Vladikavkaz Mathematical Journal
2022, Volume 24, Issue 3, P. 21–36

УДК 517.95
DOI 10.46698/x4972-4013-9236-n

EXISTENCE OF GLOBAL CLASSICAL SOLUTIONS FOR THE SAINT-VENANT EQUATIONS

R. Azib[1], S. Georgiev[2], A. Kheloufi[1] and K. Mebarki[1]

[1] Bejaia University, Bejaia 06000, Algeria;
[2] Sofia University "St. Kliment Ohridski",
15 Tzar Osvoboditel Blvd., Sofia 1504, Bulgaria

E-mail: `riyadh.azib@univ-bejaia.dz`, `azibriyadh@gmail.com`, `svetlingeorgiev1@gmail.com`, `arezki.kheloufi@univ-bejaia.dz`, `arezkinet2000@yahoo.fr`, `karima.mebarki@univ-bejaia.dz`, `mebarqi_karima@hotmail.fr`

Abstract. Nowadays, investigations of the existence of global classical solutions for non linear evolution equations is a topic of active mathematical research. In this article, we are concerned with a classical system of shallow water equations which describes long surface waves in a fluid of variable depth. This system was proposed in 1871 by Adhémar Jean-Claude Barré de Saint-Venant. Namely, we investigate an initial value problem for the one dimensional Saint-Venant equations. We are especially interested in question of what sufficient conditions the initial data and the topography of the bottom must verify in order that the considered system has global classical solutions. In order to prove our main results we use a new topological approach based on the fixed point abstract theory of the sum of two operators in Banach spaces. This basic and new idea yields global existence theorems for many of the interesting equations of mathematical physics.

Key words: Saint-Venant equations, classical solution, fixed point, initial value problem.

AMS Subject Classification: 35Q35, 35A09, 35E15.

For citation: *Azib, R., Georgiev, S., Kheloufi, A. and Mebarki, K.* Existence of Global Classical Solutions for the Saint-Venant Equations, *Vladikavkaz Math. J.*, 2022, vol. 24, no. 3, pp. 21–36. DOI: 10.46698/x4972-4013-9236-n.

1. Introduction

One of the most important works in the study of nonlinear science is the investigation of the existence of global classical solutions. In this paper, we investigate the Cauchy problem for a classical system of shallow water equations which describes long surface waves in a fluid of variable depth. This system was proposed in 1871 by Adhémar Jean-Claude Barré de Saint-Venant [1]. Namely, we consider the following initial value problem for the Saint-Venant equations:

$$
\begin{gathered}
\partial_t u + \partial_x (uv) = 0, \quad t \in (0,\infty),\ x \in \mathbb{R}, \\
\partial_t (uv) + \partial_x \left(uv^2 + \frac{1}{2} k u^2 \right) + k u \partial_x f(t,x) = 0, \quad t \in (0,\infty),\ x \in \mathbb{R}, \\
u(0,x) = u_0(x), \quad x \in \mathbb{R}, \\
v(0,x) = v_0(x), \quad x \in \mathbb{R},
\end{gathered}
\tag{1.1}
$$

where $k \in \mathbb{R}_+$ represents the gravitational constant, the initial conditions u_0, v_0 and the topography of the bottom f are given functions. Here the unknowns $u = u(t,x)$ and $v = v(t,x)$ denote respectively, the depth and the average horizontal velocity of the fluid.

There are many papers devoted to the study of shallow water flows especially by numerical methods, we can cite [2–7] and the references therein. Let us also mention some works related to the mathematical analysis of Saint-Venant equations. The correct formulation of the boundary conditions for the Saint-Venant system for simulating closed and open basins is investigated in [8]. Local and global temporal existence of classical solutions for the dissipative shallow water equations is considered in [9] and [10], respectively. In [11], the uniqueness of the classical solution of the system of 1-D Saint-Venant equations is proved. In [12], an initial-boundary value problem for a system of Saint-Venant equations is solved by two methods. In [13], Li *et al.* proved the local-in-time well-posedness of classical solutions with finite total mass and/or energy, allowing the surface height goes to zero in the far field. In [14] and [15], Cheng *et al.* proved long-time and global existence and asymptotic behavior of classical solutions for two dimensional rotating shallow water system. In [16] classical solutions for the Cauchy problem of the rotating shallow water equations with physical viscosity are obtained. In [17], local-time well-posedness and breakdown for solutions of regularized Saint-Venant equations are established. In [18], global solutions to Saint-Venant equations are investigated by a method of an additional argument. In [19], under regular initial data with small energy but possibly large oscillations, the global well-posedness of classical solution for Cauchy problem of two-dimensional chemotaxis-shallow water system is studied. In the note [20], a mathematical analysis, based on the theory of semigroups of operators on Hilbert space, of a linearized problem involving the Saint-Venant equations is given.

In this paper, we are especially interested in question of what conditions the initial data u_0, v_0 and the topography of the bottom f must verify in order that Problem (1.1) has classical solutions. Here, by a classical solution to the Saint-Venant equations we mean a solution which is along with its derivatives that appear in the equations of class $\mathscr{C}([0,\infty)\times\mathbb{R})$. In other words, (u,v) belongs to the space $\mathscr{C}^1([0,\infty)\times\mathbb{R})\times\mathscr{C}^1([0,\infty)\times\mathbb{R})$ of continuously differentiable functions on $[0,\infty)\times\mathbb{R}$. The main assumptions on the functions u_0, v_0 and f are

(H1) $u_0, v_0 \in \mathscr{C}^1(\mathbb{R})$, $0 < u_0 \leqslant B$, $0 \leqslant v_0 \leqslant B$ on $\mathbb{R}$ for some positive constant B.

(H2) $f \in \mathscr{C}([0,\infty), \mathscr{C}^1(\mathbb{R}))$, $0 \leqslant |\partial_x f| \leqslant B$ on $[0,\infty)\times\mathbb{R}$.

Our approach is based on the use of the fixed point theory for the sum of operators in Banach spaces.

The paper is organized as follows. In the next section, we describe a new topological approach which uses fixed point abstract theory of the sum of two operators. In Section 3 we give some properties of solutions of Problem (1.1). These properties concern integral representation of the solutions and some energy estimates to be made more precise later on. In Section 4 we prove existence and multiplicity of solutions for the Saint-Venant system (1.1). Finally, in Section 5 an example illustrating our main results will be given.

2. On Fixed Points for the Sum of Two Operators

In this section, some definitions and results related to fixed point theory will be given. We will recall two approaches for the existence of fixed points for the sum of two operators.

