letters 3 (1996), 693-702.

[Lo81] Lomonaco, S.J., Jr, The homotopy groups of knots I. How to compute the algebraic 2-type, Pacific J. Math. 96 (1981), 349-390.

[LR00] Long, D.D. and Reid, A.W. On the geometric boundaries of hyperbolic 4-manifolds, Geometry and Topology 4 (2000), 171-178.

[LS00'] Lück, W. and Stamm, R. Computations of K- and L-theory of cocompact planar groups, K-Theory 21 (2000), 249-292.

[Ma79] Matsumoto, T. On homotopy equivalences of $S^2 \times RP^2$ to itself, J. Math. Kyoto University 19 (1979), 1-17.

[Ma62] Mazur, B. Symmetric homology spheres, Illinois J. Math. 6 (1962), 245-250.

[Mc02] McCullough, D. Isometries of elliptic 3-manifolds,
J. London Math. Soc. 65 (2002), 167-182.

[MS86] Meeks, W.H.,III and Scott, G.P. Finite group actions on 3-manifolds,
Inventiones Math. 86 (1986), 287-346.

[Me84] Melvin, P. 2-Sphere bundles over compact surfaces,
Proc. Amer. Math. Soc. 92 (1984), 567-572.

[Me73] Meyer, W. Die Signatür von Flächenbundeln,
Math. Ann. 201 (1973), 239-264.

[Mi86] Mihalik, M.L. Ends of double extension groups,
Topology 25 (1986), 45-53.

[Mi87] Mihalik, M.L. Solvable groups that are simply connected at ∞,
Math. Z. 195 (1987), 79-87.

[Mi67] Milgram, R.J. The bar construction and abelian H-spaces,
Illinois J. Math. 11 (1967), 241-250.

[Mi68] Milnor, J.W. Infinite cyclic coverings,
in *Conference on the Topology of Manifolds* (edited by J.G.Hocking),
Prindle, Weber and Schmidt, Boston - London - Sydney (1968), 115-133.

[Mi75] Milnor, J.W. On the 3-dimensional Brieskorn manifolds $M(p,q,r)$,
in *Knots, Groups and 3-Manifolds* (edited by L.P.Neuwirth),
Annals of Mathematics Study 84,
Princeton University Press, Princeton (1975), 175-225.

[Mo73] Montesinos, J.M. Variedades de seifert que son recubricadores ciclicos ramificados de dos hojas, Boletino Soc. Mat. Mexicana 18 (1973), 1-32.

[Mo83] Montesinos, J.M. On twins in the 4-sphere I,
Quarterly J. Math. 34 (1983), 171-199;

[Mo84] Montesinos, J.M. On twins in the 4-sphere II,
Quarterly J. Math. 35 (1984), 73-84.

[Ms68] Mostow, G.D. Quasi-conformal mappings in n-space and the rigidity of hyperbolic space forms, Publ. Math. I.H.E.S. 34 (1968), 53-104.

[Mu65] Murasugi, K. On the center of the group of a link,
Proc. Amer. Math. Soc. 16 (1965), 1052-1057.
Corrigendum *ibid.* 18 (1967), 1142.

[Ne83] Neumann, W.D. Geometry of quasihomogeneous surface singularities,
in *Singularities* (edited by P.Orlik),
Proceedings of Symposia in Pure Mathematics 40,
American Mathematical Society, Providence (1983), 245-258.

[NR78] Neumann, W.D. and Raymond, F. Seifert manifolds, plumbing, μ-invariant and orientation reversing homeomorphisms,
in *Algebraic and Geometric Topology* (edited by K.C.Millet),
Lecture Notes in Mathematics 664,
Springer-Verlag, Berlin - Heidelberg - New York (1978), 163-196.

[NS85] Nicas, A. and Stark, C.W. K-Theory and surgery of codimension-two torus actions on aspherical manifolds, J. London Math. Soc. 31 (1985), 173-183.

[Ni98] Nimersheim, B. All flat three-manifolds appear as cusps of hyperbolic four-manifolds, Topology Appl. 90 (1998), 109-133.

[No64] Novikov, S.P. Homotopically equivalent smooth manifolds, Izv. AN SSSR 28 (1964), 365-474.

