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Blow-Up in a Modified Constantin-Lax-Majda Model for the Vorticity Equation

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Dedicated to Prof. L. von Wolfersdorf on the occasion of his 65th birthday

Abstract. We propose a one-dimensional model for the vorticity equation involving viscosity. Complex methods are utilzed in order to study finite time blow-up of the solutions. In particular, it is shown that the blow-up time depends monotoneously on the viscosity.

Keywords: *Vorticity equation, Hilbert transform, blow-up*

AMS subject classification: 76 C 05, 35 Q 35

1. Introduction

Physical arguments (e.g. Frisch [7: p. 115]) and numerical computations (e.g. Grauer and Sideris [8]) strongly suggest that finite time singularities develop in three-dimensional inviscid incompressible flow. The equations governing such a flow are the Euler equations

$$u_t + (u \cdot \nabla)u + \nabla p = 0 \tag{1}$$

$$\nabla \cdot u = 0 \tag{2}$$

for the velocity $u = u(x,t)$ and the pressure $p = p(x,t)$ on $\mathbb{R}^3 \times \mathbb{R}_+$. A basic question is if smooth solutions of initial value problems for (1) - (2) do exist for all time. Beale, Kato and Majda [1] have proved the following. Suppose the initial velocity field

$$u(x,0) = u_0(x) \tag{3}$$

is smooth. Then there exists a global smooth solution if and only if the vorticity $\omega = \nabla \times u$ satisfies $\int_0^T ||\omega(\cdot,t)||_\infty \, dt < \infty$ for every $T > 0$. Further, they showed that if a solution which is initially smooth loses its regularity at some later time, then the maximum vorticity necessarily grows without bound as the critical time approaches. Thus the formation of singularities in Euler equations depends on vorticity production or vortex stretching.

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The interest in these possible singularities, as pointed out by Caflisch [2] in 1993, is of physical, numerical and mathematical nature: physical because singularity formation may signify the onset of turbulence and may be a primary mechanism of energy transfer from large to small scales, numerical because special methods to solve Euler equations would be required for tackling this singularity formation, mathematical because singularities in Euler equations would prevent an establishment of global existence theorems for equations (1) and (2).

The need to understand the precise mechanism of formation of singularities in finite time has led to some model problems that mimic the Euler equations. These models should not only be simpler than (1) and (2) but also possess some of their important features.

In this direction Constantin et al. [4] proposed a very simple model for the vorticity equation. We shall briefly explain the motivation for their proposal. With $\omega := \nabla \times u$, the vorticity, the Euler equations (1) and (2) can be written in the form

$$\omega_t + (u \cdot \nabla)\,\omega = (\omega \cdot \nabla)\,u. \tag{4}$$

The initial condition $u(x,0) = u_0(x)$ is transformed into

$$\omega(x,0) = \omega_0(x) \tag{5}$$

where $\omega_0 = \nabla \times u_0$. Now u can be written in terms of ω by the Biot-Savart formula

$$u(x,t) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} \times \omega(y,t)\,dy. \tag{6}$$

By substituting (6) into (4) the latter equation is reduced to

$$\omega_t + (u \cdot \nabla)\,\omega = (D\omega)\omega \tag{7}$$

where $D\omega$ is the symmetric part of the matrix ∇u expressed in terms of ω. The operator $\omega \mapsto D\omega$ is a strongly singular integral operator. The explicit formula for D is not of interest here, but some properties are worth noting.

In two space dimensions, $(D\omega)\,\omega = 0$ which implies conservation of vorticity. In three dimensions, D is a convolution operator with a (matrix) kernel homogeneous of order -3 and vanishing mean value on the unit sphere. Constantin et al. [4] made the remarkable observation that in one space dimension there is only one such operator, the Hilbert transform

$$H\omega(x) = \frac{1}{\pi}\,\text{p.v.}\int_{-\infty}^{+\infty} \frac{\omega(y)}{x-y}\,dy.$$

By replacing the convective derivative $\omega_t + (u\cdot\nabla)\,\omega$ by the partial derivative ω_t and $D\omega$ by the Hilbert transform $H\omega$, Constantin et al. [4] arrive at a simple one-dimensional analogue of (4) and (5):

$$\omega_t = \omega H\omega \tag{8}$$

$$\omega(x,0) = \omega_0(x). \tag{9}$$

In this model the "velocity" is determined from the vorticity by

$$u(x,t) = \int_{-\infty}^{x} \omega(y,t)\,dy. \tag{10}$$

Problem (8) - (9) is explicity solvable and its solution is given by

$$\omega(x,t) = \frac{4\omega_0(x)}{(2 - t\,H\omega_0(x))^2 + t^2\,\omega_0^2(x)}. \tag{11}$$