2.1. First approach.

Theorem 2.1. *Let E be a Banach space and*

$$E_1 = \{x \in E : \|x\| \leqslant R\},$$

with $R > 0$. Consider two operators T and S, where

$$Tx = -\epsilon x, \ x \in E_1,$$

with $\epsilon > 0$ and $S : E_1 \to E$ be continuous and such that

(i) *$(I - S)(E_1)$ resides in a compact subset of E and*

(ii) *$\{x \in E : x = \lambda(I - S)x, \ \|x\| = R\} = \emptyset$, for any $\lambda \in \left(0, \frac{1}{\epsilon}\right)$. Then there exists $x^* \in E_1$ such that*

$$Tx^* + Sx^* = x^*.$$

Theorem 2.1 will be used to prove Theorem 4.1 and its proof can be found in [21]. Let us recall the proof of Theorem 2.1 for convenience.

$\lhd$ Define

$$r\left(-\frac{1}{\epsilon}x\right) = \begin{cases} -\frac{1}{\epsilon}\, x, & \text{if } \|x\| \leqslant R\epsilon, \\ \frac{Rx}{\|x\|}, & \text{if } \|x\| > R\epsilon. \end{cases}$$

Then $r\left(-\frac{1}{\epsilon}(I - S)\right) : E_1 \to E_1$ is continuous and compact. Hence and the Schauder fixed point theorem, it follows that there exists $x^* \in E_1$ so that

$$r\left(-\frac{1}{\epsilon}(I - S)x^*\right) = x^*.$$

Assume that $-\frac{1}{\epsilon}(I - S)x^* \notin E_1$. Then

$$\|(I - S)x^*\| > R\epsilon, \quad \frac{R}{\|(I - S)x^*\|} < \frac{1}{\epsilon}$$

and

$$x^* = \frac{R}{\|(I - S)x^*\|}(I - S)x^* = r\left(-\frac{1}{\epsilon}(I - S)x^*\right)$$

and hence, $\|x^*\| = R$. This contradicts the condition (ii). Therefore $-\frac{1}{\epsilon}(I - S)x^* \in E_1$ and

$$x^* = r\left(-\frac{1}{\epsilon}(I - S)x^*\right) = -\frac{1}{\epsilon}(I - S)x^*$$

or

$$-\epsilon x^* + Sx^* = x^*,$$

or

$$Tx^* + Sx^* = x^*.$$

This completes the proof. $\rhd$

2.2. Second approach. Let E be a real Banach space.

Definition 2.1. A closed, convex set $\mathscr{P}$ in E is said to be *cone* if

1) $\alpha x \in \mathscr{P}$ for any $\alpha \geqslant 0$ and for any $x \in \mathscr{P}$,

2) $x, -x \in \mathscr{P}$ implies $x = 0$.

Definition 2.2. A mapping $K : E \to E$ is said to be *completely continuous* if it is continuous and maps bounded sets into relatively compact sets.

Definition 2.3. Let X and Y be real Banach spaces. A mapping $K : X \to Y$ is said to be *expansive* if there exists a constant $h > 1$ such that

$$\|Kx - Ky\|_Y \geqslant h\|x - y\|_X$$

for any $x, y \in X$.

The following result (its proof relies on [22, Proposition 2.16]) will be used to prove Theorem 4.2.

Theorem 2.2. *Let $\mathscr{P}$ be a cone of a Banach space E; Ω a subset of $\mathscr{P}$ and U_1, U_2 and U_3 three open bounded subsets of $\mathscr{P}$ such that $\overline{U}_1 \subset \overline{U}_2 \subset U_3$ and $0 \in U_1$. Assume that $T : \Omega \to \mathscr{P}$ is an expansive mapping, $S : \overline{U}_3 \to E$ is a completely continuous and $S(\overline{U}_3) \subset (I-T)(\Omega)$. Suppose that $(U_2 \setminus \overline{U}_1) \cap \Omega \neq \varnothing$, $(U_3 \setminus \overline{U}_2) \cap \Omega \neq \varnothing$, and there exists $w_0 \in \mathscr{P}\setminus\{0\}$ such that the following conditions hold:*

(i) *$Sx \neq (I-T)(x-\lambda w_0)$, for all $\lambda > 0$ and $x \in \partial U_1 \cap (\Omega + \lambda w_0)$;*

(ii) *there exists $\varepsilon > 0$ such that $Sx \neq (I-T)(\lambda x)$, for all $\lambda \geqslant 1+\varepsilon$, $x \in \partial U_2$ and $\lambda x \in \Omega$;*

(iii) *$Sx \neq (I-T)(x-\lambda w_0)$, for all $\lambda > 0$ and $x \in \partial U_3 \cap (\Omega + \lambda w_0)$.*

Then $T+S$ has at least two non-zero fixed points $x_1, x_2 \in \mathscr{P}$ such that

$$x_1 \in \partial U_2 \cap \Omega \quad \text{and} \quad x_2 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega$$

or

$$x_1 \in (U_2 \setminus U_1) \cap \Omega \quad \text{and} \quad x_2 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega.$$

3. Integral Representation and *a Priori* Estimates for Solutions of Problem (1.1)

Let $X = X^1 \times X^1$, where $X^1 = \mathscr{C}^1([0,\infty) \times \mathbb{R})$. For $(u,v) \in X$, define the operators S_1^1, S_1^2 and S_1 as follows.

$$S_1^1(u,v)(t,x) = \int_0^x (u(t,x_1) - u_0(x_1))\,dx_1 + \int_0^t u(t_1,x)v(t_1,x)\,dt_1, \quad (t,x) \in [0,\infty) \times \mathbb{R},$$

$$\begin{aligned} S_1^2(u,v)(t,x) = \int_0^x &(u(t,x_1)v(t,x_1) - u_0(x_1)v_0(x_1))\,dx_1 \\ &+ \int_0^t \left(u(t_1,x)(v(t_1,x))^2 + \frac{1}{2}k(u(t_1,x))^2 \right) dt_1 \\ &\qquad + k \int_0^t \int_0^x u(t_1,x_1)\,\partial_x f(t_1,x_1)\,dx_1\,dt_1, \quad (t,x) \in [0,\infty) \times \mathbb{R}, \end{aligned}$$

$$S_1(u,v)(t,x) = \left(S_1^1(u,v)(t,x), S_1^2(u,v)(t,x)\right), \quad (t,x) \in [0,\infty) \times \mathbb{R}.$$

Lemma 3.1. *Suppose that* (H1) *and* (H2) *are satisfied. If $(u,v) \in X$ satisfies the equation*

$$S_1(u,v)(t,x) = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}, \tag{3.1}$$

then (u,v) is a solution of the IVP (1.1).