[Oh90] Ohba, K. On fiber bundles over S^1 having small Seifert manifolds as fibers, J. Fac. Sci. Tokyo 37 (1990), 659-702.

[OS01] Ol'shanskii, A.Yu. and Sapir, M.V. Non-amenable finitely presented torsion-by-cyclic groups, Electronic Res. Ann. AMS 7 (2001), 63-71.

[Ol53] Olum, P. Mappings of manifolds and the notion of degree, Ann. Math. 58 (1953), 458-480.

[Pa78] Pao, P. Non-linear circle actions on the 4-sphere and twisting spun knots, Topology 17 (1978), 291-296.

[Pa57] Papakyriakopolous, C. On Dehn's lemma and the asphericity of knots, Ann. Math. 66 (1957), 1-26.

[Pa68] Parshin, A.N. Algebraic curves over function fields, Math. USSR Izvestija 2 (1968), 1145-1170.

[Pa96] Patrangenaru, V. Classifying 3 and 4 dimensional homogeneous riemannian manifolds by Cartan triples, Pacific J. Math. 173 (1996), 511-532.

[Pe98] Pearson, K. Algebraic K-theory of two-dimensional crystallographic groups, K-Theory 14 (1998), 265-280.

[Pl80] Plotnick, S.P. Vanishing of Whitehead groups for Seifert manifolds with infinite fundamental group, Comment. Math. Helvetici 55 (1980), 654-667.

[Pl82] Plotnick, S.P. Homotopy equivalences and free modules, Topology 21 (1982), 91-99.

[Pl83] Plotnick, S.P. The homotopy type of four-dimensional knot complements, Math. Z. 183 (1983), 447-471.

[Pl83'] Plotnick, S.P. Infinitely many disc knots with the same exterior, Math Proc. Camb. Phil. Soc. 93 (1083), 67-72.

[Pl84] Plotnick, S.P. Finite group actions and non-separating 2-spheres, Proc. Amer. Math. Soc. 90 (1984), 430-432.

[Pl84'] Plotnick, S.P. Fibered knots in S^4 - twisting, spinning, rolling, surgery, and branching, in [GK], 437-459.

[Pl86] Plotnick, S.P. Equivariant intersection forms, knots in S^4, and rotations in 2-spheres, Trans. Amer. Math. Soc. 296 (1986), 543-575.

[PS85] Plotnick, S.P. and Suciu, A.I. k-invariants of knotted 2-spheres, Comment. Math. Helvetici 60 (1985), 54-84.

[PS87] Plotnick, S.P. and Suciu, A.I. Fibered knots and spherical space forms, J. London Math. Soc. 35 (1987), 514-526.

[Po71] Poenaru, V. A note on the generators for the fundamental group of the complement of a submanifold of codimension 2, Topology 10 (1971), 47-52.

[Pr43] Preissmann, A. Quelques propriéteés globales des espaces de Riemann, Comment. Math. Helvetici 15 (1943), 175-216.

[Qi93] Qin, Z. Complex structures on certain differentiable 4-manifolds, Topology 32 (1993), 551-566.

[Qu83] Quinn, F. The stable topology of 4-manifolds, Topology Appl. 15 (1983), 71-77.

[Rn86] Ranicki, A. Algebraic and geometric splittings of the K- and L-groups of polynomial extensions, in *Transformation Groups, Poznan 1985* (edited by S.Jackowski and K.Pawalowski), Lecture Notes in Mathematics 1217, Springer-Verlag, Berlin - Heidelberg - New York (1986), 321-363.

[Rn95] Ranicki, A. Finite domination and Novikov rings, Topology 34 (1995), 619-632.

[Rp60] Rapaport, E.S. On the commutator subgroup of a knot group, Ann. Math. 71 (1960), 157-162.

[Rt83] Ratcliffe, J.G. A fibred knot in a homology 3-sphere whose group is non-classical, in *Low Dimensional Topology* (edited by S.J.Lomonaco, Jr), CONM 20, American Mathematical Society, Providence (1983), 327-339.

[RT01] Ratcliffe, J.G. and Tschantz, S.T. On the Davis hyperbolic 4-manifold, Topology Appl. 111 (2001), 327-342.