From this formula it is clear that the solution ω blows up in a finite time T_0 if and only if there exists an x_0 such that $\omega_0(x_0) = 0$ and $H\omega_0(x_0) > 0$. Constantin et al. [4] also showed that if x_0 is a simple zero of ω_0, then for $1 \leq p < \infty$

$$\lim_{t\to T_0} \int_{-\infty}^{+\infty} |\omega(x,t)|^p dx = \infty \qquad \text{and} \qquad \lim_{t\to T_0} \int_{-\infty}^{+\infty} |u(x,t)|^p dx < M^p < \infty.$$

Thus the model vorticity equation (8) seemed to possess the most important feature of (4): finite time blow-up of vorticity with velocity remaining bounded. Now (8) - (9) with its explicit solution (11) is a challenging test problem for numerical methods designed to detect blow-up; this has been demonstrated by Stewart and Geveci [12] in 1992.

The first attempt to extend the model problem (8) - (9) to include viscous effects was made by Schochet [11], who considered the problem

$$\omega_t = \omega H\omega + \varepsilon\,\omega_{xx} \quad \text{on} \quad \mathbb{R}\times\mathbb{R}_+ \tag{12}$$

$$\omega(x,0) = \omega_0(x). \tag{13}$$

The solution to problem (12) - (13) was explicity constructed by Schochet, who found that it blows up at time T_ε with

$$T_\varepsilon < T_0, \tag{14}$$

where T_0 is the blow-up time for $\varepsilon \equiv 0$. In other words, adding diffusion makes the solution less regular! Clearly this is unsatisfactory, especially in view of the result by Constantian [3], which says that if the solution to the Euler equation is smooth, then the solutions to the slightly viscous Navier-Stokes equations with the same initial data are also smooth. Hence the simple model (12) lost most of its interest.

Improvements were suggested by De Gregorio [5, 6] who kept the convective derivative on the left-hand side and studied the equation $\omega_t + u\omega_x = \omega\,H\omega + \nu\omega_{xx}$ with viscosity $\nu \geq 0$. Note that De Gregorio defines the velocity u as a primitive of $H\omega$ and not of ω.

In the present paper we return to the Constantin et al. model (8) and introduce an alternative additive (non-local) diffusion term which results in an one-dimensional problem with an explicit solution. In contrast to Schochet's model, the inequality in (14) is now reversed, and thus the drawback mentioned above is removed.

2. A viscous model with a non-local diffusion term

In this section we derive heuristically a proposal for including viscous effects to (8). In connection with investigations of water wave phenomenons like sharp crests and breaking of waves Whitham [14] studied the problem

$$u_t = uu_x \quad \text{on } \mathbb{R} \times \mathbb{R}_+ \tag{15}$$

$$u(x,0) = u_0(x). \tag{16}$$

It is well known that solutions to problem (15) - (16) lose regularity in finite time no matter how smooth u_0 is. On the other hand, if we add viscosity to (15),

$$u_t = uu_x + \nu u_{xx}, \tag{17}$$

then a global smooth solution exists for all time. Now Whitham asked the question if there exists a viscosity term which, when added to (15), influences the solution so that it loses regularity in finite time. He conjectured that

$$u_t = uu_x - K * u_x \tag{18}$$

with the convolution kernel K having the Fourier transform $\widehat{K}(\xi) = \sqrt{\xi \tan h\, \xi}$ has the desired property. This conjecture has been completely settled by Naumkin and Shishmarëv [10] in 1991.

In a similar vein we ask the analogous question for the Constantin-Lax-Majda model: What is an appropriate viscosity term that when added to (8) will make the solution blow up at a finite time $T_\varepsilon > T_0$? Because of (14) it cannot be $\varepsilon\, u_{xx}$.

Constantin et al. [4] have shown that the blow up of (11) is different from the blow up of u_x where u is a solution to problem (15) - (16). Note that u_x satisfies along the characteristics

$$(u_x)_t = (u_x)^2 \tag{19}$$

$$u_x(x,0) = (u_0(x))_x \tag{20}$$

and hence it blows up in finite time. In other words, the equation $u_t = uu_x$ is not a good model for the breakdown of smooth solutions to (1) and (2) but $\omega_t = \omega H\omega$ is a better model. Arguing analogously one feels that $-\varepsilon H u_x$ would be a reasonable "viscosity" compared to εu_{xx}. So we propose

$$\omega_t = \omega H\omega - \varepsilon H\omega_x \quad \text{on } \mathbb{R} \times \mathbb{R}_+ \tag{21}$$

$$\omega(x,0) = \omega_0(x) \tag{22}$$

as the viscous analogue of (8) - (9). Note that $-\varepsilon H\omega_x$ is indeed a dissipative term as can be checked by solving the linear part of (21) using Fourier transform. Such a dissipative term has also been considered by Matsuno [9] in 1992.