$\lhd$ Let $(u,v) \in X$ be a solution to the equation (3.1). Then

$$S_1^1(u,v)(t,x) = 0, \quad S_1^2(u,v)(t,x) = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}. \tag{3.2}$$

We differentiate the first equation of (3.2) with respect to t and x and we find

$$\partial_t u(t,x) + \partial_x(uv)(t,x) = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}.$$

We put $t = 0$ in the first equation of (3.2) and we arrive at

$$\int_0^x (u(0,x_1) - u_0(x_1))\, dx_1 = 0, \quad x \in \mathbb{R},$$

which we differentiate with respect to x and we find

$$u(0,x) = u_0(x), \quad x \in \mathbb{R}.$$

Now, we differentiate the second equation of (3.2) with respect to t and x and we find

$$\partial_t(uv)(t,x) + \partial_x \left(uv^2 + \frac{1}{2}ku^2 \right)(t,x) + ku(t,x)\partial_x f(t,x) = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}.$$

We put $t = 0$ in the second equation of (3.2) and we get

$$\int_0^x (u(0,x_1)v(0,x_1) - u_0(x_1)v_0(x_1))\, dx_1 = 0, \quad x \in \mathbb{R},$$

which we differentiate with respect to x and we obtain

$$u(0,x)v(0,x) - u_0(x)v_0(x) = 0, \quad x \in \mathbb{R},$$

whereupon

$$v(0,x) = v_0(x), \quad x \in \mathbb{R}.$$

Thus, (u,v) is a solution to the IVP (1.1). This completes the proof. $\rhd$

Lemma 3.2. *Suppose that* (H1) *and* (H2) *are satisfied. Let* $h \in \mathscr{C}([0,\infty) \times \mathbb{R})$ *be a positive function almost everywhere on* $[0,\infty) \times \mathbb{R}$. *If* $(u,v) \in X$ *satisfies the following integral equations:*

$$\int_0^t \int_0^x (t-t_1)^2(x-x_1)^2 h(t_1,x_1) S_1^1(u,v)(t_1,x_1)\, dx_1\, dt_1 = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R},$$

and

$$\int_0^t \int_0^x (t-t_1)^2(x-x_1)^2 h(t_1,x_1) S_1^2(u,v)(t_1,x_1)\, dx_1\, dt_1 = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R},$$

then (u,v) *is a solution to the IVP* (1.1).

$\lhd$ We differentiate three times with respect to t and three times with respect to x the integral equations of Lemma 3.2 and we find

$$h(t,x)S_1(u,v)(t,x) = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R},$$

whereupon

$$S_1(u,v)(t,x) = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}.$$

Hence and Lemma 3.1, we conclude that (u,v) is a solution to the IVP (1.1). This completes the proof. $\rhd$

Now, let us prove some *a priori* estimates related to solutions of Problem (1.1). In the sequel, $X = X^1 \times X^1$, where $X^1 = \mathscr{C}^1([0,\infty)\times\mathbb{R})$ will be endowed with the norm

$$\|(u,v)\| = \max\{\|u\|_{X^1},\, \|v\|_{X^1}\}, \quad (u,v)\in X,$$

with

$$\|u\|_{X^1} = \max\left\{ \sup_{(t,x)\in[0,\infty)\times\mathbb{R}} |u(t,x)|,\ \sup_{(t,x)\in[0,\infty)\times\mathbb{R}} |u_t(t,x)|,\ \sup_{(t,x)\in[0,\infty)\times\mathbb{R}} |u_x(t,x)| \right\},$$

provided it exists. Let

$$B_1 = \max\left\{2B, B^2\left(1+\frac{1}{2}k\right), kB^2, 2B^2\right\}.$$

Lemma 3.3. *Under hypothesis* (H1) *and* (H2) *and for* $(u,v)\in X$ *with* $\|(u,v)\|\leqslant B$, *the following estimates hold:*

$$\left|S_1^1(u,v)(t,x)\right| \leqslant B_1(1+t)(1+|x|), \quad (t,x)\in[0,\infty)\times\mathbb{R},$$

and

$$\left|S_1^2(u,v)(t,x)\right| \leqslant B_1(1+t)(1+|x|), \quad (t,x)\in[0,\infty)\times\mathbb{R}.$$

$\lhd$ Suppose that (H1) and (H2) are satisfied and let $(u,v)\in X$ with $\|(u,v)\|\leqslant B$.

(i) Estimation of $|S_1^1(u,v)(t,x)|$, $(t,x)\in[0,\infty)\times\mathbb{R}$:

$$\begin{aligned}
\left|S_1^1(u,v)(t,x)\right| &= \left|\int_0^x (u(t,x_1)-u_0(x_1))\,dx_1 + \int_0^t u(t_1,x)v(t_1,x)\,dt_1\right| \\
&\leqslant \left|\int_0^x (|u(t,x_1)|+u_0(x_1))\,dx_1\right| + \int_0^t |u(t_1,x)||v(t_1,x)|\,dt_1 \\
&\leqslant 2B|x| + B^2 t \leqslant B_1(t+|x|) \leqslant B_1(1+t)(1+|x|),
\end{aligned}$$

(ii) Estimation of $|S_1^2(u,v)(t,x)|$, $(t,x)\in[0,\infty)\times\mathbb{R}$:

$$\begin{aligned}
\left|S_1^2(u,v)(t,x)\right| &= \Bigg|\int_0^x (u(t,x_1)v(t,x_1)-u_0(x_1)v_0(x_1))\,dx_1 \\
&+ \int_0^t \left(u(t_1,x)(v(t_1,x))^2 + \frac{1}{2}k(u(t_1,x))^2\right)dt_1 + k\int_0^t\int_0^x u(t_1,x_1)\partial_x f(t_1,x_1)\,dx_1\,dt_1\Bigg| \\
&\leqslant \left|\int_0^x (|u(t,x_1)||v(t,x_1)|+u_0(x_1)v_0(x_1))\,dx_1\right| + \int_0^t \left(|u(t_1,x)|(v(t_1,x))^2+\frac{1}{2}k(u(t_1,x))^2\right)dt_1 \\
&+ k\int_0^t \left|\int_0^x |u(t_1,x_1)||\partial_x f(t_1,x_1)|\,dx_1\right| dt_1 \leqslant 2B^2|x| + \left(B^2+\frac{1}{2}kB^2\right)t + kB^2 t|x| \\
&\leqslant B_1(t+|x|+t|x|) \leqslant B_1(1+|x|)(1+t).
\end{aligned}$$

This completes the proof. $\rhd$

Suppose

(H3) there exist a positive constant A and a nonnegative function $g \in \mathscr{C}([0,\infty)\times\mathbb{R})$ such that

$$8(1+t)\left(1+t+t^2\right)(1+|x|)\left(1+|x|+x^2\right)\int_0^t\left|\int_0^x g(t_1,x_1)\,dx_1\right|dt_1 \leqslant A, \quad (t,x)\in[0,\infty)\times\mathbb{R}.$$

In the last section, we will give an example for a function g that satisfies (H3). For $(u,v)\in X$, define the operators S_2^1, S_2^2 and S_2 as follows.

$$S_2^1(u,v)(t,x) = \int_0^t\int_0^x (t-t_1)^2(x-x_1)^2 g(t_1,x_1) S_1^1(u,v)(t_1,x_1)\,dx_1\,dt_1, \quad (t,x)\in[0,\infty)\times\mathbb{R},$$

$$S_2^2(u,v)(t,x) = \int_0^t\int_0^x (t-t_1)^2(x-x_1)^2 g(t_1,x_1) S_1^2(u,v)(t_1,x_1)\,dx_1\,dt_1, \quad (t,x)\in[0,\infty)\times\mathbb{R},$$

and

$$S_2(u,v)(t,x) = \left(S_2^1(u,v)(t,x), S_2^2(u,v)(t,x)\right), \quad (t,x)\in[0,\infty)\times\mathbb{R}. \tag{3.3}$$

Lemma 3.4. *Under hypothesis* (H1), (H2) *and* (H3) *and for* $(u,v)\in X$, *with* $\|(u,v)\|\leqslant B$, *the following estimate holds:*

$$\|S_2(u,v)\| \leqslant AB_1.$$

$\lhd$ Suppose that (H1), (H2) and (H3) are satisfied. Let $(u,v)\in X$, with $\|(u,v)\|\leqslant B$.