[Ro75] Robinson, D.J.S. On the cohomology of soluble groups of finite rank, J. Pure Appl. Alg. 6 (1975), 155-164.

[Ro77] Romanovskii, N.S. Free subgroups of finitely presentable groups, Algebra i Logika 16 (1977), 88-97.

[Ro84] Rosset, S. A vanishing theorem for Euler characteristics, Math. Z. 185 (1984), 211-215.

[Ro00] Roushon, S.K. L-Theory of 3-manifolds with nonvanishing first Betti number, Internat. Math. Res. Notices 2000, No 3, 107-113.

[Ru84] Rubermann, D. Invariant knots of free involutions of S^4, Topology Appl. 18 (1984), 217-224.

[Ru90] Rubermann, D. Seifert surfaces of knots in S^4, Pacific J. Math. 145 (1990), 97-116.

[RS97] Rubermann, D. and Stern, R. A fake smooth $CP^2\sharp RP^4$, Math. Research Letters 4 (1997), 375-379.

[Sa81] Sakuma, M. Periods of composite links, Math. Sem. Notes Kobe University 9 (1981), 445-452.

[Sc73] Scott, G.P. Compact submanifolds of 3-manifolds, J. London Math. Soc. 7 (1973), 437-440.

[Sc76] Scott, G.P. Normal subgroups in 3-manifold groups,
J. London Math. Soc. 13 (1976), 5-12.

[Sc83] Scott, G.P. There are no fake Seifert fibre spaces with infinite π_1,
Ann. Math. 117 (1983), 35-70.

[Sc83'] Scott, G.P. The geometries of 3-manifolds,
Bull. London Math. Soc. 15 (1983), 401-487.

[Sc85] Scott, G.P. Homotopy implies isotopy for some Seifert fibre spaces,
Topology 24 (1985), 341-351.

[Sh97] Shiga, H. On monodromies of holomorphic families of Riemann surfaces and modular transformations, Math. Proc. Camb. Phil. Soc. 122 (1997), 541-549.

[Sh63] Shimizu, H. On discontinuous groups operating on the product of upper half planes, Ann. Math. 77 (1963), 33-71.

[Si71] Siebenmann, L.C. Topological manifolds,
in *Proceedings of the International Congress of Mathematicians, Nice, 1970*, vol. 2,
Gauthier-Villars, Paris (1971), 133-163. *See* [KS].

[Si67] Siegel, J. Higher order cohomology operations in local coefficient theory,
American J. Math. 89 (1967), 909-931.

[Si91] Silver, D. Examples of 3-knots with no minimal Seifert manifolds,
Math. Proc. Camb. Phil. Soc. 110 (1991), 417-420.

[Si96] Silver, D. HNN bases and high-dimensional knots,
Proc. Amer. Math. Soc. 124 (1996), 1247-1252.

[Si97] Silver, D. Free group automorphisms and knotted tori in S^4,
J. Knot Theory R. 6 (1997), 95-103.

[Si80] Simon, J. Wirtinger approximations and the knot groups of F^n in S^{n+2},
Pacific J. Math. 90 (1980), 177-190.

[Sp49] Specker, E. Die erste Cohomologiegruppe von Überlagerungen und Homotopie Eigenschaften dreidimensionaler Mannigfaltigkeiten,
Comment. Math. Helvetici 23 (1949), 303-333.

[St65] Stallings, J. Homology and central series of groups,
J. Algebra 2 (1965), 170-181.

[St84] Stark, C.W. Structure sets vanish for certain bundles over Seifert manifolds,
Trans. Amer. Math. Soc. 285 (1984), 603-615.

[St87] Stark, C.W. L-Theory and graphs of free abelian groups,
J. Pure Appl. Alg. 47 (1987), 299-309.

[St76] Strebel, R. A homological finiteness condition,
Math. Z. 151 (1976), 263-275.

[St77] Strebel, R. A remark on subgroups of infinite index in Poincaré duality groups, Comment. Math. Helvetici 52 (1977), 317-324.

[Su85] Suciu, A. Infinitely many ribbon knots with the same fundamental group, Math. Proc. Camb. Phil. Soc. 98 (1985), 481-492.