3. Global existence versus finite time blow-up

In the following we shall consider the periodic version of (21) - (22). More precisely, we assume that the velocity is 2π-periodic in x, which implies periodicity of the initial function ω_0 and the solution ω (with respect to the space-variable x), as well as

$$\int_0^{2\pi} \omega_0(x)\,dx = 0 \qquad \text{and} \qquad \int_0^{2\pi} \omega(x,t)\,dx = 0. \tag{23}$$

In order to determine the exact solution we introduce the complex-valued functions

$$w(x,t) = H\omega(x,t) + i\,\omega(x,t) \qquad \text{and} \qquad w_0(x) = H\omega_0(x) + i\,\omega_0(x)$$

where H acts with respect to x. The functions w_0 and $w(\cdot,t)$ extend from the real axis to (periodic) bounded holomorphic functions in the lower half-plane $\mathbb{C}_-$ and tend uniformly to zero as $\operatorname{Im} z \to -\infty$. Using the identities (recall (23))

$$\begin{aligned} 2\,H(\omega H\omega) &= (H\omega)^2 - \omega^2 \\ H^2\omega &= -\omega \\ H\omega_x &= (H\omega)_x \end{aligned}$$

a straightforward calculation shows that problem (21) - (22) is transformed to the initial problem

$$w_t + i\,\varepsilon w_x = \tfrac{1}{2}w^2 \quad \text{on} \quad \mathbb{R}\times\mathbb{R}_+ \tag{24}$$

$$w(x,0) = w_0(x). \tag{25}$$

Lemma 1. *The unique solution of problem* (24) – (25) *is given by*

$$w(x,t) = \frac{2\,w_0(x - i\,\varepsilon t)}{2 - t\,w_0(x - i\,\varepsilon t)}. \tag{26}$$

Proof. Along the characteristics (24) is transformed into an ordinary differential equation. With $W(t) = w(t, x + i\,\varepsilon t)$ we get $W'(t) = \frac{1}{2}W^2$. This equation has the solution $W(t) = \frac{2\,W(0)}{2 - t\,W(0)} = \frac{2\,w(0,x)}{2 - t\,w(0,x)}$, which gives the desired result ∎

The last lemma provides us with a simple criterion for blow-up.

Lemma 2. *The solution to problem* (21) – (22) *blows up at* (x_0, t_0) *if and only if* $\omega_0(x - i\,\varepsilon t_0) = \frac{2}{t_0}$.

Proof. The solution to problem (21) - (22) is given by $\omega(x,t) = \operatorname{Im} w(x,t)$. With $z := x - i\,\varepsilon t$ we get from (26)

$$\omega(x,t) = \operatorname{Im} \frac{2w_0(z)}{2 - t\,w_0(z)}.$$

The function w_0 is holomorphic in the lower half plane and hence the solution cannot blow up if the denominator $2 - t\,w_0(z)$ does not vanish. Conversely, let $z_0 = x_0 - i\,\varepsilon t_0$ be such that $w_0(z_0) = \frac{2}{t_0}$. Since w_0 is holomorphic, by Taylor series $w_0(z) = w_0(z_0) + (z - z_0)^m g(z)$ where g is a holomorphic function with $g(z_0) = w_0^{(m)}(z_0)/m! \neq 0$ and hence

$$\omega(x,t) = \operatorname{Im} \frac{4 + 2t_0(z - z_0)^m g(z)}{2(t_0 - t) - t\,t_0\,(z - z_0)^m g(z)}.$$

If $g(z_0) \notin \mathbb{R}$ then for $t = t_0$ and $x \to x_0$,

$$\omega(x,t_0) \sim -\frac{4}{(x - x_0)^m t_0^2} \cdot \operatorname{Im} \frac{1}{g(z_0)}.$$

If $g(z_0) \in \mathbb{R} \setminus \{0\}$, then for $x \to x_0$ and $t = t_0 - \frac{1}{2}\,t_0^2\,g(z_0)(x - x_0)^m$,

$$\begin{aligned}\omega(x,t) &\sim \operatorname{Im} \frac{4}{t_0^2\,g(z_0)\,\big((x - x_0)^m - (z - z_0)^m\big)} \\ &\sim \frac{8}{t_0^4\,g(z_0)^2\,m\,\varepsilon\,(x - x_0)^{2m-1}}.\end{aligned}$$

Therefore, in both cases, the solution $w(x,t)$ is unbounded in any neighborhood of (x_0, t_0) ∎

The following technical lemma will serve to estimate the blow-up time.