(i) Estimation of $|S_2^1(u,v)(t,x)|$, $(t,x)\in[0,\infty)\times\mathbb{R}$:

$$\begin{aligned}
\left|S_2^1(u,v)(t,x)\right| &= \left|\int_0^t\int_0^x (t-t_1)^2(x-x_1)^2 g(t_1,x_1) S_1^1(u,v)(t_1,x_1)\,dx_1\,dt_1\right| \\
&\leqslant \int_0^t\left|\int_0^x (t-t_1)^2(x-x_1)^2 g(t_1,x_1)|S_1^1(u,v)(t_1,x_1)|\,dx_1\right|dt_1 \\
&\leqslant B_1\int_0^t\left|\int_0^x (t-t_1)^2(x-x_1)^2 g(t_1,x_1)(1+t_1)(1+|x_1|)\,dx_1\right|dt_1 \\
&\leqslant 4B_1(1+t)t^2(1+|x|)x^2\int_0^t\left|\int_0^x g(t_1,x_1)\,dx_1\right|dt_1 \\
&\leqslant 8B_1(1+t)(1+t+t^2)(1+|x|)(1+|x|+x^2)\int_0^t\left|\int_0^x g(t_1,x_1)\,dx_1\right|dt_1 \leqslant AB_1.
\end{aligned}$$

(ii) Estimation of $\left|\frac{\partial}{\partial t}S_2^1(u,v)(t,x)\right|$, $(t,x)\in[0,\infty)\times\mathbb{R}$:

$$
\begin{aligned}
\left|\frac{\partial}{\partial t}S_2^1(u,v)(t,x)\right| &= 2\left|\int_0^t\int_0^x (t-t_1)(x-x_1)^2 g(t_1,x_1)S_1^1(u,v)(t_1,x_1)\,dx_1\,dt_1\right| \\
&\leqslant 2\int_0^t\left|\int_0^x (t-t_1)(x-x_1)^2 g(t_1,x_1)\left|S_1^1(u,v)(t_1,x_1)\right|\,dx_1\right|dt_1 \\
&\leqslant 2B_1\int_0^t\left|\int_0^x (t-t_1)(x-x_1)^2 g(t_1,x_1)(1+t_1)(1+|x_1|)\,dx_1\right|dt_1 \\
&\leqslant 8B_1(1+t)t(1+|x|)x^2\int_0^t\left|\int_0^x g(t_1,x_1)\,dx_1\right|dt_1 \\
&\leqslant 8B_1(1+t)(1+t+t^2)(1+|x|)(1+|x|+x^2)\int_0^t\left|\int_0^x g(t_1,x_1)\,dx_1\right|dt_1\leqslant AB_1.
\end{aligned}
$$

(iii) Estimation of $\left|\frac{\partial}{\partial x}S_2^1(u,v)(t,x)\right|$, $(t,x)\in[0,\infty)\times\mathbb{R}$:

$$
\begin{aligned}
\left|\frac{\partial}{\partial x}S_2^1(u,v)(t,x)\right| &= 2\left|\int_0^t\int_0^x (t-t_1)^2(x-x_1) g(t_1,x_1)S_1^1(u,v)(t_1,x_1)\,dx_1\,dt_1\right| \\
&\leqslant 2\int_0^t\left|\int_0^x (t-t_1)^2|x-x_1| g(t_1,x_1)|S_1^1(u,v)(t_1,x_1)|\,dx_1\right|dt_1 \\
&\leqslant 2B_1\int_0^t\left|\int_0^x (t-t_1)^2|x-x_1| g(t_1,x_1)(1+t_1)(1+|x_1|)\,dx_1\right|dt_1 \\
&\leqslant 4B_1(1+t)t^2(1+|x|)|x|\int_0^t\left|\int_0^x g(t_1,x_1)\,dx_1\right|dt_1 \\
&\leqslant 8B_1(1+t)(1+t+t^2)(1+|x|)(1+|x|+x^2)\int_0^t\left|\int_0^x g(t_1,x_1)\,dx_1\right|dt_1\leqslant AB_1.
\end{aligned}
$$

As above,

$$
\left|S_2^2(u,v)(t,x)\right|,\quad \left|\frac{\partial}{\partial t}S_2^2(u,v)(t,x)\right|,\quad \left|\frac{\partial}{\partial x}S_2^2(u,v)(t,x)\right|\leqslant AB_1,\quad (t,x)\in[0,\infty)\times\mathbb{R}.
$$

Therefore

$$
\|S_2(u,v)\|\leqslant AB_1.
$$

This completes the proof. $\rhd$

4. Existence and Multiplicity of Nonnegative Solutions

In the sequel, suppose that the constants B and A which appear in the conditions (H1) and (H3), respectively, satisfy the following inequality:

(H4) $AB_1 < B$, where $B_1 = \max\left\{2B, B^2\left(1+\frac{1}{2}k\right), kB^2, 2B^2\right\}$.

Our first main result for existence of classical solutions of the IVP (1.1) is as follows.

Theorem 4.1. *Under hypotheses* (H1), (H2), (H3) *and* (H4), *the IVP* (1.1) *has at least one nonnegative solution* $(u,v) \in \mathscr{C}^1([0,\infty)\times\mathbb{R}) \times \mathscr{C}^1([0,\infty)\times\mathbb{R})$.