[Su71] Sumners, D.W. Invertible knot cobordisms, Comment. Math. Helvetici 46 (1971), 240-256.

[Su76] Suzuki, S. Knotting problems of 2-spheres in the 4-sphere, Math. Sem. Notes Kobe University 4 (1976), 241-371.

[Sw60] Swan, R.G. Periodic resolutions for finite groups, Ann. Math. 72 (1960), 267-291.

[Sw73] Swarup, G.A. On embedded spheres in 3-manifolds, Math. Ann. 203 (1973), 89-102.

[Sw74] Swarup, G.A. On a theorem of C.B.Thomas, J. London Math. Soc. 8 (1974), 13-21.

[Sw75] Swarup, G.A. An unknotting criterion, J. Pure Appl. Alg. 6 (1975), 191-296.

[Sw77] Swarup, G.A. Relative version of a theorem of Stallings, J. Pure Appl. Alg. 11 (1977), 75-82.

[Te97] Teichner, P. On the star-construction for topological 4-manifolds, in *Geometric Topology* (edited by W.H.Kazez), American Mathematical Society (1997), part 1, 300-312.

[Tl94] Teleman, A.-D. Projectively flat surfaces and Bogomolov's theorem on class VII_0 surfaces, Internat. J. Math. 5 (1994), 253-264.

[Tr89] Teragaito, M. Fibered 2-knots and lens spaces, Osaka J. Math. 26 (1989), 57-63.

[Tr90] Teragaito, M. Twisting symmetry-spins of pretzel knots, Proc. Japan Acad. 66 (1990), 179-183.

[Th77] Thomas, C.B. On Poincaré 3-complexes with binary polyhedral fundamental group, Math. Ann. 226 (1977), 207-221.

[Th80] Thomas, C.B. Homotopy classification of free actions by finite groups on S^3, Proc. London Math. Soc. 40 (1980), 284-297.

[Th76] Thurston, W.P. Some simple examples of symplectic manifolds, Proc. Amer. Math. Soc. 55 (1976), 467-468.

[Tr86] Trace, B. A note concerning Seifert manifolds for 2-knots, Math. Proc. Camb. Phil. Soc. 100 (1986), 113-116.

[Tr78] Trotter, H.F. Knot modules and Seifert matrices, in *Knot Theory, Proceedings, Plans-sur-Bex, Switzerland 1977*, (edited by J.-C.Hausmann), Lecture Notes in Mathematics 685, Springer-Verlag, Berlin - Heidelberg - New York (1978),291-299.

[Ts80] Tsukiyama, K. Self-homotopy-equivalences of a space with two nonvanishing homotopy groups, Proc. Amer. Math. Soc. 79 (1980), 134-139.

[Tu90] Turaev, V.G. Three-dimensional Poincaré complexes: homotopy classification and splitting, Math. USSR Sbornik 67 (1990), 261-282.

[Ue90] Ue, M. Geometric 4-manifolds in the sense of Thurston and Seifert 4-manifolds I, J. Math. Soc. Japan 42 (1990), 511-540.

[Ue91] Ue, M. Geometric 4-manifolds in the sense of Thurston and Seifert 4-manifolds II, J. Math. Soc. Japan 43 (1991), 149-183.

[Wd67] Waldhausen, F. Eine Klasse von 3-dimensionaler Mannigfaltigkeiten I, Inventiones Math. 3 (1967), 308-333; II, *ibid.* 4 (1967), 87-117.

[Wd78] Waldhausen, F. Algebraic K-theory of generalized free products, Ann. Math. 108 (1978), 135-256.

[Wl64] Wall, C.T.C. On simply connected 4-manifolds, J. London Math. Soc. 39 (1964), 141-149.

[Wl65] Wall, C.T.C. Finiteness conditions for CW-complexes I, Ann. Math. 81 (1965), 56-69.

[Wl67] Wall, C.T.C. Poincaré complexes: I, Ann. Math. 86 (1967), 213-245.

[Wl76] Wall, C.T.C. Classification of hermitian forms VI. Group rings, Ann. Math. 103 (1976), 1-80.