Lemma 3. *Let ω_0 be a 2π-periodic Hölder-continuous function with*

$$\int_0^{2\pi} \omega_0(x)\,dx = 0. \tag{27}$$

Then w_0 satisfies the estimate

$$|w_0(z)| \le M\,e^{-|\operatorname{Im} z|} \qquad (z \in \mathbb{C}_-)$$

where $M = \max_{x \in \mathbb{R}} |\omega_0(x) + i\,H\omega_0(x)|$.

Proof. The function $\zeta = f(z) = \exp(-i\,z)$ maps the half-strip $\{z \in \mathbb{C}_- : 0 \le \operatorname{Re} z < 2\pi\}$ onto the punctured unit disk $\dot{\mathbb{D}} = \{\zeta \in \mathbb{C} : 0 < |\zeta| < 1\}$. The transplanted function $\widetilde{w}_0(\zeta) = w_0(f^{-1}(\zeta))$ is holomorphic in $\dot{\mathbb{D}}$ and has a continuous extension onto the unit circle. Since the mean value along the boundary vanishes we have $\lim_{\zeta \to 0} \widetilde{w}_0(\zeta) = 0$. Consequently, by Schwarz' lemma,

$$|\widetilde{w}(\zeta)| \le \max\,|w_0| \cdot |\zeta| \equiv M\,|\zeta|,$$

which together with $|\zeta| = e^{-|\operatorname{Im} z|}$ yields the assertion ∎

We denote by $T_\varepsilon(\omega_0)$ the time of the first blow up,

$$T_\varepsilon(\omega_0) = \inf\left\{t > 0 : w_0(x - i\,\varepsilon t) = \frac{2}{t} \text{ for some } x \in \mathbb{R}\right\}.$$

If the set on the right-hand side is empty, $T_\varepsilon(\omega_0) := +\infty$.

In all what follows we assume that the initial function ω_0 is not a constant. In order to study the dependence of the blow-up time $T_\varepsilon(\omega_0)$ on ε and ω_0 we consider the images of the closed lower half-planes

$$\mathbb{C}_y := \{z \in \mathbb{C}_- : \operatorname{Im} z \le -y\}, \qquad (y \ge 0)$$

under the mapping w_0. More precisely, $R_y := w_0(\mathbb{C}_y) \cup \{0\}$.

Lemma 4. *Let ω_0 satisfy the assumptions of Lemma 3. Then the origin lies in the interior of all sets R_y, and R_y contracts to 0 as $y \to +\infty$. The sets R_y form a strictly nested family,*

$$R_{y_2} \subset \operatorname{int} R_{y_1} \text{ if } y_2 > y_1 \ge 0. \tag{28}$$

The blow-up time $T_\varepsilon(\omega_0)$ is characterized by

$$T_\varepsilon(\omega_0) = \inf\{t > 0 \colon\ 2/t \in R_{\varepsilon t}\}. \tag{29}$$

Proof. First of all we note that R_y is the image of the closed disk $D_y := \{z \in \mathbb{C} \colon |z| \le \exp(-y)\}$ under the mapping $\widetilde{w}_0$ (see proof of Lemma 3). The first assertion follows from $\widetilde{w}_0(0) = 0$ and Lemma 3.

The second assertion is a consequence of the open mapping principle for holomorphic functions.

In order to prove the third assertion, we recall that the solution blows up at time t if and only if $2/t = w_0(x - i\,\varepsilon t)$ for some $x \in \mathbb{R}$.

Since the R_y are nested and R_0 is bounded, the point $2/t$ lies outside $R_{\varepsilon t}$ for sufficiently small t. More precisely, there is no blow-up for all t with $t < T := \inf\{t \in \mathbb{R}_+ \colon 2/t \in R_{\varepsilon t}\}$.