$\lhd$ Choose $\epsilon \in (0,1)$, such that $\epsilon B_1(1+A) < B$. For $(u,v) \in X = \mathscr{C}^1([0,\infty)\times\mathbb{R}) \times \mathscr{C}^1([0,\infty)\times\mathbb{R})$, we will write

$$(u,v) \geqslant 0 \quad \text{if} \quad u(t,x) \geqslant 0 \quad \text{and} \quad v(t,x) \geqslant 0 \quad \text{for any} \quad (t,x) \in [0,\infty)\times\mathbb{R}.$$

Let $\tilde{\tilde{\tilde{Y}}}$ denotes the set of all equi-continuous families in X with respect to the norm $\|\cdot\|$, $\tilde{\tilde{Y}} = \overline{\tilde{\tilde{\tilde{Y}}}}$ be the closure of $\tilde{\tilde{\tilde{Y}}}$, $\widetilde{Y} = \tilde{\tilde{Y}} \cup \{(u_0, v_0)\}$ and

$$Y = \left\{(u,v) \in \widetilde{Y}:\ (u,v) \geqslant 0,\ \|(u,v)\| \leqslant B\right\}.$$

Note that Y is a compact set in X. For $(u,v) \in X$, define the operators

$$T(u,v)(t,x) = -\epsilon(u,v)(t,x), \quad (t,x) \in [0,\infty)\times\mathbb{R},$$

$$S(u,v)(t,x) = (u,v)(t,x) + \epsilon(u,v)(t,x) + \epsilon S_2(u,v)(t,x), \quad (t,x) \in [0,\infty)\times\mathbb{R}.$$

For $(u,v) \in Y$ and by using Lemma 3.4, it follows that

$$\|(I-S)(u,v)\| = \|\epsilon(u,v) - \epsilon S_2(u,v)\| \leqslant \epsilon\|(u,v)\| + \epsilon\|S_2(u,v)\| \leqslant \epsilon B_1 + \epsilon A B_1 = \epsilon B_1(1+A) < B.$$

Thus, $S : Y \to X$ is continuous and $(I-S)(Y)$ resides in a compact subset of X. Now, suppose that there is a $(u,v) \in X$ so that $\|(u,v)\| = B$ and

$$(u,v) = \lambda(I-S)(u,v)$$

or

$$\frac{1}{\lambda}(u,v) = (I-S)(u,v) = -\epsilon(u,v) - \epsilon S_2(u,v),$$

or

$$\left(\frac{1}{\lambda} + \epsilon\right)(u,v) = -\epsilon S_2(u,v)$$

for some $\lambda \in \left(0, \frac{1}{\epsilon}\right)$. Hence, $\|S_2(u,v)\| \leqslant AB_1 < B$,

$$\epsilon B < \left(\frac{1}{\lambda} + \epsilon\right) B = \left(\frac{1}{\lambda} + \epsilon\right)\|(u,v)\| = \epsilon\|S_2(u,v)\| < \epsilon B,$$

which is a contradiction. Hence and Theorem 2.1, it follows that the operator $T+S$ has a fixed point $(u^*, v^*) \in Y$. Therefore

$$\begin{aligned}(u^*,v^*)(t,x) = T(u^*,v^*)(t,x) + S(u^*,v^*)(t,x) = -\epsilon(u^*,v^*)(t,x) \\ + (u^*,v^*)(t,x) + \epsilon(u^*,v^*)(t,x) + \epsilon S_2(u^*,v^*)(t,x), \quad (t,x) \in [0,\infty)\times\mathbb{R},\end{aligned}$$

whereupon

$$0 = S_2(u^*, v^*)(t, x), \quad (t, x) \in [0, \infty) \times \mathbb{R}.$$

Lemma 3.2 yields that (u^*, v^*) is a solution to the IVP (1.1). This completes the proof. $\rhd$

In the sequel, suppose that the constants B and A which appear in the conditions (H1) and (H3), respectively, satisfy the following inequality:

(H5) $AB_1 < \frac{L}{5}$, where $B_1 = \max\left\{2B, B^2\left(1 + \frac{1}{2}k\right), kB^2, 2B^2\right\}$ and L is a positive constant that satisfies the following conditions:

$$r < L < R_1 \leqslant B, \quad R_1 > \left(\frac{2}{5m} + 1\right) L,$$

with r and R_1 are positive constants and $m > 0$ is large enough.

Our second main result for existence and multiplicity of classical solutions of the IVP (1.1) is as follows.

Theorem 4.2. *Under Hypotheses* (H1), (H2), (H3) *and* (H5), *the IVP* (1.1) *has at least two nonnegative solutions* $(u_1, v_1), (u_2, v_2) \in \mathscr{C}^1([0, \infty) \times \mathbb{R}) \times \mathscr{C}^1([0, \infty) \times \mathbb{R})$.

$\lhd$ Set $X = \mathscr{C}^1([0, \infty) \times \mathbb{R}) \times \mathscr{C}^1([0, \infty) \times \mathbb{R})$ and let

$$\widetilde{P} = \{(u, v) \in X : \ (u, v) \geqslant 0 \text{ on } [0, \infty) \times \mathbb{R}\}.$$

With $\mathscr{P}$ we will denote the set of all equi-continuous families in $\widetilde{P}$. For $(u, v) \in X$, define the operators T_1 and S_3 as follows:

$$T_1(u, v)(t, x) = (1 + m\epsilon)(u, v)(t, x) - \left(\epsilon\frac{L}{10}, \epsilon\frac{L}{10}\right), \quad (t, x) \in [0, \infty) \times \mathbb{R},$$

$$S_3(u, v)(t, x) = -\epsilon S_2(u, v)(t, x) - m\epsilon(u, v)(t, x) - \left(\epsilon\frac{L}{10}, \epsilon\frac{L}{10}\right), \quad (t, x) \in [0, \infty) \times \mathbb{R},$$

where ϵ is a positive constant, $m > 0$ is large enough and the operator S_2 is given by formula (3.3). Note that any fixed point $(u, v) \in X$ of the operator $T_1 + S_3$ is a solution to the IVP (1.1). Now, let us define

$$U_1 = \mathscr{P}_r = \{(u, v) \in \mathscr{P} : \ \|(u, v)\| < r\},$$

$$U_2 = \mathscr{P}_L = \{(u, v) \in \mathscr{P} : \ \|(u, v)\| < L\},$$

$$U_3 = \mathscr{P}_{R_1} = \{(u, v) \in \mathscr{P} : \ \|(u, v)\| < R_1\},$$

$$\Omega = \overline{\mathscr{P}_{R_2}} = \{(u, v) \in \mathscr{P} : \ \|(u, v)\| \leqslant R_2\}, \qquad R_2 = R_1 + \frac{A}{m}B_1 + \frac{L}{5m}.$$

1) For $(u_1, v_1), (u_2, v_2) \in \Omega$, we have

$$\|T_1(u_1, v_1) - T_1(u_2, v_2)\| = (1 + m\epsilon)\|(u_1, v_1) - (u_2, v_2)\|,$$

whereupon $T_1 : \Omega \to X$ is an expansive operator with a constant $h = 1 + m\epsilon > 1$.

2) Let $(u, v) \in \overline{\mathscr{P}_{R_1}}$, then Lemma 3.4 yields

$$\|S_3(u, v)\| \leqslant \epsilon\|S_2(u, v)\| + m\epsilon\|(u, v)\| + \epsilon\frac{L}{10} \leqslant \epsilon\left(AB_1 + mR_1 + \frac{L}{10}\right).$$

Therefore $S_3(\overline{\mathscr{P}_{R_1}})$ is uniformly bounded. Since $S_3 : \overline{\mathscr{P}_{R_1}} \to X$ is continuous, we have that $S_3(\overline{\mathscr{P}_{R_1}})$ is equi-continuous. Consequently $S_3 : \overline{\mathscr{P}_{R_1}} \to X$ is completely continuous.