[Wl78] Wall, C.T.C. Free actions of finite groups on spheres, in *Algebraic and Geometric Topology, Proceedings of Symposia in Pure Mathematics XXXII*, (edited by R.J.Milgram), American Mathematical Society, Providence (1978), Part 1, 115-124.

[Wl85] Wall, C.T.C. Geometries and geometric structures in real dimension 4 and complex dimension 2, in *Geometry and Topology* (edited by J.Alexander and J.Harer), Lecture Notes in Mathematics 1167, Springer-Verlag, Berlin - Heidelberg - New York (1985), 268-292.

[Wl86] Wall, C.T.C. Geometric structures on complex analytic surfaces, Topology 25 (1986), 119-153.

[Wa72] Wang, H.C. Topics in totally disconnected groups, in *Symmetric Spaces* (edited by W.M.Boothby and G.L.Weiss), Marcel Dekker, New York (1972), 460-485.

[We83] Weinberger, S. The Novikov conjecture and low-dimensional topology, Comment. Math. Helvetici 58 (1983), 355-364.

[We87] Weinberger, S. On fibering four- and five-manifolds, Israel J. Math. 59 (1987), 1-7.

[Yj69] Yajima, T. On a characterization of knot groups of some spheres in R^4, Osaka J. Math. 6 (1969), 435-446.

[Ym93] Yamanoshita,T. On the space of self homotopy equivalences of the projective plane, J. Math. Soc. Japan 45 (1993), 489-494.

[Yn69] Yanagawa, T. On ribbon 2-knots - the 3-manifold bounded by the 2-knots, Osaka J. Math. 6 (1969), 447-464.

[Yn77] Yanagawa, T. On cross sections of higher dimensional ribbon knots, Math. Sem. Notes Kobe University 7 (1977), 609-628.

[Yo80] Yoshikawa, K. On 2-knot groups with the finite commutator subgroup, Math. Sem. Notes Kobe University 8 (1980), 321-330.

[Yo82] Yoshikawa, K. On a 2-knot group with nontrivial center, Bull. Austral. Math. Soc. 25 (1982), 321-326.

[Yo82'] Yoshikawa, K. A note on Levine's condition for knot groups, Math. Sem. Notes Kobe University 10 (1982), 633-636.

[Yo86] Yoshikawa, K. Knot groups whose bases are abelian, J. Pure Appl. Alg. 40 (1986), 321-335.

[Yo88] Yoshikawa, K. A ribbon knot which has no free base, Proc. Amer. Math. Soc. 102 (1988), 1065-1070.

[Yo92] Yoshikawa, K. Certain abelian subgroups of two-knot groups, in *Knots 90* (edited by A.Kawauchi), W. de Gruyter, Berlin - New York (1992), 231-240.

[Yo97] Yoshikawa, K. The centres of fibered two-knot groups, AMS/IP Studies in Advanced Math. 2 (1997), 473-477.

[Ze65] Zeeman, E.C. On twisting spun knots, Trans. Amer. Math. Soc. 115 (1965), 471-495.

[Zn79] Zimmermann, B. Periodische Homöomorphismen Seifertscher Faserräume, Math. Z. 166 (1979), 287-297.

[Zn80] Zimmermann, B. Uber Gruppen von Homöomorphismen Seifertscher Faserräume und flacher Mannigfaltigkeiten, Manuscripta Math. 30 (1980), 361-373.

[Zn82] Zimmermann, B. Das Nielsensche Realisierungsproblem für hinreichend grosse 3-Mannigfaltigkeiten, Math. Z. 180 (1982), 349-359.

[Zn85] Zimmermann, B. Zur Klassifikation höherdimensionaler Seifertscher Faserräume, in *Low Dimensional Topology* (edited by R.Fenn), London Mathematical Society Lecture Note Series 95, Cambridge University Press, Cambridge (1985), 214-255.

[Zn86] Zimmermann, B. Finite group actions on Haken 3-manifolds, Quarterly J. Math. Oxford 37 (1986), 499-511.

[Zn90] Zimmermann, B. On the Hantzsche-Wendt manifold, Monatshefte Math. 110 (1990), 321-327.

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Greek characters

Non-alphabetic symbols