On the other hand, a continuity argument shows that $2/t \in R_{\varepsilon t}$ if $t = T$. It follows that $2/t = w_0(x - i\,y)$ for some x and some $y \ge \varepsilon t$. Now $y > \varepsilon t$ would imply that $2/t \in \operatorname{int} R_{\varepsilon t}$ and hence $2/t \in R_{\varepsilon t}$ for some $t < T$, which is impossible by the definition of T. Consequently $y = \varepsilon t$ ∎

It has already been mentioned that $T_0(\omega_0)$ is finite if and only if there exists an x_0 such that $\omega_0(x_0) = 0$ and $H\omega_0(x_0) > 0$. The next result shows that the solution necessarily blows up for $\varepsilon = 0$ if the mean value of ω_0 vanishes, which is always the case for periodic velocity.

Theorem 1. *Let ω_0 be a non-constant Hölder-continuous 2π-periodic function. If $\int_0^{2\pi} \omega_0(x)\,dx = 0$ then $T_0(\omega_0)$ is finite.*

Proof. Since R_0 is a compact set, the point $\frac{2}{t}$ lies outside R_0 if t is sufficiently small. The origin is an interior point of R_0 and hence $2/t$ belongs to R_0 if t is large. Lemma 4 proves the assertion ∎

The next theorem shows that the viscous term increases the blow-up time and even prevents blow-up if ε is sufficiently large.

Theorem 2. *Let ω_0 satisfy the assumptions of Theorem 1.*

(i) *The blow-up time $T_\varepsilon(\omega_0)$ is a monotoneously increasing function of ε. In particular, if $0 < \varepsilon \le \delta$, then $T_0(\omega_0) < T_\varepsilon(\omega_0) \le T_\delta(\omega_0)$.*

(ii) *For each initial function ω_0 there exists a positive ε_* such that $T_\varepsilon(\omega_0) = +\infty$ if $\varepsilon > \varepsilon_*$.*

(iii) *For all $\varepsilon > 0$ and $C \in \mathbb{R}$ there exists a constant $T_* = T_*(\varepsilon, C)$ such that for all ω_0 with Hölder norm $\|\omega_0\|_\alpha \le C$ either $T_\varepsilon(\omega_0) \le T_*$ or $T_\varepsilon(\omega_0) = +\infty$.*

Moral. What survived sufficiently long will persist forever.

Proof of Theorem 2. (i) If $\varepsilon < \delta$, then $R_{\delta t} \subset \operatorname{int} R_{\varepsilon t}$ for all t and hence the point $\frac{2}{t}$ (which lies outside $R_{\varepsilon t}$ for small t) meets the domain $R_{\varepsilon t}$ at an earlier time than $R_{\delta t}$.

(ii) According to Lemma 3 the intersection of $R_{\varepsilon t}$ with the real axis is contained in the interval $\{x \in \mathbb{R} : |x| < M \exp(-\varepsilon t)\}$, and hence the solution cannot blow up if $\frac{2}{t} > M \exp(-\varepsilon t)$ for all $t > 0$. The latter condition is satisfied for all sufficiently large ε.

(iii) By what was said above, the blow-up time (if it is finite) is subject to $\frac{2}{T_\varepsilon} \le M \exp(-\varepsilon T_\varepsilon)$ which gives an upper bound for T_ε ∎

Example. For the initial function $\omega_0(x) = \cos x$ we get $w_0(x) = H\omega_0(x) + i\,\omega_0(x) = \sin x + i \cos x$ which has the analytical extension $w_0(z) = i \exp(-i\,z)$ onto $\mathbb{C}_-$. Thus the solution ω is the imaginary part of

$$w(x,t) = \frac{2\,w_0(x - i\,\varepsilon t)}{2 - t\,w_0(x - i\,\varepsilon t)} = -\frac{2}{t + 2\,i\,\exp(i\,x)\exp(\varepsilon t)}.$$

The blow-up time T_ε is determined by the condition $i\,\exp(i\,x)\exp(\varepsilon T_\varepsilon) = -\frac{T_\varepsilon}{2}$, which splits into

$$\exp(\varepsilon T_\varepsilon) = \frac{T_\varepsilon}{2} \qquad \text{and} \qquad \exp(i\,x) = i.$$

The solution blows up if and only if $0 \leq \varepsilon \leq \frac{1}{2}\varepsilon$, the blow-up time satisfies $2 \leq T_\varepsilon \leq 2\varepsilon$.

The figures show the behaviour of the solution for $\varepsilon = 0.21 > \frac{1}{2}\varepsilon$ (left, no blow-up) and $\varepsilon = 0.17 < \frac{1}{2}\varepsilon$ (right, blow-up at $T_\varepsilon \approx 3.845$).

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