3) Let $(u_1, v_1) \in \overline{\mathscr{P}_{R_1}}$. Set

$$(u_2, v_2) = (u_1, v_1) + \frac{1}{m} S_2(u_1, v_1) + \left(\frac{L}{5m}, \frac{L}{5m} \right).$$

Note that $S_2^1(u_1, v_1) + \frac{L}{5} \geqslant 0$, $S_2^2(u_1, v_1) + \frac{L}{5} \geqslant 0$ on $[0, \infty) \times \mathbb{R}$. We have $u_2, v_2 \geqslant 0$ on $[0, \infty) \times \mathbb{R}$ and

$$\|(u_2, v_2)\| \leqslant \|(u_1, v_1)\| + \frac{1}{m} \|S_2(u_1, v_1)\| + \frac{L}{5m} \leqslant R_1 + \frac{A}{m} B_1 + \frac{L}{5m} = R_2.$$

Therefore $(u_2, v_2) \in \Omega$ and

$$-\epsilon m(u_2, v_2) = -\epsilon m(u_1, v_1) - \epsilon S_2(u_1, v_1) - \epsilon \left(\frac{L}{10}, \frac{L}{10} \right) - \epsilon \left(\frac{L}{10}, \frac{L}{10} \right)$$

or

$$(I - T_1)(u_2, v_2) = -\epsilon m(u_2, v_2) + \epsilon \left(\frac{L}{10}, \frac{L}{10} \right) = S_3(u_1, v_1).$$

Consequently, $S_3(\overline{\mathscr{P}_{R_1}}) \subset (I - T_1)(\Omega)$.

4) Assume that for any $(w_0, z_0) \in \mathscr{P}^* = \mathscr{P} \setminus \{0\}$ there exist $\lambda \geqslant 0$ and $(u, v) \in \partial \mathscr{P}_r \cap (\Omega + \lambda(w_0, z_0))$ or $(u, v) \in \partial \mathscr{P}_{R_1} \cap (\Omega + \lambda(w_0, z_0))$ such that

$$S_3(u, v) = (I - T_1)((u, v) - \lambda(w_0, z_0)).$$

Then

$$-\epsilon S_2(u, v) - m\epsilon(u, v) - \epsilon \left(\frac{L}{10}, \frac{L}{10} \right) = -m\epsilon((u, v) - \lambda(w_0, z_0)) + \epsilon \left(\frac{L}{10}, \frac{L}{10} \right)$$

or

$$-S_2(u, v) = \lambda m(w_0, z_0) + \left(\frac{L}{5}, \frac{L}{5} \right).$$

Hence,

$$\|S_2(u, v)\| = \left\| \lambda m(w_0, z_0) + \left(\frac{L}{5}, \frac{L}{5} \right) \right\| > \frac{L}{5}.$$

This is a contradiction.

5) Let $\varepsilon_1 = \frac{2}{5m}$. Assume that there exist $(u_1, v_1) \in \partial \mathscr{P}_L$ and $\lambda_1 \geqslant 1 + \varepsilon_1$ such that $\lambda_1(u_1, v_1) \in \overline{\mathscr{P}_{R_1}}$ and

$$S_3(u_1, v_1) = (I - T_1)(\lambda_1(u_1, v_1)). \tag{4.1}$$

Since $(u_1, v_1) \in \partial \mathscr{P}_L$ and $\lambda_1(u_1, v_1) \in \overline{\mathscr{P}_{R_1}}$, it follows that

$$\left(\frac{2}{5m} + 1 \right) L < \lambda_1 L = \lambda_1 \|(u_1, v_1)\| \leqslant R_1.$$

Moreover,

$$-\epsilon S_2(u_1, v_1) - m\epsilon(u_1, v_1) - \epsilon \left(\frac{L}{10}, \frac{L}{10} \right) = -\lambda_1 m\epsilon(u_1, v_1) + \epsilon \left(\frac{L}{10}, \frac{L}{10} \right)$$

or

$$S_2(u_1, v_1) + \left(\frac{L}{5}, \frac{L}{5} \right) = (\lambda_1 - 1) m(u_1, v_1).$$

From here,

$$2\frac{L}{5} \geqslant \left\| S_2(u_1,v_1) + \left(\frac{L}{5}, \frac{L}{5}\right) \right\| = (\lambda_1 - 1)m\|(u_1,v_1)\| = (\lambda_1 - 1)mL$$

and

$$\frac{2}{5m} + 1 \geqslant \lambda_1,$$

which is a contradiction.

Therefore all conditions of Theorem 4.2 hold. Hence, the IVP (1.1) has at least two solutions (u_1, v_1) and (u_2, v_2) so that

$$\|(u_1,v_1)\| = L < \|(u_2,v_2)\| \leqslant R_1$$

or

$$r \leqslant \|(u_1,v_1)\| < L < \|(u_2,v_2)\| \leqslant R_1. \ \triangleright$$

5. An Example

Below, we will illustrate our main results. Let

$$h(s) = \log\frac{1+s^{11}\sqrt{2}+s^{22}}{1-s^{11}\sqrt{2}+s^{22}}, \quad l(s) = \arctan\frac{s^{11}\sqrt{2}}{1-s^{22}}, \quad s \in \mathbb{R},\ s \neq \pm 1.$$

Then

$$h'(s) = \frac{22\sqrt{2}s^{10}(1-s^{22})}{(1-s^{11}\sqrt{2}+s^{22})(1+s^{11}\sqrt{2}+s^{22})}, \quad l'(s) = \frac{11\sqrt{2}s^{10}(1+s^{22})}{1+s^{44}}, \quad s \in \mathbb{R},\ s \neq \pm 1.$$

Therefore

$$-\infty < \lim_{s\to\pm\infty}\left(1+s+s^2+s^3+s^4+s^5+s^6\right)h(s) < \infty,$$

$$-\infty < \lim_{s\to\pm\infty}\left(1+s+s^2+s^3+s^4+s^5+s^6\right)l(s) < \infty.$$

Hence, there exists a positive constant C_1 so that

$$\left(1+s+s^2+s^3+s^4+s^5+s^6\right)\left(\frac{1}{44\sqrt{2}}\log\frac{1+s^{11}\sqrt{2}+s^{22}}{1-s^{11}\sqrt{2}+s^{22}} + \frac{1}{22\sqrt{2}}\arctan\frac{s^{11}\sqrt{2}}{1-s^{22}}\right) \leqslant C_1,$$

$s \in \mathbb{R}$. Note that $\lim\limits_{s\to\pm1} l(s) = \frac{\pi}{2}$ and by [23, p. 707, Integral 79], we have

$$\int \frac{dz}{1+z^4} = \frac{1}{4\sqrt{2}}\log\frac{1+z\sqrt{2}+z^2}{1-z\sqrt{2}+z^2} + \frac{1}{2\sqrt{2}}\arctan\frac{z\sqrt{2}}{1-z^2}.$$

Let

$$Q(s) = \frac{s^{10}}{(1+s^{44})(1+s+s^2)^2}, \quad s \in \mathbb{R},$$

and

$$g_1(t,x) = Q(t)Q(x), \quad t \in [0,\infty),\ x \in \mathbb{R}.$$

Then there exists a constant $C_2 > 0$ such that

$$8(1+t)\left(1+t+t^2\right)(1+|x|)\left(1+|x|+|x|^2\right)\int_0^t\left|\int_0^x g_1(t_1,x_1)\,dx_1\right|dt_1 \leqslant C_2, \quad (t,x)\in[0,\infty)\times\mathbb{R}.$$

Let

$$g(t,x) = \frac{A}{C_2}g_1(t,x), \quad (t,x)\in[0,\infty)\times\mathbb{R}.$$

Then

$$8(1+t)\left(1+t+t^2\right)(1+|x|)\left(1+|x|+|x|^2\right)\int_0^t\left|\int_0^x g(t_1,x_1)\,dx_1\right|dt_1 \leqslant A, \quad (t,x)\in[0,\infty)\times\mathbb{R},$$

i. e., (H3) holds. Now, consider the initial value problem

$$\begin{gathered}\partial_t u + \partial_x(uv) = 0, \quad t>0,\ x\in\mathbb{R},\\ \partial_t(uv) + \partial_x\left(uv^2+u^2\right) + \frac{4xu}{1+x^2} = 0, \quad t>0,\ x\in\mathbb{R},\\ u(0,x) = v(0,x) = \frac{3}{1+x^{16}}, \quad x\in\mathbb{R},\end{gathered} \tag{5.1}$$

so that (H1) and (H2) hold, with $B = 10$, for example. Take $B = 10$ and $A = \frac{1}{10^4}$. Then

$$B_1 = \max\left\{20, 2\cdot 10^2\right\} = 2\cdot 10^2$$

and

$$AB_1 = \frac{1}{10^4}\cdot 2\cdot 10^2 < B.$$

So, Condition (H4) is fulfilled. Thus, the conditions (H1), (H2), (H3) and (H4) are satisfied. Hence, by Theorem 4.1, it follows that Problem 6 has at least one nonnegative solution $(u,v)\in\mathscr{C}^1([0,\infty)\times\mathbb{R})\times\mathscr{C}^1([0,\infty)\times\mathbb{R})$.

In the sequel, take

$$R_1 = B = 10, \quad L = 5, \quad r = 4, \quad m = 10^{50}, \quad A = \epsilon = \frac{1}{10^4}.$$

Clearly,

$$r < L < R_1 \leqslant B, \quad \epsilon > 0, \quad R_1 > \left(\frac{2}{5m}+1\right)L, \quad AB_1 < \frac{L}{5},$$

i. e., (H5) holds. Thus, the conditions (H1), (H2), (H3) and (H5) are satisfied. Hence, by Theorem 4.2, it follows that the initial value Problem 6 has at least two nonnegative solutions $(u_1,v_1),(u_2,v_2)\in\mathscr{C}^1([0,\infty)\times\mathbb{R})\times\mathscr{C}^1([0,\infty)\times\mathbb{R})$.

Acknowledgements: The authors R. Azib, A. Kheloufi and K. Mebarki acknowledge support of “Direction Générale de la Recherche Scientifique et du Développement Technologique (DGRSDT)”, MESRS, Algeria.

References

1. *Saint-Venant, A. J. C. Théorie du mouvement non-permanent des eaux, avec application aux crues des rivières et a l'introduction des marées dans leur lit,* Comptes rendus de l'Académie des Sciences de Paris, 1871, vol. 73, pp. 147–154.
2. *Baklanovskaya, V. F. and Chechel', I. I.* A Numerical Method for Solving St. Venant's Equations (Chamber Model), *USSR Computational Mathematics and Mathematical Physics*, 1976, vol. 16, no. 5, pp. 125–141. DOI: 10.1016/0041-5553(76)90143-9.
3. *Alcrudo, F. and Garcia-Navarro, P.* A High-Resolution Godunov-Type Scheme in Finite Volumes for the 2D Shallow-Water Equations, *International Journal for Numerical Methods in Fluids*, 1993, vol. 16, no. 6, pp. 489–505. DOI: 10.1002/fld.1650160604.
4. *Garcia-Navarro, P., Alcrudo, F. and Savirón, J. M.* 1-D Open-Channel Flow Simulation Using TVD-McCormack Scheme, *Journal of Hydraulic Engineering*, 1992, vol. 118, no. 10, pp. 1359–1372. DOI: 10.1061/(ASCE)0733-9429(1992)118:10(1359).
5. *Lien, F.-S.* A Pressure-Based Unstructured Grid Method for All-Speed Flows, *International Journal for Numerical Methods in Fluids*, 2000, vol. 33, no. 3, pp. 355–374. DOI: 10.1002/1097-0363(20000615)33:3<355::AID-FLD12>3.0.CO;2-X.
6. *Nguyen, D. K., Shi, Y.-E., Wang, S. S. and Nguyen, T. H.* 2D Shallow-Water Model Using Unstructured Finite-Volumes Methods, *Journal of Hydraulic Engineering*, 2006, vol. 132, no. 3, pp. 258–269. DOI: 10.1061/(ASCE)0733-9429(2006)132:3(258).
7. *Zarmehi, F., Tavakoli, A. and Rahimpour, M.* On Numerical Stabilization in the Solution of Saint-Venant Equations Using the Finite Element Method, *Computers and Mathematics with Applications*, 2011, vol. 62, no. 4, pp. 1957–1968. DOI: 10.1016/j.camwa.2011.06.039.
8. *Baklanovskaya, V. F., Paltsev, B. V. and Chechel, I. I.* Boundary Value Problems for the St. Venant System of Equations on a Plane, *USSR Computational Mathematics and Mathematical Physics*, 1979, vol. 19, no. 3, pp. 157–174. DOI: 10.1016/0041-5553(79)90137-X.
9. *Ton, B. A.* Existence and Uniqueness of a Classical Solution of an Initial Boundary Value Problem of the Theory of Shallow Waters, *SIAM Journal on Mathematical Analysis*, 1981, vol. 12, no. 2, pp. 229–241. DOI: 10.1137/0512022.
10. *Kloeden, P. E.* Global Existence of Classical Solutions in the Dissipative Shallow Water Equations, *SIAM Journal on Mathematical Analysis*, 1985, vol. 16, no. 2, pp. 301–315. DOI: 10.1137/0516022.
11. *Lai, H. V., Long, L. M. and Trung, M. D.* On Uniqueness of a Classical Solution of the System of Non-Linear 1-D Saint Venant Equations, *Vietnam Journal of Mechanics, NCST of Vietnam*, 1999, vol. 21, no. 4, pp. 231–238. DOI: 10.15625/0866-7136/10004.
12. *Muzaev, I. D. and Tuaeva, Zh. D.* Two Methods for Solving an Initial-Boundary Value Problem for a System of Saint-Venant Equations, *Vladikavkaz. Math. J.*, 1999, vol. 1, no. 1, pp. 43–47 (in Russian).
13. *Li, Y., Pan, R. and Zhu, S.* On Classical Solutions to 2D Shallow Water Equations with Degenerate Viscosities, *Journal of Mathematical Fluid Mechanics*, 2017, vol. 19, no. 1, pp. 151–190. DOI: 10.1007/s00021-016-0276-3.
14. *Cheng, B. and Tadmor, E.* Long-Time Existence of Smooth Solutions for the Rapidly Rotating Shallow-Water and Euler Equations, *SIAM Journal on Mathematical Analysis*, 2008, vol. 39, no. 5, pp. 1668–1685. DOI: 10.1137/070693643.
15. *Cheng, B. and Xie, C.* On the Classical Solutions of Two Dimensional Inviscid Rotating Shallow Water System, *Journal of Differential Equations*, 2011, vol. 250, no. 2, pp. 690–709. DOI: 10.1016/j.jde.2010.09.017.
16. *Duan, B., Luo, Z. and Zheng, Y.* Local Existence of Classical Solutions to Shallow Water Equations with Cauchy-Data Containing Vacuum, *SIAM Journal on Mathematical Analysis*, 2012, vol. 44, no. 2, pp. 541–567. DOI: 10.1137/100817887.
17. *Liu, J.-G., Pego, R. L. and Pu, Y.* Well-Posedness and Derivative Blow-Up for a Dispersionless Regularized Shallow Water System, *Nonlinearity*, 2019, vol. 32, no. 11, pp. 4346–4376. DOI: 10.1088/1361-6544/ab2cf1.
18. *Alekseenko, S. N., Dontsova, M. V. and Pelinovsky, D. E.* Global Solutions to the Shallow Water System with a Method of an Additional Argument, *Applicable Analysis*, 2017, vol. 96, no. 9, pp. 1444–1465. DOI: 10.1080/00036811.2016.1208817.
19. *Yang, Y.* Global Classical Solutions to Two-Dimensional Chemotaxis-Shallow Water System, *Discrete and Continuous Dynamical Systems — B*, 2021, vol. 26, no. 5, pp. 2625–2643. DOI: 10.3934/dcdsb.2020198.
20. *Licht, C. C., Ha, T. T. and Vu, Q. P.* On Some Linearized Problems of Shallow Water Flows, *Differential and Integral Equations*, 2009, vol. 22, no. 3–4, pp. 275–283.

21. *Georgiev, S. G. and Zennir, K.* Existence of Solutions for a Class of Nonlinear Impulsive Wave Equations, *Ricerche di Matematica*, 2022, vol. 71, no. 1, pp. 211–225. DOI: 10.1007/s11587-021-00649-2.
22. *Djebali, S. and Mebarki, K.* Fixed Point Index Theory for Perturbation of Expansive Mappings by k-Set Contractions, *Topological Methods Nonlinear Analysis*, 2019, vol. 54, no. 2A, pp. 613–640. DOI: 10.12775/TMNA.2019.055.
23. *Polyanin, A. and Manzhirov, A. Handbook of Integral Equations,* CRC Press, 1998, pp. 796.

Received September 14, 2021

Riyadh Azib
Bejaia University, Laboratory of Applied Mathematics,
Faculty of Exact Sciences,
Bejaia 06000, Algeria,
Ph.D student of the Department of Mathematics
E-mail: riyadh.azib@univ-bejaia.dz, azibriyadh@gmail.com
https://orcid.org/0000-0001-8162-8315

Svetlin Georgiev
Sofia University "St. Kliment Ohridski",
Faculty of Mathematics and Informatics,
15 Tzar Osvoboditel Blvd., Sofia 1504, Bulgaria,
Professor of the Department of Differential Equations
E-mail: svetlingeorgiev1@gmail.com
https://orcid.org/0000-0001-8015-4226

Arezki Kheloufi
Bejaia University, Laboratory of Applied Mathematics,
Faculty of Exact Sciences,
Bejaia 06000, Algeria,
Professor of the Department of Mathematics
E-mail: arezki.kheloufi@univ-bejaia.dz, arezkinet2000@yahoo.fr
https://orcid.org/0000-0001-5584-1454

Karima Mebarki
Bejaia University, Laboratory of Applied Mathematics,
Faculty of Exact Sciences,
Bejaia 06000, Algeria,
Professor of the Department of Mathematics
E-mail: karima.mebarki@univ-bejaia.dz, mebarqi_karima@hotmail.fr
https://orcid.org/0000-0002-6679-5059

СУЩЕСТВОВАНИЕ ГЛОБАЛЬНЫХ КЛАССИЧЕСКИХ РЕШЕНИЙ ДЛЯ УРАВНЕНИЙ СЕН-ВЕНАНА

Азиб Р.[1], Георгиев С.[2], Хелоуфи А.[1], Мебарки К.[1]

[1] Университет Беджая, Алжир, 06000, Беджая;
[2] Софийский университет имени святого Климента Охридского,
Болгария, 1504, София, Бульвар Царя-Освободителя, 15

E-mail: riyadh.azib@univ-bejaia.dz, azibriyadh@gmail.com,
svetlingeorgiev1@gmail.com, arezki.kheloufi@univ-bejaia.dz, arezkinet2000@yahoo.fr,
karima.mebarki@univ-bejaia.dz, mebarqi_karima@hotmail.fr

Аннотация. В настоящее время исследования существования глобальных классических решений нелинейных эволюционных уравнений являются предметом активных математических исследований. В этой статье нас интересует классическая система уравнений мелкой воды, описывающая длинные поверхностные волны в жидкости переменной глубины. Эта система была предложена в 1871 г. Адмаром Жан-Клодом Барром де Сен-Венаном. А именно, мы исследуем начальную задачу для одномерных уравнений Сен-Венана. Нас особенно интересует вопрос, при каких достаточных условиях должны верифицироваться начальные данные и топография дна, чтобы рассматриваемая система имела глобальные классические решения. Для доказательства наших основных результатов мы используем новый топологический подход, основанный на абстрактной теории суммы двух операторов в банаховых пространствах с фиксированной точкой. Эта основная и новая идея приводит к глобальным теоремам существования для многих интересных уравнений математической физики.

Ключевые слова: уравнения Сен-Венана, классическое решение, неподвижная точка, начальная задача.

AMS Subject Classification: 35Q35, 35A09, 35E15.

Образец цитирования: *Azib, R., Georgiev, S., Kheloufi, A. and Mebarki, K.* Existence of Global Classical Solutions for the Saint-Venant Equations // Владикавк. мат. журн.—2022.—Т. 24, № 3.—С. 21–36 (in English). DOI: 10.46698/x4972-4013-9236-n.